Cartan subalgebras of locally finite and Borcherds-Kac-Moody Lie algebras

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1 Introduction

In [D-CPS], the authors develop a theory of Cartan subalgebras in the context of locally reductive Lie algebras. In this paper, we answer two of their questions in the negative, namely that in a root reductive Lie algebra, there are nilpotent, self-normalizing Lie subalgebras that are not Cartan subalgebras; and that a Cartan subalgebra of a locally reductive Lie algebra need not be nilpotent.

In finite dimensional Lie algebras, there are several equivalent definitions of a Cartan subalgebra. In infinite dimension, most of them are no longer equivalent. Depending on the context, different definitions are used. We give arguments showing that one of these definitions seems more reasonable than the others in the context of locally finite and Borcherds-Kac-Moody Lie algebras.

2 Infinite dimensional Lie algebras

Arbitrary finite dimensional Lie algebras are hard enough to investigate. Only the study of some subclasses, albeit important ones such as the semisimple ones, has been truly successful so far. This is obviously even more true for Lie algebras of arbitrary dimension, infinite and finite, as they have little in common. So it is reasonable to concentrate on some subclasses with enough similarities in pattern to yield results of interest.

Two natural generalizations of finite dimensional Lie algebras to infinite dimension have received particular attention:

1. the Borcherds-Kac-Moody (BKM) algebras; and
2. the locally finite Lie algebras.

The BKM algebras are a generalization of the finite dimensional semisimple Lie algebras and locally finite Lie algebras of arbitrary finite dimensional ones. Let us remind the reader of their definitions. Throughout this paper the base field is taken to be the complex field $\mathbb{C}$.

Definition 1. *A Lie algebra G is a BKM algebra if it satisfies the following properties:*

1. *G is $\mathbb{Z}$-graded, i.e. $G = \sum_{i\in\mathbb{Z}} G_i$, where $[G_iG_j] \leq G_{i+j}$ and $\dim G_i < \infty$ for all $i \neq 0$;*
2. *there is an anti-linear involution ω of G such that $\omega(G_i) = G_{i+1}$ and ω is multiplication by -1 on a real form of the Lie subalgebra G_0;*

3. *there is a non-degenerate Hermitian form* $(.,.)$ *on* G *which is positive definite on each subspace* G_i *with* $i \neq 0$ *and which is contravariant, i.e. for all* $g, x, y \in G$, $([gx], y) = -(x, [\omega(g)y])$.

It can be shown that the Lie algebra G is then a generalization of symmetrizable Kac-Moody algebras [Bor].

Theorem 1 (Bor)**.** *Let* I *be a discrete indexing set;* $H_{\mathbf{R}}$ *a real vector space with a non-degenerate symmetric real valued bilinear form* $(.,.)$ *and elements* $h_i, i \in I$ *such that*

1. $(h_i, h_j) \leq 0$ *if* $i \neq j$,

2. *if* $(h_i, h_i) > 0$, *then* $\frac{2(h_i,h_j)}{(h_i,h_i)} \in \mathbf{Z}$ *for all* $j \in I$.

Set $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. *Let* A *be the symmetric real valued matrix with entries* $a_{ij} = (h_i, h_j)$. *The Borcherds-Kac-Moody Lie algebra* $G = G(A, H)$ *associated to the Cartan matrix* A *and the abelian Lie algebra* H *is the Lie algebra generated by the abelian Lie algebra* H *and elements* $e_i, f_i, i \in I$ *satisfying the following defining relations:*

1. $[e_i, f_j] = \delta_{ij} h_i$;

2. $[h, e_i] = (h, h_i)e_i$, $[h, f_i] = -(h, h_i)f_i$;

3. $(\mathrm{ad}\,(e_i))^{1-\frac{2a_{ij}}{a_{ii}}} e_j = 0 = (\mathrm{ad}\,(f_i))^{1-\frac{2a_{ij}}{a_{ii}}} f_j$ *if* $a_{ii} > 0$ *and* $i \neq j$;

4. $[e_i, e_j] = 0 = [f_i, f_j]$ *if* $a_{ij} = 0$.

Definition 2. *A BKM algebra* $G = G(H, A)$ *for which* $\dim H < \infty$, *the matrix* A *is of finite size, and* $a_{ii} > 0$ *is a symmetrizable Kac-Moody (KM) algebra.*

This definition of a Kac-Moody Lie algebra is based on the one given in Chapter 1 of [Ka] and hence does not include Cartan matrices of infinite size.

Definition 3. *A Lie algebra* G *is said to be locally finite if any finite subset of* G *is contained in a finite dimensional Lie subalgebra.*

The locally finite Lie algebras of particular interest have countably infinite dimension. At least, it is reasonable to consider them first. Their construction is somewhat more amenable [BB].

Proposition 1 (BB)**.** *A Lie algebra* G *is locally finite of countable dimension if and only if it is the nested union of finite dimensional Lie algebras.*

Proof. If the locally finite Lie algebra G is of countable dimension then there is a countable basis $\mathcal{B}$ which can be taken to be $\{x_1, \cdots, x_n, \cdots\}$. Let G_j be the Lie subalgebra of G generated by the finite subsets $\{x_1, \cdots, x_j\}$. By definition of G, $\dim G_j < \infty$ and by construction, $G_j \leq G_{j+1}$ for all $j \geq 1$. So, $G = \cup_{i \in \mathbb{N}} G_i$. The converse is obvious. $\square$

However locally finite Lie algebras with countable bases are too large and disparate a class. Among them, one particular subclass is of particular interest, the locally reductive ones. They too, like the BKM algebras, are a natural generalization of semisimple finite dimensional Lie algebras.

Definition 4. *A Lie algebra G is said to be locally reductive if it is isomorphic to the union $\cup_{i\in\mathbb{N}}G_i$ of nested finite dimensional reductive Lie algebras G_i which are reductive in G_{i+1}.*

Since a locally reductive Lie algebra is locally finite, the only locally reductive Lie algebras that are KM algebras are the finite dimensional reductive ones. Indeed a KM algebra is infinite dimensional if and only if its set of roots contains an imaginary root [Chapter 5, Ka] since by assumption Cartan matrices of KM algebras have finite size. More precisely,

let $G = G(H, A)$ be a KM algebra. A root $\alpha \in H^* - \{0\}$ is imaginary if and only if for any $x \in G_\alpha = \{x \in G : [h, x] = \alpha(h)x,\ h \in H\}$, there exists a root $\beta \in H^* - \{0\}$ and an element $y \in G_\beta$ such that $(\operatorname{ad} x)^n y \neq 0$ for all $n \in \mathbb{Z}_+$. Since $(\operatorname{ad} x)^n y \in G_{n\alpha+\beta}$, these elements are linearly independent. Therefore the Lie subalgebra $< x, y >$ generated by the elements x and y is not finite dimensional.

However the larger class of BKM algebras and that of locally reductive Lie algebras have non-trivial intersection since there is no finiteness condition on the size of the Cartan matrix.

Example 1.

The infinite dimensional Lie algebra $G = sl(\infty)$ is both BKM and locally reductive. On the one hand, $G = G(H, A)$ where the abelian Lie subalgebra H has a countably infinite dimensional basis $\{h_i, i \in \mathbb{N}\}$ and Cartan matrix given by

$$(h_i, h_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } j = i + 1, \\ 0 & \text{otherwise .} \end{cases} .$$

On the other, $G = \cup_{n\in\mathbb{N}} sl(n)$ where $sl(n) \leq sl(n + 1)$ via the inclusion

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

This last example brings us to a smaller subclass. A major problem concerning locally reductive Lie algebras is due to the possibly complex nature of the injection of a finite dimensional reductive Lie algebra into another. It is always true that if H_{i+1} is a Cartan subalgebra (CSA) of the finite dimensional reductive Lie algebra G_{i+1}, then $H_{i+1} \cap G_i$ being a toral subalgebra, is contained in a CSA of G_i but is not necessarily a CSA of G_i. In the event when it is, the root spaces of G_i with respect to H_i need not remain root spaces for H_{i+1}. This is well illustrated by the next example.

Example 2.

Let $G_1 \cong sl(2)$ and $G_2 \cong sl(4)$. Consider the injection $G_1 \to G_2$ is given by: $X \mapsto \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$. The subalgebra $H_2 =< E_{11} - E_{22}, E_{22} - E_{33}, E_{33} - E_{44} >$ is a CSA of G_2 and $H_1 = H_2 \cap G_1 =< E_{11} - E_{22} + E_{33} - E_{44} >$ is a CSA of $sl(2)$. The root spaces of G_1 are $E_{12} + E_{34}$ and $E_{21} + E_{43}$. These are not root spaces of G_2 with respect to H_2.

In fact if T_2 is a CSA of G_2 and $T_1 = T_2 \cap G_1$ a CSA of G_1, then a root space of T_1 in G_1 is never contained in a root space of T_2. To see this, by considering conjugate CSAs under inner automorphisms induced by elements in G_1, we may assume that $T_1 = H_1$. We want to find a toral subalgebra of dimension 3 having $E_{12} + E_{34}$ as eigenvector. Let $h \in G_2$ be such that

$$[h, E_{12} + E_{34}] = c(E_{12} + E_{34})$$

for some scalar $c \in \mathbb{C}$ and

$$[h, E_{11} - E_{22} + E_{33} - E_{44}] = 0.$$

Then,

$$h \in < E_{11} - E_{22} + E_{33} - E_{44}, E_{22} - E_{33}, E_{13} + E_{24} > .$$

However the latter subalgebra contains only a 2-dimensional toral subalgebra.

Some locally reductive Lie algebras do have a structure comparatively easier to investigate in the sense that we can find CSAs of their nested subalgebras without the previously discussed complications.

Definition 5. *A locally reductive Lie algebra $G = \cup_i G_i$ is said to be root reductive if for all i, the finite dimensional reductive Lie algebras G_i satisfy the following:*

1. *there is a Cartan subalgebra (CSA) H_i of G_i such that $H_i \cap G_{i-1} = H_{i-1}$ is a CSA of G_{i-1};*
2. *any root space $(G_{i-1})_\alpha$ with respect to the CSA H_{i-1} of G_{i-1} is also a root space $(G_i)_\alpha$ of G_i with respect to the CSA H_i.*

Example 2 shows that the diagonal Lie algebra $sl(2^\infty)$ which is the nested union of the subalgebras $G_i = sl(2^i)$ with the embeddings given as above is not root reductive. In fact this is an immediate consequence of the structure of root reductive Lie algebras as shown in [P].

Proposition 2 (P)**.** *Let G be a root reductive. Then, the derived subalgebra $[GG]$ is a direct sum of finite dimensional simple Lie algebras and of copies of $sl(\infty)$, $o(\infty)$ and $sp(\infty)$ with countable multiplicities. Moreover, there is a splitting exact sequence of Lie algebras:*

$$0 \longrightarrow [GG] \longrightarrow G \longrightarrow G/[GG] \longrightarrow 0,$$

where the Lie algebra $G/[GG]$ is abelian.

