The pairing on K-groups in the fields of normalization by rank n

S.V. Vostokov

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92-97

§ 1. Introduction

1. The explicit formula for Hilbert's symbol which was obtained in the article [V1] gave an impulse to obtaining explicit formulae both in various fields and for various objects (formal groups, for example, see [V3], [V4], [BV]). As the local class fields theory developed, the scope of local fields was extended for which the methods of [V1] are applicable.

The present article is devoted to complete fields to discrete normalization by rank n, which have the characteristic 0. In those fields we construct in explicit form the pairing on topological K-groups, for which the sum of dimensions is equal to $n+1$. In particular, the explicit formula for Hilbert pairing in the fields of normalization by rank n also is obtained. Such fields may be represented as a chain of the complete discretely valued field $s: k^{(0)}, \cdots, k^{(n)} = k$, where the field $k^{(i-1)}$ is a residue field for the field $k^{(i)}$, $1 \le i \le n$. We suppose also that the first residue field has a characteristic p and the last residue field $k^{(0)}$ is perfect. In the ascertained terminology, the fields, whose last field is finite are called to be the multidimensional local fields. Their theory was developed in the articles by Parshin A.N. and Kato K. (see [P], [K]).

A.N. Parshin gave the complete classification off multidimensional local fields. We shall quote it for the case of differently-characteristic fields with 0–characteristic.

The field $k\{\{t_1\}\}\cdots\{\{t_{n-1}\}\}$ where $t_1, \cdots, t_{n-1}$ are independent variables will be called standard n-dimensional local field. Parshin's classification is the following: every n-dimensional local field of null characteristic whose first residue field has a characteristic p (different-characteristic n-dimensional field) contains a standard field as a finite subextension, and besides, such a field is contained in a finite standard extension, i.e. there are numerical local fields k and k' and the systems of local parameters $t_1, \cdots, t_{n-1}$ and $t_1', \cdots, t_{n-1}'$ such that

$$E = k\{\{t_1\}\}\cdots\{\{t_{n-1}\}\} \subset K \subset k'\left\{\left\{t_1'\right\}\right\}\cdots\left\{\left\{t_{n-1}'\right\}\right\} = K',$$

and K/E, K'/K are finite extensions.

We introduce the basic notations.

K - is a complete field relative to discrete normalization by rank n with the characteristic 0.

F - is a first residue field of of K, i.e. $F = k^{(n-1)}$,

π - is a prime element of K by rank 1 relative to discrete normalization.

$t_1, \cdots, t_{n-1}, \pi$ are local parameters of K (a lift of prime elements of residue fields $k^{(i)}$, which define the field K).

$\overline{v}_K = \left(v^{(1)}, \cdots, v^{(n)}\right) : K^* \to (\mathbf{Z})^n$ is a normalization by rank n correspondent to the chosen local parameters; $(\mathbf{Z})^n$ is ordered as $(m_1, \cdots, m_n) < \left(m_1', \cdots, m_n'\right)$ if $m_n = m_n', \cdots, m_{i+1} = m_{i+1}'$, but $m_i < m_1'$ for some i. where o_K is a ring of normalization by rank n

$\mathfrak{M}_K$ is the unique maximal ideal of o_K

$v = v^{(n)}$ is a normalization by rank 1 of K as a discrete valued field

$e = v(p)$ is a ramification index of K relative to normalization by rank 1

$\overline{e} = \left(e^{(1)}, \cdots, e^{(n)}\right) = \overline{v}_K(n)$

ζ is fixed p^m of unity, which is contained in K.

F_0 is a maximal perfect subfield in the residue field F of the field K which is supposed to be non-algebraically closed.

$o_0 = W(F_0)$ is a ring of Witt vectors for F_0

$\mathcal{R}$ is Teichmüller system of the representatives of field elements F_0 in the ring o_0

k_0 is a field of relations for o_0

Δ is the Frobenius automorphism in k_0

$\mathfrak{p}(\alpha) = \alpha^{\Delta} - \alpha$ is the Cartie operator over the completion of the maximal unramified extension of the ring o_0.

$\alpha \equiv \beta \bmod (\mathfrak{p}, p^m)$ in the ring o_0 means that $\alpha = \beta + \mathfrak{p}(\gamma) + p^m\gamma'$, where $\gamma, \gamma' \in o_0$. $\alpha \approx \beta$ or $\alpha \equiv \beta \bmod (K^*)^{p^m}$ means that the elements α and β from K are different by p^m-power in K.

$K_{st} = k_0\{\{t_1\}\} \cdots \{\{t_n\}\}$ is a standard absolutely unramified field of normalization by rank $n+1$ ($t_1, \cdots, t_n$ are independent variables). $o = o_0\{\{t_1\}\} \cdots \{\{t_{n-1}\}\}$ is a ring of normalization by rank n in the field $k_0\{\{t_1\}\} \cdots \{\{t_{n-1}\}\} \subset K$ $\quad o' = o((t_n))$ is a ring of Laurent series $\mathcal{H}_m = (o')^*$ is a group of invertible elements of the ring o'.

The Frobenius operator Δ in the ring $o\{\{t_n\}\}$ is

$$\left(\sum_{\overline{\eta}} a_{\overline{\eta}} t_1^{r_1} \cdots t_n^{r_n}\right)^{\Delta} = \sum_{\overline{\eta}} a_{\overline{\eta}}^{\Delta} t_1^{pr_1} \cdots t_n^{pr_n}, \; a_{\overline{\eta}} \in o_0,$$

(one should note that the operator Δ depends on the choice of local parameters $t_1, \cdots, t_n$).

$\deg f$ denotes the order of Laurent

$$f(x) = a_m x^m + a_{m+1} x^{m+1} + \cdots, \text{ i.e. } m = \deg f.$$

The congruence

$$f \equiv g \bmod (p^r, \deg s)$$

means that the coefficients whose powers are less that s, are congruos $\bmod p^r$.

Remark. The above definitions are given in the sense that they are lexicographically ordered for the series with a few variables.

