The new proof of the main theorem about Exceptional and Rigid Sheaves on $\mathbb{P}^2$

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Abstract

The author gives the new method of proof of the constructibility of exceptional bundles and exceptional collections on $\mathbb{P}^2$.

Introduction.

Exceptional bundles on $\mathbb{P}^2$ were first investigated by Dreset and Le Potier in [3] where they gave a description of these bundles in terms of their slopes.

Using these results A. L. Gorodentsev and A. N. Rudakov [1] gives a constructive description of the set of exceptional bundles on $\mathbb{P}^2$ by showing that they can all be obtained from invertible sheaves by the canonical operations of mutation. This description uses the fact that exceptional bundles on $\mathbb{P}^2$ can be put together into so-called exceptional triples such that the ranks in each triple yield a solution to the Markov equation. Moreover, the mutations of the solutions of the Markov equation one to one correspond to the canonical mutations of the exceptional triples. Because of the set of all solutions of the Markov equation can be obtained from the one of them by mutations, the authors get that any exceptional triple of bundles is obtained from one of them by mutations.

Further, using the stability of exceptional bundles on the projective plane A. L. Gorodentsev and A. N. Rudakov prove that any exceptional bundle E on $\mathbb{P}^2$ is included in an exceptional triple. Whence E is obtained from the line bundles by mutations.

Besides, the authors generalize the exceptional triples and their mutations to exceptional collections of bundles on $\mathbb{P}^n$. But the question about constructive description of exceptional bundles on $\mathbb{P}^n$ when $n > 2$ is open.

In the present paper the author gives the new proof of the constructibility of exceptional bundles on $\mathbb{P}^2$, which does not use the Markov equation and the stability of exceptional bundles. The author hopes for generalization this proof to exceptional bundles on $\mathbb{P}^n$.

Notations.

Let $r(F)$ be the rank of a coherent sheaf F on $\mathbb{P}^n$;

$\mathcal{O}$ be the trivial line bundle on $\mathbb{P}^n$;

$\mathcal{O}(1)$ be the line bundle corresponding to the generator of the Picar group of $\mathbb{P}^n$;

$\mathcal{O}(n) \stackrel{def}{=} \mathcal{O}(1)^{\otimes n}$;
$F(n)$ is the tensor product of a sheaf F and $\mathcal{O}(n)$;
F^* is the dual sheaf, that is the sheaf of local homomorphisms $\mathcal{H}om_{\mathcal{O}}(F, \mathcal{O})$;
$\operatorname{Hom}(E, F)$ is the space of global maps from E to F;
$h^i(E, F)$ denote the dimension of space $\operatorname{Ext}^i(E, F)$;
$\chi(E, F)$ is the Euler characteristic of any two sheaves, which equals $\sum(-1)^i h^i(E, F)$;
$\chi(E)$ is the Euler characteristic of sheaf, which equals $\chi(\mathcal{O}, E)$;
We identify a bundle with the sheaf of its local sections. Sometimes we will arrange a long cohomology sequence associated to an exact triple into a table. For example, the application of functor $\operatorname{Ext}^\cdot(F, \cdot)$ to the exact triple

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives

k	$\operatorname{Ext}^k(F, A)$	$\rightarrow$	$\operatorname{Ext}^k(F, B)$	$\rightarrow$	$\operatorname{Ext}^k(F, C)$
0	$*$		?		$*$
1	0		?		0
2	$*$		?		$*$

This table calculates $\operatorname{Ext}^1(F, B)$. In particular, $\operatorname{Ext}^1(F, B) = 0$.

1 Preliminary Information.

In this section using the results of A. L. Gorodentsev and A. N. Rudakov ([2], [1]) we provide the initial information about exceptional bundles and exceptional collections on $\mathbb{P}^n$.

DEFINITION. A coherent sheaf E on $\mathbb{P}^n$ is called *exceptional* if

$$\operatorname{Ext}^0(E, E) = \mathbb{C} \quad \text{and} \quad \operatorname{Ext}^i(E, E) = 0 \quad \text{for} \quad i > 0.$$

It is easy to see that these cohomological conditions imply that E is homogeneous, and therefore an exceptional sheaf on $\mathbb{P}^n$ is automatically locally free. Because of this we shall speak about exceptional bundles on $\mathbb{P}^n$.

DEFINITION. An ordered collection of bundles $\sigma = (E_0, E_1, E_2, \dots, E_k)$ is called an *exceptional collection* if all the bundles are exceptional and

$$\operatorname{Ext}^k(E_i, E_j) = 0 \quad \text{if} \quad k \geq 1,$$

$$\operatorname{Ext}^k(E_j, E_i) = 0 \quad \text{if} \quad k \geq 0$$

when $0 \leq i < j \leq k$.

It is clear that if $\sigma = (E_0, E_1, E_2, \dots, E_k)$ is an exceptional collection then the collections $\sigma^* = (E_k^*, \dots, E_1^*, E_0^*)$ and $\sigma(m) = (E_0(m), E_1(m), \dots, E_k(m))$ are also exceptional.

DEFINITION. Let (A, B) be an exceptional pair on $\mathbb{P}^n$. Suppose there exist the following exact sequences of bundles:

$$0 \longrightarrow L_A B \longrightarrow \operatorname{Hom}(A, B) \otimes A \xrightarrow{can} B \longrightarrow 0,$$

$$0 \longrightarrow A \xrightarrow{can} \operatorname{Hom}(A, B)^* \otimes B \longrightarrow R_B A \longrightarrow 0,$$

where the map $can \in \mathrm{Hom}(A, B)^* \otimes \mathrm{Hom}(A, B)$ corresponds to the identity endomorphism of the vector space $\mathrm{Hom}(A, B)$. In this case the bundle $L_A B$ is called the *left shift* of B and the bundle $R_B A$ is called the *right shift* of A. The pairs $(L_A B, A)$ and $(B, R_B A)$ are called the left and right mutations of (A, B).

1.1 PROPOSITION. *Let $\sigma = (E_0, \ldots, E_i, E_{i+1}, \ldots, E_k)$ be an exceptional collection.*

1. *If the left mutation of (E_i, E_{i+1}) is defined then the collection*

$$L_i\sigma = (E_0, \ldots, L_{E_i}E_{i+1}, E_i, \ldots, E_k)$$

is also exceptional, and the right mutation of the pair $(L_{E_i}E_{i+1}, E_i)$ is defined and equals (E_i, E_{i+1}).

2. *If the right mutation of (E_i, E_{i+1}) is defined then the collection*

$$R_{i+1}\sigma = (E_0, \ldots, E_{i+1}, R_{E_{i+1}}E_i, \ldots, E_k)$$

is also exceptional, and the left mutation of the pair $(E_{i+1}, R_{E_{i+1}}E_i)$ is defined and equals (E_i, E_{i+1}).

The collections $L_i\sigma$ and $R_{i+1}\sigma$ are called the *left* and *right mutations* of σ.

DEFINITION. Let $\sigma = (E_0, E_1, E_2, \ldots, E_n)$ be an exceptional collection of bundles on $\mathbb{P}^n$. It is called *full* provided the bounded derived category $D^b(\mathbb{P}^n)$ of sheaves on $\mathbb{P}^n$ is generated by σ, i.e., the set of all objects of $D^b(\mathbb{P}^n)$ can be obtained from the elements of σ by taking the direct sum and forming cones of all possible morphisms.

For example, the collection of the line bundles on $\mathbb{P}^n$ $\sigma_0 = (\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ is the full exceptional collection. Further, we call this collection the *basic collection*.

We see that the left mutation $L_i\sigma_0$ is defined for $i = 0, \ldots, n-1$ and the right mutation $R_i\sigma$ is defined for $i = 1, \ldots, n$.

DEFINITION. Suppose that for an exceptional collection of bundles on $\mathbb{P}^n$

$$\sigma = (E_0, E_1, E_2, \ldots, E_k)$$

there exist exceptional bundles $E_{k+1}, \ldots, E_n$ such that the collection

$$\sigma' = (E_0, \ldots, E_k, E_{k+1}, \ldots, E_n)$$

is exceptional and full. Assume, in addition, that σ' is obtained from the basic collection σ_0 by mutations. Then the collection σ is called *constructible*.

1.2 PROPOSITION. *Let $\sigma = (E_0, E_1, E_2, \ldots, E_k)$ be a constructible exceptional collection of bundles on $\mathbb{P}^n$. Then*

1. *the left mutation $L_i\sigma$ is defined for $i = 0, \ldots, k-1$ and the right mutation $R_i\sigma$ is defined for $i = 1, \ldots, k$,*

2. $\dim \mathrm{Hom}(E_i, E_{i+1}) > 2$ *for $i = 0, 1, \ldots, k-1$.*

1.3 REMARK *Let* $\sigma = (E_0, E_1, E_2, \dots, E_k)$ *be a constructible exceptional collection, then*

1. *each of the bundles* $E_1, E_2, \dots, E_k$ *can be obtained by the right shift over* E_0*, i.e., there exists the constructible exceptional collection* $\sigma_l = (E_{-1}, \dots, E_{-k}, E_0)$ *such that*
$$R_1 \cdots R_{k-1} R_k \sigma_l = \sigma;$$
2. *each of the bundles* $E_0, E_1, \dots, E_{k-1}$ *can be obtained by the left shift over* E_k*.*

1.4 PROPOSITION. *Let* $\sigma = (E_0, E_1, E_2, \dots, E_n)$ *be a full constructible exceptional collection of bundles on* $\mathbb{P}^n$*. Then*

1. *the left and right mutations of* σ *are full exceptional collections.*
2. *the following relations are valid:*
$$R_n \cdots R_2 R_1 \sigma = (E_1, \dots, E_n, E_0(n+1)),$$
$$L_0 \cdots L_{n-2} L_{n-1} \sigma = (E_n(-n-1), E_0, \dots, E_{n-1}).$$

Because of the last properties we can consider an infinite periodic collection of exceptional bundles on $\mathbb{P}^n$
$$\dots E_{-1}, E_0, \dots, E_n, E_{n+1}, \dots$$
such that $\forall k \quad E_{k+n+1} = E_k(n+1)$ and $(E_k, \dots, E_{k+n})$ is the full exceptional collection. This collection is colled a helix.

1.5 PROPOSITION. *If* $\dots E_{-1}, E_0, \dots, E_n, E_{n+1}, \dots$ *is a helix, then*

1. *for any pair* $i \le j$ *the canonical map* $can : \mathrm{Hom}(E_i, E_j) \otimes E_i \to E_j$ *is the epimorphism;*
2. *for any pair* $i \le j$ *the canonical map* $can : E_i \to \mathrm{Hom}(E_i, E_j)^* \otimes E_j$ *is the monomorphism.*

1.6 LEMMA. *Let* (A, B, C) *be a constructible exceptional collection on* $\mathbb{P}^n$ *then*
$$L_A L_B C = L_{B'} L_A C, \qquad \text{where} \qquad B' = L_A B,$$
$$R_C R_B A = R_{B''} R_C A, \qquad \text{where} \qquad B'' = R_C B.$$

Looking at this lemma we see that the mutations of constructible exceptional collections define the action of the braid group on the set of all constructible exceptional collections with fixed number of elements. In particular, the braid group acts on the set of all full constructible exceptional collections.

