Groups acting on tree-graded spaces and splittings of relatively hyperbolic groups

Cornelia Druţu[a*] and Mark Sapir[b,†]

Dedicated to Alexander Yurievich Olshanskii's 60th birthday

Abstract

Tree-graded spaces are generalizations of R-trees. They appear as asymptotic cones of groups (when the cones have cut points). Since many questions about endomorphisms and automorphisms of groups, solving equations over groups, studying embeddings of a group into another group, etc. lead to actions of groups on the asymptotic cones, it is natural to consider actions of groups on tree-graded spaces. We develop a theory of such actions which generalizes the well known theory of groups acting on R-trees. As applications of our theory, we describe, in particular, relatively hyperbolic groups with infinite groups of outer automorphisms, and co-Hopfian relatively hyperbolic groups.

Contents

*The authors are grateful to Institut des Hautes Études Scientifiques in Paris and to Max-Planck-Institut-für-Mathematik in Bonn for their hospitality during the work on this paper.

†The work of the second author was supported in part by the NSF grants DMS 0245600 and DMS-0455881.

1 Introduction

1.1 Actions of groups on real trees

An $\mathbb{R}$–tree is a geodesic metric space where every two points are connected by a unique arc. The theory of actions of groups on $\mathbb{R}$-trees, started by Bruhat-Tits [Ti] and Bass-Serre [Ser] in the case of simplicial trees, and continued in the general situation by Morgan-Shalen and Rips, and then by Sela, Bestvina-Feighn, Gaboriau-Levitt-Paulin, Chiswell, Dunwoody and others, turned out to be an important part of group theory and topology. See Bestvina [Be_2] for a survey of the theory.

Most applications of the theory are based on the following remarkable fact first observed by Bestvina [Be_1] and Paulin [Pau]. If a finitely generated group has "many" actions by isometries on a Gromov-hyperbolic metric space, then it acts non-trivially (i.e. without a global fixed point) by isometries on the asymptotic cone of that space which is an $\mathbb{R}$-tree. (The word "many" is explained in Lemma 4.28 below. Basically it means that there are infinitely many actions that are pairwise non-conjugate by isometries of the space.) Using theorems of Rips, Bestvina-Feighn and others, this often allows to write the group as the fundamental group of a non-degenerate graph of groups (see Section 2.5 for precise formulations).

The Bestvina-Paulin's observation applies, for example, given a hyperbolic group G and infinitely many pairwise non-conjugate homomorphisms from a finitely generated group Λ into G. Then there are "many" actions of Λ on the Cayley graph of G: every homomorphism $\phi\colon \Lambda \to G$ defines an action $g \cdot x = \phi(g)x$. In particular, this situation occurs if one of the following holds (see [Be_2, RS, Sel_1, Sel_3, Sel_4]):

1. An equation $w(a, b, ..., x, y, ...) = 1$ has infinitely many pairwise non-conjugate solutions in a group G (here $a, b, ... \in G$, $x, y, ...$ are variables, w is a word); then the group $\Lambda = G * \langle x, y, ...\rangle/(w = 1)$ has infinitely many pairwise non-conjugate homomorphisms into G;

2. $\mathrm{Out}(G)$ is infinite;

3. G is not co-Hopfian, i.e. it has a non-surjective but injective endomorphism ϕ; then we can consider powers of ϕ;

4. G is not Hopfian, i.e. it has a non-injective but surjective endomorphism ϕ; then we can consider powers of ϕ.

In cases 2, 3, 4 above $\Lambda = G$. Note that in all these cases, the group Λ acts non-trivially on the asymptotic cone of the group G even if G is not hyperbolic.

1.2 Tree-graded spaces

The asymptotic cones of non-hyperbolic groups need not be trees. Still, in several important cases they are *tree-graded spaces* in the sense of [DS]. Recall the definition.

Definition 1.1. Let $\mathbb{F}$ be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic subsets (called *pieces*). Suppose that the following two properties are satisfied:

(T_1) Every two different pieces have at most one common point.

(T_2) Every simple geodesic triangle (a simple loop composed of three geodesics) in $\mathbb{F}$ is contained in one piece.

Then we say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$.

In the property (T_2), we allow for trivial geodesic triangles, consequently $\mathcal{P}$ covers $\mathbb{F}$. In order to avoid extra singleton pieces, we make the convention that a piece cannot contain another piece.