There are some other types of locally finite Lie algebras that have been studied, though not to a large extent. Among them, we find the ideally finite ones. They were constructed and their structure investigated by Stewart in [S1] and [S2].

Definition 6. *A Lie algebra is said to be ideally finite if it can be generated by a collection of finite dimensional ideals.*

Ideally finite Lie algebras are clearly locally finite and include all the finite dimensional Lie algebras but not any infinite dimensional locally reductive Lie algebras nor BKM algebras.

Any statements made in this paper will be verified in the context of the above mentioned classes.

3 Characterizations of Cartan subalgebras

A major and early tool conceived of by Killing in [Ki] and Cartan in [C] in the investigation of the structure of finite dimensional Lie algebras are Lie subalgebras, today known as Cartan subalgebras (CSA). They lead to some non-trivial results in the arbitrary case, but are of particular importance in the context of semisimple finite dimensional Lie algebras. The reason for their usefulness is that finite dimensional Lie algebras decompose into generalized eigenspaces with respect to the adjoint action of a CSA in such a way that the 0-th generalized eigenspace coincides with the CSA itself. This is known as the Cartan decomposition. Of equal significance is that CSAs are all conjugate under the action of inner automorphisms and so the Cartan decomposition is an invariant aspect of finite dimensional Lie algebras. In particular, the set of non-zero eigenvalues or roots is unique for each isomorphism class of finite dimensional Lie algebras. Hence, it is reasonable to try to define Lie subalgebras of infinite dimensional Lie algebras generalizing the concept of CSAs. We leave the meaning of this statement vague for the moment.

CSAs can be defined via one of several equivalent characterizations. It is natural to try and extend one of these charaterizations to infinite dimension and investigate whether any of them leads to a Lie subalgebra that will continue to play the same role.

Let us start by listing the several definitions of a CSA in finite dimension. It is an elementary fact that any linear map f on a finite dimensional vector space is the sum of a semisimple linear map f_s and of a nilpotent one f_n and the two maps commute. This is known as the Chevalley-Jordan decomposition. We also remind the reader that a regular element x of a finite dimensional Lie algebra is one for which $\dim G_0(x)$ is minimal, where

$$G_0(x) = \{y \in L : \exists n \geq 0, (\operatorname{ad} x)^n y = 0\}.$$

Proposition 3. *Let G be a finite dimensional Lie algebra. A Lie subalgebra H of G is a CSA if any of the following equivalent properties hold:*

1. *there is a regular element $x \in G$ such that $H = G_0(x)$;*
2. *H is nilpotent and self-normalizing, i.e. $H = N_G(H)$;*
3. *$H = \{y \in G : [\operatorname{ad} y, (\operatorname{ad} x)_s] = 0 \ \forall x \in H\}$;*
4. *H is nilpotent and if $H \leq L$ is a Lie subalgebra of G, K is an ideal in L and the Lie algebra L/K is nilpotent, then $L = K + H$;*
5. *H is nilpotent and for any Lie subalgebra $H \leq K \leq G$, $K = N_G(K)$.*

As mentioned above, the CSAs are most useful for the finite dimensional semisimple Lie algebras. A key property in this case is that the Lie algebras are splitting. The Chevalley-Jordan decomposition implies that if G is a finite dimensional Lie algebra and $x \in G$, then $(\operatorname{ad} x) = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$. However, in general there are no elements $x_s, x_n \in G$ such that $\operatorname{ad} x_s = (\operatorname{ad} x)_s$ and $\operatorname{ad} x_n = (\operatorname{ad} x)_n$.

Definition 7. *An element of Lie algebra G is said to be semisimple (resp. nilpotent) if it acts* ad *-diagonally (resp.* ad *-nilpotently) on G. A Lie algebra G*

is said to be splitting if for any $x \in G$, there are elements $x_s, x_n \in G$ such that $\operatorname{ad} x_s = (\operatorname{ad} x)_s$ *and* $\operatorname{ad} x_n = (\operatorname{ad} x)_n$.

For splitting Lie algebras, there is yet another characteristic property satisfied by CSAs. First we remind the reader of the definition of a toral subalgebra.

Definition 8. *A toral subalgebra H of G is a Lie subalgebra all of whose elements are semisimple.*

Proposition 4. *Let G be a finite dimensional splitting Lie algebra. Then, the Lie subalgebra H is a CSA if and only if H is the centralizer $Z(T)$ of a maximal toral subalgebra T of G.*

When the Lie algebra G is semisimple, $Z(T) = T$ for maximal toral subalgebras T. So Proposition 4 reduces to the following.

Proposition 5. *Let G be a finite dimensional semisimple Lie algebra. Then, the Lie subalgebra H is a CSA if and only if H is a maximal toral subalgebra of G.*

Each defining property of a CSA has to be stated in a slightly modified form to continue to have meaning in infinite dimension. Definition (1) is the original one which appears in [Ki] and [C]. For the concept of a regular element to make sense it is necessary that $\dim G_0(x) < \infty$. When the Lie algebra G is infinite dimensional, it cannot be expected to contain elements for which this is the case. In finite dimension, the above definition of a CSA H is equivalent to $H = G_0(H) = \cap_{h \in H} G_0(h)$. This becomes satisfactory in a general context once we have restated the definition of $G_0(H)$. If H is a Lie subalgebra of G, set

$$G_0(H) = \{x \in G : \exists n, \forall h_1, \cdots, h_n \in H, (\operatorname{ad} h_1) \cdots (\operatorname{ad} h_n) x = 0\}.$$

Then, property (1) holds if and only if $H = G_0(H)$.

The equivalence of properties (1) and (2) (see [Bou]) is well known and (2) is often taken in textbooks to be the definition of a CSA since it states the essential qualities of a CSA. Nilpotency is often too strong a condition in infinite dimension and its natural generalization is that of local nilpotency.

Definition 9. *A Lie subalgebra of a Lie algebra G is said to be locally nilpotent if any finite number of elements in G generate a nilpotent Lie subalgebra of G.*

Property (3) is less well known. A proof of its equivalence to property (1) can be found in [NP]. We list it here as it is the definition used in the context of locally reductive Lie algebras and leads to the definition of a CSA that we propose for the arbitrary case. Property (3) however makes no sense in an arbitrary infinite dimensional Lie algebra since the Chevalley-Jordan decomposition of a homomorphism acting on an infinite dimensional vector space usually does not hold. Indeed this the case for the most well known class of infinite dimensional Lie algebra, namely the infinite dimensional KM algebras since – as stated above – they have imaginary roots. For a linear map on a vector space of arbitrary dimension to decompose in this manner, it needs to act locally finitely.

Definition 10. *A homomorphism f on a vector space V is said to act locally finitely if for any vector $v \in V$, the vector subspace generated by the element $f^i(v)$, $i \geq 0$, is finite dimensional.*

Lemma 1. *Let V be a vector space of countable dimension. Let f be a linear map on the vector space V acting locally finitely. Then, there is a unique linear map f_s (resp. f_n) acting semisimply (resp. locally nilpotently) on V such that $f = f_s + f_n$ and $[f_s, f_n] = 0$.*

Proof. Let $v_1, \cdots, v_n, \cdots$ be a basis for V. Let V_i be the vector subspace generated by the vectors $f^j v_i$. By assumption, for each i, $\dim V_i < \infty$. Therefore V_i is the direct sum of generalized eigenspaces for the restriction of the linear map f on V_i. As a result, V is the sum of generalized eigenspaces V_λ for f, where $V_\lambda = \{v \in V : \exists n = n(v), (f - \lambda I)^n v = 0\}$. Usual techniques imply that the sum is direct. Set f_s to be the semisimple linear map on V which acts as multiplication by the scalar $\lambda \in \mathbb{C}$ on the weight space V_λ. Set $f_n = f - f_s$. By definition of V_λ, for all $x \in V_\lambda$, there is an integer $r = r(x) \geq 0$ such that $f_n^r x = 0$. So the linear map f_n acts locally nilpotently on V_λ. Since f acts locally finitely, for all $v \in V$, there are only finitely many eigenvalues $\lambda_1, \cdots, \lambda_s \in \mathbb{C}$ such that $v = \sum_{i=1}^s v_i$, where $v_i \in V_{\lambda_i}$. Therefore the linear map f_n acts locally nilpotently on V. $\square$

Property (4) and its equivalence to property (2) appears in [BG-H] in the context of the development of the theory of formations for solvable Lie algebras. It is far less known than the other definitions given but is nevertheless used for the class of ideally finite Lie algebras constructed by Stewart in [S1]. Characterization (5) is only slightly different from (3). However we will see later that in infinite dimension they are far from being always equivalent. Let us re-write Proposition 1 in a form that makes sense for arbitrary Lie algebras though equivalences may not hold for any Lie algebra of arbitrary dimension.

Proposition 6. *Let G be a finite dimensional Lie algebra. A Lie subalgebra H of G is a CSA if any of the following equivalent properties hold:*

1. *$H = G_0(H)$;*
2. *H is locally nilpotent and $H = N_G(H)$;*
3. *$H = \{y \in G : [\operatorname{ad} y, (\operatorname{ad} x)_s] = 0\ \forall x \in H_{fin}\}$, where $H_{fin} = \{x \in H : \operatorname{ad} x$ acts locally finitely on $G\}$;*
4. *H is locally nilpotent and if $H \leq L$ is a Lie subalgebra of G, K is an ideal in L and the Lie algebra L/K is nilpotent, then $L = K + H$;*
5. *H is locally nilpotent and for any Lie subalgebra $H \leq K \leq G$, $K = N_G(K)$.*

When $\dim G < \infty$, the semisimple elements of a toral subalgebra are simultaneously diagonalizable. This is no longer necessarily the case when $\dim G = \infty$. It is well known that in infinite dimension commuting diagonalizable linear maps need not be simultaneously diagonalizable. The next example in $gl(\infty)$ illustrates this to be the case in the context of BKM algebras and root reductive Lie algebras. It is a modified version of an example constructed by Penkov in [P] for a different purpose (see below). First note that toral subalgebras in $gl(\infty)$ are abelian [NP]. This is shown to be more generally the case in [S2].

Lemma 2 (S2)**.** *Toral subalgebras of locally finite Lie algebras are abelian.*

Example 2.

Consider the root reductive Lie algebra $gl(\infty)$. It is the nested union $\cup_{n\geq 1} gl(n)$, where the inclusion $gl(n) \to gl(n+1)$ is given in the manner described in Example 1, and hence its elements are square matrices of countably infinite size with only finitely many non-zero entries.

Set E_{ij} to be the matrix with (i,j)-th entry equal to 1 and all other entries equal to 0. Consider the abelian subalgebra

$$H = \langle E_{ii} + E_{1i} : i \geq 2, i \rangle.$$

Since for any $i \geq 1$, $\exp(\operatorname{ad} E_{1i})(E_{ii}) = E_{ii} + E_{1i}$, H is a toral subalgebra.