DEFINITION. We shall say that a subcategory in $D^b(\mathbb{P}^n)$ is generated by an exceptional collection $\sigma = (E_0, E_1, E_2, \dots, E_k)$ if any its object can be obtained from the elements of

σ by taking the direct sum and forming cones of all possible morphisms. This subcategory we denote by $\langle E_0, E_1, \ldots, E_k \rangle_{cat}$.

1.7 PROPOSITION. *Let $(E_0, E_1, E_2, \ldots, E_n)$ be a full exceptional collection and F be a sheaf. Then*

1. $\Big(\mathrm{Ext}^k(F, E_i) = 0 \quad \forall k \quad \text{and} \quad i = 0, 1, \ldots, s\Big) \iff \Big(F \in \langle E_{s+1}, E_{s+2}, \ldots, E_n \rangle_{cat}\Big)$,
2. $\Big(\mathrm{Ext}^k(E_i, F) = 0 \quad \forall k \quad \text{and} \quad i = s+1, \ldots, n\Big) \iff \Big(F \in \langle E_0, E_1, \ldots, E_s \rangle_{cat}\Big)$.

DEFINITION. Let $\sigma = (E_0, E_1, E_2, \ldots, E_k)$ be a constructible exceptional collection. The collection $\sigma^\vee = (E_k^\vee, \ldots, E_1^\vee, E_0)$, where

$$E_1^\vee = L_{E_0} E_1, \; E_2^\vee = L_{E_0} L_{E_1} E_2, \; \ldots, \; E_k^\vee = L_{E_0} L_{E_1} \cdots L_{E_{k-1}} E_k$$

is called right dual to σ. The collection ${}^\vee\sigma = (E_k, {}^\vee E_{k-1}, \ldots, {}^\vee E_0)$, where

$${}^\vee E_{k-1} = R_{E_k} E_{k-1}, \; \ldots, \; {}^\vee E_0 = R_{E_k} \ldots R_{E_1} E_0$$

is called left dual to σ.

1.8 REMARK. *It is easy to see that the left dual collection to $\sigma^\vee$ and the right dual collection to ${}^\vee\sigma$ are equal to σ. Besides,*

$$\mathrm{Ext}^s(E_i, E_j^\vee) = \begin{cases} 0 & \text{when} \quad i \neq j \\ \mathbb{C} & \text{when} \quad i = j = s \end{cases}.$$

In the previous notation the following theorem hold.

1.9 THEOREM. *Let $\sigma = (E_0, E_1, E_2, \ldots, E_k)$ be a constructible exceptional collection of bundles on $\mathbb{P}^n$. Then for any sheaf Q belonging to the subcategory generated by σ ($Q \in \langle E_0, E_1, \ldots, E_k \rangle_{cat}$) there exist two spectral sequences associated to the right and left dual collections to σ. The E_1-term of the first sequence has the form*

$$E_1^{p,q} = \mathrm{Ext}^q(E_{-p}, Q) \otimes E_{-p}^\vee.$$

The E_1-term of the second sequence has the form

$$E_1^{p,q} = \mathrm{Ext}^{k-q}(Q, E_{-p})^* \otimes {}^\vee E_{-p}.$$

Both these sequences converge to Q on the main diagonal, i.e., $E_\infty^{p,q} = 0$ for $p + q \neq 0$ and $Gr(Q) = (E_\infty^{0,0}, E_\infty^{-1,1}, \ldots, E_\infty^{-n,n})$.

2 Subcategory Generated by Pair.

In this section we study the subcategory generated by a constructible exceptional pair on $\mathbb{P}^n$ and prove that any rigid sheaf belonging to this subcategory is the direct sum of exceptional bundles.

DEFINITION. A sheaf F on $\mathbb{P}^n$ is called *rigid* if $\mathrm{Ext}^1(F,F)=0$. It is called *superrigid* if $\mathrm{Ext}^i(F,F)=0$ when $i>0$.

Let us introduce some notation using in this section. Consider a constructible exceptional pair (E_0, E_1) on $\mathbb{P}^n$. By definition, put

$$E_{n+1} = R_{E_n}E_{n-1} \quad \text{for} \quad n>0 \quad \text{and} \quad E_{-(n+1)} = L_{E_{-n}}E_{1-n} \quad \text{for} \quad n \geq 0.$$

Denote by e_n the images of E_n in $K \stackrel{def}{=} K_0(\mathbb{P}^n)\otimes\mathbb{Q}$. The vector space K inherits the bilinear form $\chi(\cdot,\cdot)$. Denote it by $(\cdot,\cdot)$. By definition of the exceptional pair and 1.2 we have

$$(e_0,e_0)=(e_1,e_1)=1, \quad (e_1,e_0)=0, \quad (e_0,e_1)=h>2.$$

Furthermore, by definition of the mutation, we obtain that

$$e_{n+1}=he_n-e_{n-1}, \quad (e_n,e_n)=1, \quad (e_{n+1},e_n)=0, \quad (e_n,e_{n+1})=h.$$

Let F be a sheaf belonging to $\langle E_0,E_1\rangle_{cat}$. Denote by f the image of F in K. It is obvious that f belongs to the linear span $\langle e_0,e_1\rangle \subset K$.

In the above notation the following lemma is valid.

2.1 LEMMA. *The inequality $(f,f)>0$ holds if and only if there exist an integer n and numbers a, b such that $f=ae_n+be_{n+1}$ with $a\cdot b\geq 0$.*

PROOF. It is easy to prove by induction on n that

$$e_n = x_ne_1 - x_{n-1}e_0 \quad (n\geq 1) \quad \text{and} \quad e_{-n}=x_{n+1}e_0-x_ne_1 \quad (n\geq 0),$$

where $x_0=0,\ x_1=1,\ x_{n+1}=hx_n-x_{n-1}>0$.

It can be proved that the sequence $\frac{x_n}{x_{n-1}}$ decreases and the sequence $\frac{x_n}{x_{n+1}}$ increases. Let us calculate the following limit

$$l_+ = \lim_{n\to\infty}\frac{x_n}{x_{n-1}} = h - \lim_{n\to\infty}\frac{x_n}{x_{n+1}} = h-l_- = h-\frac{1}{l_+}.$$

Therefore, l_+ and l_- are the roots of the equation $x^2-hx+1=0$.

Thus we obtain that for all n

$$\frac{x_n}{x_{n-1}} > l_+ > l_- > \frac{x_n}{x_{n+1}}.$$

This implies that for any vector $f=xe_1-e_0$ with $x>l_+$ there exist a positive integer n and nonnegative a, b such that $f=ae_n+be_{n+1}$. Analogously, if $f=e_0-xe_1$ with $0<x<l_-$, then $f=ae_{-n}+be_{1-n}$ for some positive integer n and nonnegative a, b.

On the other hand, if $f=\pm(xe_1-e_0)$ with positive x, then $(f,f)=x^2-hx+1$, i.e., $(f,f)\leq 0$ if and only if $x\in[l_-,l_+]$. This completes the proof.

2.2 Corollary. *1. Suppose that for any integer $n \geq 0$ there exist positive numbers a_n, b_{n+1} such that $f = b_{n+1}e_{n+1} - a_n e_n$; then $f^2 \leq 0$.*

2. Suppose that for any integer $n < 0$ there exist positive numbers a_n, b_{n+1} such that $f = a_n e_n - b_{n+1}e_{n+1}$; then $f^2 \leq 0$.

2.3 Corollary. *Suppose that $F \in \langle E_0, E_1\rangle_{cat}$ and $\chi(F,F) > 0$; then there exists an integer n such that $\chi(E_n, F) \geq 0$ and $\chi(E_{n+1}, F) \leq 0$.*

Proof. It follows from the previous lemma that there exists an integer m and non-negative a, b such that $f = ae_m + be_{m+1}$. It can easily be checked that $(e_{m+1}, f) = b$ and $(e_{m+2}, f) = -a$. On the other hand, $(e_k, f) = \chi(E_k, F)$ for all integer k. Hence the statement holds provided $n = m+1$.

2.4 Lemma. *Suppose a sheaf F belongs to the subcategory $\langle E_0, E_1\rangle_{cat}$; then*

1. $\mathrm{Ext}^k(F,F) = \mathrm{Ext}^k(E_n, F) = \mathrm{Ext}^k(F, E_n) = 0 \quad \forall n \in \mathbb{Z}$ *and* $k \geq 2$,
2. *if* $\chi(E_0, F) > 0$ *and* $\chi(E_1, F) < 0$, *then F is decomposable.*

Proof. Since $\langle E_0, E_1\rangle_{cat} = \langle E_n, E_{n+1}\rangle_{cat} \quad \forall n \in \mathbb{Z}$, it is sufficiently to check the first statement for $n = 0$.

Consider the spectral sequence associated to the right dual pair $(E_1^\vee, E_0) = (E_{-1}, E_0)$ (by our notation) to (E_0, E_1) and converging to F (1.9). Its E_1-term has the form:

$$\begin{array}{lcl} E_1^{-1,n} = \mathrm{Ext}^n(E_1, F) \otimes E_{-1} & \xrightarrow{d} & E_1^{0,n} = \mathrm{Ext}^n(E_0, F) \otimes E_0 \\ \cdots & \cdots & \cdots \\ E_1^{-1,1} = \mathrm{Ext}^1(E_1, F) \otimes E_{-1} & \xrightarrow{d} & E_1^{0,1} = \mathrm{Ext}^1(E_0, F) \otimes E_0 \\ E_1^{-1,0} = \mathrm{Ext}^0(E_1, F) \otimes E_{-1} & \xrightarrow{d} & E_1^{0,0} = \mathrm{Ext}^0(E_0, F) \otimes E_0 \end{array} . \qquad (1)$$

The fact that this sequence converges on the main diagonal implies that $E_1^{-1,q} \cong E_1^{0,q}$ when $q \geq 2$. Since $E_{-1} \not\cong E_0$, we obtain that $\mathrm{Ext}^q(E_1, F) = \mathrm{Ext}^q(E_0, F) = 0$ for $q \geq 2$.

Similarly, to prove the equality $\mathrm{Ext}^k(F, E_0) = 0 \quad \forall k > 1$ we can consider the spectral sequence associated to the left dual pair to (E_0, E_1).

Besides, spectral sequence (1) splits into 3 exact triples:

$$0 \longrightarrow F_1 \longrightarrow V_1 \otimes E_{-1} \longrightarrow W_1 \otimes E_0 \longrightarrow 0,$$

$$0 \longrightarrow V_0 \otimes E_{-1} \longrightarrow W_0 \otimes E_0 \longrightarrow F_2 \longrightarrow 0,$$

$$0 \longrightarrow F_2 \longrightarrow F \longrightarrow F_1 \longrightarrow 0,$$

where $V_q = \mathrm{Ext}^q(E_1, F)$; $W_q = \mathrm{Ext}^q(E_0, F)$.