2. For any invertible in the ring $o\{\{t_n\}\}$ series f the next function is well defined

$$l(f) = \frac{1}{p} \log f^p / f^{\Delta} \tag{1.1}$$

because $f^p \equiv f^{\Delta} \bmod p$ (see also [V1], prop. 1 and [V5]). For any series g from the ideal $xo[[x]]$ the Artin-Masse function is well defined

$$E(g) = \exp\left(\sum_{m \geq 0} g^{\Delta^m} / p^m\right)$$

Besides the following assertion holds (see [V5]).

Proposition 1. *The functions l and E give the reciprocally inverse isomorphisms between the multiplicative $\mathbf{Z}_p$-module $1 + t_n o[[t_n]]$ and the additive $\mathbf{Z}_p$-module $t_n o[[t_n]]$. In particular any series $\varepsilon(t_n)$ from $1 + t_n o[[t_n]]$ may be uniquely represented in the form*

$$\varepsilon(t_n) = E(\eta(t_n)) \tag{1.2}$$

where $\eta(t_n) = l(\varepsilon)$.

For the element α from the field K let us denote the series in the ring $o\{\{t_n\}\}$ as $\underline{\alpha}(t_n)$. It is obtained from the expansion of the element α into series in the prime π from K whose coefficients are in o. Thus

$$\underline{\alpha}(t_n)|_{t_n=\pi} = \alpha \tag{1.3}$$

Let

$$s_r(t_n) = \underline{\zeta}^{p^r}(t_n) - 1 \tag{1.4}$$

$$u_r(t_n) = s_r/s_{r-1} = p + \sum_{i=1}^{p-1} C_p^{i+1} S_{r-1}^i \tag{1.5}$$

(we shall omit an index for $r = m$ in the series $s_r(t_n)$ and $u_r(t_n)$).

3. Expansion into series.

We denote the prime element of the field K relative the discrete normalization by rank 1 as π. The set of multi-indexes $I \subset \mathbf{Z}^{(n)}$ is said to be admissible if for any fixed set of integer $i_{l+1}, \cdots, i_n$, $1 \le l \le n$, in the set of all multi-indexes $\overline{r} = (r_1, \cdots, r_l, r_{l+1}, \cdots, r_n)$ from I for which the indexes $r_{l+1}, \cdots, r_n$ coincide with the indexes $i_{l+1}, \cdots, i_n$ respectively, the index r_l is bounded from below, i.e. there exists the integer i, such that $r_l \ge i$ for any $\overline{r} = (r_1, \cdots, r_l, i_{l+1}, \cdots, i_n)$ from I.

Any element α from K is uniquely represented as the following series

$$\alpha = \sum_{\overline{r} \in I} \theta_{\overline{r}} t_1^{r_1} \cdots t_{n-1}^{r_{n-1}} \pi^{r_n} \tag{1.6}$$

where I is the admissible set and $\theta \in \mathcal{R}$ (see [VZF], Lemma 2).

Let $o' = o((t_n))$ be a ring of L series, where $o = o_0\{\{t_1\}\} \cdots \{\{t_{n-1}\}\}$. We shall write

$$T^{\overline{r}} = t_1^{r_1} \cdots t_n^{r_n}$$

Any element $\alpha(t_n)$ of the ring o' is uniquely represented in the form

$$\alpha(t_n) = \sum_{\overline{r} \in I} a_{\overline{r}} T^{\overline{r}}, \ a_{\overline{r}} \in o_0 \tag{1.7}$$

where I is an admissible set.

We shall denote the multiplicative group of the ring o' as $\mathcal{H}_m$. There is the surjective homomorphism (uncanonical)

$$\begin{gathered} \eta_m : \mathcal{H}_m \to K^* \\ \alpha(T) \mapsto \alpha = \alpha(t_1, \cdots, t_{n-1}, \pi) \end{gathered} \tag{1.8}$$

4. Consider the series $u(t_n)$ constructed in (1.5). As in [V1] we proof

Proposition 2 *Let* $\varepsilon(t_n) = 1 + a_1 t_n + \cdots$ *be an invertible series from the ring* $o[[t_n]]$. *If* $\varepsilon(\pi) = 1$ *for prime* π *from* K *then there is the series* $\psi(t_n) \in o[[t_n]]$ *such that* $\varepsilon(\pi) = 1 + u\psi$.

In the group $\mathcal{H}_m$ we consider the subgroup

$$\mathcal{U}_m = \langle 1 + u(t_n)\varphi(t_n) \rangle \tag{1.9}$$

where $\varphi \in o[[t_n]]$, $\varphi(0) = 0$.

Proposition 3. *There is the exact sequence*

$$1 \to \mathcal{U}_m \to \mathcal{H}_m \overset{\eta_m}{\to} K^* \to 1$$

where η_m *is the homomorphism from (1.8).*

Proof: From the definition of the epimorphism η_m and that of the series $u(t_n)$ any element from the subgroup $\mathcal{U}_m$ it turns into 1 under η_m. Conversely, if $\eta_m(h(t_n)) = 1$ for some series $h \in \mathcal{H}_m$ then $h(t_1, \cdots, t_{n-1}, \pi) = 1$. To finish the proof the proposition 2.2 should be used.

§ 2. Primary elements

Remember, an element $\omega \in K$ is said to be p^m-primary if the extension $K(p^m\sqrt{\omega})/K$ in the unique unramified (i.e. may be obtained when the residue field $k^{(0)}$ is extended). This elements in a multi-dimensional local field have been constructed in [V5].

Let $\widetilde{K}$ be a maximal abelian unique unramified p extension of the field K. If for the maximal complete subfield $F_0 = k^{(0)}$ in the residue field $\overline{K}$ we have $F_0/\mathfrak{p}(F_0) \cong \bigoplus_{\chi} \mathbf{Z}/p\mathbf{Z}$ then the Galois group of the maximal abelian p-extension of the field F_0 is isomorphic to $\prod_{\chi} \mathbf{Z}_p$ and hence

$$\mathrm{Gal}\left(\widetilde{K}/K\right) \cong \mathrm{Gal}\left(F_0^{p,ab}/F_0\right) \cong \prod_{\chi} \mathbf{Z}_p.$$

Let $o_0 = W(F_0)$ be a Witt vector ring and o_0^{nr} be an integer ring of the supplement of the maximal unramified extension k_0 (i.e. the field of quotient for the ring o_0).