Tree-graded spaces have many properties similar to the properties of $\mathbb{R}$–trees (see our paper [DS] and Section 2.2 below).

As we mentioned in [DS, Lemma 2.30], any complete geodesic metric space with cut-points has non-trivial tree-graded structure. Namely, it is tree-graded with respect to maximal connected subsets without cut-points. Having cut-points in asymptotic cones is a very weak form of hyperbolicity and, more generally, of relative hyperbolicity. As proved in [KKL], having a cut-point in an asymptotic cone is equivalent to having super-linear divergence of geodesics in the Cayley graph of the group. Note that Gromov hyperbolicity is equivalent to the superlinear divergence of any pair of geodesic rays with common origin [Alo]. Here are examples of groups and other metric spaces whose asymptotic cones have cut-points:

- (strongly) relatively hyperbolic groups[1] and metrically (strongly) relatively hyperbolic spaces [DS];
- Mapping class groups of punctured surfaces such that 3 times the genus plus the number of punctures is at least 5 [B];
- Teichmüller spaces with Weil-Petersson metric if 3 times the genus of the corresponding surface plus the number of punctures is at least 6 [B];
- Fundamental groups of graph-manifolds which are not Sol nor Nil manifolds [KKL];
- Many right angled Artin groups ([BDM]).

Note that mapping class groups, Teichmüller spaces and fundamental groups of graph-manifolds are not relatively hyperbolic [BDM] neither as groups nor as metric spaces (that is, there exists no finite family of finitely generated subgroups with respect to which they are hyperbolic, and no family of subsets with respect to which they are metrically relatively hyperbolic in the sense of [DS]).

1.3 Main results

The above examples give a motivation for the study of actions of groups on tree-graded spaces. In this paper, we show that a group acting "nicely" on a tree-graded space also acts "nicely" on an $\mathbb{R}$–tree, and so the group splits as a graph of groups with "reasonable" edge and vertex stabilizers. In order to explain the last phrase, we need the following definitions.

In [DS], we proved that for every point x in a tree-graded space $(\mathbb{F}, \mathcal{P})$, the union of geodesics $[x, y]$ intersecting every piece by at most one point is an $\mathbb{R}$–tree called a *transversal* tree of $\mathbb{F}$. A geodesic $[x, y]$ from a transversal tree is called *transversal geodesic*.

Notation: For every group G acting on a tree-graded space $(\mathbb{F}, \mathcal{P})$,

[1] In this paper, when speaking about relative hyperbolicity we shall always mean strong relative hyperbolicity.

- $\mathcal{C}_1(G)$ is the set of stabilizers of subsets of $\mathbb{F}$ all of whose finitely generated subgroups stabilize pairs of distinct pieces in $\mathcal{P}$;
- $\mathcal{C}_2(G)$ is the set of stabilizers of pairs of points of $\mathbb{F}$ not from the same piece;
- $\mathcal{C}_3(G)$ is the set of stabilizers of triples of points of $\mathbb{F}$, neither from the same piece nor on the same transversal geodesic.

Here is our main result about groups acting on tree-graded spaces.

Theorem 1.2 (Theorem 3.1). *Let G be a finitely generated group acting by isometries on a tree-graded space $(\mathbb{F}, \mathcal{P})$. Suppose that the following hold:*

(i) *every isometry $g \in G$ permutes the pieces;*

(ii) *no piece in $\mathcal{P}$ is stabilized by the whole group G; likewise no point in $\mathbb{F}$ is fixed by the whole group G.*

Then one of the following four situations occurs:

(I) *the group G acts by isometries on an $\mathbb{R}$–tree non-trivially, with stabilizers of non-trivial arcs in $\mathcal{C}_2(G)$, and with stabilizers of non-trivial tripods in $\mathcal{C}_3(G)$;*

(II) *there exists a point $x \in \mathbb{F}$ such that for any $g \in G$ any geodesic $[x, g \cdot x]$ is covered by finitely many pieces: in this case the group G acts non-trivially on a simplicial tree with stabilizers of pieces or points of $\mathbb{F}$ as vertex stabilizers, and stabilizers of pairs (a piece, a point inside the piece) as edge stabilizers;*

(III) *the group G acts non-trivially on a simplicial tree with edge stabilizers from $\mathcal{C}_1(G)$;*

(IV) *the group G acts on a complete $\mathbb{R}$–tree by isometries, non-trivially, such that stabilizers of non-trivial arcs are locally inside $\mathcal{C}_1(G)$-by-Abelian subgroups, and stabilizers of tripods are locally inside subgroups in $\mathcal{C}_1(G)$; moreover if G is finitely presented then the stabilizers of non-trivial arcs are in $\mathcal{C}_1(G)$.*

Using results of Rips, Bestvina-Feighn, Sela and Guirardel (see Section 2.5), we apply Theorem 1.2 to deduce from actions on tree-graded spaces splittings of the group.