To show that $\mathrm{Ext}^1(F_1, F_2) = 0$ we use the cohomology tables associated with these exact triples.

k	$V_0 \otimes \mathrm{Ext}^k(E_0, E_{-1})$	$\rightarrow$	$W_0 \otimes \mathrm{Ext}^k(E_0, E_0)$	$\rightarrow$	$\mathrm{Ext}^k(E_0, F_2)$
0	0		$*$		?
1	0		0		?
2	0		0		?
...	...		...		...
n	0		0		?

k	$V_0 \otimes \mathrm{Ext}^k(E_{-1}, E_{-1})$	$\to$	$W_0 \otimes \mathrm{Ext}^k(E_{-1}, E_0)$	$\to$	$\mathrm{Ext}^k(E_{-1}, F_2)$
0	$*$		$*$		?
1	0		0		?
2	0		0		?
...	...		...		...
n	0		0		?

k	$W_1^* \otimes \mathrm{Ext}^k(E_0, F_2)$	$\to$	$V_1^* \otimes \mathrm{Ext}^k(E_{-1}, F_2)$	$\to$	$\mathrm{Ext}^k(F_1, F_2)$
0	$*$		$*$		?
1	0		0		?
2	0		0		?
...	...		...		...
n	0		0		?

Thus we get, $\mathrm{Ext}^k(F_1, F_2) = 0$ when $k > 0$ and $F = F_1 \oplus F_2$ (but either F_1 or F_2 can be equal to the zero sheaf).

In the same way we can check that $\mathrm{Ext}^k(F_2, F_1) = 0$ when $k > 1$

k	$\mathrm{Ext}^k(F_2, F_2)$	$\to$	$W_0^* \otimes \mathrm{Ext}^k(E_0, F_2)$	$\to$	$V_0^* \otimes \mathrm{Ext}^k(E_{-1}, F_2)$
0	?		$*$		$*$
1	?		0		0
2	?		0		0
...	...		...		...
n	?		0		0

This table implies that $\mathrm{Ext}^k(F_2, F_2) = 0$ when $k \geq 2$. The analogous fact for F_1 is checked in the similar way.

Since $F = F_1 \oplus F_2$, the above calculation implies the first statement of the lemma.

Suppose now that $\chi(E_0, F) > 0$ and $\chi(E_1, F) < 0$. It follows from the first statement that

$$\chi(E_0, F) = h^0(E_0, F) - h^1(E_0, F).$$

Hence $\mathrm{Ext}^0(E_0, F) \neq 0$, i.e., $F_2 \neq 0$. Similarly,

$$\chi(E_1, F) = h^0(E_1, F) - h^1(E_1, F).$$

Therefore $\mathrm{Ext}^1(E_1, F) \neq 0$ and $F_1 \neq 0$. This completes the proof.

2.5 PROPOSITION. *Let F be a rigid sheaf belonging to the subcategory generated by an exceptional pair (E_0, E_1); then*

1. *The sheaf F is superrigid.*

2. *The sheaf F is the direct sum $F = x_n E_n \oplus x_{n+1} E_{n+1}$, where x_n, x_{n+1} are nonnegative integers and (E_n, E_{n+1}) is an exceptional pair of bundles obtained from (E_0, E_1) by mutations.*

PROOF. The first statement follows from the assumptions and the previous lemma. Hence $\chi(F,F) = h^0(F,F) > 0$. By corollary 2.3 there exists an integer n such that $\chi(E_n, F) \geq 0$ and $\chi(E_{n+1}, F) \leq 0$.

Assume now that F is indecomposable, then it follows from lemma 2.4 that either $\chi(E_n, F)$ or $\chi(E_{n+1}, F)$ equals 0. Without loss of generality it can be assumed that $\chi(E_n, F) > 0$ and $\chi(E_{n+1}, F) = 0$.

On the other hand, lemma 2.1 implies that the images F in K has the form $f = ae_m + be_{m+1}$ for some integer m and nonnegative a, b. It is easy to prove by induction on n that

$$\mathrm{Ext}^0(E_k, E_m) \neq 0 \iff k \leq m, \tag{2}$$

$$\mathrm{Ext}^1(E_k, E_m) \neq 0 \iff k > m+1. \tag{3}$$

Therefore,

$$\chi(E_k, E_m) = (e_k, e_m) = \begin{cases} 0 & \text{when } k - m = 1 \\ h_{km} > 0 & \text{when } k \leq m \\ h_{km} < 0 & \text{when } k > m+1 \end{cases}.$$

Thus the indecomposable superrigid sheaf F from $\langle E_0, E_1 \rangle_{cat}$ has the form E_n for an integer n. Hence in the general case we get $F = \bigoplus_{i=1}^{s} x_{n_i} E_{n_i}$. Now we see that conditions (2), (3) and the fact that F is superrigid imply that $s \leq 2$, and $|n_1 - n_2| \leq 1$, i.e.,

$$F = x_n E_n \oplus x_{n+1} E_{n+1}.$$

This concludes the proof.

3 Universal Extension.

DEFINITION. Let E and F be a objects of an Abelian category. The extensions determined by the element of the groups

$$\mathrm{Ext}^1(F, \mathrm{Ext}^1(F, E)^* \otimes E) \cong \mathrm{Ext}^1(F, E)^* \otimes \mathrm{Ext}^1(F, E)$$

and

$$\mathrm{Ext}^1(\mathrm{Ext}^1(F, E) \otimes F, E) \cong \mathrm{Ext}^1(F, E)^* \otimes \mathrm{Ext}^1(F, E)$$

that corresponds to the identity endomorphism of the vector space $\mathrm{Ext}^1(F, E)$:

$$0 \longrightarrow \mathrm{Ext}^1(F, E)^* \otimes E \longrightarrow F' \longrightarrow F \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow E \longrightarrow F'' \longrightarrow \mathrm{Ext}^1(E, E) \otimes F \longrightarrow 0 \tag{5}$$

are called the *universal extensions.*

3.1 REMARK. *For sequence (4) the coboundary homomorphism*

$$\delta : \mathrm{Hom}(E, E) \otimes \mathrm{Ext}^1(F, E) \longrightarrow \mathrm{Ext}^1(F, E)$$

is epimorphism.

For sequence (5) the coboundary homomorphism

$$\delta : \mathrm{Ext}^1(F,E) \otimes \mathrm{Hom}(F,F) \longrightarrow \mathrm{Ext}^1(F,E)$$

is the epimorphism also.

3.2 LEMMA. *Let E be an exceptional and F be a rigid sheaves on $\mathbb{P}^2$ such that $\mathrm{Ext}^q(E,F) = 0$ for $q > 0$ and $\mathrm{Ext}^1(F,E) \neq 0$; then the sheaf F' from the universal extension (4) satisfies the following conditions:*

1. $\mathrm{Ext}^1(F',E) = 0$ *and* $\mathrm{Ext}^q(F',E) \cong \mathrm{Ext}^q(F,E)$ *when* $q = 0, 2$,
2. *the map* $\mathrm{Hom}(F',E) \longrightarrow \mathrm{Ext}^1(F,E) \otimes \mathrm{Hom}(E,E)$ *that is obtained after the application of the functor* $\mathrm{Ext}^{\cdot}(\cdot,E)$ *to (4) is trivial,*
3. $\mathrm{Ext}^1(F',F) = 0$ *and* $\mathrm{Ext}^2(F',F) \cong \mathrm{Ext}^2(F,F)$,
4. $\mathrm{Ext}^1(F',F') = 0$ *and* $\mathrm{Ext}^2(F',F') \cong \mathrm{Ext}^2(F,F) \oplus \mathrm{Ext}^1(F,E)^* \otimes \mathrm{Ext}^2(F,E)$.

PROOF. The proof is immediate if we consider the following cohomology tables:

k	$\mathrm{Ext}^k(F,E)$	$\to$	$\mathrm{Ext}^k(F',E)$	$\to$	$\mathrm{Ext}^1(F,E)\otimes\mathrm{Ext}^k(E,E)$
0	$\mathrm{Ext}^0(F,E)$		?		$\mathrm{Ext}^1(F,E)$
1	$\mathrm{Ext}^1(F,E)$		?		0
2	$\mathrm{Ext}^2(F,E)$		?		0

,

k	$\mathrm{Ext}^k(F,F)$	$\to$	$\mathrm{Ext}^k(F',F)$	$\to$	$\mathrm{Ext}^1(F,E)\otimes\mathrm{Ext}^k(E,F)$
0	*		?		*
1	0		?		0
2	$\mathrm{Ext}^2(F,F)$		?		0

,

k	$\mathrm{Ext}^1(F,E)^*\otimes\mathrm{Ext}^k(F',E)$	$\to$	$\mathrm{Ext}^k(F',F')$	$\to$	$\mathrm{Ext}^k(F',F)$
0	*		?		*
1	0		?		0
2	$\mathrm{Ext}^1(F,E)^*\otimes\mathrm{Ext}^2(F,E)$		?		$\mathrm{Ext}^2(F,F)$

.

It is not hard to prove the dual statement:

3.3 LEMMA. *Let E be an exceptional and F be a rigid sheaves on $\mathbb{P}^2$ such that $\mathrm{Ext}^q(F,E) = 0$ for $q > 0$ and $\mathrm{Ext}^1(E,F) \neq 0$; then the sheaf F' from the universal extension*

$$0 \longrightarrow F \longrightarrow F' \longrightarrow \mathrm{Ext}^1(E,F) \otimes E \longrightarrow 0 \qquad (6)$$

satisfies the following conditions:

1. $\mathrm{Ext}^1(E,F') = 0$ *and* $\mathrm{Ext}^q(E,F') \cong \mathrm{Ext}^q(E,F)$ *when* $q = 0, 2$,
2. *the map* $\mathrm{Hom}(E,F') \longrightarrow \mathrm{Ext}^1(E,F) \otimes \mathrm{Hom}(E,E)$ *that is obtained after the application of the functor* $\mathrm{Ext}^{\cdot}(E,\cdot)$ *to (6) is trivial.*

3. $\mathrm{Ext}^1(F, F') = 0$ *and* $\mathrm{Ext}^2(F, F)' \cong \mathrm{Ext}^2(F, F)$,

4. $\mathrm{Ext}^1(F', F') = 0$ *and* $\mathrm{Ext}^2(F', F') \cong \mathrm{Ext}^2(F, F) \oplus \mathrm{Ext}^1(E, F)^* \otimes \mathrm{Ext}^2(E, F)$.

3.4 LEMMA. *Let* $0 \longrightarrow \mathrm{Ext}^1(F, E)^* \otimes E \xrightarrow{\varphi} F' \longrightarrow F \longrightarrow 0$ *be the universal extension of an exceptional bundle* E *and a rigid sheaf* F *on* $\mathbb{P}^2$; *then* F' *cannot be equal to* $E \oplus F''$ *provided* $\mathrm{Ext}^1(F, E) \neq 0$.

PROOF. Suppose $F' = E \oplus F''$. Denote by π the projection $\pi : F' \to E \to 0$. Since the extension is universal, we obtain that the map

$$\mathrm{Hom}(F', E) \longrightarrow \mathrm{Ext}^1(F, E) \otimes \mathrm{Hom}(E, E)$$

is trivial. In particular, $\pi \cdot \varphi = 0$. Therefore F is the direct sum $F = E \oplus F_1$ and $\mathrm{Ext}^1(F, E) \subset \mathrm{Ext}^1(F, F) = 0$. This contradition concludes the proof.