Let us describe the eigenspaces for the element $E_{ii}+E_{1i}$. It has three distinct eigenvalues: 0, -1 and 1.

1. The 0-eigenspace has basis E_{jk}, $j \neq i$ $k \neq 1, i$, $E_{j1} - E_{ji}$, $j \neq i$;
2. the 1-eigenspace has basis The $E_{ik} + E_{1k}$, $k \neq 1$, $E_{i1} + E_{11} - E_{ii} - E_{1i}$;
3. the -1-eigenspace has basis E_{ji}, $j \neq i$.

It follows that the centralizer of H in G, or the 0-eigenspace for H is equal to H itself. If the subalgebra H were ad-diagonalizable, then the vector E_{11} would be a sum of H-eigenvectors. However the above list shows that it only appears in expressions for eigenvectors in the 0-eigenspace for the element $E_{ii} + E_{1i}$, for all $i > 2$, in other words, for eigenvectors in the 0-eigenspace for H. So the element $E_{11} \in H$, contradicting the definition of H. Hence the abelian Lie subalgebra H is not ad-diagonalizable.

Therefore, the version of Proposition 5 that is more appropriate in some infinite dimensional contexts is the following.

Proposition 7. *Let G be a finite dimensional semisimple Lie algebra. Then, the Lie subalgebra H is a CSA if and only if it is a maximal* ad*-diagonalizable Lie subalgebra.*

4 Towards a definition of a CSA for several classes of Lie algebras

In this section we will attempt to select a characterization of a CSA that seems to be the most appropriate in a general context.

As the main raison d'être of a CSA H in finite dimension is the Cartan decomposition which has the special property that $G_0(H) = H$, it is reasonable to see whether this can be a motivating factor in infinite dimension.

In mathematical terms, if G is a finite dimensional Lie algebra and H a CSA of G, then as the Lie subalgebra H is nilpotent, by Lie's Theorem there is a basis for G with respect to which for all $x \in H$, $\operatorname{ad} x$ is an upper triangular matrix. In particular for all $x \in [HH]$, $\operatorname{ad} x$ is a strictly upper triangular matrix and so 0 is its only eigenvalue. Hence,

$$G = H \oplus (\oplus_{\alpha \in (H/[HH])^*, \alpha \neq 0}) G_\alpha, \tag{1}$$

where $G_\alpha = \{x \in G : \forall h \in H, \exists n \geq 0, (\operatorname{ad} h - \alpha(h)I)^n x = 0\}$ is the generalized eigenspace for H with eigenvalue α.

4.1 Maximal ad-diagonalizable subalgebras

Let us first consider a BKM algebra $G = G(H, A)$. We have to be somewhat careful. Let Q be the free abelian group generated by elements α_i, $i \in I$. With the bilinear form defined by $(\alpha_i, \alpha_j) = a_{ij}$, the group Q becomes the root lattice and

$$G = H \oplus (\oplus_{\alpha \in Q} G_\alpha).$$

Moreover the set of roots does not contain 0 and is contained in $Q^+ \cup (-Q^+)$, where $Q^+ = \sum_i \mathbb{Z}_+ \alpha_i$. The reason for taking roots in Q rather than in the dual space H^* is that in infinite dimension the simple roots may be dependent if they are considered to be elements of H^*. However there may be roots $\alpha \in Q$ such that $[H, G_\alpha] = 0$.

Lemma 3. *Let $G = G(H, A)$ be a BKM algebra with an indecomposable Cartan matrix A. Then there is a root $\alpha \in Q$ such that $[H, G_\alpha] = 0$ if and only if the matrix A is of affine type and the centre $Z(G)$ of G is trivial.*

Proof. Suppose that the root $\alpha \in Q$ is as stated. Then, $(\alpha, \alpha_i) = 0$ for all $i \in I$. Hence as the Cartan matrix A is indecomposable, $I = supp(\alpha)$, the support of the root α. It follows that the matrix A is of affine type. Let $c \in H$ be the image of the root α in H. Then, $(c, H) = 0$ and so $c = 0$ since the bilinear form is non-degenerate on H. Moreover as $Z(G) = \mathbb{C}c$ and so $Z(G) = 0$. The converse clearly holds. $\square$

Therefore,

Corollary 1. *Let $G = G(H, A)$ be a BKM algebra. Then, $G = H \oplus (\oplus_{\alpha \in Q} G_\alpha)$. Moreover, if A is of affine type, suppose that $Z(G) \neq 0$. Then, $H = G_0(H)$.*

BKM algebras $G = G(H, A)$ generalize semisimple finite dimensional Lie algebras and the Lie subalgebra H is maximal ad-diagonalizable, the defining characteristic of a CSA in the finite dimensional semisimple context.

And indeed this is the definition of a CSA given by Kac and Petersen in [KP] for Kac-Moody algebras. It can be directly extended to BKM algebras and is the one used in this more general setup [R1].

Definition 11. *Let $G = G(H, A)$ be a BKM algebra for which $\dim H < \infty$. A maximal ad-diagonalizable Lie subalgebra is said to be a BKM-CSA.*

In [KP] a BKM-CSA is called a splitting Cartan subalgebra.

Proposition 8. *The BKM-CSA subalgebras of a BKM algebra are all conjugate under the action of inner automorphisms.*

This was shown in [KP] for Kac-Moody algebras and this result was extended to BKM algebras in [R1].

Is Definition 11 a reasonable one for CSAs in the context of the other class of Lie algebras generalizing the finite dimensional semisimple ones, namely the locally reductive Lie algebras? The answer is no, though at a first glance, it may seem to be so.

Indeed consider a root reductive Lie algebra $G = \cup_i G_i$, where for each i, G_i is a finite dimensional reductive Lie algebra. Let H_i be a CSA of G_i. For each

i, H_i may be so chosen that $H_i \leq H_{i+1}$. Hence set $H = \cup_i H_i$. Then, the Lie subalgebra H is maximal ad-diagonalizable. More precisely,

$$G = H \oplus (\oplus_{\alpha \in \Delta} G_\alpha),$$

where there is some i_α such that $G_\alpha = \cup_j G_{\alpha^j}$ for some root α^j of G_i, for all $i \geq i_\alpha$, $G_{\alpha^j} \leq G_i$ and $G_{\alpha^j} \leq G_{\alpha_{j+1}}$. Since all root spaces in G_{i+1} have dimension 1, $G_{\alpha_j} = G_{\alpha_{j+1}}$ and so $\dim G_\alpha = 1$. The root α is the inverse limit of the roots α^j. In particular, $0 \notin \Delta$, i.e. $H = G_0(H)$. It can also be shown that all maximal ad-diagonalizable Lie subalgebras of G are isomorphic [D-CPS]. Hence it would seem that Definition 11 is adequate for root reductive Lie algebras. However, if one considers the larger class of locally reductive Lie algebras, this is no longer the case. Note first that for arbitrary Lie algebras, just as we have to be careful with the definition of the 0-eigenspace, so do we for that of all eigenspaces. For any Lie algebra G, subalgebra H of G, and $\alpha \in (H/[HH])^*$, set

$$G_\alpha = \{x \in G : \exists n, \forall h_1, \cdots, h_n \in H, (\mathrm{ad}\, h_1 - \alpha(h_1)I) \cdots (\mathrm{ad}\, h_n - \alpha(h_n)I)x = 0\}.$$

From [P] we know that

Lemma 4 (P)**.** *If G is a locally finite Lie algebra and H a Lie subalgebra for which $H = G_0(H)$, then the the sum of the subspaces G_α, where $\alpha \in (H/[HH])^*$, is direct and an H-submodule of G.*

The following result proved in [P] shows that the only locally finite semisimple Lie algebras having a generalized form of Cartan decomposition are the root reductive ones, giving a negative answer to the above question.

Proposition 9 (P)**.** *Let G be a locally finite semisimple Lie algebra of countable dimension and H a Lie subalgebra for which $H = G_0(H)$. If $G = H \oplus (\oplus_\alpha G_\alpha)$, then the Lie algebra G is root reductive.*

This result also allows us to deduce that root reductive Lie algebras and BKM algebras with a positive definite contravariant bilinear form are the same objects.

Proposition 10. *A Lie algebra is a BKM algebra G with a positive definite Hermitian form $(.,.)$ which is contravariant with respect to an anti-linear involution ω if and only if G is a root reductive Lie algebra.*

Proof. Let G be a BKM algebra with the stated property. It follows that the map is positive definite on the Lie subalgebra H. In particular, all the roots of G have positive norm. Consider a finite subset of simple roots $S = \{\alpha_1, \cdots, \alpha_n\}$. The corresponding submatrix of the Cartan matrix is a positive definite Cartan matrix. Hence the Lie subalgebra generated by the elements e_i, f_i, $1 \leq i \leq n$ is finite dimensional. Therefore, the Cartan decomposition of G implies that any finite subset of elements of G generates a finite dimensional Lie algebra. So the BKM algebra G is locally finite and so must by Proposition 9 be a root reductive Lie algebra.

Conversely, suppose that G is a root reductive Lie algebra. Then, Proposition ? tells us that the derived subalgebra $[GG]$ is a direct sum of finite dimensional simple Lie algebras and of copies of $sl(\infty)$, $o(\infty)$ and $sp(\infty)$ with countable multiplicities. Each of these summands are BKM algebras with the desired property. Therefore so is G. $\square$

Proposition 10 together with Theorem 1 in [R1] gives an alternative proof to that given in [D-CPS] of the conjugacy of all maximal ad-diagonalizable Lie subalgebras of a root reductive Lie algebra under the action of inner automorphisms.

4.2 Generalized Cartan subalgebras

Proposition 9 tells us that Definition 11 cannot be taken as that of a CSA in the context of locally reductive Lie algebras since no such subalgebras exist. Cartan subalgebras in the infinite dimensional context will not lead to a decomposition of the full Lie algebra. Hence that is not what should be motivating their definition. Perhaps, a condition that a CSA H should satisfy is $H = G_0(H)$. At least in the context of locally finite Lie algebras this we have seen implies the existence of a H-submodule of G consisting of a direct sum of root spaces for H.

Let us consider relaxing the definition and considering maximal toral subalgebras. We have seen in the above example that they may not be ad-diagonalizable. The existence of these Lie subalgebras requires the concept of semisimple elements. This is indeed the case for locally reductive Lie algebras. Indeed let $G = \cup_i G_i$, where $G_i \leq G_{i+1}$ are finite dimensional reductive Lie algebras. Consider an arbitrary element $x \in G$. Let i be such that $x \in G_i$ and $x = x_s + x_n$, where $x_s, x_n \in G_i$ are respectively its semisimple and nilpotent parts. Since the Lie algebras G_j are all finite dimensional reductive, x_s and x_n remain respectively semisimple and nilpotent (see [Bou]) as elements of G_j for all $j \geq i$. Therefore, the following definition and Lemma make sense.

