A characterization of the Veronese surface

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Abstract. Here we prove a slight modification of a conjecture of Beltrametti–Sommese proving that the Veronese surface and a general intersection of 3 quadrics are the only smooth surface of $\mathbf{CP}^5$ which are 2–spanned.

The aim of this short note is the proof of a slight modification of a conjecture raised in [BS], conjecture 2.6, proving a slightly different and slightly more general result (see theorem 1). The main tools will be an enumerative formula, a well–known theorem on projective curves and the classification given in [BS], §5, for embedded surfaces of sectional genus at most 5. To state the result we need to introduce a few notations. We work over the complex number field. Let S be a smooth, complete surface , $L \in \mathrm{Pic}(S)$ and $W \subseteq H^0(S,L)$. A finite subscheme Z of S is called *curvilinear* if it is contained in a smooth curve, i.e. if for all point $P \in \mathrm{Supp}(Z)$ Z is given around P by equations $x = y^m = 0$, x and y suitable local coordinates around P. According to [BFS] and [BS], we say that W k–spans S, k an integer ≥0, if for all curvilinear subschemes Z of S with length(Z) = k+1, the restriction map from W to $H^0(Z, L|Z)$ is surjective. L is called k–spanned if $H^0(S,L)$ k–spans S. A smooth surface $S \subset \mathbf{P}^n$ is called k–spanned if the linear system on S determined by the embedding in $\mathbf{P}^n$ k–spans S. There are other, perhaps more natural, definitions of k–spannedness (see [BFS], §4), but not only the one given here is the one used heavily in [BFS] and [BS], but also it seems the weakest one among the natural possible definitions, and so the one with which theorem 1 is stronger. It is easy to check that if W is (k+1)–spanned, then it is k–spanned. In [BFS], 0.4.1, it was noted that W is 1–spanned if and only if it embedds S in a projective space (this is true even for non complete linear systems). In [BS], 2.4, as a corollary of a more general result, it was proved that if L k–spans S for some k≥2, then $H^0(S,L) \geq 6$. In [BS], 2.6, it was conjectured that if L k–spans S for some k≥2 and $H^0(S,L) = 6$, then $S \cong \mathbf{P}^2$ and L gives the Veronese embedding. It is easy to check that the Veronese surface in $\mathbf{P}^5$ is 2–spanned; this follows also from general results in BFS : L is the tensor power of two very ample line bundles. This note contains only the proof of the following theorem.

Theorem 1 *Assume that W 2–spans the smooth, complete surface S. Then dim(W)≥6 and if dim(W) = 6, then either* $S \cong \mathbf{P}^2$ *and W gives the Veronese embedding into* $\mathbf{P}^5$*, or S is the intersection of 3 quadric hypersurfaces. In the latter case S is 2–spanned if and only if it contains no line; the set of such smooth complete intersection containing at least a line form a non–empty hypersurface in the variety of all smooth complete intersection of 3 quadric hypersurfaces in* $\mathbf{P}^5$.

In particular note that the Veronese surface is the only smooth surface in $\mathbf{P}^5$ such that all smooth embedded deformations in $\mathbf{P}^5$ are again 2–spanned.

Note that by definition if $S \subset \mathbf{P}^n$ is a 2–spanned surface, then there is no line $D \subset S$ and no line $R \not\subset S$ with length$(R \cap S) \geq 3$ (a "trisecant line"). Thus dim(W)>4. Consider S embedded in $\mathbf{P}^n$ by W. Then for any hyperplane H, $H \cap S$ contains no "trisecant line". For general H, $S \cap H$ is a smooth, irreducible space curve (Bertini's theorem). Set d:= deg(S) and let g be the genus of $S \cap H$.

First assume dim(W) = 5. By the genus formula for plane curves and the genus' formula for smooth plane curves , one sees immediately that for any point P of $S \cap H$, there is a "trisecant line" to $S \cap H$ passing through P, unless (d, g) = (3, 0) or (4, 1). If d = 3, then S is a minimal degree surface in $\mathbf{P}^4$, hence a scroll, hence it contains many lines, contradiction. Assume (d, g) = (4, 1). Since $S \cap H$ is linearly normal, so is S. Note that $S \cap H$ is the intersection of two quadrics. From the exact sequence

$$0 \longrightarrow \mathcal{I}_S(1) \longrightarrow \mathcal{I}_S(2) \longrightarrow \mathcal{I}_{S \cap H, H}(2) \longrightarrow 0 \qquad (1)$$

and Bezout theorem, one sees that S is the intersection of two quadrics. There are several ways to check that any such S contains a line, contradicting the 2–spannedness. One can work in the Grassmannian G of lines in $\mathbf{P}^4$, take as X the codimension 3 ([GH], p.739) cicle of lines contained in a fixed smooth quadric and note that $X^2 \neq 0$. Or one can use the fact that S is a Del Pezzo surface, obtained blowing–up 5 suitable points; a line on S is the strict transform on S of the conic through these 5 points ([D] in positive characteristic). Note that this part of the theorem, i.e. the inequality $\dim(W)\geq 6$, works even if the base field has positive characteristic.