The dual statement is formulated in the following way.

3.5 LEMMA. *Let*

$$0 \longrightarrow F \longrightarrow F' \longrightarrow \mathrm{Ext}^1(E, F) \otimes E \longrightarrow 0$$

be the universal extension of an exceptional bundle E *and a rigid sheaf* F *on* $\mathbb{P}^2$; *then* F' *cannot be equal to* $E \oplus F''$ *provided* $\mathrm{Ext}^1(E, F) \neq 0$.

4 Conditions of Decomposability of Sheaves.

4.1 LEMMA. *Let* F *be a sheaf and* E *be a simple sheaf* ($\mathrm{Hom}(E, E) \cong \mathbb{C}$) *on a complex variety* X. *Suppose that* $\mathrm{Hom}(F, E) \neq 0$ *and* $\mathrm{Hom}(E, F) \neq 0$, *then* $F = F' \oplus E$ *whenever one of the following conditions holds: for some finite-dimensional vector space* V *there exists either an epimorphism* $\alpha : V \otimes E \to F$ *or a monomorphism* $\beta : F \to V \otimes E$.

PROOF. Since the statements of the lemma are dual, we can prove the first of them only.

Let φ be a nontrivial morphism from F to E, then the composition $\varphi \cdot \alpha : V \otimes E \to E$ is nontrivial as well. Let us show that there exists an inclusion $i : E \to V \otimes E$ such that $\varphi \cdot \alpha \cdot i \neq 0$.

Consider an arbitrary inclusion $i' : E \to V \otimes E$ suth that the following exact triple splits

$$0 \longrightarrow E \xrightarrow{i'} E \oplus V' \otimes E \xrightarrow{\pi} V' \otimes E \longrightarrow 0.$$

If $\varphi \cdot \alpha \cdot i' = 0$, then there exists the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{i'} & E \oplus V' \otimes E & \xrightarrow{\pi} & V' \otimes E & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle \varphi\cdot\alpha}\downarrow & & {\scriptstyle \psi}\downarrow & & \\ & & 0 & \longrightarrow & E & \longrightarrow & E & \longrightarrow & 0 \end{array}.$$

Denote by π^{-1} the morphism from $V' \otimes E$ to $V \otimes E$ such that $\pi \cdot \pi^{-1} = id$. It is clear that $\psi = \pi^{-1} \cdot \varphi \cdot \alpha \neq 0$. Since $\dim V' < \dim V$, by inductive hypothesis there exists inclusion $i'' \colon E \to V' \otimes E$ such that $\pi^{-1} \cdot \varphi \cdot \alpha \cdot i'' \neq 0$. Hence $i = \pi^{-1} \cdot i''$ is the required map.

Thus we have the sequence:

$$E \xrightarrow{i} V \otimes E \xrightarrow{\alpha} F \xrightarrow{\varphi} E.$$

By definition, put $\theta = \varphi \cdot \alpha \cdot i \neq 0$. Since θ is the nontrivial endomorphism of the simple sheaf E, we get that $\theta = \lambda \cdot id_E$ for $\lambda \in \mathbb{C}^*$. Without loss of generality we can assume that $\lambda = 1$.

Thus we have the morphisms $\alpha \cdot i : E \to F$ and $\varphi : F \to E$ such that $\varphi \cdot (\alpha \cdot i) = id_E$. Therefore, $F = F' \oplus E$.

4.2 LEMMA. *Let F be a nonzero sheaf on $\mathbb{P}^n$ and $\sigma = (E_{-n}, \dots, E_0)$ be a full constructible exceptional collection on $\mathbb{P}^n$. Suppose $\mathrm{Ext}^q(F, E_{-i}) = 0$ when $q \geq i > 0$, and $\mathrm{Ext}^q(F, E_0) = 0$ when $q > 0$; then $\mathrm{Hom}(F, E_0) \neq 0$ and either $\mathrm{Hom}(E_0, F) = 0$, or $F = F' \oplus xE_0$ for some sheaf F' and a positive integer x such that $\mathrm{Hom}(E_0, F') = 0$ and*

$$F' \in \langle E_{-n}, \dots, E_{-1} \rangle_{cat}.$$

PROOF. Consider the spectral sequence associated to the left dual collection (which we denote here by $(E_0, E_1, E_2, \dots, E_n)$) to σ and converging to F (1.9). The diagram of the E_1-term looks as follows:

$V_{00} \otimes E_0$	$V_{01} \otimes E_1$	$V_{02} \otimes E_2$	$\dots$	$V_{0n} \otimes E_n$
$E_1^{-n,n-1} = 0$	$E_1^{1-n,n-1} = 0$	$V_{12} \otimes E_2$	$\dots$	$V_{1n} \otimes E_n$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$E_1^{-n,0} = 0$	$E_1^{1-n,0} = 0$	$E_1^{2-n,0} = 0$	$\dots$	$E_1^{0,0} = 0$

where $V_{ij} = \mathrm{Ext}^i(F, E_{-j})^*$. The diagram implies that $E_\infty^{-i,i} = 0$ when $i = 0, 1, \dots, n-1$. Since

$$Gr(F) = (E_\infty^{0,0}, E_\infty^{-1,1}, \dots, E_\infty^{-n,n}) = E_\infty^{-n,n},$$

we see that $V_{0,0} = \mathrm{Hom}(F, E_0)^* \neq 0$. On the other hand, $E_\infty^{-n,n} = \ker(d_n^{-n,n})$. Therefore there exists an inclusion $F \hookrightarrow V_{0,0} \otimes E_0$.

Assume that $\mathrm{Hom}(E_0, F) \neq 0$, then the previous lemma implies that E_0 is the direct summand of F. Without loss of generality it can be assumed that $F = F' \oplus xE_0$ and $\mathrm{Hom}(E_0, F') = 0$ (in the opposite case E_0 is the direct summand of F').

Further, by the assumptions of the lemma we have $\mathrm{Ext}^q(E_0, F') = 0\ \forall q$. Therefore proposition 1.7 proves the statement.

By the same argument, the following lemma is proved.

4.3 LEMMA. *Let F be a nonzero sheaf on $\mathbb{P}^n$ and $\sigma = (E_0, \dots, E_n)$ be a full constructible exceptional collection on $\mathbb{P}^n$. Suppose $\mathrm{Ext}^q(E_i, F) = 0$ when $q \geq i > 0$, and $\mathrm{Ext}^q(E_0, F) = 0$ when $q > 0$; then $\mathrm{Hom}(E_0, F) \neq 0$ and either $\mathrm{Hom}(F, E_0) = 0$, or*

$F = F' \oplus xE_0$ *for some sheaf* F' *and a positive integer* x *such that* $\mathrm{Hom}(F', E_0,) = 0$ *and*

$$F' \in \langle E_1, \ldots, E_n \rangle_{cat}.$$

4.4 LEMMA. *Let* $\sigma = (E_0, E_1, E_2)$ *be a full constructible exceptional collection and* F *be a nonzero sheaf on* $\mathbb{P}^2$. *Suppose,* $\mathrm{Ext}^q(E_2, F) = 0$ *when* $q > 0$; *then* $\mathrm{Ext}^2(E_i, F) = 0$ *for* $i = 0, 1$ *and one of the following possibilities takes place:*

1. $F = F_0 \oplus F_1$, *where* $\mathrm{Ext}^q(E_2, F_1) = 0 \quad \forall q$, *i.e.,* $F_1 \in \langle E_0, E_1 \rangle_{cat}$;
2. $\mathrm{Ext}^1(E_0, F) = \mathrm{Ext}^1(E_1, F) = 0$;
3. $\mathrm{Hom}(E_i, F) = 0$ *with* $i = 0, 1, 2$.

PROOF. Consider the spectral sequence converging to F and associated to the right dual collection (E_{-2}, E_{-1}, E_0) to σ (1.9). By assumptions we have the following E_1-diagram:

$E_1^{-2,2} = 0$	$\mathrm{Ext}^2(E_1, F) \otimes E_{-1}$	$\mathrm{Ext}^2(E_0, F) \otimes E_0$
$E_1^{-2,1} = 0$	$\mathrm{Ext}^1(E_1, F) \otimes E_{-1}$	$\mathrm{Ext}^1(E_0, F) \otimes E_0$
$\mathrm{Ext}^0(E_2, F) \otimes E_{-2}$	$\mathrm{Ext}^0(E_1, F) \otimes E_{-1}$	$\mathrm{Ext}^0(E_0, F) \otimes E_0$

This diagram yields, $\mathrm{Ext}^2(E_1, F) \otimes E_{-1} \cong \mathrm{Ext}^2(E_0, F) \otimes E_0$. Since $E_{-1} \not\cong E_0$, we get,

$$\mathrm{Ext}^2(E_1, F) = \mathrm{Ext}^2(E_0, F) = 0.$$

Besides, the E_1-term of the spectral sequence splits into the following exact triples:

$$0 \longrightarrow F_1 \longrightarrow \mathrm{Ext}^1(E_1, F) \otimes E_{-1} \longrightarrow \mathrm{Ext}^1(E_0, F) \otimes E_0 \longrightarrow 0, \tag{7}$$

$$0 \longrightarrow \mathrm{Ext}^0(E_2, F) \otimes E_{-2} \longrightarrow \mathrm{Ext}^0(E_1, F) \otimes E_{-1} \longrightarrow Q \longrightarrow 0, \tag{8}$$

$$0 \longrightarrow Q \longrightarrow \mathrm{Ext}^0(E_0, F) \otimes E_0 \longrightarrow F_0 \longrightarrow 0, \tag{9}$$

$$0 \longrightarrow F_0 \longrightarrow F \longrightarrow F_1 \longrightarrow 0. \tag{10}$$

It follows from (8) that the map

$$\mathrm{Ext}^0(E_1, F) \otimes \mathrm{Ext}^2(E_{-1}, E_{-1}) \longrightarrow \mathrm{Ext}^2(E_{-1}, Q)$$

is the epimorphism. Since E_{-1} is exceptional (in particular, $\mathrm{Ext}^2(E_{-1}, E_{-1}) = 0$) we get

$$\mathrm{Ext}^2(E_{-1}, Q) = 0.$$

From (9) follows the exact sequence

$$\mathrm{Hom}(E_0, F) \otimes \mathrm{Ext}^1(E_{-1}, E_0) \longrightarrow \mathrm{Ext}^1(E_{-1}, F_0) \longrightarrow \mathrm{Ext}^2(E_{-1}, Q).$$

Since (E_{-1}, E_0) is the exceptional pair, $\mathrm{Ext}^1(E_{-1}, E_0) = 0$ and by the equality proved before we obtain,

$$\mathrm{Ext}^1(E_{-1}, F_0) = 0.$$

Next, it follows from exact sequence (9) and the fact that E_0 is the exceptional bundle that

$$\mathrm{Ext}^2(E_0, F_0) = 0.$$

Finally, sequence (7) and the equalities proved before imply

$$\mathrm{Ext}^1(F_1, F_0) = 0.$$

Thus we see that $F = F_1 \oplus F_0$, and from (7) one follows that $F_1 \in \langle E_{-1}, E_0\rangle_{cat} = \langle E_0, E_1\rangle_{cat}$.