If the group G acting on a tree-graded space is finitely presented then the following theorem holds.

Theorem 1.3 (see Theorem 3.23). *Let G be a finitely presented group acting by isometries on a tree-graded space $(\mathbb{F}, \mathcal{P})$ such that*

(i) *every isometry from G permutes the pieces;*

(ii) *no piece in $\mathcal{P}$ is stabilized by the whole group G; likewise no point in $\mathbb{F}$ is fixed by the whole group G;*

(iii) *the collection of subgroups $\mathcal{C}(G) = \mathcal{C}_1(G) \cup \mathcal{C}_2(G)$ satisfies the ascending chain condition.*

Then one of the following three cases occurs:

(1) G splits over a $\mathcal{C}(G)$-by-cyclic group;

(2) G can be represented as the fundamental group of a graph of groups whose vertex groups are stabilizers of pieces and stabilizers of points in $\mathbb{F}$, and edge groups are stabilizers of pairs (a piece, a point in the piece);

(3) the group G has a $\mathcal{C}_1(G)$-by-(free Abelian) subgroup of index at most 2.

With the help of a yet unpublished result of Guirardel, Theorem 2.34, the following weaker version of Theorem 1.3 for all finitely generated groups is proved. It is still sufficient for our applications to relatively hyperbolic groups.

Theorem 1.4 (Theorem 3.25). *Let G be a finitely generated group acting by isometries on a tree-graded space $(\mathbb{F}, \mathcal{P})$ such that properties **(i)** and **(ii)** from Theorem 1.3 hold, moreover*

(iii) *the subgroups in $\mathcal{C}_2(G)$ are (finite of uniformly bounded size)-by-Abelian and the subgroups in $\mathcal{C}_1(G) \cup \mathcal{C}_3(G)$ have uniformly bounded size.*

Then one of the following three cases occurs:

(1) G splits over a [(finite of uniformly bounded size)-by-Abelian]-by-(virtually cyclic) subgroup;

(2) same as case (2) from Theorem 1.3;

(3) the group G has a subgroup of index at most 2 which is a [(finite of uniformly bounded size)-by-Abelian]-by-(free Abelian) subgroup.

Note that if G is torsion-free then one can weaken the assumption on $\mathcal{C}_2(G)$ from Theorem 1.4 by using Sela's theorem from [Sel$_2$] instead of Guirardel's Theorem 2.34. The conclusion in that case is stronger:

Theorem 1.5 (see Theorem 3.24). *Let G be a finitely generated group acting by isometries on a tree-graded space $(\mathbb{F}, \mathcal{P})$ such that properties **(i)** and **(ii)** from Theorem 3.23 hold, and in addition*

(iii) *the collection of subgroups $\mathcal{C}_2(G)$ satisfies the ascending chain condition and every subgroup in $\mathcal{C}_1(G) \cup \mathcal{C}_3(G)$ is trivial.*

Then one of the following three cases occurs:

(1) G splits over a $\mathcal{C}_2(G)$-by-cyclic group or an Abelian-by-cyclic group;

(2) same as case (2) from Theorem 1.3;

(3) the group G has a metabelian subgroup of index at most 2.

1.4 The case of relatively hyperbolic groups

As the authors together with D. Osin showed in [DS], relatively hyperbolic groups provide natural examples of groups with tree-graded asymptotic cones. We describe now some applications of Theorems 1.3 and 1.4 to relatively hyperbolic groups.

One of the strongest known results about homomorphisms into relatively hyperbolic groups is due to Dahmani.

Definition 1.6. Following Dahmani [Dah], we say that a homomorphism ϕ from a group Λ into a relatively hyperbolic group G has an *accidental parabolic* if either $\phi(\Lambda)$ is parabolic (in which case ϕ is called a *parabolic homomorphism*) or Λ splits over a subgroup C such that $\phi(C)$ is either parabolic or finite.[2]

Dahmani proved in [Dah] that if Λ is finitely presented, and G is relatively hyperbolic then there are finitely many subgroups of G, up to conjugacy, that are images of Λ in G by homomorphisms without accidental parabolics.