Lemma 1. For any element $a \in o_0$ there is an element $A \in o_0^{nr}$, which satisfies the next equation

$$\mathfrak{p}(A) = A^{\Delta} - A = a. \tag{2.1}$$

Besides for any automorphism φ from $\mathrm{Gal}\left(\widetilde{K}/K\right)$ the element $a_\varphi = A^\varphi - A$ is an p-adic integer, which does not depend from a choice of the root A.

Proof: The equation $\mathfrak{p}(x) = \overline{a}$ where $\overline{a}$ is a residue of a in the field F_0 is solvable in the extension $F_0^{p,ab}$. Using the method of successive approximations we obtain the solution of the equation (2.1). If $\varphi \in \mathrm{Gal}\left(\widetilde{K}/K\right)$ then $a_\varphi^\Delta - a_\varphi = a^\varphi - a$. An element a belongs to o_0, then $a^\varphi = a$. Hence $a_\varphi^\Delta = a_\varphi$ so $a_\varphi \in \mathbf{Z}_p$. Obviously a_φ does not depend from the choice of A. The lemma is proved.

Let ζ be a primitive $p^m - th$ root of unity, which is contained in the field K and let $\underline{\zeta}(x)$ be a series from the ring $o[[x]]$ that is obtained from the expansion the element ζ into a power series in the prime element π. As before $s(x) = \underline{\zeta}(x)^{p^m} - 1$.

Theorem *1. Any p^m -primary element α of the field K may be represented (with precision to p^m -th powers) as follows*

$$\alpha \approx w(a) = E(as(x))|_{X=\pi} \tag{2.2}$$

for some $a \in o_0$. Vice verra, for any $a \in o_0$ the element $w(a)$ of the form (2.) is p^m -primary. Besides the element $w(a)$ does not depend from the expansion of the root ζ into series in the prime element π. Further

$$\sqrt[p^m]{w(a)}^{\varphi-1} = \zeta^{A^{\varphi}-A} = \zeta^{a_\varphi} \tag{2.3}$$

for any automorphism φ of the Galois group $\mathrm{Gal}\left(\widetilde{K}/K\right)$. At last

$$w(a) \approx 1 \Leftrightarrow a \equiv \mathfrak{p}(a_0) \bmod p^m, \quad a_0 \in o_0 \tag{2.4}$$

Corollary. Let Ω be a group of p^m -primary element of the field K and μ_{p^m} be a group of $p^m - th$ roots of unity. If $F_0 = k^{(0)}$ is a finite field and φ is the Frobenius automorphism in $\widetilde{K}/K$, then

$$\sqrt[p^m]{w(a)}^{\varphi-1} = \zeta^{tr\, a}$$

and there is an isomorphism χ such that

$$\begin{aligned} \chi : \Omega/\Omega \cap K^{*p^m} &\xrightarrow{\sim} \mu_{p^m} \\ \chi(w(a)) &= \zeta^{tr\, a}, \end{aligned} \tag{2.5}$$

where tr is the trace operator in $k_0/\mathbf{Q}_p$.

Proof: Let $q = p^f$ denote the number of elements of the field F_0. Then $\varphi = \Delta^f$ and hence $a_\varphi = A^\varphi - \Delta = A^{\Delta^f} - A = a + a^\Delta + \cdots + a^{\Delta^{f-1}} = tr\, a$. Hence the formula (2.2) follows from the equality (2.5) of the theorem. The second statement of corollary follows from the formula (2.4) of the theorem.

Proof of theorem: 1. We shall firstly verify that if α is a p^m -primary element of K then it may be represented in the form (2.2). The next judgement belong I. Fesenko whom the author is profoundly grateful.

Really, for the series $s_1(x) = \underline{\zeta}_1^p(x) - 1$, where $\zeta_1^p = 1$, the next comparison is obviously holds

$$s_1(x) \equiv \left(\underline{\zeta}_1(x) - 1\right)^p = c^p t_1^{pe_1'} \cdots t_{n-1}^{pe_{n1}'} x^{pe_n'} + \cdots \bmod p$$

where $\zeta_1^p = 1$, the next comparison is obviously holds

$$s_1(x) \equiv \left(\underline{\zeta}_1(x) - 1\right)^p = c^p t_1^{pe_1'} \cdots t_{n-1}^{pe_{n-1}'} x^{pe_n'} + \cdots \bmod p$$

where $\underline{\zeta}_1(x) = 1 + c\, t_1^{e_1'} \cdots t_{n-1}^{e_{n-1}'} x^{e_n'} + \cdots$. Hence, using the properties of the function E we have

$$E(s_1(x)) \underset{p}{\approx} E\left(c^p t_1^{pe_1'} \cdots t^{pe_{n-1}'} x^{pe_n'} \right)$$

The element α is α is p^m -primary, so it is p-primary, hence $\alpha = 1 + a_* t_1^{pe_1'} \cdots t_{n-1}^{pe_{n-1}'} \pi^{pe_n'} + \cdots$ for some a_* of o_0. Then, there is $a_0 \in o_0$ such that

$$E(a_0 s_1(x))|_{x=\pi} \equiv \left(c^p a_0 t_1^{pe_1'} \cdots t_{n-1}^{pe_{n-1}'} x^{pe_n'} \right)|_{x=\pi} \equiv \alpha \bmod K_1^{*p}$$

i.e. $\alpha = E(a_0 s_1(x))|_{x=\pi} \cdot \beta^p$ for some β, $\overline{v}_K(\beta - 1) > \overline{e}_1$.