Definition 12. *Let $G = \cup_i G_i$, where $G_i \leq G_{i+1}$ are finite dimensional reductive Lie algebras be a locally reductive Lie algebra. Then $x \in G$ is said to be semisimple (resp. nilpotent) if x is a semisimple (resp. nilpotent) element of G_i for i such that $x \in G_i$.*

Lemma 5. *Let $G = \cup_i G_i$, where $G_i \leq G_{i+1}$ are finite dimensional reductive Lie algebras be a locally reductive Lie algebra. For any $x \in G$, there exist unique semisimple and nilpotent elements $x_s, x_n \in G$ such that $x = x_s + x_n$ and $[x_s, x_n] = 0$.*

Maximal toral subalgebras are not adequate however as a definition of a CSA even in the context of the nice class of root reductive Lie algebras since they are not self-centralizing and so are not equal to their 0-generalized eigenspace. This is shown by the following counterexample, given in [P], which we reproduce here – in an equivalent but different form – as [P] is not published.

Example 3.

Consider the root reductive Lie algebra $gl(\infty)$. It is the nested union $\cup_{n\geq 1} gl(n)$, where the inclusion $gl(n) \to gl(n+1)$ is given as stated above for $sl(\infty)$, and hence its elements are square matrices of countably infinite size with only finitely non-zero entries.

Set E_{ij} to be the matrix with (i,j)-th entry equal to 1 and all other entries equal to 0. Consider the abelian subalgebra

$$H = \{E_{ii} + E_{1i}, E_{jj} + E_{j2} : i \geq 3, i\text{odd}, j \geq 4, j\text{even}\}.$$

Since for any $i \geq 1$, $\exp(\operatorname{ad} E_{1i})(E_{ii}) = E_{ii} + E_{1i}$ and $\exp(\operatorname{ad}(-E_{i2}))(E_{ii}) = E_{ii} + E_{i2}$, H is a toral subalgebra.

We show that H is a maximal toral subalgebra. Let $H \leq T$ be a toral subalgebra of $gl(\infty)$. Consider $H_n = H \cap gl(n)$ and $T_n = T \cap gl(n)$. Then, $H_n \leq T_n$ and as T_n is a toral subalgebra of $gl(n)$, it is contained in a CSA of $gl(n)$. As $\dim H_n = n-2$ and the rank of $gl(n)$ is equal to n, $n-2 \leq \dim T_n \leq n$. As T_n is a toral subalgebra of the finite dimensional reductive Lie algebra $gl(n)$, it is abelian.

So let us find $Z_n = Z_{gl(n)}(H_n)$, the centralizer of H_n in $gl(n)$. Let $A = (a_{ij}) \in Z_n$.

Then,

1. for all $l \geq 3$ and l odd, $A(E_{ll} + E_{1l}) = (E_{ll} + E_{1l})A$; and
2. for all $l \geq 4$ and l even, $A(E_{ll} + E_{l2}) = (E_{ll} + E_{l2})A$.

So

for all $l \geq 3$ odd,

1. $a_{il} + a_{i1} = 0$ for $i \neq l$,$i > 1$,
2. $a_{1l} + a_{11} = a_{ll}$,
3. $a_{ll} + a_{l1} = a_{ll}$,
4. $a_{11} = a_{l1}$,

and for all $l \geq 4$ even,

1. $a_{il} = 0$ for $i \neq l$,
2. $a_{ll} = a_{ll} + a_{2l}$,
3. $a_{l2} = a_{l2} + a_{22}$,
4. $0 = a_{li} + a_{2i}$ for i odd,
5. $a_{l2} = a_{ll}$.

Therefore

$$a_{l1} = 0\, \forall l \text{ odd } \ l \geq 3$$

and so

$$a_{lk} = 0\, \forall l,k \text{ odd } \ l,k \geq 3$$

and

$$a_{11} = 0,$$

and

$$a_{2l} = 0\, \forall l \text{ even } \ l \geq 2.$$

This turn leads to

$$a_{1l} = a_{ll}\, \forall l \text{ odd } \ l \geq 3.$$

Therefore, the centralizer Z_n is generated by H_n and the elements E_{12}; $E_{il} - E_{i1}$, $i \neq l$,$i > 1$, $l \geq 3$ odd.

Therefore as T_n is a toral subalgebra $T_n = H_n$ since the elements E_{12} and $E_{il} - E_{i1}$ for $i \neq l$, $i > 1$ are nilpotent.

However $E_{12} \in Z_n$ for all $n \geq 0$ and so the maximal toral subalgebra H is not self-normalizing.

The above example together with Proposition 4 in the context of splitting Lie algebras, which is precisely the case for locally reductive ones, leads to the consideration of the centralizers of maximal toral subalgebras. This is what the authors do in [NP] and [D-CPS].

Toral subalgebras only make sense when semisimple elements can be defined, which even in finite dimension is not generally the case. Hence defining CSAs as centralizers of maximal toral subalgebras would not be satisfactory. In the previously mentioned two papers, the authors give the following equivalent characterizations for the the centralizer of maximal toral subalgebras (for $gl(\infty)$ in [NP] and more generally in [D-CPS]).

Note first that since the semisimple and nilpotent parts of an endomorphism on finite dimensional vector spaces are polynomials of that endomorphism, if T is a maximal toral subalgebra, then $Z_G(T)$ contains the semisimple and nilpotent parts of its elements.

Proposition 11 (D-CPS, D-CP)**.** *Let $G = \cup_i G_i$ be a locally reductive Lie algebra, H a Lie subalgebra of G and H_s the set of semisimple elements in H. Then the following are equivalent.*

1. *$H = Z_G(T)$, where T is a maximal toral subalgebra of G,*
2. *$H = Z(H_s)$ and H_s is a Lie subalgebra,*
3. *$H = Z(H_s)$,*
4. *$H = \cap_i G_0(H_i)$, where $H_i = G_i \cap H$.*

Any of the above properties can be taken to be the definition of a Cartan subalgebra of a locally reductive Lie algebra. In [NP], [D-CPS] and [D-CP], property (3) is chosen. Note that in the first two papers the added condition that H be locally reductive is added. However this is a consequence of $H = Z(H_s)$ in the context of locally finite Lie algebras. And indeed it is omitted in [D-CP].

Definition 13. *A Cartan subalgebra of a locally reductive Lie algebra is defined to be a Lie subalgebra H such that $H = Z(H_s)$, where H_s is the subalgebra of semisimple elements in H.*

We have seen that in a KM algebra G, the semisimple and nilpotent parts of the homomorphisms $\mathrm{ad}\, x$ for an element $x \in G$ is not in general defined. Therefore toral subalgebras H_s of a Lie subalgebra H make no sense, and hence we have to replace Definition 13 by its more general form given in Proposition 6. We show in this paper that, under the evidence available, it is one of the most reasonable definition for a Cartan subalgebra of a locally finite or a BKM Lie algebra. At the end of this article, we will briefly discuss the case of an arbitrary Lie algebra.

Definition 14. *Let G be a Lie algebra. A Lie subalgebra H of G is said to be a generalized Cartan subalgebra (GCSA) of G if H is locally nilpotent and*

$$H = \{y \in G : [\operatorname{ad} y, (\operatorname{ad} x)_s] = 0, \quad \forall x \in H_{fin}\},$$

where $H_{fin} = \{x \in H : \quad \operatorname{ad} x$ acts locally finitely on $G\}$.

Does this definition continue to be meaningful in the context of BKM algebras? Are their BKM-CSAs GCSAs? The answer is yes.

Lemma 6. *Let G be a BKM algebra and H a BKM-CSA of G. Then, the Lie subalgebra H is a GCSA of G.*

Proof. Let H be a maximal ad-diagonalizable Lie subalgebra of L. Hence, for all $h \in H$, $\operatorname{ad} h$ acts locally finitely on G and $\operatorname{ad} h = (\operatorname{ad} h)_s$. Moreover, $H_0 = H$ and so H is a GCSA. □

The converse does not hold when the BKM algebra G has infinite dimensional BKM-CSAs. For example, consider $sl(\infty)$. Example 2 together with Proposition 11 implies that the Lie subalgebra $\mathbb{C}E_{12} + \{E_{ii} + E_{1i} : i \geq 3, i\text{odd}, j \geq 4, j\text{even}\}$ is a GCSA but is not a BKM-CSA since the element E_{12} acts nilpotently and not diagonally on $sl(\infty)$.

However except for affine Lie algebras with trivial centres, it does hold when the BKM-CSAs are assumed to be finite dimensional, in particular it does for all Lorentzian BKM algebras.

Proposition 12. *Let $G = G(A, T)$ be a BKM algebra with an indecomposable Cartan matrix A. Assume that G is of finite rank (i.e. $\dim T < \infty$). If the Cartan matrix A is of affine type and $Z(G) = 0$, then a Lie subalgebra H of G is a GCSA if and only if H is a BKM-CSA or H is conjugate under an inner automorphism to the Lie subalgebra $< T, e_{n\alpha} : (\alpha, \alpha) = 0; n \in \mathbb{Q} >$. Otherwise, H is a GCSA if and only if H is a BKM-CSA.*

For any $h \in G_{fin}$, we know from Lemma 1 that the operator $(\operatorname{ad} h)_s$ is well defined. This is a derivation of G and hence to prove Proposition 12, we need some information about derivations of BKM algebras. The proof of the next result is simply a direct generalization of the proof of Theorem 5.2 in [MZ] to the context of BKM algebras from that of Kac-Moody ones.

Let Z be the centre of the BKM algebra $G = G(T, A)$, Δ its set of roots with respect to the BKM-CSA T, $\Delta^{\pm}$ the set of positive and negative roots respectively, G_α the α-root space for $\alpha \in \Delta$ and $N_{\pm} = \oplus_{\alpha \in \Delta^{\pm}} G_\alpha$. Let $\Pi = \{\alpha_i : i \in I\}$, where I is a countable indexing set, be a subset of simple roots. Set $Q = \mathbb{Z}_+\Pi \cup \mathbb{Z}_-\Pi$.