Instead of homomorphic images, we consider the set of homomorphisms. Recall that a group Λ satisfies *property FA* of Serre [Ser] if every action of Λ on a simplicial tree has a global fixed point. The class of groups with property FA includes all finite groups, finite index subgroups in $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, and, more generally all groups with Kazhdan property T, $\mathrm{Aut}(F_n)$, $n \geq 3$, mapping class groups of surfaces with positive genus or spheres with at least 5 punctures [Bo, CV], subgroups of finite index in Chevalley groups of rank ≥ 2 over rings of integers [Ti], etc.

Theorem 1.7 (See Corollary 4.37). *If a finitely generated group Λ satisfies property FA then for every relatively hyperbolic group G there are only finitely many pairwise non-conjugate non-parabolic homomorphisms $\Lambda \to G$.*

If the group Λ is the fundamental group of a non-degenerate graph of groups (i.e. it does not have property FA), then the structure of homomorphisms into relatively hyperbolic groups is more complicated. Note that if a group G splits over an Abelian subgroup C, say $G = A *_C B$, then it typically has many outer automorphisms that are identity on A and conjugate B by elements of C. Hence in order to get a finiteness result for homomorphisms up to conjugacy, we need to modify the definition of accidental parabolics as follows.

Definition 1.8. We say that a homomorphism $\phi\colon \Lambda \to G$ has a *weakly accidental parabolic* if one of the following two cases occurs:

- $\phi(\Lambda)$ is either parabolic or virtually cyclic;
- Λ splits over a subgroup C such that $\phi(C)$ is either parabolic or virtually cyclic.

Here we formulate some of our results in not the strongest possible form. To simplify the formulation, we impose the restriction that peripheral subgroups do not contain free non-Abelian subgroups (see Section 4.4 where this condition is removed). From now till the end of the section, G is a (strongly) relatively hyperbolic group.

Theorem 1.9 (An immediate corollary of Theorem 4.40 and Remark 4.39.). *Suppose that the peripheral subgroups of G do not contain free non-Abelian subgroups. Let Λ be a finitely generated subgroup of G which is neither virtually cyclic nor parabolic. Assume moreover that Λ does not split over a parabolic subgroup nor over a virtually cyclic subgroup.*

Then there are finitely many conjugacy classes in G of injective homomorphisms $\Lambda \to G$ whose image is not parabolic.

Dahmani's theorem from [Dah] and Theorem 1.9 immediately imply the following corollary.

Corollary 1.10. *Let Λ be a finitely presented group and assume that the peripheral subgroups of G do not contain non-Abelian free subgroups. Then there are finitely many conjugacy classes in G of homomorphisms $\psi\colon \Lambda \to G$ without weakly accidental parabolics and with Hopfian images.*

[2] It is easy to see that this definition is equivalent to the definition in [Dah].

Proof. By Dahmani [Dah], up to conjugacy in G, there are finitely many subgroups that are images of Λ by homomorphisms $\psi\colon \Lambda \to G$ without accidental parabolics. So we may assume that the image of ψ is fixed. Then $\psi\colon \Lambda \to G$ is the composition of a fixed surjective homomorphism $\psi_1\colon \Lambda \to \psi_1(\Lambda)$ and a homomorphism ψ_2 of $\Lambda_1 = \psi_1(\Lambda)$ onto itself. If Λ_1 is Hopfian then ψ_2 is necessarily injective and we can apply Theorem 1.9. □

We do not know if a non-parabolic finitely generated subgroup of a relatively hyperbolic group can be non-Hopfian provided peripheral subgroups are Hopfian. Hence it might be possible to strengthen Corollary 1.10 at least in the case of Hopfian peripheral subgroups. If peripheral subgroups are not Hopfian, the whole group can be non-Hopfian as well: consider a free product $A * B$ where A is non-Hopfian.

Theorem 1.9 also immediately implies the following corollary. For every subgroup $\Lambda < G$ let $N_G(\Lambda)$ (resp. $C_G(\Lambda)$) be the normalizer (resp. the centralizer) of Λ in G. Clearly there exists a natural embedding ε of $N_G(\Lambda)/C_G(\Lambda)$ into the group of automorphisms $\mathrm{Aut}(\Lambda)$.