To obtain the following equality

$$E(a_0 s_1(x))]|_{x=\pi} = E\left(pa_0 \log \underline{\zeta}_1(x) \right)|_{x=\pi} \cdot \gamma^p$$

(for some $\gamma \in K^*$) we shall use lemma 9 from [V1]. Further, for $\zeta = \sqrt[p^{m-1}]{\zeta_1}$ we have

$$\begin{aligned} &E\left(pa_0 \log \underline{\zeta}_1(x) \right)|_{x=\pi} = E\left(pa_0 \log \underline{\zeta}(x)^{p^{m-1}} \right)|_{x=\pi} = \\ &= E(p^m a_0 \log \underline{\zeta}(x))|_{x=\pi} = E(a_0 s(x))|_{x=\pi} \cdot (\gamma')^p,\ \gamma' \in K^*. \end{aligned}$$

Hence we obtain $\alpha = w(a_0) \cdot \alpha_1^p$, the element α_1 will be p^{m-1} -primary. By induction one could suppose that

$$\alpha_1 \equiv E\left(p^{m-1} a_1 \log \underline{\zeta}_{m-1}(x) \right)|_{x=\pi} \equiv E(a_1 s(x))|_{x=\pi} \bmod K^{*p^{m-1}}.$$

Then $\alpha \equiv w(a_0 + a_1) \bmod K^{*p^m}$ and the decomposition (2.2) is proved.

Let now consider an element of the form

$$w(a) = E(as(x))|_{x=\pi}.$$

We shall verify that this element is p^m -primary. Suppose

$$H(a) = E\left(p^m A^\Delta l(\underline{\zeta}(x)) \right)|_{x=\pi}$$

where A is a root of the equation $\mathfrak{p}(x) = a$. The following equality

$$H(a) = E(p^m a \log \underline{\zeta}(x))|_{x=\pi} \tag{2.6}$$

is verified by analogy with lemma 9 from [V2]. To obtain the relation

$$w(a) \underset{p^m}{\approx} H(a) \tag{2.7}$$

one could use lemma 9 of [V1].

It is more easy to verify the $H(a)$ is primary then to do that for $w(a)$. Namely as follows from (2.6) the element $H(a)$ belongs to the field K. But $H(a)$ is p^m -th power of the element $E(A^\Delta) l(\underline{\zeta}(x))|_{x=\pi}$, which obviously belongs to the unique unramified extension

$\widetilde{K}$ of the field K, because the element A is in $\widetilde{K}$. Thus, p^m-primaryness of the $H(a)$ (and also $w(a)$) is proved.

Now we shall verify the element (2.) does not depend from the choice of expansion of the root ζ. As (2.7) holds, it is sufficiently to verify the analogical independence of the element $H(a.)$ Let $\underline{\zeta}^{(1)}(x)$ be a series corresponding to some another expansion of ζ. Then we have

$$H(a) = E\Big(p^m A^{\Delta} l(\underline{\zeta}(x))\Big)|_{x=\pi} = E\Big(p^m A^{\Delta} l\Big(\underline{\zeta}^{(1)}(x)\Big)\Big)|_{x=\pi} \cdot \eta^{p^m} \tag{2.8}$$

where $\eta = E\Big(A^{\Delta} l(\underline{\zeta}(x))/\underline{\zeta}^{(1)}(x)\Big)|_{x=\pi}$. It remains to verify that $\eta \in K$. Let φ be an arbitrary automorphism of the Galois group $\mathrm{Gal}\Big(\widetilde{K}/K\Big)$. Then we have $A^{\Delta^{\varphi}} - A^{\Delta} = a_{\varphi}^{\Delta} = a_{\varphi} \in \mathbf{Z}_p$. As the function E is $\mathbf{Z}_p$-multiplicative and the functions E and l are reciprocally inverse we obtain (see Proposition 1) $\eta^{\varphi} = \eta \cdot E\Big(l\Big(\underline{\zeta}(x)/\underline{\zeta}^{(1)}(x)\Big)\Big)^{a_{\varphi}}|_{x=\pi} = \eta \cdot \left(\frac{\zeta(\pi)}{\underline{\zeta}^{(1)}(\pi)}\right)^{a_{\varphi}} = \eta \cdot \left(\frac{\zeta}{\zeta}\right)^{a_{\varphi}} = \eta$. Hence $\eta \in K$. The independence of the element $H(a)$ (and simultaneously with it $w(a)$) from the expansion in a series in the prime element π follows from the last relation and the formula (2.8).

We shall verify (2.3) for the element $H(a)$. Using the formula (2.7), the definition of $H(a)$ we obtain

$$\sqrt[p^m]{w(a)}^{\,\varphi-1} = \sqrt[p^m]{H(a)}^{\,\varphi-1} = E\Big(\Big(A^{\Delta^{\varphi}} - A^{\Delta}\Big) l(\underline{\zeta}(x))\Big)|_{x=\pi} =$$
$$= E\Big(a_{\varphi}^{\Delta} l(\underline{\zeta}(x))\Big)|_{x=\pi}.$$

But $a_{\varphi} \in \mathbf{Z}_p$ hence $a_{\varphi}^{\Delta} = a_{\varphi}$, besides we can use the proposition 1 for the functions E and l :

$$E\Big(a_{\varphi}^{\Delta} l(\underline{\zeta}(x))\Big)|_{x=\pi} = E(l(\underline{\zeta}(x)))|_{x=\pi}^{a_{\varphi}} = \underline{\zeta}(\pi)^{a_{\varphi}} = \zeta^{a_{\varphi}}.$$

Thus the equality (2.3) is obtained. It remains to verify the condition (2.4). From the formula (2.7) and the definition of the element $H(a)$ we have

$$w(a) \approx 1 \Leftrightarrow H(a) \approx 1 \Leftrightarrow \eta = H\Big(A^{\Delta} l(\underline{\zeta}(x))\Big)|_{x=\pi} \in K.$$

The last condition is equivalent to following: for any automorphism φ of the Galois group $\mathrm{Gal}\Big(\widetilde{K}/K\Big)$ $\eta^{\varphi} = \eta$. But

$$\eta^{\varphi-1} = E\Big(\Big(A^{\Delta\varphi} - A^{\Delta}\Big) l(\underline{\zeta}(x))\Big)|_{x=\pi} = \zeta^{a_{\varphi}},$$

where $A^{\Delta\varphi} - A^{\Delta} = a_{\varphi}^{\Delta} = a_{\varphi} \in \mathbf{Z}_p$. Hence $\eta \in K \Leftrightarrow a_{\varphi} \equiv 0 \bmod p^m \Leftrightarrow A^{\varphi} - A \equiv 0 \bmod p^m$, and the element A may be represented as $A = a_0 + p^m A_1$, where $a_0 \in o_0$, $A_1 \in o_0^m$. It immediately follows that

$$a = \wp(A) = \wp(a_0) + p^m \wp(A_1) \equiv \wp(a_0) \bmod p^m$$

and (2.4) is proved. Thus the proof of the theorem is completed.