Suppose that $F_1 = 0$, then by sequence (7), $\mathrm{Ext}^1(E_1, F) = \mathrm{Ext}^1(E_0, F) = 0$. On the other hand, in the case $F_0 = 0$ we get the following exact sequence

$$0 \longrightarrow \mathrm{Ext}^0(E_2, F) \otimes E_{-2} \longrightarrow \mathrm{Ext}^0(E_1, F) \otimes E_{-1} \longrightarrow \mathrm{Ext}^0(E_0, F) \otimes E_0 \longrightarrow 0.$$

Since $\mathrm{Ext}^1(E_0, E_{-2}) = 0$, we obtain that

$$\mathrm{Ext}^0(E_1, F) \otimes E_{-1} \cong \mathrm{Ext}^0(E_2, F) \otimes E_{-2} \oplus \mathrm{Ext}^0(E_0, F) \otimes E_0.$$

This is impossible, since E_{-1} is indecomposable and $E_{-1} \ncong E_{-2}$, $E_{-1} \ncong E_0$. Therefore in the case $F_0 = 0$ we have

$$\mathrm{Ext}^0(E_i, F) = 0 \qquad \text{for} \quad i = 0, 1, 2.$$

In the same way we can prove the following statements.

4.5 LEMMA. *Let $\sigma = (E_0, E_1, E_2)$ be a full constructible exceptional collection and F be a nonzero sheaf on $\mathbb{P}^2$. Suppose, $\mathrm{Ext}^q(F, E_0) = 0$ when $q > 0$; then $\mathrm{Ext}^2(F, E_i) = 0$ for $i = 1, 2$ and one of the following possibilities takes place:*

1. *$F = F_0 \oplus F_1$, where $\mathrm{Ext}^q(F_1, E_0) = 0 \quad \forall q$, i.e., $F_1 \in \langle E_1, E_2\rangle_{cat}$;*
2. *$\mathrm{Ext}^1(F, E_1) = \mathrm{Ext}^1(F, E_2) = 0$;*
3. *$\mathrm{Hom}(F, E_i) = 0$ with $i = 0, 1, 2$.*

4.6 LEMMA. *Let $\sigma = (E_{-2}, E_{-1}, E_0)$ be a full constructible exceptional collection and F be a nonzero sheaf on $\mathbb{P}^2$. Suppose, $\mathrm{Ext}^q(F, E_0) = 0$ when $q < 2$, then $\mathrm{Ext}^0(F, E_{-i}) = 0$ for $i = 1, 2$ and $F = F_1 \oplus F_2$, with F_1 and F_2 from the exact sequences:*

$$0 \longrightarrow F_1 \longrightarrow \mathrm{Ext}^1(F, E_{-1})^* \otimes E_1 \longrightarrow \mathrm{Ext}^1(F, E_{-2})^* \otimes E_2 \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Ext}^2(F, E_0)^* \otimes E_0 \longrightarrow \mathrm{Ext}^2(F, E_{-1})^* \otimes E_1 \longrightarrow \mathrm{Ext}^2(F, E_{-2})^* \otimes E_2 \longrightarrow F_2 \longrightarrow 0,$$

where $E_1 = R_{E_0}E_{-1}$, $E_2 = E_{-2}(3)$

4.7 LEMMA. *Suppose that a constructible exceptional collection* (E_{-2}, E_{-1}, E_0) *and a nonzero rigid sheaf* F *satisfy the following conditions:*

1) $\mathrm{Ext}^q(E_0, F) = 0$ *for* $q > 0$,
2) $\mathrm{Ext}^q(F, E_{-2}) = 0$ *for* $q < 2$,
3) $\mathrm{Hom}(F, E_{-i}) = 0$ *for* $i=0,1,2$,
4) $\mathrm{Ext}^1(F, E_{-i}) \neq 0$ *for* $i = 0,\ 1$.
Then one of the following statements is valid
(1) either $F \in \langle E_{-1}, E_0\rangle_{cat}$
(2) or the sheaf F' *from the universal extension*

$$0 \longrightarrow \mathrm{Ext}^1(F, E_0)^* \otimes E_0 \longrightarrow F' \longrightarrow F \longrightarrow 0 \tag{11}$$

is rigid and isomorphic to the direct sum

$$F' = \mathrm{Ext}^1(F', E_{-1})^* \otimes R_{E_0}E_{-1} \oplus F_2,$$

where F_2 *satisfies the conditions:*
a) $\mathrm{Ext}^q(E_{-i}(3), F_2) = 0 \quad \text{for } q > 0, \quad i = 0, 1, 2;$
b) there exists an epimorphism $\mathrm{Hom}(E_{-2}(3), F_2) \otimes E_{-2}(3) \to F_2$.

PROOF. Suppose that $\mathrm{Ext}^2(F, E_{-2}) = 0$, then by assumption 2) $\mathrm{Ext}^q(F, E_{-2}) = 0\ \forall q$. Therefore, $F \in \langle E_{-1}, E_0\rangle_{cat}$ (1.7).

Now suppose, $\mathrm{Ext}^2(F, E_{-2}) \neq 0$. Applying lemma 3.2 to universal extension (11), we obtain

$$\mathrm{Hom}(F', E_0) = \mathrm{Ext}^1(F', E_0) = \mathrm{Ext}^1(F', F') = 0.$$

It follows from the long cohomology exact sequence associated to (11) and assumption 2) that

$$\mathrm{Ext}^1(F', E_{-2}) = 0, \quad \mathrm{Ext}^2(F', E_{-2}) \cong \mathrm{Ext}^2(F, E_{-2}) \neq 0, \qquad \mathrm{Ext}^1(F', E_{-1}) \neq 0.$$

The previous lemma yields that

$$F' = \mathrm{Ext}^1(F', E_{-1})^* \otimes R_{E_0}E_{-1} \oplus F_2$$

and there exists an epimorphism

$$\mathrm{Ext}^2(F, E_{-2})^* \otimes E_2 \longrightarrow F_2 \longrightarrow 0.$$

It follows in the standard way that $\mathrm{Ext}^q(F_2, E_{-i}) = 0$ when $q < 2$ and $i = 0, 1, 2$. Therefore by the Serre duality we have

$$\mathrm{Ext}^q(E_{-i}(3), F_2) = 0 \quad \text{for } q > 0, \quad i = 0, 1, 2.$$

4.8 LEMMA *Let* E *be a constructible exceptional bundle and* F *be a sheaf on* $\mathbb{P}^2$ *such that* $\mathrm{Ext}^1(E, F) = 0$ *and* $\mathrm{Ext}^q(E, F) \neq 0$ *when* $q = 0,\ 2$. *Then* $F = F_0 \oplus F_2$, *where* $\mathrm{Ext}^q(E, F_0) = 0$ *for* $q > 0$ *and* $\mathrm{Ext}^q(E, F_2) = 0$ *for* $q < 2$.

PROOF. Consider the full constructible exceptional collection containing E $\sigma = (E_0, E, E_1)$. Denote by $\sigma^\vee = (A, B, C)$ the right dual collection to σ and consider the spectral sequence associated to $\sigma^\vee$ and converging to F. Its E_1-term has the form:

$\mathrm{Ext}^2(E_1, F)\otimes A$	$\mathrm{Ext}^2(E, F)\otimes B$	$\mathrm{Ext}^2(E_0, F)\otimes C$
$\mathrm{Ext}^1(E_1, F)\otimes A$	$E_1^{-1,1} = 0$	$\mathrm{Ext}^1(E_0, F)\otimes C$
$\mathrm{Ext}^0(E_1, F)\otimes A$	$\mathrm{Ext}^0(E, F)\otimes B$	$\mathrm{Ext}^0(E_0, F)\otimes C$

We see that the spectral sequence splits into the exact triples:

$$0 \longrightarrow \mathrm{Ext}^0(E_1, F)\otimes A \longrightarrow \mathrm{Ext}^0(E, F)\otimes B \longrightarrow Q \longrightarrow 0,$$

$$0 \longrightarrow Q \longrightarrow \mathrm{Ext}^0(E_0, F)\otimes C \longrightarrow Q' \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Ext}^1(E_1, F)\otimes A \longrightarrow Q' \longrightarrow F_0 \longrightarrow 0,$$

$$0 \longrightarrow G \longrightarrow \mathrm{Ext}^2(E, F)\otimes B \longrightarrow \mathrm{Ext}^2(E_0, F)\otimes C \longrightarrow 0,$$

$$0 \longrightarrow G' \longrightarrow \mathrm{Ext}^2(E_1, F)\otimes A \longrightarrow G \longrightarrow 0,$$

$$0 \longrightarrow F_2 \longrightarrow G' \longrightarrow \mathrm{Ext}^1(E_0, F)\otimes C \longrightarrow 0,$$

$$0 \longrightarrow F_0 \longrightarrow F \longrightarrow F_2 \longrightarrow 0.$$

It can be shown by the standard method that $\mathrm{Ext}^1(F_2, F_0) = 0$ and $\mathrm{Ext}^q(E, F_0) = 0$ for $q > 0$, $\mathrm{Ext}^q(E, F_2) = 0$ for $q < 2$.

4.9 COROLLARY. *Let E_1 and E_2 be nonisomorphic exceptional bundles on $\mathbb{P}^2$. Suppose that* $\mathrm{Ext}^1(E_2, E_1) = \mathrm{Ext}^1(E_1, E_2) = 0$*; then one of the spaces either* $\mathrm{Ext}^2(E_1, E_2)$ *or* $\mathrm{Ext}^2(E_2, E_1)$ *is trivial.*

PROOF. Assume the converse. Since E_i are indecomposable, the previous lemma implies that

$$\mathrm{Hom}(E_1, E_2) = \mathrm{Hom}(E_2, E_1) = 0.$$

But $\mathrm{Ext}^2(E_1, E_2)^* \cong \mathrm{Ext}^0(E_2, E_1(-3))$ and there exists a monomorphism $E_1(-3) \hookrightarrow E_1$, i.e., $\mathrm{Hom}(E_2, E_1) \neq 0$. This contradiction proves the statement.