Corollary 1.11 (See Corollary 4.41 and Remark 4.39). *Suppose that $\Lambda \leq G$ is neither virtually cyclic nor parabolic, and that it does not split over a parabolic or virtually cyclic subgroup. Suppose that peripheral subgroups of G do not contain free non-Abelian subgroups.*

Then $\varepsilon(N_G(\Lambda)/C_G(\Lambda))$ has finite index in $\mathrm{Aut}(\Lambda)$. In particular, if $\mathrm{Out}(\Lambda)$ is infinite then Λ has infinite index in its normalizer.

In the case when $\mathrm{Out}(G)$ is infinite or G is not co-Hopfian, we need weaker or no extra assumptions on the peripheral subgroups to obtain splittings of G.

Theorem 1.12 (Theorem 4.43, Remark 3.26). *Suppose that the peripheral subgroups of G are not relatively hyperbolic with respect to proper subgroups. If $\mathrm{Out}(G)$ is infinite then one of the followings cases occurs:*

(1) G splits over a virtually cyclic subgroup;

(2) G splits over a parabolic (finite of uniformly bounded size)-by-Abelian-by-(virtually cyclic) subgroup;

(3) G can be represented as a non-trivial amalgamated product or HNN extension with one of the vertex groups a maximal parabolic subgroup of G.

Remark 1.13. Note that if a peripheral subgroup is relatively hyperbolic then, according to Lemma 4.23, it can be replaced in the list of peripheral subgroups by its own peripheral subgroups. Still in an arbitrary relatively hyperbolic group this process of getting smaller and smaller peripheral subgroups might not terminate. Indeed, there exist relatively hyperbolic groups that are not hyperbolic relative to any list of subgroups non-relatively hyperbolic with respect to proper subgroups. An example is the inaccessible group of Dunwoody [Dun_1] (the argument showing it is in [BDM]). It is not known if torsion free examples of this sort exist.

Theorem 1.14 (Theorem 4.45). *Suppose that a relatively hyperbolic group G is not co-Hopfian. Let ϕ be an injective but not surjective homomorphism $G \to G$. Then one of the following holds:*

- *$\phi^k(G)$ is parabolic for some k.*
- *G splits over a parabolic or virtually cyclic subgroup.*

Z. Sela has proved in [Sel_1] that a hyperbolic group which is non-elementary and torsion free is co-Hopfian if and only if it is freely indecomposable.

In the context of relatively hyperbolic groups, various weaker versions of Theorems 1.9, 1.12 and 1.14 (with various restrictions on parabolic subgroups) have been proved before by K. Ohshika and L. Potyagailo [OP], D. Groves ([Gro_1, Gro_2, Gro_3]), I. Belegradek and A. Szczepański [BS]. Instead of the asymptotic cones of the relatively hyperbolic group itself, Belegradek and Szczepański used actions of relatively hyperbolic groups on locally compact hyperbolic spaces. This led them in, say, the case when $\mathrm{Out}(G)$ is infinite or G is not co-Hopfian, to an action of G on an $\mathbb{R}$–tree (the asymptotic cone of the hyperbolic space) with parabolic stabilizers. But since they do not have control on the stabilizers, they have to assume that all subgroups of peripheral subgroups are finitely generated (i.e. the peripheral subgroups are thin).

Note that co-Hopf geometrically finite Kleinian groups (these are hyperbolic relative to Abelian subgroups) have been described by T. Delzant and L. Potyagailo in [DP]. A description of all homomorphisms of a finitely presented group into a relatively hyperbolic group via Makanin-Razborov diagrams has been provided in the case of limit groups [Ali], and more generally in the case of torsion-free groups that are hyperbolic relative to Abelian subgroups [Gro_3].

1.5 Organization of the paper

The plan of the paper is the following.

In Section 2, we present some general facts about tree-graded spaces. In particular, we define an $\mathbb{R}$–tree appearing as a natural quotient of a tree-graded space. It is essentially the union of all transversal trees, and can be described as the factor-space $\mathbb{F}/\approx$, where $x \approx y$ if in any geodesic joining x and y non-trivial subarcs contained in pieces compose a dense subset. We also define a natural construction that can be applied to any tree-graded space $(\mathbb{F}, \mathcal{P})$ and gives a tree-graded space $(\mathbb{F}, \mathcal{P}')$ with "bigger" pieces. In that section we also recall results about groups acting on $\mathbb{R}$–trees (theorems of Bestvina-Feighn, Levitt, Sela and Guirardel). We prove a version of Levitt's theorem about actions by homeomorphisms for finitely generated groups (Theorem 2.40).