§ 3. Maps $\gamma_{(T)}$ and $\Gamma_{(T)}$.

We shall construct the mapping from $(m+1)$-th copy of the group $\mathcal{H}_m$ into the ring o_0. Afterwards we shall realize this map as that from $(m+1)$-th copy of the group K^* into the group of p^m-primary elements Ω. Let $(T) = (t_1, \cdots, t_n)$ be a system of local parameters in the field $k_0\{\{t_1\}\}\cdots\{\{t_n\}\}$. In the ring $o\{\{t_n\}\}$, where $o = o_0\{\{t_1\}\}\cdots\{\{t_{n-1}\}\}$, the function

$$l(\alpha) = \frac{1}{p}\log \alpha^p/\alpha^{\Delta}$$

is defined for any inverse series $\alpha(T)$. Further, for a series $\alpha(T)$ of the ring $o\{\{t_n\}\}$ we shall denote the $i-th$ logarithmic derivative as $\delta_i(\alpha)$, i.e.

$$\delta_i(\alpha) = \frac{\partial}{\partial t_i}\alpha, \quad 1 \le i \le n.$$

Remember that the series $\underline{\zeta}(t_n)^{p^m} - 1$ was denoted as $s(t_n)$ (see (1.4)).

For $p \neq 2$ we shall define the map

$$\begin{aligned} &\gamma_{(T)} : \mathcal{H}_m \times \cdots \times \mathcal{H}_m \to 0_0 \bmod (\wp, p^m) \\ &\gamma_{(T)}(\alpha_1, \cdots, \alpha_{n+1}) = \operatorname{res} \phi(\alpha_1, \cdots, \alpha_{n+1})/s \end{aligned} \tag{3.1}$$

where

$$\phi(\alpha_1, \cdots, \alpha_{n+1}) = l(\alpha_{n+1})D_{n+1} - l(\alpha_n)D_n + \cdots + (-1)^n l(\alpha_1)D_1,$$

$$D_i = \frac{1}{p^{n-i+1}}\begin{vmatrix} \delta_1(\alpha_1)\cdots\delta_n(\alpha_1) \\ ----------- \\ \delta_1(\alpha_{i-1})\cdots\delta_n(\alpha_{i-1}) \\ \delta_1(\alpha_{i+1}^{\Delta})\cdots\delta_n(\alpha_{i+1}^{\Delta}) \\ ----------- \\ \delta_1(\alpha_{n+1}^{\Delta})\cdots\delta_n(\alpha_{n+1}^{\Delta}) \end{vmatrix}, \ 1 \le i \le n+1.$$

Further in particular case will be used. Let the sets of indexes $I = (i_1 < i_2 < \cdots < i_{u-1})$ and $J = (j_1 < j_2 < \cdots < j_{v-1})$, where $u + v = n - 1$, be such that they compose some permutation $(1, 2, \cdots, n)$ together with some κ, i.e. $\mathrm{z} I \cup (\kappa) \cup J = (1, 2, \cdots, n)$. Then we have

$$\begin{aligned} &\phi(t_{i_1}, \cdots, t_{i_{n-1}}, \alpha, t_{j_1}, \cdots, t_{j_{v-1},\beta}) = \\ &= (-1)^{\mathcal{H}} t_1^{-1} \cdots \widehat{t_\kappa^{-1}} \cdots t_n^{-1}\Big(l(\beta)\delta_\kappa(\alpha) + (-1)^{n-\kappa+v} l(\alpha)\frac{1}{p}\delta_\kappa\big(\beta^{\Delta}\big)\Big) \end{aligned}$$

where $\mathcal{H}$ is the number of inversions in the permutation (I, κ, J).

<u>**Proposition 4.**</u> *The mapping $\gamma_{(T)}$ is well-defined (i.e. is invariant relatively the choice of local parameters $t_1, \cdots, t_n$ of the field K). This one is multiplicative in all its arguments, skew-symmetry, proportional, i.e.*

$$\gamma_{(T)}(\cdots, \alpha_i, \cdots, -\alpha_i, \cdots) \approx 1.$$

Besides this mapping satisfies the Steinberg relation

$$\gamma_{(T)}(\cdots, \alpha_i, \cdots, 1 - \alpha_i, \cdots) \approx 1.$$

The proposition can be verified similarly to the proposition 2 of [V5]. Consider the subgroup

$$\mathcal{U}_m = \langle 1 + u(t_n)\psi(t_n) \rangle$$

(see (1.9) in the group $\mathcal{H}_m$.

<u>**Proposition 5.**</u> *The subgroup $\mathcal{U}_m$ belongs to the kernel of the mapping $\gamma_{(T)}$ in all its arguments, so $\gamma_{(T)}$ induces for $p \neq 2$ nontrivial mapping (denoted by the same letter)*

$$\gamma_{(T)}\left(\mathcal{H}_m/\mathcal{U}_m\mathcal{H}_m^{p^m}\right)^{n+1} \to o_0 \bmod (\wp, p^m).$$

Proof: We verify first that

$$\gamma_{(T)}(t_1, \cdots, t_n, \varepsilon) \equiv 0 \bmod (\wp, p^m) \tag{3.2}$$

for any series $\varepsilon(t_n)$ from $\mathcal{U}_m$. By definition of $\gamma_{(T)}$ we have

$$\gamma_{(T)}(t_1, \cdots, t_n, \varepsilon) = \mathrm{res}_{(T)} t_1^{-1} \cdots t_n^{-1} l(u\psi)/s$$

where $\varepsilon = 1 + u\psi$. Taking into account the congruence (5) from [V5] this we obtain (3.2). Further, any element α of $\mathcal{H}_m$ can be represented as the product of local parameters. Hence, from the relations proved above the congruence (3.2) and from multiplicativity and invariance of $\gamma_{(T)}$ the congruence follows

$$\gamma_{(T)}(\alpha_1, \cdots, \alpha_n, \varepsilon) \equiv 0 \bmod (\wp, p^m)$$

where $\alpha_1, \cdots, \alpha_n \in \mathcal{H}_m$, and $\varepsilon \in \mathcal{U}_m$.