5 Indecomposable Superrigid Sheaves on $\mathbb{P}^2$.

5.1 LEMMA. *Let E be a constructible exceptional bundle and F be an indecomposable superrigid sheaf on $\mathbb{P}^2$ such that* $\mathrm{Ext}^q(F, E) = \mathrm{Ext}^q(E, F) = 0$ *when $q > 0$ and* $\mathrm{Hom}(E, F) \neq 0$, $\mathrm{Hom}(F, E) \neq 0$*; then $E \cong F$.*

PROOF. Taking into account lemma 4.4 , we obtain that for any constructible exceptional collection (A_{-1}, A_0, E)

$$\mathrm{Ext}^q(A_i, F) = 0 \qquad \text{for} \quad q > 0,\ i = 0, -1. \tag{12}$$

Similarly, by lemma 4.5 it follows that for a constructible exceptional collection (E, B_0, B_1)

$$\mathrm{Ext}^q(F, B_i) = 0 \qquad \text{for} \quad q > 0,\ i = 0, 1. \tag{13}$$

Since the bundles A_{-1}, A_0 are obtained by the left shifts over E (1.3), we have the exact triples

$$0 \longrightarrow A_0 \longrightarrow \mathrm{Hom}(E, B_1') \otimes E \longrightarrow B_1' \longrightarrow 0, \quad (14)$$

$$0 \longrightarrow A_{-1} \longrightarrow \mathrm{Hom}(E, B_0') \otimes E \longrightarrow B_0' \longrightarrow 0, \quad (15)$$

where (E, B_0', B_1') is a constructible exceptional collection. Using equalities (13) we apply the functor $\mathrm{Ext}^{\cdot}(F, \cdot)$ to exact triple (14) to obtain

$$\mathrm{Ext}^2(F, A_i) = 0 \qquad i = -1, 0. \quad (16)$$

For the similar reason, $\mathrm{Ext}^2(B_i, F) = 0$.

If $\mathrm{Ext}^1(F, A_0) = 0$, then by lemma 4.2 it follows that either $\mathrm{Hom}(E, F) = 0$ or E is the direct summand of F. By assumption $\mathrm{Hom}(E, F) \neq 0$ and the sheaf F is indecomposable. Therefore, $E \cong F$.

Suppose now, that $\mathrm{Ext}^1(F, A_0) \neq 0$ and consider the universal extension:

$$0 \longrightarrow \mathrm{Ext}^1(F, A_0)^* \otimes A_0 \longrightarrow F' \longrightarrow F \longrightarrow 0. \quad (17)$$

Combining our assumptions, lemma 3.2 and (16), we get,

$$\mathrm{Hom}(F', A_0) \cong \mathrm{Hom}(F, A_0), \quad \mathrm{Ext}^1(F', A_0) \cong \mathrm{Ext}^2(F', A_0) = 0,$$

$$\mathrm{Ext}^1(F', F) = \mathrm{Ext}^2(F', F) = 0, \quad \mathrm{Ext}^1(F', F') = \mathrm{Ext}^2(F', F') = 0.$$

Besides, applying the functors $\mathrm{Ext}^{\cdot}(\cdot, E)$ and $\mathrm{Ext}^{\cdot}(\cdot, A_{-1})$ to sequence (17) we get the following cohomology tables:

k	$\mathrm{Ext}^k(F, E)$	$\to$	$\mathrm{Ext}^k(F', E)$	$\to$	$\mathrm{Ext}^1(F, A_0) \otimes \mathrm{Ext}^k(A_0, E)$
0	*		?		*
1	0		?		0
2	0		?		0

k	$\mathrm{Ext}^k(F, A_{-1})$	$\to$	$\mathrm{Ext}^k(F', A_{-1})$	$\to$	$\mathrm{Ext}^1(F, A_0) \otimes \mathrm{Ext}^k(A_0, A_{-1})$
0	*		?		0
1	*		?		0
2	0		?		0

Therefore the superrigid sheaf F' and the constructible exceptional collection

$$(A_{-1}, A_0, E)$$

satisfy the assumption of lemma 4.2 . Hence either $F' = F'' \oplus xE$ or $\mathrm{Hom}(E, F') = 0$. But the application the functor $\mathrm{Ext}^{\cdot}(E, \cdot)$ to (17) yields that $\mathrm{Hom}(E, F)' \cong \mathrm{Hom}(E, F) \neq 0$. Thus

$$F' = F'' \oplus xE \quad \text{and} \quad \mathrm{Hom}(E, F'') = 0.$$

Since F' is superrigid, we get $\mathrm{Ext}^q(E, F'') \subset \mathrm{Ext}^q(F', F') = 0$ when $q > 0$. Therefore, $F'' \in \langle A_{-1}, A_0 \rangle_{cat}$.

Taking into account proposition 2.5 , we obtain

$$F'' = yA_n \oplus zA_{n+1}$$

for some constructible exceptional pair obtained from (A_{-1}, A_0) by mutations and nonnegative integers y, z. Without loss of generality we can assume that $y > 0$. Suppose that $n < 0$, then it follows from equality (1) that $\mathrm{Hom}(A_0, A_n) = 0$. Hence A_n is the direct summand of F. Since F is indecomposable, we obtain that $F \cong A_n$. But it is impossible, because of by assumption $\mathrm{Hom}(E, F) \neq 0$ and $\mathrm{Hom}(E, A_n) = 0$.

The case $n = 0$ with $y > 0$ is impossible also (3.4). On the other hand, $\mathrm{Ext}^1(F', A_0) = 0$. Hence by equality (2), $n \leq 1$.

Thus we see that $F = yA_1 \oplus xE$, where (A_0, A_1, E) is constructible exceptional collection. Now sequence (17) has the form:

$$0 \longrightarrow \mathrm{Ext}^1(F, A_0)^* \otimes A_0 \longrightarrow yA_1 \oplus xE \longrightarrow F \longrightarrow 0. \tag{18}$$

It is easy to check that $x = h^0(E, F) = \chi(E, F)$. Moreover, if we assume that $E \ncong F$, then for any exceptional pair $(A_n, A_{n+1}) \in \langle A_0, A_1 \rangle_{cat}$ there exists the exact triple.

$$0 \longrightarrow h_n A_n \longrightarrow y_{n+1} A_{n+1} \oplus xE \longrightarrow F \longrightarrow 0.$$

Denote by a_n, e, f the images of the sheaves A_n, E and F in $K = K_0(\mathbb{P}^2) \otimes \mathbb{Q}$ respectively. Then for each integer n we have

$$f - xe = y_{n+1} a_{n+1} - h_n a_n$$

with positive integers y_{n+1} and h_n. By lemma 2.1 these equalities imply that

$$(f - xe, f - xe) \leq 0.$$

Note that $(f - xe, f - xe) = f^2 + x^2(e, e) - x(e, f) - x(f, e) = f^2 - xy$, where $y = (f, e) = \chi(F, E)$. Let us prove that $f^2 - xy > 0$.

Since (A_n, A_{n+1}, E) is constructible exceptional collection, by 1.5 we obtain that the canonical map $\mathrm{Hom}(A_{n+1}, E) \otimes A_{n+1} \longrightarrow E$ is the epimorphism. Besides, the sheaf F is the quotient of $y_{n+1} A_{n+1} \oplus xE$. Therefore the canonical map $\mathrm{Hom}(A_{n+1}, F) \otimes A_{n+1} \longrightarrow F$ is epimorphism as well.

Suppose that $\mathrm{Hom}(F, A_{n+1}) \neq 0$, then it follows from lemma 4.1 that A_{n+1} is the direct summand of F. But this is impossible. Hence $\mathrm{Hom}(F, A_{n+1}) = 0$. Moreover, the last equality hold for all integers n. In particular, $\mathrm{Hom}(F, A_0) = \mathrm{Hom}(F, A_1) = 0$.

Let us apply the functor $\mathrm{Ext}^{\cdot}(F, \cdot)$ to (18). We see that

$$0 \longrightarrow x\mathrm{Hom}(F, E) \longrightarrow \mathrm{Hom}(F, F) \xrightarrow{\delta} \mathrm{Ext}^1(F, A_0)^* \otimes \mathrm{Ext}^1(F, A_0)$$

is the exact sequence. Since sequence (18) does not split, the coboundary homomorphism δ is nontrivial. Therefore $xh^0(F, E) < h^0(F, F)$, i.e., $xy < f^2$. This contradiction concludes the proof.

5.2 COROLLARY. *Let E_1 and E_2 be nonisomorphic exceptional bundles on $\mathbb{P}^2$. Suppose that* $\mathrm{Ext}^1(E_2, E_1) = \mathrm{Ext}^1(E_1, E_2) = 0$*; then either* $\mathrm{Ext}^0(E_1, E_2) = 0$ *or* $\mathrm{Ext}^0(E_2, E_1) = 0$.

PROOF. Since E_2 is indecomposable, lemma 4.8 implies that one of the spaces either $\mathrm{Hom}(E_1, E_2)$ or $\mathrm{Ext}^2(E_1, E_2)$ is trivial. Suppose, $\mathrm{Hom}(E_1, E_2) \neq 0$ and $\mathrm{Ext}^2(E_1, E_2) = 0$.

Similarly we see that one of the spaces ether $\mathrm{Hom}(E_2, E_1)$ or $\mathrm{Ext}^2(E_2, E_1)$ is trivial. Assume, $\mathrm{Hom}(E_2, E_1) \neq 0$ and $\mathrm{Ext}^2(E_2, E_1) = 0$; then by the previous lemma, $E_1 \cong E_2$.

5.3 LEMMA. *Let E be a constructible exceptional bundle and F be an indecomposable superrigid sheaf on $\mathbb{P}^2$ such that*

$$\mathrm{Ext}^q(E, F) = 0 \text{ when } \quad q > 0, \quad \mathrm{Ext}^2(F, E) = 0, \ \mathrm{Ext}^1(F, E) \neq 0;$$

then

1) $\mathrm{Hom}(F, E) = 0$,

2) there exist nonnegative integers x_1, x_2 and the full constructible exceptional collection containing E (E, E_1, E_2) such that the following sequence

$$0 \longrightarrow \mathrm{Ext}^1(F, E)^* \otimes E \longrightarrow x_1 E_1 \oplus x_2 E_2 \longrightarrow F \longrightarrow 0 \tag{19}$$

is exact,

3) $\mathrm{Ext}^q(E_2, F) = 0$ *when* $q > 0$ *and* $x_2 = h^0(E_2, F)$,

4) $\mathrm{Ext}^2(F, E_2) = 0$ *and either* $\mathrm{Ext}^1(F, E_2) \neq 0$ *or F is the constructible exceptional bundle.*

5) the canonical map $\mathrm{Hom}(E_2, F) \otimes E_2 \longrightarrow F$ *is epimorphism provided $x_2 > 0$,*

6) there exist nonnegative integers y_0, y_1 and the full constructible exceptional collection containing E_2 (E'_0, E'_1, E_2) such that the following sequence

$$0 \longrightarrow y_0 E'_0 \oplus y_1 E'_1 \longrightarrow x_2 E_2 \longrightarrow F \longrightarrow 0 \tag{20}$$

is exact,

7) $\mathrm{Ext}^q(E'_i, F) = 0$ *when* $q > 0$, $i = 0, 1$ *and* $\mathrm{Ext}^q(F, E'_i) = 0$ *when* $q \neq 1$, $i = 0, 1$.