Lemma 7. *Let D be a derivation of the BKM algebra $G = G(T, A)$. Then, for any element $n \in N_+ \oplus N_-$ such that $(D - (\operatorname{ad} n))T \subset T$, $(D - (\operatorname{ad} n))T \subset Z$ and $(D - (\operatorname{ad} n))(G_\alpha) \leq G_\alpha$ for all $\alpha \in \Delta$. Moreover, there exists such an element n.*

Proof. No complex vector space is the countable union of proper subspaces. So there is an element $h \in T$ such that $\alpha(h) = \beta(h)$ for $\alpha, \beta \in Q$ implies that $\alpha = \beta$. Write $D(h) = h_1 + \sum_{\alpha \in \Delta} e_\alpha$, where $h_1 \in T$ and $e_\alpha \in G_\alpha$. Set

$n = -\sum_\alpha \alpha(h)^{-1} e_\alpha$. Then, $(D - (\operatorname{ad} n))(h) = h_1$. Hence, without loss of generality, we may assume that

$$D(h) \in T.$$

So, for any $t \in T$,

$$0 = D([h,t]) = [D(h),t] + [h, D(t)] = [h, D(t)].$$

By definition of h it follows that

$$D(T) \subset T.$$

Moreover for all $\alpha \in \Delta$, $e_\alpha \in G_\alpha$, $t \in T$,

$$\alpha(t)D(e_\alpha) = D([t, e_\alpha]) = \alpha(D(t))e_\alpha + [t, d(e_\alpha)].$$

Writing $D(e_\alpha) = c + \sum_{\beta \in D} x_\beta$, where $x_\beta \in G_\beta$, $c \in T$, we get:

$$\alpha(t)(c + \sum_{\beta \in D} x_\beta) = \alpha(D(t))e_\alpha + \sum_{\beta \in \Delta} \beta(t) x_\beta.$$

Considering $t = h$, the definition of h implies that

$$D(e_\alpha) \in G_\alpha$$

. This in turn gives and

$$\alpha(D(t)) = 0$$

for all roots $\alpha \in D$ and $t \in T$. This forces

$$D(T) \subset Z.$$

$\square$

We now prove Proposition 12.

Proof. Let H be a GCSA of the BKM algebra $G = G(T, A)$. Consider the subset H_{fin} of elements acting locally finitely on G. Let $x \in H_{fin}$. Write

$$x = t + \sum_{\alpha \in \Delta} x_\alpha,$$

where $x_\alpha \in G_\alpha$. From Theorem 2.3.33 in [R2] we know that unless G is a Heisenberg algebra, for any imaginary root α, there is a root β such that $(\operatorname{ad} x_\alpha)^n x_\beta \neq 0$ for all integers $n \geq 0$.

Case 1: G is not a Heisenberg algebra.

Since the operator $\operatorname{ad} x$ acts locally finitely on G, the above forces all roots $\alpha \in \Delta$ such that $x_\alpha \neq 0$ to be real. As a consequence, $x \in G_{KM}$, where G_{KM} is the Kac-Moody subalgebra of G generated by the BKM-CSA T and the simple root spaces $G_{\pm\alpha_i}$, where $\alpha_i \in \Pi$ is a real root. Hence $(\operatorname{ad} x)G_{KM} \subset G_{KM}$, and so $(\operatorname{ad} x)_s(G_{KM}) \subset G_{KM}$. The semisimple operator $(\operatorname{ad} x)_s$ is a derivation of G. Therefore Lemma 7 implies that there is an element $n \in G_{KM}$ such that

$$(\operatorname{ad} x)_s - (\operatorname{ad} n)(T) \leq Z$$

and

$$(\operatorname{ad} x)_s - (\operatorname{ad} n)(G_\alpha) \subset G_\alpha$$

for all $\alpha \in \Delta$. From section 2.3 in [R2] we know that $\dim G_{\pm\alpha_i} = 1$ for all real simple roots α_i. Let $0 \neq e_i \in G_{\alpha_i}$ and $f_i \in G_{\alpha_i}$ be Chevalley generators of G_{KM}. Then, $D(e_i) = r_i e_i$ and $D(f_i) = s_i f_i$ for some scalars $r_i, s_i \in \mathbb{C}$. Setting $h_i = [e_i, f_i]$, Then,

$$D(h_i) = (r_i + s_i)h_i.$$

If $D(h_i) \neq 0$, then $h_i \in Z$ and hence $(h_i, h_i) = 0$, contradicting the fact that α_i is a real root. So $r_i + s_i = 0$. Therefore there is an element $h \in T$ such that

$$D = (\operatorname{ad} x)_s - \operatorname{ad}(n + h)$$

satisfies

$$D([G_{KM}, G_{KM}]) = 0, \quad D(G_\alpha) \subset G_\alpha \,\forall \alpha \in \Delta \quad D(T) \subset Z.$$

Since the derivation $(\operatorname{ad} x)_s$ acts semisimply on G_{KM}, it follows that the element $x_s = n + h$ is a semisimple element of G_{KM} and hence of G. As $x_s \in \phi(T)$ for some inner automorphism ϕ of G and ϕ is a product of $\exp(\operatorname{ad} e_\alpha)$, where $\alpha \in \Delta$ is a real root and so $e_\alpha \in G_{KM}$, we can deduce that $D(\phi(G_\alpha)) \subset \phi(G_\alpha)$ and $D(\phi(T)) \subset Z$. So without loss of generality, we may assume that $n = 0$. Hence, for all $t \in T$, $(\operatorname{ad} t)(\operatorname{ad} x_s) = (\operatorname{ad} x_s)(\operatorname{ad} t)$. In other words from the definition of the GCSA H, we get

$$T \subset H.$$

So,

$$H \leq Z_G(T)$$

since the elements of T are semisimple. Let $y = t + \sum_\alpha y_\alpha \in Z_G(T)$, where $y_\alpha \in G_\alpha$. Then, $\alpha(T) = 0$ for all roots $\alpha \in \Delta$ for which $y_\alpha \neq 0$. Since the bilinear form on T is non-degenerate, this forces $[G_\alpha G_\alpha] = 0$ and $(\alpha, \alpha_i) = 0$ for all $i \in I$. Therefore by Lemma 3, if G is not an affine Lie algebra with trivial centre, then

$$H = T.$$

Otherwise, H is as stated.

Case 1: G is a Heisenberg algebra.

Since the Cartan matrix A is indecomposable, it follows that the dimension of the derived algebra $[GG]$ is 3 and so has generators $e, f, h = [ef]$. As $\operatorname{ad}(ae + bf)^2 = 0$, the operator $\operatorname{ad}(ae + bf)$ is nilpotent for all scalars $a, b \in \mathbb{C}$. In particular $\exp(\operatorname{ad} e)$ and $\exp(\operatorname{ad} f)$ are automorphisms of G. Let α be the root corresponding to the root vector e. If $t \in T$ is such that $\alpha(t) \neq 0$, then

$$\exp(\operatorname{ad} sf)\exp(\operatorname{ad} re)t = t + \alpha(t)rsh - \alpha(t)re + \alpha(t) + se.$$

It follows that any element

$$x = t + ae + bf$$

with $\alpha(t) \neq 0$ is semisimple. If $\alpha(t) = 0$, then the above implies that $t = x_s$ and $ae + bf = x_n$ are respectively the semisimple and nilpotent parts of x. Suppose that $x \in H$. If $\alpha(t) \neq 0$, then what precedes implies that we may assume $x = t$.

Let $y = t_1 + a_1 e + b_1 f$, where $a_1, b_1 \in \mathbb{C}$, be another element in H, then $[xy] = 0$ and so $a_1 = b_1 = 0$. Therefore,

$$H = T.$$

If $\alpha(t) = 0$, then as t is the semisimple part of x, $y = t_1 + a_1 e + b_1 f \in H$ implies that $[y, t] = 0$. From what precedes, we may assume that $\alpha(t_1) = 0$. It follows that $H = G$ and $\alpha(T) = 0$, contradicting the non-degeneracy of the bilinear form on T.

□

Therefore, the definition of a Cartan subalgebra given in Definition 14 remains reasonable in the context of BKM algebras. It is so in the context of locally reductive Lie algebras whereas toral or ad-diagonalizable subalgebras are not. In the context of BKM algebras, the known interesting examples are either root reductive (see Proposition 10) or have finite dimensional BKM-CSAs and among these the only occurrence when a GCSA is not necessarily a BKM-CSA is for affine type BKM algebras with trivial centre, in which case, by Proposition 12, the conjugacy classes of GCSAs is can be easily described.

Corollary 2. *Let G be a BKM Lie algebra with a Cartan matrix of finite size. Then, there is only one class of GCSAs unless G is affine with a trivial centre, in which case there are two.*

We next look at the other characterizations of Cartan subalgebras in finite dimension and see whether they do not in fact provide better alternatives to Definition 14.

4.3 Locally nilpotent and self-normalizing Lie subalgebras

So let us consider defining property (2) given in Proposition 4 of a CSA. It is the one used usually in finite dimension nowadays as it is the most efficient. In [BP], the authors in fact take this to be their definition of a Cartan subalgebra in an arbitrary setting.

In the context of BKM algebras and locally finite Lie algebras, we have the following property:

Proposition 13. *Let G be either a locally finite Lie algebra or a BKM Lie algebra of finite rank. Then, the GCSAs of G are locally nilpotent.*

Proof. Let H be a GCSA of G. Suppose first that the Lie algebra G is locally finite. Let $x, y \in H$, then $(\mathrm{ad}\, x) = (\mathrm{ad}\, x)_n + (\mathrm{ad}\, x)_s$ and $(\mathrm{ad}\, y) = (\mathrm{ad}\, y)_n + (\mathrm{ad}\, y)_s$. It follows that $[(\mathrm{ad}\, x), (\mathrm{ad}\, y)_s] = 0$ and so $[(\mathrm{ad}\, x)_n, (\mathrm{ad}\, y)_s] = 0 = [(\mathrm{ad}\, x)_s, (\mathrm{ad}\, y)_s] = 0$. Similarly, $[(\mathrm{ad}\, y)_n, (\mathrm{ad}\, x)_s] = 0$. Therefore, in $Der(G)$, $(\mathrm{ad}\, H) \leq N \oplus S$, where N contains only nilpotent elements and S is an abelian toral subalgebra of $Der(G)$. As a result the Lie subalgebra $(\mathrm{ad}\, H)$ must be locally nilpotent. It follows that so is the GCSA H.

When G is a BKM algebra, the result follows from Proposition 12. □

In fact it is shown in [NP] for gl_∞ and in [D-CPS] for arbitrary locally reductive Lie algebras, that

Lemma 8. *The GCSAs of locally reductive Lie algebras are self-normalizing.*

As for BKM-CSAs, the Cartan decomposition of BKM algebras implies that

Lemma 9. *The GCSAs of BKM algebras are self-normalizing.*

Note that together with Proposition 12, Lemma 8 implies the GCSAs of BKM algebras with finite dimensional BKM-CSAs satisfy a stronger condition than local nilpotence.

Corollary 3. *The GCSAs of BKM algebras of finite rank are abelian.*

In fact, it is shown in [D-CPS] that for root reductive Lie algebras, though GCSAs may not always be abelian, they are always nilpotent.

Lemma 10 (D-CPS)**.** *The GCSAs of root reductive Lie algebras are nilpotent of depth at most 2.*

We construct an example showing that the converse is false. For this we need the following observation.