Now assume $\dim(W) = 6$. By a theorem of Le Barz ([LB], p.182), the cicle of trisecant lines to $S\cap H$ has degree $t(d,g) := (d-2)(d-3)(d-4)/6 - g(d-4)$; since by assumption $S\cap H$ has a finite number of trisecant lines (indeed none) this number represents the number of trisecant lines to $S\cap H$, counted with suitable multiplicities. Thus $t(d,g) = 0$. If $d = 4$, then S is a minimal degree surface, hence either a Veronese surface or a scroll. Since a scroll contains infinitely many lines, the latter case is impossible. Thus we may assume $d>4$. Since $t(d,g) = 0$, we find $g = (d-2)(d-3)/6$. Let $m_1, \varepsilon_1, \mu_1$, be defined by: $m_1 := [(d-1)/4]$, $\varepsilon_1 := d-4m -1$, $\mu_1 := 1$ if $\varepsilon_1 = 3$, $\mu_1 := 0$ otherwise. Set $\pi_1(d,4) := m_1(m_1-1)/2 + m_1(\varepsilon_1+1) + \mu_1$. First assume $g>\pi_1(d,4)$. By [ACGH], p.123, if $d>10$ $S\cap H$ is contained in a minimal degree surface T of H, $\deg(T) = 3$. T is either a smooth scroll or the cone over a rational normal curve in $\mathbf{P}^3$. First assume that T is a cone. Let P be the vertex of T. Since $S\cap H$ has no "trisecant line", it is smooth, and $g>0$, $S\cap H$ does not contain P and intersects each line of T in two points. The minimal desingularization of T is the Hirzebruch surface F_3. $\mathrm{Pic}(F_3)$ has as basis h, f (f is the fiber while h is contracted to P) with the relations: $h^2 = -3$, $h\cdot f = 1$, $f^2 = 0$. Furthermore the pull-back of the hyperplane class of T is h+3f. The inverse image of $S\cap H$ in F_3 must have 2h+6f as class. Thus $g = 2$, contradicting the classification (up to a case) of all 2-spanned polarized surfaces with sectional genus at most 5, given in [BS], §5. Now assume that T is a smooth scroll. One check easily that T is isomorphic to the Hirzebruch surface F_1, with h and f as basis of Pic(T), with the relations $h^2 = -1$, $h\cdot f = 1$, $f^2 = 0$, and with h+2f as class of the hyperplane section. Since $H\cap S$ has $g>0$ and no "trisecant line", the class of $H\cap S$ is represented by 2h+bf for some integer b. Hence $d = b+2$. By the adjunction formula, we find $g = b-2$, contradiction. The case $d = 10$ cannot occur, since g is an integer.

Now assume d = 9, hence g = 7. Set m:= $[(d-1)/3]$, ε:= d-1-3m, $\pi(d,4) := 3m(m-1)/2 + m\varepsilon$([ACGH], p.116). Note that $\pi(9,4) = 7 = g$. By [ACGH], th.2.5(iii) p.122, $H \cap S$ has infinetely many trisecant lines, contradiction.

Now assume that $g \le \pi_1(d,4)$. Checking the 4 possible congruence classes of d mod(4), one sees that $d \le 8$, hence $g \le 5$. Furthermore d = 7 is excluded since g is an integer. If (d,g) = (5,1) or (6,2), the hyperplane bundle of $H \cap S$ is non-special, hence $S \cap H$ and S are linearly normal. If d = 8, $S \cap H$ (hence S) must be linearly normal both if the hyperplane section is special or not. Thus we may apply the results in [BS], §5. By [BS], §5, to prove theorem 1 it is sufficient to consider the case of a smooth S, S intersection of 3 quadric hypewrsurfaces in $\mathbf{P}^5$. Such a surface S is not 2-spanned if and only if it contains a line, since a line R is "trisecant" to S if and only if $R \subset S$ (Bezout's theorem). Let G:= G(1,5) be the Grassmannian of lines in $\mathbf{P}^5$, dim(G) = 8, and X(Q) the class of the lines contained in a fixed smooth quadric Q. X(Q) has codimension 3 in G ([GH], p.739). For every $g \in \mathrm{Aut}(\mathbf{P}^5)$ = Aut(G), we have g(X(Q)) = X(g(Q)). Hence by Bertini's theorem ([K]) the general intersection of 3 quadrics contains no line. Using Bertini's theorem one checks easily that there are smooth S containing a line, hence not 2-spanned; for completenss' sake, we give the details. Take a line R and let A be the set of quadrics containg R; take Q' and Q" general in A and set V:= $Q' \cap Q''$; V is smooth; since for every $x \in R$, the set A(x) of quadrics $Q \in A$ with $T_xQ = T_xV$ has codimension 2 in A, the general $Q \in A$ is not tangent to V along R; by Bertini's theorem for general $Q \in A$, $Q \cap V$ is smooth. Any such surface S is a K3 surface; any line $R \subset S$ has self-intersection -2 and x+1 as Hilbert polynomial; thus any such S contains at most finitely many lines. Counting dimensions, we see that the set of such S containing at least a line has codimension 1 in the set of smooth complete intersection of 3 quadrics.

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