From this and skew-symmetry of $\gamma_{(T)}$ follows the assertion of the proposition. The exact sequence (see proposition 3)

$$1 \to \mathcal{U}_m \to \mathcal{H}_m \stackrel{\eta_m}{\to} K^* \to 1$$

determines the isomorphism

$$\eta_m^* : K^* \cong \mathcal{H}_m/\mathcal{U}_m \tag{3.3}$$

which is defined as follows: every element $\alpha \in K^*$ is expanded into the series in the local parameters $t_1, \cdots, t_{n-1}, \pi$ afterwards the prime π of the field K is replaced by variable t_n. According to proposition 5 the homomorphism η_m^* is well-defined and is the isomorphism. With the help of the constructed isomorphism the mapping $\gamma_{(T)}$ is transferred in the field K. Namely, for $p \neq 2$, we shall define the mapping

$$\Gamma_{(T)} : (K^*)^{n+1} \to \Omega/\Omega^{p^m}$$

$$\Gamma_{(T)}(\alpha_1, \cdots, \alpha_{n+1}) = w(\gamma_{(T)})$$

where $\gamma_{(T)} = \gamma_{(T)}(\eta_n^*(\alpha_1), \cdots, \eta_m^*(\alpha_{n+1}))$.

<u>**Proposition 6.**</u> *The mapping $\Gamma_{(T)}$ for $p \neq 2$ is $\mathbf{Z}_p$-multiplicative, in all its arguments, invariant in relation to the choice of local parameters $t_1, \cdots, t_{n-1}, \pi$ of K, independent of the expansition way of the elements $\alpha_1, \cdots, \alpha_{n+1}$ into series on the system $t_1, \cdots, t_{n-1}, \pi$. Besides one satisfies Steinberg relation*

$$\Gamma(\cdots, \alpha_i, \cdots, 1 - \alpha_i \cdots)_{(T)} \approx 1$$

for any $\alpha_i \neq 1$ of K^*.

Proof: The $\mathbf{Z}_p$-multiplicativity, invariance and Steinberg relation follow from the corresponding properties of the mapping $\gamma_{(T)}$ (see proposition 4). Independence from the expansion of elements into series follows from the proposition 5 and isomorphism η_m^*.

§ 4. The pairing on topological K-groups

1. Let $K_u(K)$ be $u-th$ Milnor group of the field K, $u \geq 0$. We shall choose the most strong topology on $K_u(K)$ in which the map $(K^*)^u \to K_u(K)$ is sequentially continuous on every argument and if $x_n \to x$, $y_n \to y$, then $x_n + y_n \to x + y$ and $-x_n \to -x$ in $K_u(K)$. Let $\Lambda_u(K)$ be a subgroup in $K_u(K)$ which is the intersection of all the neighbourhoods of zero in $K_u(K)$. Then we define the topological Parshin's K-group as follows

$$K_u^{\mathrm{top}}(K) = K_u(K)/\Lambda_u(K)$$

(see [P]).

In this section we shall define nondegenerated pairing on K-groups which has values in the group of p^m-primary elements Ω. Let K contain all the $p^m - th$ roots of the unity. The mapping $\Gamma_{(T)}$ constructed in the preceeding section, induces the pairing in K-groups $K_u^{\mathrm{top}}(K)$ and $K_v^{\mathrm{top}}(K)$, where $u + v = n + 1$:

$$\langle\ ,\ \rangle_{(T)}^{(u)} : K_u^{\mathrm{top}}(K)/p^m \times K_v^{\mathrm{top}}(K)/p^m \to \Omega/\Omega^{p^m},$$

which is defined on the symbols $\alpha = \{\alpha_1, \cdots, \alpha_u\}$ and $\beta = \{\beta_1, \cdots, \beta_v\}$ as follows

$$\langle \alpha, \beta \rangle_{(T)}^{(u)} = \Gamma_{(T)}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_v).$$

The pairing $\langle\ ,\ \rangle_{(T)}^{(u)}$ is extended on the remaining elements by linearity.

Theorem 2. *The pairing $\langle\ ,\ \rangle_{(T)}^{(u)}$ for $p \neq 2$ is well-defined, non-degenerate on both arguments.*

Remark From skew-symmetry of $\Gamma_{(T)}$ follows the property

$$\langle \alpha, \beta \rangle_{(T)}^{(u)} = \langle \beta, \alpha \rangle_{(T)}^{(-1)^{uv}}.$$

Proof of the theorem. *From the properties of $\Gamma_{(T)}$ (see proposition 6) it follows that $\Gamma_{(T)}$ defines the well-defined pairing $\langle\ ,\ \rangle_{(T)}^{(u)}$ on K-groups. Non-degenerative is verified in the same way as in proposition 3 of [V5].*

§ 5. The pairing $\langle\ ,\ \rangle_{(T)}$ and Hilbert symbol

1. Let us consider the K-groups $K_n(K)$ and $K_1(K) \cong K^*$ and the pairing

$$\langle\ ,\ \rangle_{(T)} = \langle\ ,\ \rangle_{(T)}^{(n)} : K_n(K)/p^m \times K^*/K^{*p^m} \to \Omega/\Omega^{p^m} \tag{5.1}$$

in context of preceeding section. If the field K is a multidimensional local field with null characteristic (i.e. the last residue field $k^{(0)}$ of K is finite) then the character χ defined on the group of p^m-primary elements as

$$\chi(w) = \sqrt[p^m]{w}^{\,\Delta - 1} = \zeta^{\mathrm{tr}\ a},$$

where Δ is Frobenius automorphism in the unramified extension $K(\sqrt[p^m]{w})/K$ and tr is the trace operator in $k_0/\mathbf{Q}_p$, sets the isomorphism

$$\chi : \Omega/\Omega^{p^m} \cong \mu_{p^m} = \langle \zeta \rangle. \tag{5.2}$$

Thus the pairing (5.1) induces the mapping

$$\chi\Big(\langle\ ,\ \rangle_{(T)}\Big) : K_n(K)/p^m \times K^*/K^{*p^m} \to \mu_{p^m}.$$

Equally to the constructed map (5.2) in the multidimensional local field the Hilbert norm residue symbol is defined on the same set

$$\begin{aligned}(\ ,\ )_m : K_n(K)/p^m \times K^*/K^{*p^m} &\to \mu_{p^m}\\ (\{\alpha_1,\cdots,\alpha_n\},\beta)_m &= \sqrt[p^m]{\beta}^{\,\psi(\{\alpha_1,\cdots,\alpha_n\})-1}\end{aligned}$$

where ψ is a canonical reciprocity homomorphism

$$\psi : K_n(K) \to \mathrm{Gal}\Big(K^{\mathrm{ab}}/K\Big)$$

(see [P], [K]).