Lemma 11. *Let H be a nilpotent Lie subalgebra of the locally finite Lie algebra G such that $H = H_s \leq Z(H)$, where H_s is the set of semisimple elements in H and $Z(H)$ is the centre of H. Then, $N_G(H) \leq Z_G(H_s)$, where $N_G(H)$ is the normalizer of H in G and $Z_G(H_s)$ the centraliser of H_s in G.*

Proof. Let $x \in N_G(H)$. As $H \leq Z_G(H_s)$, $[H_s[H,x]] = 0$ and so $[H,x] \leq Z_G(H_s)$. If $x \notin Z_G(H_s)$, then the Lie algebra being locally finite, we may assume that there is an element $h \in H$ for which $[h,x] = \lambda x$, where $\lambda \neq 0$. Therefore, $x \in [H,x] \leq H$ but $[H_s,x] \neq 0$, contradicting the fact that $[H,H_s] = 0$. $\square$

Example 4.
Consider the root reductive Lie algebra $G = gl(\infty)$. Let H be the Lie subalgebra of G generated by the elements

$$\begin{aligned}
y_1 &= E_{31} + E_{32} + E_{33} \\
y_2 &= E_{54} + E_{56} + E_{52} - E_{53} \\
y_l &= \sum_{k=2l-4}^{2l-1} E_{2l+1,2k} + E_{2,2l+1} - E_{3,2l+1} \quad \text{for } l \geq 3 \\
x_l &= E_{1,2l+1} - E_{3,2l+1} \quad \text{for } l \geq 2 \text{ and } l \equiv 0 \pmod 2 \\
x_l &= E_{1,2l+1} - E_{2,2l+1} \quad \text{for } l \geq 2 \text{ and } l \equiv 1 \pmod 2 \\
z_l &= E_{4l,4l} - E_{4l+2,4l} \quad \text{for } l \geq 1
\end{aligned}$$

The subalgebra H is nilpotent of depth 1 since

$$[HH] = \langle [x_l, y_l] : l \geq 2 \rangle \leq Z(H).$$

We show that $H = N_G(H)$. Let $x = \sum_{ij} a_{ij} E_{ij} \in N_G(H)$.

$$[x, z_l] = \sum_i (a_{i,4l} - a_{i,4l+2}) E_{i,4l} - \sum_k a_{4l,k} (E_{4l,k} - E_{4l+2,k}).$$

Since the element z_l is semisimple, Lemma 11 implies that $[x, z_l] = 0$. Therefore, as

$$a_{i,4l} = a_{i,4l+2} \quad \text{if } i \neq 4l, 4l+2;$$

$$a_{4l,k} = 0 \quad \text{if } k \neq 4l;$$

$$a_{4l+2,4l} - a_{4l+2,4l+2} = -a_{4l,4l}.$$

Since $z_l \in H$, without loss of generality, we may consider $x - a_{4l,4l} z_l$ instead of x and thus assume that $a_{4l,4l} = 0$. It follows that

$$a_{4l,k} = 0 \quad \forall k \tag{i}$$

and

$$a_{i,4l} = a_{i,4l+2} \quad \forall i \tag{ii}$$

$$\begin{aligned}[x, y_l] = &\sum_i a_{i,2l+1} (\sum_{k=2l-4}^{2l-1} E_{i,2k}) - (\sum_{j=2l-4}^{2l-1} a_{2j,k}) E_{2l+1,k} \\ &+ \sum_i (a_{i2} - a_{i3}) E_{i,2l+1} - \sum_k a_{2l+1,k} (E_{2k} - E_{3k})\end{aligned}$$

Since $x_l, y_l \in H$, using equalities (i) and (ii), for $l \geq 3$,

$$\begin{aligned}a_{2l+1,2l+1} - a_{4l-6,4l-8} - a_{4l-2,4l-8} &= a_{2l+1,2l+1} - a_{4l-6,4l-4} - a_{4l-2,4l-4} \\ &= \begin{cases} a_{22} - a_{23} - a_{2l+1,2l+1} & \text{if } l \equiv 0 \pmod 2 \\ a_{33} - a_{32} - a_{2l+1,2l+1} & \text{if } l \equiv 0 \pmod 2 \end{cases}\end{aligned}$$

and

$$a_{22} - a_{23} + a_{12} - a_{13} = a_{33} - a_{32},$$

$$a_{33} - a_{32} - a_{12} + a_{13} = a_{22} - a_{23}.$$

Since the above holds for all integers $l \geq 3$ and as the element $x \in gl_\infty$ is a finite linear combinations of the basis vectors E_{ij}, we get

$$a_{22} = a_{23} \tag{iii}$$

$$a_{32} = a_{33} \tag{iv}$$

$$a_{12} - a_{13} = a_{33} - a_{32} \tag{v}$$

and

$$-a_{12} + a_{13} = a_{22} - a_{23} \tag{vi}$$

For $l = 2$ we get

$$a_{64} = 2a_{55} \tag{vii}$$

and

$$a_{6k} = 0 \quad \text{for } k \neq 5, 4, 6 \tag{$viii$}$$

Also

$$a_{4l-6,k} = -a_{4l-2,k} \quad \text{for } k \neq 2l+1, 4l-8, 4l-6, 4l-4, 4l-2. \tag{ix}$$

Again x being a finite linear combination of the basis vectors, using equalities (i) and (ii), the above implies that

$$a_{2l,2k} = 0 \quad \forall l \geq 2,\ \forall k \tag{x}$$

and that

$$a_{2l+1,2l+1} = 0 \quad \forall l \geq 2 \tag{xi}$$

As $[x_l, y_l] \in H$,

$$a_{1,2l+1} = \begin{cases} -a_{3,2l+1} - a_{2l+1,2k} & \forall k = 2l-4, 2l-2 \quad \text{if } l \text{ is even} \\ -a_{2,2l+1} - a_{2l+1,2k} & \forall k = 2l-4, 2l-2 \quad \text{if } l \text{ is even} \end{cases} \tag{xii}$$

As $z_{l-2}, z_{l-1} \in H$,

$$a_{4l-8,2l+1} = -a_{4l-6,2l+1}$$

and

$$a_{4l-4,2l+1} = -a_{4l-2,2l+1}.$$

Also

$$a_{i2} = a_{i3} \quad \forall i \neq 2l+1, 1, 2, 3$$

and as $y_l \in H$, we may consider $x - (a_{2,2l+1} - a_{2l+1,4l-4})y_l$ instead of x assume that

$$a_{2l+1,k} = 0 \quad \forall k \tag{$xiii$}$$

(since (xi) holds) and that

$$a_{2,2l+1} = 0 \quad \text{if } l \text{ is even} \tag{xiv}$$

and

$$a_{3,2l+1} = 0 \quad \text{if } l \text{ is odd} \tag{xv}$$

and

$$a_{4l-6,2l+1} + a_{4l-2,2l+1} = a_{2l+1,3} - a_{2l+1,2}$$

$$a_{65} = a_{53} - a_{52}$$

So equalities (xi), (vii) and $(viii)$ imply that

$$a_{4l-6,k} = -a_{4l-2,k} \quad \forall l \geq 3, \forall k$$

and

$$a_{6k} = 0 \quad \forall k \neq 4, 6.$$

Hence as the element x is a finite linear combination of basis vectors,

$$a_{4l+2,k} = 0 \quad \forall l \geq 1,\ \forall k \tag{xvi}$$

Since $x_l \in H$, from equalities (xii) and $(xiii)$, we may replace the element x by $x - a_{1,2l+1}x_l$ and thus assume that

$$a_{1,2l+1} = a_{3,2l+1} = 0 \quad \text{if } l \text{ is even}$$

and

$$a_{1,2l+1} = a_{2,2l+1} = 0 \quad \text{if } l \text{ is odd}$$

It follows from equalities (i), $(xiii) - -(xvi)$ that

$$x = \sum_{1 \le i,k \le 3,k} a_{ik} E_{ik}.$$

Using (x),

$$[x, x_l] = \begin{cases} \sum_i (a_{i1} - a_{i3}) E_{i,l+1} & \text{if } l \text{ is even} \\ \sum_i (a_{i1} - a_{i2}) E_{i,l+1} & \text{if } l \text{ is odd} \end{cases}$$

As $x_l \in H$,

$$a_{11} - a_{13} = a_{33} - a_{31} \tag{xvii}$$

and

$$a_{11} - a_{12} = a_{22} - a_{21}. \tag{xviii}$$

Otherwise,

$$a_{21} = a_{23} \tag{xix}$$

$$a_{31} = a_{32} \tag{xx}$$

Since the element y_1 is semisimple, by Lemma 11,

$$0 = [x, y_1]$$

gives

$$a_{13} = 0 \tag{xxi}$$

$$a_{23} = 0 \tag{xxii}$$

$$a_{33} = a_{1k} + a_{2k} + a_{3k} \quad \text{for } k = 1, 2, 3 \tag{xxiii}$$

Equalities (iii), (iv), (v), (vi), $(xvii) - -(xxiii)$ then imply that

$$x = a_{33} y_1 \in H.$$

In conclusion,

$$H = N_G(H).$$

From [NP] we know that all GCSAs of the Lie algebra $gl(\infty)$ are abelian. In conclusion, since the subalgebra H is not abelian, it follows that it is not a GCSA. Equivalently $Z(H_s) \neq H$, where H_s is the subalgebra consisting of the semisimple elements in H.

We have remarked earlier that the most useful definition of a Cartan subalgebra in finite dimension, namely that of a nilpotent self-normalizing subalgebra becomes in the context of arbitrary dimension a locally nilpotent self-normalizing subalgebra. Example 4 shows that the proposal in [BP] of taking this as the definition of a GCSA is not advisable as this class is in the context of root reductive and hence of BKM algebras too large and unintersting.

In fact, Example 4 goes further. Keeping the far stronger condition of a nilpotent (and not just locally nilpotent) self-normalizing subalgebra as the definition is itself not a good idea as this smaller class is itself too large in the context of root reductive and hence BKM algebras. This answers a question posed in [D-CPS]:

Lemma 12. *There are root reductive Lie algebras with self-normalizing nilpotent subalgebras that are not GCSAs.*

More precisely, in [D-CPS] the authors show that the GCSAs of $gl(\infty)$ and $sl(\infty)$ are all abelian, and those of $o(\infty)$ and $sp(\infty)$ have depth at most 2 and these latter two contain GCSAs of depth 2. We have thus reached the following conclusion:

Proposition 14. *There are root reductive Lie algebras G with nilpotent self-normalizing subalgebras H of depth $k > j$, where j (≤ 2) is the maximal depth of a GCSA of G.*

We have not been able to construct an abelian subalgebra of $gl(\infty)$ or equivalently of $sl(\infty)$ that is self-normalizing and yet not a GCSA. Therefore the next question remains an open:

Open Question: Let G be a root reductive Lie algebra and j (≤ 2) be the maximal depth of a GCSA of G. Are all self-normalizing nilpotent subalgebras of G of depth at most equal to j GCSAs of G?

Since GCSAs of root reductive Lie algebras are nilpotent and not just locally nilpotent and root reductive Lie algebras belong to the class of locally reductive Lie algebras, it is natural to ask if this remains more generally the case for the GCSAs of this larger class of Lie algebras. This question is asked in [D-CPS]. We next construct an example showing that as expected nilpotency is too strong a condition in infinite dimension.