In Section 3, we develop our theory of actions on tree-graded spaces and prove Theorem 1.2. The proof proceeds as follows. Let G be a group acting on a tree-graded space $(\mathbb{F}, \mathcal{P})$ such that conditions (i) and (ii) from Theorem 1.2 hold. First we analyze the situation (Case A) when G acts on the $\mathbb{R}$–tree $\mathbb{F}/\approx$ non-trivially. Then we show that the stabilizers of arcs of that $\mathbb{R}$–tree are from $\mathcal{C}_2(G)$. This gives Case (I) from Theorem 1.2.

In Case B, G acts on $\mathbb{F}/\approx$ with a global fixed point. The corresponding $\approx$–equivalence class R stabilized by G is also a tree-graded space with a collection of pieces $\mathcal{R} \subset \mathcal{P}$. We define an increasing transfinite sequence of tree-graded structures on R, $\mathcal{P}_0 = \mathcal{R} < \mathcal{P}_1 = \mathcal{P}_0' < ... < \mathcal{P}_\alpha$, using the construction from Section 2.5 and corresponding equivalence relations $\sim_0$, $\sim_1$,... on R (two points a, b of R are $\sim_\beta$–equivalent if a geodesic $[a, b]$ is covered by finitely many pieces from $\mathcal{P}_\beta$). This sequence of tree-graded structures must stabilize for some cardinal α. We consider two subcases: Case B1 is when G stabilizes a piece in $\mathcal{P}_\alpha$, and Case B2 when G does not stabilize a piece there.

In Case B1, we take the minimal cardinal $\delta \leq \alpha$ such that G fixes a piece A in $\mathcal{P}_\delta$. We show that δ is not a limit cardinal, so $\delta - 1$ exists. Then we construct a simplicial tree and an action of G on it. The vertices of the tree are pieces of $\mathcal{P}_{\delta-1}$ intersecting non-trivially a copy in $\mathbb{F}$ of the Cayley graph of G, and points of intersection of pieces. Edges connect a piece and an intersection point contained in that piece. If $\delta = 1$ then we get Case (II) of Theorem 1.2, if

$\delta > 1$ then we get Case (III).

In Case B2, the pieces of $\mathcal{P}_\alpha$ do not intersect. Then the $\mathcal{P}_\alpha$–pieces have a natural structure of pre-tree in the sense of Bowditch [Bow] and the group G acts on this pre-tree by automorphisms. The pre-tree is G-isomorphic (as a pre-tree with a G-action) to an $\mathbb{R}$–tree. We prove that by first embedding it G-equivariantly into an $\mathbb{R}$–tree, and then showing that the embedding is surjective. (For this embedding we might have used a result of Bowditch and Crisp [BC], but we found an easy proof of the existence of such an embedding before we were aware of the reference [BC]).) Then we use our version of Levitt's theorem (Theorem 2.40), and obtain an action of G on an $\mathbb{R}$–tree by isometries with "good" arc stabilizers (so that Case (IV) of the theorem holds).

In Section 4, we consider applications of our results to relatively hyperbolic groups. First we recall results from [DS] about relatively hyperbolic groups and asymptotically tree-graded spaces and prove some modifications of these results. Then we consider the isometry groups of asymptotic cones of relatively hyperbolic groups and describe stabilizers of pieces, pairs of pieces and triples of points of the cone. This allows us to apply Theorems 1.2 and 1.4 and deduce Theorems 1.12 and 1.14.

Acknowledgement. We are grateful to I. Belegradek, F. Dahmani, V. Guirardel, I. Kapovich, M. Kapovich, G. Levitt, F. Paulin and Z. Sela for many helpful conversations.

2 Preliminaries

2.1 Notation and definitions

Throughout this paper, we shall use the following notation.

For every path $\mathfrak{p}$, we denote the start of $\mathfrak{p}$ by $\mathfrak{p}_-$ and the end of $\mathfrak{p}$ by $\mathfrak{p}_+$.