__Theorem 3.__ *For any $\alpha \in K_n(K)$ and any $\beta \in K^*$, where $p \neq 2$, the following equality holds*

$$(\alpha,\beta)_m = \chi\Big(\langle\alpha,\beta\rangle_{(T)}\Big) \qquad (5.3)$$

which sets Hilbert symbol the explicit form with the help of the formula (3.1).

Proof: According to isomorphisms (5.2) the mapping $\chi\Big(\langle\ ,\ \rangle_{(T)}\Big)$ coincides with the pairing $\langle\, ,\, \rangle_\Gamma$, which was constructed in the theorem 3 of the article [V5]. Then the theorem 4 from this article sets (5.3).

Now we consider the complete discrete valued field K by characteristic 0 with the perfect residue field F by characteristic $p > 0$. The field F is not algebraically closed. In the context of this article the field K is the field of normalization by rank 1. The reciprocity map for such a field is constructed as follows (see [F]).

Let L/K be a totally ramified Galois extension by power p^m, then

$$(\ ,L/K) : U_{1,K}/N_{L/K}U_{1,L} \xrightarrow{\sim} \mathrm{Hom}\left(\mathrm{Gal}\Big(\widetilde{K}/K\Big), \mathrm{Gal}(L/K)^{\mathrm{ab}}\right) \qquad (5.4)$$

where $U_{1,K}$ is the group of principal units, $\widetilde{K}$ is the maximal abelian unramified p-extension of K, and Hom are continuous $\mathbf{Z}_p$-homomorphisms. We shall note that

$$\mathrm{Gal}\Big(\widetilde{K}/K\Big) \cong \prod_x \mathbf{Z}_p$$

where $F/\wp F = \bigoplus_x \mathbf{Z}/p\mathbf{Z}$ the direct sum of x items $\mathbf{Z}/p\mathbf{Z}$. Besides, it's clear, that the group, which is in the right side of (5.4) is noncanonically isomorphic to $\bigoplus_x \mathrm{Gal}(L/K)^{\mathrm{ab}}$.

We set Hilbert symbol of p^m-power in the field K which contains all the $p^m - th$ roots of the unity

$$\begin{aligned}(\ ,\ )_m : U_{1,K} \times K^* &\to \mathrm{Hom}\Big(\mathrm{Gal}\Big(\widetilde{K}/K\Big), \mu_{p^m}\Big)\\ (\alpha,\beta)_m &: \varphi \mapsto \sqrt[p^m]{\beta}^{\,\sigma_\alpha(\varphi)-1}\end{aligned}$$

where $\varphi \in \mathrm{Gal}\left(\widetilde{K}/K\right)$, $\sigma_\alpha(\varphi)$ is the extension $(\alpha, M/k)(\varphi)$ on $\mathrm{Gal}\left(\widetilde{K}(\sqrt[p^m]{\beta})/\widetilde{K}\right)$, M/K is totally ramified extension, such that

$$\begin{array}{ccc} K & - & M \\ | & & | \\ \widetilde{K} & - & \widetilde{K} \quad (\sqrt[p^m]{\beta}). \end{array}$$

It is easy to see that, the symbol $(\ ,\ )_m$ is multiplicative in both arguments, skew-symmetrical, i.e. $(\alpha,\beta)_m = (\beta,\alpha)_m^{-1}$, if $\alpha,\beta \in U_{1,K}$, and satisfied Steinberg relation $(1-\alpha,\alpha)_m = 1$ for any $\alpha \equiv 0 \bmod \pi$. Besides it's clear that the symbol $(\ ,\ )_m$ induces the nondegenerative pairing

$$U_{1,K}/U_{1,K}^{p^m} \times K^*/\Omega K^{*^{p^m}} \to \mathrm{Hom}\left(\mathrm{Gal}\left(\widetilde{K}/K\right), \mu_{p^m}\right)$$

where Ω is the group of p^m-primary elements of K. Using the skew-symmetry property one could extend the symbol $(\ ,\ )_m$ on $K^* \times K^*$, for (π,α), $\alpha \in U_{1,K}$, we set

$$(\pi,\alpha) = (\alpha,\pi)^{-1}$$

and $(\pi,\pi) = (-1,\pi)$. Then the pairing $(\ ,\ )_m$ induces the nondegenerative Hilbert pairing

$$(\ ,\ )_m : K^*/K^{*^{p^m}} \times K^*/K^{*^{p^m}} \to \mathrm{Hom}\left(\mathrm{Gal}\left(\widetilde{K}/K\right), \mu_{p^m}\right).$$

Let $\lambda : \Omega/\Omega^{p^m} \to \mathrm{Hom}\left(\mathrm{Gal}\left(\widetilde{K}/K\right), \mu_{p^m}\right)$

$$\lambda(w)(\varphi) = \sqrt[p^m]{w}^{\,\varphi-1}.$$

Then the mapping λ is an isomorphism and Hilbert symbol $(\ ,\ )_m$ coincides with the composition

$${}_{p^m}\mathrm{Br}(K) \xrightarrow{\mu} \Omega/\Omega \cap K^{*^{p^m}} \xrightarrow{\lambda} \mathrm{Hom}\left(\mathrm{Gal}\left(\widetilde{K}/K\right), \mu_{p^m}\right)$$

where μ is defined as follows. Any cyclic pair (α,β) can be uniquely represented as $(\alpha,\beta) = (\pi, w)$, where w is p^m-primary element (see [W]). Then $\mu(\alpha,\beta) = w$. The pairing $\langle\ ,\ \rangle_{(T)}^{(u)}$ constructed in § 4 in our case has the form. Let $\alpha,\beta \in K^*$ then

$$\Gamma_\pi : K^*/K^{*^{p^m}} \times K^*/K^{*^{p^m}} \to \Omega/\Omega^{p^m}$$
$$\Gamma_\pi(\alpha,\beta) = w(\gamma_{\alpha,\beta})$$

where

$$\gamma_{\alpha,\beta} = \mathrm{res}_x\left(l(\underline{\alpha})l(\underline{\beta})' - l(\underline{\alpha})\underline{\beta}'/\underline{\beta} + l(\underline{\beta})\underline{\alpha}'/\underline{\alpha}\right)/s \tag{5.5}$$