We define the finite dimensional semisimple Lie algebras G_i by induction on i. Set

$$G_1 = X_{r_1}$$

to be a finite dimensional simple Lie algebra of type X and

$$G_i = X_{r_i} \oplus \cdots \oplus X_{r_i},$$

to be the direct sum of m_i copies of the simple Lie algebra X_{r_i}. Let H_i be a nilpotent Lie subalgebra of the Lie algebra G_i. Consider the Lie algebra

$$L_i = X_{r_i+2} \oplus \cdots \oplus X_{r_i+2},$$

where the simple Lie algebra X_{r_i+2} appears m_i times. Consider G_i as a Lie subalgebra of L_i, the inclusion being the obvious one, namely if e_i, f_i (in the usual notation), $1 \leq i \leq r_i - 1$ are the generators of of the Lie algebra G_i, then e_i, f_i, $1 \leq i \leq r_i + 1$ are the generators of L_i. For each i, let H_i be a nilpotent subalgebra containing only nilpotent elements and that

$$H_i \leq H_{i+1} \qquad (i).$$

Assume that there exits s_i semisimple elements $x_1, \cdots, x_{s_i}$ in L_i such that

$$\cap_{j=1}^{s_i} Z_{G_i}(x_j) = H_i. \qquad (ii)$$

Then, define

$$G_{i+1} = G_{i+1}^0 \oplus \cdots \oplus G_{i+1}^{s_i},$$

where for each $k \geq i$, G_{i+1}^k is isomorphic to the Lie algebra L_i and G_{i+1}^0 is isomorphic to X_{r_i+2}. Equivalently, G_{i+1} is the direct sum of $s_i m_i + 1$ copies of the simple Lie algebra X_{r_i+2}. Let e_j and f_j be the Chevalley generators of the Lie algebra L_i and e_j^k, f_j^k their images in the Lie subalgebra G_{i+1}^k for each $k \geq 1$. Define the Lie algebra homomorphism

$$\phi_i : G_i \to G_{i+1}$$

as follows:

$$e_j \mapsto \sum_{k=0}^{s_i} e_j^k$$

$$f_j \mapsto \sum_{k=0}^{s_i} f_j^k.$$

The following property of the map ϕ is not hard to verify.

Lemma 13. *The map ϕ is a monomorphism.*

We will identify G_i with its image $\phi_i(G_i)$ in G_{i+1}. We keep the same notation x_j for the image of the semisimple element $x_j \in L_i$ in G_{i+1}^j. Set

$$z_i = \sum_{j=1}^{s_i} x_j \in G_{i+1}.$$

We will write $Z_{G_i}(z_i)$ for the set of elements in G_i commuting with the element z_i.

Lemma 14.

$$Z_{G_i}(z_i) = H_i.$$

Proof. Let $y \in Z_{G_i}(z_i)$. Then, $y = y_0 + \cdots y_{s_i}$, where $y_k \in G_{i+1}^k$. Then, $[y, z_i] = 0$ implies that $[y_k, x_k] = 0$ for all $1 \leq k \leq s_i$. Therefore assumption (ii) on the elements x_k forces $y \in H_i$. □

Assume that

$$H_{i+1} \cap G_{i+1}^1 \oplus \cdots \oplus G_{i+1}^{s_i} = H_i \tag{iii}$$

and

$$H_{i+1} = (H_{i+1} \cap G_{i+1}^0) \oplus \cdots \oplus H_{i+1} \cap G_{i+1}^{s_i})$$

Let $y_1, \cdots, y_{s_{i+1}} \in X_{r_i+4}$ be such that

$$\cap_{j=1}^{s_{i+1}} Z_{G_{i+1}^0} y_j = H_{i+1} \cap G_{i+1}^0. \tag{iv}$$

Then,

$$Z_{G_{i+1}} z_i \cap (\cap_{j=1}^{s_{i+1}} Z_{G_{i+1}^0} y_j) = H_{i+1}.$$

Define the Lie algebra G to be the nested union of the Lie algebras G_i and H_n to be the union of the Lie subalgebras H_i. Consider the Lie subalgebra H generated by the Lie subalgebra H_n and the semisimple elements z_i. By construction of the elements z_i and by assumption (iii),

$$[z_i, z_j] = 0$$

for all i, j and hence Lemma 14 and condition (iv) imply the following Corollary.

Corollary 4. *The Lie subalgebra H of the Lie algebra G is a GCSA subalgebra of G.*

Therefore, in order to find a non-nilpotent Cartan subalgebra of a locally semisimple Lie algebra, it suffices to take a locally finite dimensional simple Lie subalgebra $G_1 = X_{r_1}$ and for each i to find a nilpotent Lie subalgebra $H_i \leq X_{r_1+2i}$ containing uniquely nilpotent elements such that $H_{i-1} \leq H_i$ with the added property that if d_i is the nilpotent depth of H_i then $d_i > d_{i-1}$, and for which there exists finitely many semisimple elements $x_j \in X_{r_1+2(i+1)}$ such that $\cap_j Z_{X_{r_1+2i}}(x_j) = H_i$. We now construct such a sequence of Lie algebras and nilpotent subalgebras with these properties.

Example 5.

We keep the above notation. In $sl(n)$, we will write $e_{\alpha_i+\cdots\alpha_j} := [e_i[\cdots e_j]]$ and $f_{\alpha_i+\cdots\alpha_j} := [f_i[\cdots f_j]]$ for $j \geq i$ and T_n for the Cartan subalgebra generated by the elements $[e_i, f_i]$.

Set $G_1 = sl(3)$ and we define the subalgebras $H_i \leq sl(3+2i)$ as follows:

$$H_i = \langle e_{2j+1} - e_{\alpha_{2j+1}+\alpha_{2j+2}} : 0 \leq j \leq i \rangle.$$

Then for each i,

$$H_{i-1} \leq H_i$$

and the subalgebra H_i is nilpotent of depth

$$d_i = i - 1.$$

It only remains to find finitely many semisimple elements in $sl(5+2i)$ such that the intersection of their centralizers in $sl(3+2i)$ is equal to H_i. First consider for all $1 \leq k \leq i+1$,

$$x_k = f_{2k} + t_k,$$

where

$$\alpha_j(t_k) = \begin{cases} -1 & \text{if } j = 2k-1 \\ 1 & \text{if } j = 2k \\ 0 & \text{otherwise} \end{cases}$$

and $t_k \in T_{5+3i}$. Then,

$$Z_{sl(3+2i)}(x_k) = \langle e_{2k-1} - e_{\alpha_{2k-1}+\alpha_{2k}}, e_{2k+1}, f_{2k+1} + f_{\alpha_{2k}+\alpha_{2k+1}}, f_{\alpha_{2k-1}+\alpha_{2k}}, \\ \alpha_{2k}(h) f_{2k} + h, e_j, f_j : j \neq 2k-1, 2k, 2k+1, h \in T_{3+2i} \rangle$$

So,

$$\cap_{k=1}^{i+1} Z_{sl(3+2i)}(x_k) = \langle H_i, f_{\alpha_1+\cdots+\alpha_{2m}}, f_{\alpha_{2j+1}+f_{2j+2}} + f_{\alpha_{2j}+\alpha_{2j+1}+\alpha_{2j+2}}, h + \alpha_{2k}(h) f_{2k} : \\ 1 \leq m \leq i+1, 1 \leq j < i+1, h \in T_{2i+3}, \alpha_{2j}(h) = 0 \quad \forall j \neq k, 1 \leq j \leq i+1 \rangle.$$

Next consider

$$y_k = e_{\alpha_{2k+2}+\cdots+\alpha_{2i+3}} + e_{\alpha_{2k+3}+\cdots+\alpha_{2i+3}} + t,$$

where
$$\alpha_j(t) = \begin{cases} 1 & \text{if } j = 2i+3 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le k \le i$ and $t \in T_{i+1}$.

Then,

$$\begin{aligned} 0 &= [y_k, e_{2j-1} - e_{\alpha_{2j-1}+\alpha_j}] \\ &= [y_k, f_{\alpha_1+\cdots+\alpha_{2m}}] \end{aligned}$$

for all k, j, m;
$$0 = [y_k, h + \alpha_{2j}(h) f_{2j}]$$

for all $j \neq k+1$ and

$$0 = [y_k, f_{\alpha_{2j+1}+f_{2j+2}} + f_{\alpha_{2j}+\alpha_{2j+1}+\alpha_{2j+2}}]$$

for all $j \neq k+1$. Hence

$$(\cap_{k=0}^{i} Z_{sl(3+2i)}(y_k)) \cap (\cap_{k=1}^{i+1} Z_{sl(3+2i)}(x_k)) = \langle H_i, f_{\alpha_1+\cdots+\alpha_{2m}}, 1 \le m \le i+1 \rangle.$$

Finally set
$$y_{i+1} = e_{\alpha_1+\cdots+\alpha_{2i+3}} + t,$$

where t is as defined above. Therefore we get

$$(\cap_{k=0}^{i+1} Z_{sl(3+2i)}(y_k)) \cap (\cap_{k=1}^{i+1} Z_{sl(3+2i)}(x_k)) = H_i$$

as desired.

This answers the question posed in [D-CPS] in the negative.

Proposition 15. *There are locally reductive Lie algebras containing a GCSA which are not nilpotent.*

The example constructed is locally semisimple and not locally simple. So, the question posed by I. Penkov whether GCSAs of locally simple Lie algebras are nilpotent remains open. We will address this problem and the previously mentioned open question in a subsequent paper.

4.4 0-generalized eigenspace of a Lie subalgebra

We now turn to the original definition of a Cartan subalgebra, namely Property 1 of Proposition 4 used in [C]. Given that the equality of a Lie subalgebra with its 0-eigenspace is the very property that is at the centre of the pioneering work on Lie algebras of Killing and Cartan, it is reasonable to investigate if it remains the property of GCSAs of a Lie algebra of arbitrary dimension or whether it could be taken as a reasonable definition for generalized Cartan subalgebras instead of the one given above.

Example 4 also implies the next result:

Proposition 16. *A nilpotent self-normalizing Lie subalgebra H of a root reductive Lie algebra G is not necessarily equal to $G_0(H)$.*

From [D-CPS] (see Proposition 11, Property 4), we know that if G is a locally reductive Lie algebra and H a Lie subalgebra of G, then a necessary and sufficient condition for H to be a GCSA of G is $H = G_0(H)$. In fact this holds more generally for all locally finite Lie algebras.

Proposition 17. *Let G be a locally finite Lie algebra. Then a Lie subalgebra H of G is a GCSA if and only if $H = G_0(H)$.*

Proof. From Lemma 1 we know that for all $x \in G$, $(\operatorname{ad} x)_s$ is well defined. Therefore,

$$G_0(H) = \cap_{h \in H} G_0(h) = \cap_{h \in H} Z_G((\operatorname{ad} h)_s)$$

and hence the result follows. □

We next investigate what happens for BKM algebras. Corollary 1 tells us that a BKM-CSA H or equivalently a GCSA of a BKM algebra $G = G(A, T)$ with $\dim T < \infty$ satisfies $H = G_0(H)$ unless the Cartan matrix A is of affine type and $Z(G) = 0$, in which case it does not.