Let X be a metric space. Given two quasi-geodesics in X, $\mathfrak{p}\colon [0,a] \to X$ and $\mathfrak{q}\colon [0,b] \to X$, such that $\mathfrak{p}_+ = \mathfrak{q}_-$, we denote by $\mathfrak{p} \sqcup \mathfrak{q}$ the map from $[0, a+b]$ to X that is equal to $\mathfrak{p}$ on $[0,a]$ and to $\mathfrak{q}[t-a]$ on $[a, a+b]$.

If a, b are two points in X then $[a,b]$ denotes any geodesic $\mathfrak{q}$ with $\mathfrak{q}_- = a, \mathfrak{q}_+ = b$.

If x is a point in X and $r \geq 0$ then $B(x,r)$ denotes the ball of radius r around x in X.

For every $Y \subseteq X$, $r \geq 0$, $\mathcal{N}_r(Y)$ and $\overline{\mathcal{N}}_r(Y)$ denote the open and respectively the closed r-tubular neighborhood of Y in X.

Recall that a group is said to have some property *locally* if any finitely generated subgroup of it has this property.

2.2 Tree-graded metric spaces

Lemma 2.1 ([DS], Proposition 2.17). *The property (T_2) in the definition of tree-graded spaces (see Definition 1.1) can be replaced by the assumption that $\mathcal{P}$ covers $\mathbb{F}$ together with the following property (which can be viewed as an extreme version of the bounded coset penetration property of [Fa]):*

> *(T_2') for every topological arc $\mathfrak{c} : [0,d] \to \mathbb{F}$ and $t \in [0,d]$, let $\mathfrak{c}[t-a, t+b]$ be a maximal sub-arc of $\mathfrak{c}$ containing $\mathfrak{c}(t)$ and contained in one piece. Then every other topological arc with the same endpoints as $\mathfrak{c}$ must contain the points $\mathfrak{c}(t-a)$ and $\mathfrak{c}(t+b)$.*

Throughout the rest of the section, $(\mathbb{F}, \mathcal{P})$ is a tree-graded space.

The following statement is an immediate consequence of [DS, Corollary 2.11].

Lemma 2.2. *Let A and B be two pieces in $\mathcal{P}$. There exist a unique pair of points $a \in A$ and $b \in B$ such that any topological arc joining A and B contains a and b. In particular* $\operatorname{dist}(A, B) = \operatorname{dist}(a, b)$.

Definition 2.3. For every topological arc $\mathfrak{g}$ in $\mathbb{F}$ we define its *strict saturation*, denoted by $\operatorname{Sat}_0 \mathfrak{g}$, as the union of $\mathfrak{g}$ with all the pieces intersecting $\mathfrak{g}$ non-trivially.

Notation: For every arc $\mathfrak{g}$ in $\mathbb{F}$, we denote by $I(\mathfrak{g})$ the collection of non-trivial sub-arcs which appear as intersections of $\mathfrak{g}$ with pieces from $\operatorname{Sat}_0 \mathfrak{g}$.

The following statement immediately follows from property (T_1).

Lemma 2.4. *Every subarc in $I(\mathfrak{g})$ is a maximal subarc of $\mathfrak{g}$ contained in a piece.*

Definition 2.5. The *saturation* of $\mathfrak{g}$, denoted by $\operatorname{Sat} \mathfrak{g}$, is the union of $\operatorname{Sat}_0 \mathfrak{g}$ and all the pieces intersecting $\mathfrak{g}$ by single points outside the arcs from $I(\mathfrak{g})$.

Obviously, $\operatorname{Sat}_0 \mathfrak{g} \subseteq \operatorname{Sat} \mathfrak{g}$.

Definition 2.6. Let $\mathfrak{g}$ be a geodesic segment, ray or line in $\mathbb{F}$. We denote by $\operatorname{Cutp}(\mathfrak{g})$ the complementary set in $\mathfrak{g}$ of the union of all the interiors of subarcs from $I(\mathfrak{g})$. We call it the *set of cut-points on* $\mathfrak{g}$.

Lemma 2.7 (Lemma 2.23, (3) and Corollary 2.10 in [DS]). *If $\mathfrak{c}_1 \sqcup \mathfrak{c}_2 \sqcup \ldots. \sqcup \mathfrak{c}_k$ is a polygonal line then $\operatorname{Sat}_0 \mathfrak{c}_1 \cup \operatorname{Sat}_0 \mathfrak{c}_2 \cup \ldots. \cup \operatorname{Sat}_0 \mathfrak{c}_k$ is strongly convex, i.e. it contains all topological arcs with endpoints in it.*

Property (T_2') and Lemma 2.7 immediately imply the following statement.