($\underline{\alpha}(x)$ denotes the series $\eta_m^*(\alpha)$, see (3.3)). Along with the pairing Γ_π which defines the mapping into the group of p^m-primary elements it is natural to consider the pairing

$$\langle\ ,\ \rangle_\pi : K^* \times K^* \to \mathrm{Hom}\left(\mathrm{Gal}\left(\widetilde{K}/K\right), \mu_{p^m}\right)$$

which is the composit of Γ_π and the isomorphism

$$\lambda : \Omega/\Omega^{p^m} \to \mathrm{Hom}\left(\mathrm{Gal}\left(\widetilde{K}/K\right), \mu_{p^m}\right)$$
$$\lambda(w)(\varphi) = \sqrt[p^m]{w}^{\,\varphi-1}$$

i.e. $\langle\alpha,\beta\rangle_\pi = \lambda(\Gamma_\pi(\alpha,\beta))$.

<u>**Theorem 4.**</u> *The pairing Γ_π and also $\langle\ ,\ \rangle_\pi$ is bilinear, skew-symmetrical, well-defined. Both pairings satisfy the Steinberg relation, i.e.* $\Gamma_\pi(\alpha,1-\alpha)\approx 1$, $\langle\alpha,1-\alpha\rangle_\pi = 1$, $\alpha\neq 0;\ 1$. *Besides, if the field F is not algebraically closed, then both pairings set nondegenerated with the precision to p^m-th powers mappings. Both pairings have a non property.*

Proof: All the properties of the pairings Γ_π and $\langle\ ,\ \rangle_\pi$ are proved in § 3. On the norm property see theorem 7 in the paper [V5].

One should verify the main result of this section. Let K be a complete discrete valued field whose residue field is perfect, not algebraically closed, and has a characteristic $p>0$. Let the field K contain all the p^m-th roots of unity and $(\ ,\ )_m$-Hilbert symbol in the field K. We shall consider the isomorphism

$$\lambda:\Omega/\Omega^{p^m}\to \mathrm{Hom}\left(\mathrm{Gal}\left(\widetilde{K}/K\right),\mu_{p^m}\right)$$
$$\lambda(w)=\sqrt[p^m]{w}^{\,\varphi-1}.$$

In the group of principal units we shall use Shafarevich canonical decomposition (see [Sh]), which in our case may be represented as follows. Let e be ramification index of K and $e_1=\frac{e}{p-1}$. Let, further, K contain all the p^m-th roots of unity and π - a prime element of K.

<u>**Proposition 7.**</u> *Any principal unity ε of the field K up to terms of p^m-th powers is uniquely represented as*

$$\varepsilon = w(a_\varepsilon)E(w_\varepsilon(x))|_{x=\pi} \tag{5.6}$$

where $w_\varepsilon(x)=\sum c_ix^i$, $1\le i\le pe_1$, $(i,p)=1$. *Besides* $\varepsilon\in K^{*p^m}\iff a_\varepsilon\equiv 0 \bmod (\wp,p^m)$, $w_\varepsilon(x)\equiv 0 \bmod p^m$.

<u>**Theorem 5.**</u> *Let $p\neq 2$, then*

$$(\alpha,\beta)_m=\lambda\big(w(\gamma_{\alpha,\beta})\big),$$

where

$$\Gamma_{\alpha,\beta}=\mathrm{res}_x\Big(l(\underline{\alpha})l(\underline{\beta})'-l(\underline{\alpha})\underline{\beta}'+\underline{\beta}+l(\underline{\beta})\underline{\alpha}'/\underline{\alpha}\Big)/s.$$

Proof: Consider first the case $\alpha=\pi$, $\beta=\varepsilon$. Mark that as the pairing $\langle\alpha,\beta\rangle_\pi = \lambda(\Gamma_\pi(\alpha,\beta))=\lambda\big(w(\lambda_{\alpha,\beta})\big)$ has the property of independence, any expansion of the unity ε into power series on the prime π may be used. Hence we shall represent ε in the canonical form (5.6). Then from (6.6) we obtain

$$\langle\pi,\varepsilon\rangle_\pi=\lambda(w(a_\varepsilon)).$$

On the other hand, the unity $E(c\pi^i)$, $p\nmid i$, is the norm in the extension $K(\sqrt[p^m]{\pi})$ (see, [Sh]). Hence for Hilbert symbol we obtain

$$(\pi,E(w_\varepsilon(x))|_{x=\pi})_m=1.$$

So

$$(\pi,\varepsilon)_m=(\pi,w(a_\varepsilon))_m=\lambda(w(a_\varepsilon))$$

and equality

$$(\pi, \varepsilon)_m = \langle \pi, \varepsilon \rangle_m$$

is verified.

If ε, η be the principal units of the fielf K, then supposing $\tau = \pi\varepsilon$ and using invarience property of the pairing $\langle \, , \rangle_\pi$ we obtain

$$\langle \varepsilon, \eta \rangle_\pi = \langle \pi\varepsilon, \eta \rangle_\pi \cdot \langle \pi, \eta \rangle_\pi^{-1} = \langle \tau, \eta \rangle_\pi \cdot \langle \pi, \eta \rangle_\pi^{-1}.$$

From this and the properties of $\langle \, , \rangle_\pi$ follows the general case. The theorem is proved. More concrete exposition will appear in Adv. in Soviet Math., 1993.

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