In fact, one can easily construct examples of BKM algebras G with subalgebras H satisfying $H = G_0(H)$. It suffices to take a BKM algebra with a root α of norm 0 such that $[G_\alpha, G_{-\alpha}]^\perp$, the orthogonal complement of the subspace $[G_\alpha, G_{-\alpha}]$ in H is $[G_\alpha, G_{-\alpha}]$ itself. The point is that a fundamental feature of infinite dimensional BKM algebras is the existence of elements that do not act in a locally finite manner.

Example 6.

Let $G = G(A, T)$, where $\dim T = 2$ and $A = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$. Set $H =< e_1, f_1, h_1 >$. Then, $(h, h_1) = 0$ implies that $h \in \mathbb{C}h_1$. Therefore $Z_G(h_1) = H + T$ and so $G_0(H) \leq H$. Since $(\operatorname{ad} e_1)^2(H) = 0 = (\operatorname{ad} f_1)^2$, $H = G_0(H)$.

Therefore in the context of BKM algebras, the condition $H = G_0(H)$ does not give classes of Lie subalgebras of much interest or usefulness.

4.5 Cartan subalgebras and ideally finite Lie algebras

We now consider the fourth and fifth characterizations of a CSA given in Proposition 3. In fact, it is shown in [St] and [GHT] that they remain equivalent for arbitrary Lie algebras. As mentioned earlier, characterization (4) was taken by Stewart in [S1] and [S2] as the definition of a Cartan subalgebra in the context of ideally finite Lie algebras.

Definition 15 (S)**.** *Let G be an ideally finite Lie algebra. A Lie subalgebra H is an IF-CSA of G if $H \leq L$ is a Lie subalgebra of G, K is an ideal in L and the Lie algebra L/K is nilpotent, then $L = K + H$.*

Proposition 18 (St)**.** *[GHT] Let G be a Lie algebra and H a Lie subalgebra of G. Then H is a IF-CSA if and only if H is locally nilpotent and for any Lie subalgebra $H \leq K \leq L$, $K = N_G(K)$.*

Though in finite dimension, it seems pointless to state properties (2) and (5) as two different characterizations of a CSA, they are far from being equivalent in infinite dimension, even among one of nicest classes known, namely the root reductive ones. Indeed from [D-CPS] (see Proposition 11) we know that for any GCSA H of a root reductive Lie algebra G, $H = N_G(H)$. However as the next Example shows this equality need not hold for all Lie subalgebras $H \leq L \leq G$.

Example 7.

We return to Example 1. So let $G = gl(\infty)$ and $H =< E_{ii} + E_{1i} : i \geq 2 >$. Set $h_i = E_{ii} + E_{1i}$. Then H is a a GCSA of G since $\cap_{i\geq 2} Z_G(h_i) = H$ (see Example 1). Consider the Lie subalgebra $L =< E_{ii}, E_{1i} : i \geq 2 >$. Then, $H \leq L$. We show that the elements $E_{11} - E_{ii}$, $i \geq 2$, normalize L: $[E_{11} - E_{ii}, E_{jj}] = 0$ for all $j \geq 2$ and $[E_{11} - E_{ii}, E_{1j}] = 2\delta_{ij} E_{1j}$. Therefore, $N_G(L) \neq L$.

Therefore a GCSA of a locally reductive Lie algebra or of a BKM algebra is not necessarily a IF-CSA. However the converse holds for locally reductive Lie algebras.

First note that

Proposition 19. *Let G be a Lie algebra. If H is an IF-CSA, then $H = G_0(H)$.*

Proof. From Proposition 18 we know that $H = N_G(H)$. Hence the definition of $G_0(H)$ implies that $G_0(H) \leq H$. Since H is locally nilpotent, equality holds. □

In particular, together with Proposition 17 it implies the following:

Corollary 5. *Let G be a locally reductive Lie algebra. If H is an IF-CSA, then H is a GCSA.*

Therefore the definition of an IF-CSA is too strong as Example 7 tells us that in the context of root reductive Lie algebras, there are GCSAs which are not IF-CSAs. So the definition of a GCSA given seems the more sensible choice in the context of two of the major classes of Lie algebras generalizing the finite dimensional reductive ones, namely the BKM algebras and the locally reductive Lie algebras. Indeed GCSAs of locally reductive Lie algebras are centralizers of maximal toral subalgebras and this is the most natural definition for split Lie algebras. Moreover the classification and description of GCSAs of root reductive Lie algebras in [D-CPS] is evidence that this is the class that should be considered.

Let us see whether the definition of a GCSA remains reasonable for ideally finite Lie algebras, a class of Lie algebras that generalizes not only semisimple finite dimensional Lie algebras but also solvable ones.

First we need to observe that contrary to the case of locally reductive Lie algebras, toral subalgebras of ideally finite Lie algebras are ad-diagonalizable. More generally,

Lemma 15. *Let G be an ideally finite Lie algebra and H a subalgebra of G such that for all $x, y \in H$, $[(\operatorname{ad} x)_s, (\operatorname{ad} y)_s] = 0$. Then, G is $(\operatorname{ad} H)_s$-diagonalizable.*

Proof. Let $x \in G$. Then, there is a finite dimensional ideal I such that $x \in I$. As I is an ideal, $[H, I] \leq I$. Since I is finite dimensional, Lemma 1 implies that $(\operatorname{ad} x)_s(I) \leq I$ for all $x \in H$. Therefore, x is the sum of simultaneous eigenvectors for the elements $(\operatorname{ad} x)_s$ as x runs through H. □

We next generalize Theorem 3.3 in [S2].

Proposition 20. *Let G be an ideally finite Lie algebra and H a Lie subalgebra of G. Then H is a GCSA if and only if H is an IF-CSA.*

Proof. Let H be a GCSA. Let K be a Lie subalgebra such that $H \leq K \leq G$. From Proposition 18 it suffices to show that $N = N_G(K) = K$. Suppose that there is so subalgebra K for which $N \neq K$. Let $x \in N - K$. Set $L = \langle K, x \rangle$. Then, K is an ideal in L. Both K and L are H-modules and hence Corollary 5 implies that L is the direct sum of weight spaces for $(\mathrm{ad}\, H)_s$ and so is L/K. By definition of N, $[h, x] \in K$ for all $h \in H$. Since $\dim L/K = 1$, there is a unique weight $\lambda : (\mathrm{ad}\, H)_s \to \mathbb{C}$ such that the weight spaces $L_\lambda \neq K_\lambda$. We may assume that $x \in L_\lambda - K_\lambda$. Hence, $(\mathrm{ad}\, h)_s(x) = \lambda(\mathrm{ad}\, (h)_s)x$. On the other hand, as $[h, x] \in K$ for all $h \in H$, Lemma 1 implies that $(\mathrm{ad}\, h)_s x \in K$ for all $h \in H$. Therefore $\lambda = 0$ and so $(\mathrm{ad}\, h)_s(x) = 0$ for all $h \in H$. So by definition of H, $x \in H \leq K$, contradicting the fact that $x \notin K$. Therefore H is an IF-CSA.

The converse is an immediate consequence of Propositions 18 and 19. □

Proposition 20 gives strong evidence that the definition of a generalized Cartan subalgebra given in Definition 14 is the best possibility in an arbitrary setting or at the very least among the two major classes of Lie algebras generalizing the finite dimensional semisimple ones and also among an important class of Lie algebras generalizing finite dimensional solvable ones.

5 Summary

Let G be a Lie algebra and H a Lie subalgebra of G. We will use the following terminology:

1. NCSA if H is locally nilpotent and if for any Lie subalgebra $H \leq K \leq L$, $K = N_G(K)$;
2. TCSA if $H = Z_G(T)$, where T is a maximal toral subalgebra of G;
3. CCSA if $H = G_0(H)$.

Proposition 21. *Let G be a Lie algebra. Then,*

$$\textit{NCSA} \iff \textit{IF-CSA}$$

1. *Suppose the Lie algebra G is locally finite. Then,*

$$\textit{GCSA} \iff \textit{CCSA}$$

   (a) *Suppose the Lie algebra G is locally reductive.*

$$\textit{NCSA} \implies \textit{GCSA} \iff \textit{TCSA} \iff \textit{CCSA}$$

   If H is a GCSA, then $N_G(H) = H$ but H is not necessarily a NCSA. If G is root reductive, then

$$\textit{BKM-CSA} \implies \textit{GCSA}\ .$$

   (b) *Suppose the Lie algebra G is ideally finite.*

$$\textit{NCSA} \iff \textit{GCSA}$$

2. Suppose $G = G(A)$ is a BKM algebra and H a Lie subalgebra of G.

$$
\begin{aligned}
&H \quad \textit{BKM-CSA or} \quad A \quad \textit{of affine type}\ , Z(G) = 0, \\
&H =< T, e_{n\alpha} : n \in \mathbb{Q}, (\alpha, \alpha) = 0, T \ \textit{BKM-CSA} \ > \\
&\Longleftrightarrow H \quad \textit{GCSA}\ .
\end{aligned}
$$

Moreover, a BKM-CSA is not necessarily a NCSA.

Moreover, there are root reductive Lie algebras with non-nilpotent self-normalizing subalgebras that are not GCSAs and the GCSAs of all locally reductive Lie algebras are not necessarily nilpotent.

Some Remarks about arbitrary Lie algebras

Before finishing, we make some comments about the arbitrary case. Since the definition of a GCSA is reasonable in the context of Borcherds-Kac-Moody Lie algebras, the class of Lie algebras it is applied to need not be locally finite or more generally split. However if there are no elements acting locally finitely in a Lie algebra G, then the definition only makes sense if G is locally nilpotent, though trivially. Otherwise GCSAs do not exist.

Lemma 16. *Let G be a Lie algebra with no locally finite Lie subalgebra. Suppose that there exists a GCSA in G. Then, G is locally nilpotent and has a unique GCSA, namely G itself.*

Proof. Suppose that H is a GCSA in G. Then, as there are no elements in H acting locally finitely on G, $H = G$ and hence G must be locally nilpotent. □

Therefore there are infinite dimensional Lie algebras for which the definition of a generalized Cartan subalgebra given in [BP], namely that of a locally nilpotent self-centralizing subalgebra, may be more adequate. However we have seen that this definition corresponds to a too large and uninteresting class of subalgebras in the context of locally finite or Borcherds-Kac-Moody algebras. It thus seems unlikely to be able to develop any meaningful unified theory of Cartan subalgebras in the context of arbitrary Lie algebras. The definition given in this paper is nevertheless reasonable for several classes of infinite dimensional Lie algebras.

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