PROOF. Consider the universal extension

$$0 \longrightarrow \mathrm{Ext}^1(F, E)^* \otimes E \longrightarrow F' \longrightarrow F \longrightarrow 0. \tag{21}$$

By lemma 3.2 the sheaf F' satisfies the following conditions:

$$\mathrm{Ext}^q(F', E) = \mathrm{Ext}^q(F', F') = \mathrm{Ext}^q(F', F) = 0 \quad \text{when} \quad q > 0; \quad \mathrm{Hom}(F', E) \cong \mathrm{Hom}(F, E). \tag{22}$$

It is easy to see that $\mathrm{Ext}^q(E, F') = 0$ when $q > 0$ and $\mathrm{Hom}(E, F') \neq 0$. Without loss of generality we can assume that $F' = \bigoplus_{i=1}^{s} x_i F_i$, where F_i are indecomposable superrigid sheaves, $x_i > 0$ and $F_i \ncong F_j$ when $i \neq j$. Suppose that $\mathrm{Hom}(E, F_i) = 0$ for some index i. Then it follows from (21) that the sheaf F_i is the direct summand of F and, since F is indecomposable, $F \cong F_i$. But this is impossible because of $\mathrm{Ext}^1(F, E) \neq 0$ and $\mathrm{Ext}^1(F_i, E) \subset \mathrm{Ext}^1(F', E) = 0$. Thus,

$$\mathrm{Hom}(E, F_i) \neq 0 \quad \text{for any} \quad i = 1, 2, \ldots, s.$$

Suppose that $\mathrm{Hom}(F, E) \neq 0$. Then from (22) we have $\mathrm{Hom}(F', E) \neq 0$. Therefore there is an index i such that $\mathrm{Hom}(F_i, E) \neq 0$,$\mathrm{Hom}(E, F_i) \neq 0$ and $\mathrm{Ext}^q(E, F_i) = \mathrm{Ext}^q(F_i, E) = 0$

when $q > 0$. The previous lemma implies that in this case $F_i \cong E$. Taking into account lemma 3.4 , we see that this case is impossible. Therefore,

$$\mathrm{Hom}(F, E) = 0,$$

i.e., the first statement of the lemma is valid.

Thus we have, $\mathrm{Ext}^q(F', E) = 0$ for any q. Hence $F' \in \langle E_1, E_2\rangle_{cat}$ for some exceptional pair such that (E, E_1, E_2) is the full constructible exceptional collection. Using proposition 2.5 , we can assume that $F' = x_1 E_1 \oplus x_2 E_2$ with nonnegative integers x_1, x_2, i.e., sequence (19) takes place.

It follows from (22) that $\mathrm{Ext}^q(E_2, F) \subset \mathrm{Ext}^q(F', F) = 0$ when $q > 0$. Besides, applying the functor $\mathrm{Ext}^{\cdot}(E_2, \cdot)$ to sequence (19), we obtain that

$$x_2 \mathrm{Hom}(E_2, E_2) \cong \mathrm{Hom}(E_2, F).$$

Since E_2 is simple, this equality implies that $x_2 = h^0(E_2, F)$. this proves statement 3 of our lemma.

Applying the functor $\mathrm{Ext}^{\cdot}(F, \cdot)$ to sequence (19) we get

$$\mathrm{Ext}^2(F, E_2) = 0.$$

Suppose that $\mathrm{Ext}^1(F, E_2) = 0$ and $\mathrm{Ext}^0(F, E_2) \neq 0$, then the constructible exceptional bundle E_2 and the indecomposable superrigid sheaf F are isomorphic (5.1). But this case is impossible, since $\mathrm{Ext}^1(E_2, E) = 0$ and $\mathrm{Ext}^1(F, E) = 0$. Therefore either

$$\mathrm{Ext}^1(F, E_2) \neq 0$$

or $\mathrm{Ext}^q(F, E_2) = 0 \ \forall q$. In the last case we see that F belongs to the subcategory generated by a constructible exceptional pair and consequently it is constructible exceptional bundle. This completes the proof of statement 4 of the lemma.

Combining proved statements 2, 3 and 4 we see that the constructible exceptional bundle E_2 and indecomposable superrigid sheaf F satisfy the assumptions of our lemma. Therefore $\mathrm{Hom}(F, E_2) = 0$ and there exist the full constructible exceptional collection (E_2, E_3, E_4) and nonnegative integers x_3, x_4 such that the following sequence

$$0 \longrightarrow \mathrm{Ext}^1(F, E)_2^* \otimes E_2 \longrightarrow x_3 E_3 \oplus x_4 E_4 \longrightarrow F \longrightarrow 0$$

is exact. By proposition 1.5 we get that the canonical maps $\mathrm{Hom}(E_2, E_i) \otimes E_2 \longrightarrow E_i$ are epimorphisms for $i = 3, 4$. Therefore the last exact sequence yields statement 5 of the lemma.

Now let us consider the following exact sequence

$$0 \longrightarrow \ker(can) \longrightarrow \mathrm{Hom}(E_2, F) \otimes E_2 \xrightarrow{can} F \longrightarrow 0. \tag{23}$$

Recall that

$$\mathrm{Ext}^q(E_2, F) = 0 \quad \text{when } q > 0 \quad \text{and} \quad \mathrm{Ext}^q(F, E_2) = 0 \quad \text{when } q \neq 1.$$

Since the map $\mathrm{Hom}(E_2, F) \otimes \mathrm{Hom}(E_2, E_2) \longrightarrow \mathrm{Hom}(E_2, F)$ that is obtained after the application the functor $\mathrm{Ext}^{\cdot}(E_2, \cdot)$ to sequence (23) is isomorphism, $\mathrm{Ext}^q(E_2, \ker(can)) = 0$

for any q. That is $\ker(can) \in \langle E'_0, E'_1\rangle_{cat}$ for some exceptional pair such that (E'_0, E'_1, E) is the full constructible exceptional collection.

By the standard method we can show that $\ker(can)$ is the superrigid sheaf and the following relations are valid:

$$\mathrm{Ext}^q(\ker(can), F) = 0 \quad \text{when } q > 0; \quad \mathrm{Ext}^q(F, \ker(can)) = 0 \quad \text{when } q \neq 1.$$

Therefore $\ker(can)$ is the direct sum of exceptional bundles. Without loss of generality it can be assumed that

$$\ker(can) \cong y_0 E'_0 \oplus y_1 E'_1.$$

This completes the proof.

5.4 LEMMA *Let the exceptional bundle E and the indecomposable superrigid sheaf F satisfy the assumptions of the previous lemma; then F is the constructible exceptional bundle.*

PROOF. By the previous lemma we have the following exact sequences:

$$0 \longrightarrow x_0 E \longrightarrow x_1 E_1 \oplus x_2 E_2 \longrightarrow F \longrightarrow 0,$$

$$0 \longrightarrow y_0 E'_0 \oplus y_1 E'_1 \longrightarrow x_2 E_2 \longrightarrow F \longrightarrow 0$$

with nonnegative integers x_0, x_1, x_2, y_0, y_1 such that (E, E_1, E_2) and (E'_0, E'_1, E_2) are the full constructible exceptional collections. Moreover, E'_0 and F satisfy the assumptions of the previous lemma as well. It is easy to prove that $y_o = h^1(F, E'_0)$. Therefore, there exist the following exact sequences:

$$0 \longrightarrow y_0 E' \longrightarrow z_1 E''_1 \oplus z_2 E''_2 \longrightarrow F \longrightarrow 0,$$

$$0 \longrightarrow w_0 E'''_0 \oplus w_1 E'''_1 \longrightarrow z_2 E''_2 \longrightarrow F \longrightarrow 0,$$

and so on...

Note that the ranks of the bundles from these exact sequences satisfy the following inequalities:

$$r(x_0 E) \geq r(y_0 E'_0 \oplus y_1 E'_1) \geq r(y_0 E') \geq r(w_0 E'''_0 \oplus w_1 E'''_1) \geq \ldots$$

Since the rank of a sheaf is nonnegative, we see that in this sequence of the inequalities there is an equality. For example suppose that $r(x_0 E) = r(y_0 E'_0 \oplus y_1 E'_1)$. In this case we obtain that $x_1 = 0$, i.e.,

$$0 \longrightarrow x_0 E \longrightarrow x_2 E_2 \longrightarrow F \longrightarrow 0$$

and $F \in \langle E, E_2\rangle_{cat}$. Hence by proposition 2.5 F is the constructible exceptional bundle. This completes the proof.

The results of this section can be summarized as follows.

5.5 PROPOSITION. *Let E be a constructible exceptional bundle and F be an indecomposable superrigid sheaf on $\mathbb{P}^2$. Suppose that*

$$\mathrm{Hom}(E, F) \neq 0, \ \mathrm{Ext}^q(E, F) = 0 \quad \text{when } q > 0 \quad \text{and} \quad \mathrm{Ext}^2(F, E) = 0;$$

then F is the constructible exceptional bundle.

PROOF. Let us consider 3 cases.

Case 1: $\mathrm{Ext}^q(F, E) = 0 \ \ \forall q$.

In this case F belongs to the subcategory generated by a constructible exceptional pair (E_0, E_1) such that (E, E_0, E_1) is the full constructible exceptional collection. Therefore our statement follows from proposition 2.5 .

Case 2: $\mathrm{Ext}^q(F, E) = 0 \ \ q > 0, \quad \mathrm{Hom}(F, E) \neq 0$.

It follows from lemma 5.1 that $F \cong E$ and consequently F is the constructible exceptional bundle.

Case 3: $\mathrm{Ext}^2(F, E) = 0, \quad \mathrm{Ext}^1(F, E) \neq 0$.

In the last case the proof follows from lemma 5.4 .

6 Proof of the Main Theorem.

It follows from Serre's theorem ([4]) that for a nonzero bundle F on $\mathbb{P}^2$ and integers $i \ll 0$ the following condition is valid

$$\mathrm{Ext}^q(\mathcal{O}(i), F) = 0 \ \text{ for } q > 0, \quad \mathrm{Ext}^0(\mathcal{O}(i), F) \neq 0. \tag{24}$$

Denote by $(\nu_l(F) - 1)$ the maximal integer satisfying this condition. On the other hand, for all integers $i \gg 0$ we have

$$\mathrm{Hom}(\mathcal{O}(i), F) = 0. \tag{25}$$

Denote by $\nu_r(F)$ the minimal integer satisfying (25).

6.1 LEMMA. *Let F be an indecomposable rigid bundle on $\mathbb{P}^2$, then one of the following statements hold:*

(1) the bundle F is constructible exceptional bundle,

(2) $\mathrm{Ext}^2(\mathcal{O}(n), F) = \mathrm{Ext}^2(\mathcal{O}(n+1), F) = 0,$

$\mathrm{Ext}^1(\mathcal{O}(n), F) \neq 0, \ \mathrm{Ext}^1(\mathcal{O}(n+1), F) \neq 0,$ *where* $n = \nu_l(F)$.

PROOF. By choice n we see that either $\mathrm{Ext}^q(\mathcal{O}(n), F) = 0 \ \forall q$ or at least one of the spaces $\mathrm{Ext}^2(\mathcal{O}(n), F)$, $\mathrm{Ext}^1(\mathcal{O}(n), F)$ is nontrivial. In the first case the sheaf F belongs to the subcategory generated by the constructible exceptional pair $(\mathcal{O}(n-2), \mathcal{O}(n-1))$. Therefore F is the constructible exceptional bundle (2.5).

Let us show that $\mathrm{Ext}^2(\mathcal{O}(n), F) = 0$ in the second case . Consider the constructible exceptional collection $(\mathcal{O}(n-2), T(n-3), \mathcal{O}(n-1))$, where T is the tangent bundle on $\mathbb{P}^2$. We have the exact sequence

$$0 \longrightarrow T(n-3) \longrightarrow \mathrm{H}^0(\mathcal{O}(1)) \otimes \mathcal{O}(n-1) \longrightarrow \mathcal{O}(n) \longrightarrow 0. \tag{26}$$

Combining lemma 4.4 , the indecomposability of F and the relations: $\mathrm{Hom}(\mathcal{O}(n-1), F) \neq 0$, $\mathrm{Ext}^q(\mathcal{O}(n-1), F) = \mathrm{Ext}^q(\mathcal{O}(n-2), F) = 0$, when $q > 0$, we get

$$\mathrm{Ext}^q(T(n-3), F) = 0 \ \text{ for } q > 0.$$

Therefore from exact sequence (26) it follows that $\mathrm{Ext}^2(\mathcal{O}(n), F) = 0$. Thus we have $\mathrm{Ext}^1(\mathcal{O}(n), F) \neq 0$.

Now consider the collection $(T(n-3), R, \mathcal{O}(n-1))$, where $R = R_{T(n-3)}\mathcal{O}(n-2)$. As before, we see that

$$\mathrm{Ext}^q(R, F) = 0 \quad \text{when} \quad q > 0. \tag{27}$$

By proposition 1.4 we obtain that $\mathcal{O}(n+1) = R_{\mathcal{O}(n-1)}R$, i.e., there exists the exact triple

$$0 \longrightarrow R \longrightarrow \mathrm{Hom}(R, \mathcal{O}(n-1))^* \otimes \mathcal{O}(n-1) \longrightarrow \mathcal{O}(n+1) \longrightarrow 0.$$

Using this sequence and (27), we get $\mathrm{Ext}^2(\mathcal{O}(n+1), F) = 0$. Hence bu choice n we see that either $\mathrm{Ext}^1(\mathcal{O}(n+1), F) \neq 0$ or $\mathrm{Ext}^q(\mathcal{O}(n+1), F) = 0 \ \ \forall q$ (i.e., F is constructible exceptional bundle). This concludes the proof.

6.2 PROPOSITION. *Any indecomposable rigid bundle F on $\mathbb{P}^2$ is constructible exceptional bundle.*

PROOF. The proof is by induction on $\Delta(F) = \nu_r(F) - \nu_l(F) \geq 0$.

Suppose that $\Delta(F) = 0$, then it follows from the definitions of $\nu_l(F)$, $\nu_r(F)$ and the previous lemma that either F is constructible exceptional bundle or

$$\mathrm{Ext}^q(\mathcal{O}(n), F) = 0 \ \text{ when } q = 0,\ 2 \quad \text{and } \mathrm{Ext}^1(\mathcal{O}(n), F) \neq 0$$

(here $n = \nu_l(F) = \nu_r(F)$). By Serre's duality we have

$$\mathrm{Ext}^q(F, \mathcal{O}(n-3)) = 0, \ \text{when } q = 0,\ 2 \quad \text{and } \mathrm{Ext}^1(F, \mathcal{O}(n-3)) \neq 0. \tag{28}$$

Besides,

$$\mathrm{Ext}^q(\mathcal{O}(n-3), F) = 0, \ \text{when } q > 0 \quad \text{and } \mathrm{Ext}^0(\mathcal{O}(n-3), F) \neq 0. \tag{29}$$

Consider the universal extension

$$0 \longrightarrow \mathrm{Ext}^1(F, \mathcal{O}(n-3))^* \otimes \mathcal{O}(n-3) \longrightarrow F' \longrightarrow F \longrightarrow 0.$$

Combining (28), (29), lemma 3.2 and the fact that F is rigid, we obtain that

$$\mathrm{Ext}^q(F', \mathcal{O}(n-3)) = 0, \ \forall q, \quad \mathrm{Ext}^1(F', F') = 0; \tag{30}$$

$$\mathrm{Ext}^2(F', F') \cong \mathrm{Ext}^2(F, F) \tag{31}$$

From equalities (30) and proposition 2.5 one follows that F' is superrigid. In particular, $\mathrm{Ext}^2(F', F') = 0$. Therefore $\mathrm{Ext}^2(F, F) = 0$ (see (31)). Thus we obtain that F is superrigid. Hence the line bundle $\mathcal{O}(n-3)$ and the indecomposable superrigid sheaf F satisfy the assumptions of proposition 5.5 , i.e., F is constructible exceptional bundle.

In the general case without loss of generality we can assume that $\nu_l(F) = 4$. Suppose F is not constructible exceptional bundle. Hence lemma 6.1 implies that

$$\mathrm{Ext}^q(\mathcal{O}(3), F) = 0 \ \text{when } q > 0, \quad \mathrm{Hom}(\mathcal{O}(3), F) \neq 0,$$

$$\mathrm{Ext}^2(\mathcal{O}(i), F) = 0, \quad \text{and } \mathrm{Ext}^1(\mathcal{O}(i), F) \neq 0 \quad \text{when } i = 4,\ 5.$$

By Serre's duality we have

$$\mathrm{Ext}^q(F, \mathcal{O}) = 0 \text{ when } q > 0, \quad \mathrm{Ext}^2(F, \mathcal{O}) \neq 0,$$

$$\mathrm{Ext}^0(F,\mathcal{O}(i))=0, \text{ and } \mathrm{Ext}^1(F,\mathcal{O}(i))\neq 0 \quad \text{when } i=1,\ 2.$$

Besides, by definition $\nu_l(F)$ $\mathrm{Ext}^q(\mathcal{O}(2),F)=0$ for $q>0$.

Hence the application of lemma 4.7 to the collection $(\mathcal{O},\mathcal{O}(1),\mathcal{O}(2))$ and the sheaf F yields that the rigid sheaf F' from the universal extension

$$0\longrightarrow \mathrm{Ext}^1(F,\mathcal{O}(2))^*\otimes\mathcal{O}(2)\longrightarrow F'\longrightarrow F\longrightarrow 0 \tag{32}$$

is isomorphic to the direct sum $F'=x_1E_1\oplus F_2$, where $E_1(-1)$ is the tangent bundle on $\mathbb{P}^2$ and the rigid sheaf F_2 satisfies the conditions:

$$\mathrm{Ext}^q(\mathcal{O}(i),F_2)=0 \quad \text{for } q>0,\quad i=3,\ 4,\ 5; \tag{33}$$

$$\text{there exists an epimorphism} \quad \mathrm{Hom}(\mathcal{O}(3),F_2)\otimes\mathcal{O}(3)\to F_2. \tag{34}$$

It is clear that each indecomposable direct summand of F_2 $(E_2,E_3,\dots,E_k)$ is rigid and satisfies conditions (33), (34).

Let us show that for $j=2,3,\dots,k$

$$3<\nu_r(E_j)\leq\nu_r(F).$$

In fact the left inequality follows from (34). To prove the right inequality note that if $\mathrm{Hom}(\mathcal{O}(i),E_j)\neq 0$ for $i>2$, then exact sequence (32) implies $\mathrm{Hom}(\mathcal{O}(i),F)\neq 0$.

Taking into account (33), we may assume that $\mathrm{Hom}(\mathcal{O}(5),E_j)\neq 0$ when $j=2,3,\dots,k$, since in opposite case E_j is constructible exceptional bundle (2.5). Therefore using (33), we get $\nu_l(E_j)\geq 6$. Thus,

$$\Delta(E_j)=\nu_r(E_j)-\nu_l(E_j)\leq\nu_r(F)-6<\Delta(F)$$

for each indecomposable direct summand of F.

By the induction hypothesis, all E_j are the constructible exceptional bundles. Since $F'=\bigoplus x_jE_j$ is rigid, $\mathrm{Ext}^1(E_i,E_j)=0\ \forall i,j$. Now using corollary 4.9 , we may assume that $\mathrm{Ext}^2(E_i,E_j)=0$ for $i<j$. Therefore, $\mathrm{Ext}^q(F',E_k)=0$ when $q>0$.

The application of the functor $\mathrm{Ext}^{\cdot}(\mathcal{O}(2),\cdot)$ to sequence 32 yields

$$\mathrm{Ext}^q(\mathcal{O}(2),E_k)=0 \quad \text{when } q>0.$$

It follows from the cohomology table

q	$\mathrm{Ext}^q(F,E_k)$	$\to$	$\mathrm{Ext}^q(F',E_k)$	$\to$	$\mathrm{Ext}^1(F,\mathcal{O}(2))\otimes\mathrm{Ext}^q(\mathcal{O}(2),E_k)$
0	?		$*$		$*$
1	?		0		0
2	?		0		0

that

$$\mathrm{Ext}^2(F,E_k)=0. \tag{35}$$

Using (3.2), we get $\mathrm{Ext}^1(F',F)=0$, in particular

$$\mathrm{Ext}^1(E_j,F)=0 \quad \text{for } j=1,2,\dots,k. \tag{36}$$

It is easily shown that

$$\mathrm{Ext}^0(E_j, F) \neq 0 \quad \text{for } j = 1, 2, ..., k. \tag{37}$$

Now if we combine (36), (37), lemma 4.8 and the fact that F is indecomposable, we get

$$\mathrm{Ext}^2(E_j, F) = 0 \quad \text{for } j = 1, 2, ..., k. \tag{38}$$

Therefore $\mathrm{Ext}^2(F', F) = 0$ and lemma 3.2 yields that F is superrigid. Finally, using (35), (36), (37),(38) we can apply proposition 5.5 to the constructible exceptional bundle E_k and indecomposable superrigid sheaf F. This completes the proof.

6.3 THEOREM. *1. Any rigid bundle on $\mathbb{P}^2$ is the direct sum of constructible exceptional bundles.*

2. Any superrigid bundle F on $\mathbb{P}^2$ has the form

$$F = x_0F_0 \oplus x_1F_1 \oplus x_2F_2,$$

where $x_i \geq 0$ and (F_0, F_1, F_2) is the constructible exceptional collection.

3. Any exceptional collection of bundles on $\mathbb{P}^2$ is constructible.

PROOF. The first statement follows from the previous proposition.

Let F be a superrigid sheaf on $\mathbb{P}^2$. Then $F = \bigoplus_{i=0}^{s} x_iF_i$, where F_i are constructible exceptional bundles. Taking into account corollary 5.2 , we may assume that $\mathrm{Hom}(F_i, F_j) = 0$ when $i > j$. On the other hand, since F is superrigid, we have

$$\mathrm{Ext}^q(F_i, F_j) = 0 \;\; \forall q \quad \text{when } i > j.$$

Therefore $s \leq 2$ and $\tau = (F_0, F_1, F_2)$ is exceptional collection. To prove the constructibility of τ note that F_0 is constructible. Hence there exists the constructible exceptional collection (F_0, E_1, E_2) containing F_0. By proposition 1.7 , $F_1, F_2 \in \langle E_1, E_2\rangle_{cat}$. Now using proposition 2.5 , we have that the pair (F_1, F_2) is obtained from (E_1, E_2) by mutations. Thus τ is constructible.

Finally, for any exceptional collection $\tau = (F_0, F_1, F_2)$ we may construct the superrigid sheaf $F = F_0 \oplus F_1 \oplus F_2$ and the constructibility follows from the second statement. This completes the proof.

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