Corollary 2.8. *Two topological arcs with the same endpoints have the same strict saturation, the same saturation and the same set of cut-points.*

In particular a topological arc joining two points in a piece is contained in the piece.

Corollary 2.8 implies that we can define the following notions.

Definition 2.9. Let x, y be any two distinct points in $\mathbb{F}$. We define the *strict saturation* of the pair of points x, y, which we denote by $\operatorname{Sat}_0 \{x, y\}$, as the common strict saturation of all the topological arcs joining x and y. The *saturation* of the pair of points x, y, $\operatorname{Sat} \{x, y\}$, is defined similarly.

We likewise define the *set of cut-points separating x and y*, which we denote by $\operatorname{Cutp} \{x, y\}$, as the set of cut-points of some (any) topological arc joining x and y.

Remark 2.10. If an isometry fixes two points x and y in $\mathbb{F}$ then it stabilizes $\operatorname{Sat}_0 \{x, y\}$ and $\operatorname{Sat} \{x, y\}$, and it fixes $\operatorname{Cutp} \{x, y\}$ pointwise.

Remark 2.11. Note that $I(\mathfrak{g})$ together with the singletons not included in any sub-arc of $I(\mathfrak{g})$ define a structure of tree-graded space on $\mathfrak{g}$ induced by the tree-graded structure of $\mathbb{F}$.

Lemma 2.12. *For every $\epsilon > 0$ let x, y, x', y' be points in X such that $\operatorname{dist}(x, x') < \epsilon$, $\operatorname{dist}(y, y') < \epsilon$. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two geodesics connecting x with y and x' with y' respectively, and let $\mathfrak{g}_\epsilon = \mathfrak{g} \setminus (B(x, \epsilon) \cup B(y, \epsilon))$.*

Then $\mathfrak{g}_\epsilon \subset \operatorname{Sat}_0 \mathfrak{g}'$. In particular there exists an injective map $\iota : I(\mathfrak{g}_\epsilon) \to I(\mathfrak{g}')$ preserving the order of the arcs and such that:

(1) $\mathfrak{a}$ and $\iota(\mathfrak{a})$ are in the same piece for each $\mathfrak{a} \in I(\mathfrak{g}_\epsilon)$;

(2) $\mathfrak{a}$ and $\iota(\mathfrak{a})$ have the same endpoints for all but at most two extremal intervals $\mathfrak{a} \in I(\mathfrak{g}_\epsilon)$;

(3) *the sum of lengths of the sub-arcs in $I(\mathfrak{g})$ differs from the sum of lengths of sub-arcs in $I(\mathfrak{g}')$ by at most 2ϵ.*

Proof. Let $[x,x']$ and $[y,y']$ be two arbitrary geodesics. Let $\bar{x}$ be the farthest from x point in $\mathfrak{g}\cap[x,x']$. The point $\bar{y}$ is defined similarly for y and $\mathfrak{g}\cap[y,y']$. Denote by $\bar{\mathfrak{g}}$ the sub-arc of $\mathfrak{g}$ between $\bar{x}$ and $\bar{y}$. Then $[x',\bar{x}]\cup\bar{\mathfrak{g}}\cup[\bar{y},y']$ is a topological arc joining x' and y' and containing $\mathfrak{g}_\epsilon$. The statements of the lemma now follow from (T_2'). □

Lemma 2.13. *Let U be a union of pieces of a tree-graded space $(\mathbb{F},\mathcal{P})$. Suppose that for every two points x,y in U, $\mathrm{Sat}_0\{x,y\}\subseteq U$. Then every geodesic in $\mathbb{F}$ connecting two points from the closure $\overline{U}$ is contained in $\overline{U}$, moreover its interior is contained in U.*

Proof. Let x,y be two points in $\overline{U}$. Then x is the limit point of a sequence $x_n\in U$ and y is the limit point of a sequence $y_n\in U$. Let $[x_n,y_n]$ be a geodesic in $\mathbb{F}$. Then $[x_n,y_n]$ is inside $\mathrm{Sat}_0\{x_n,y_n\}\subseteq U$. Let $[x,y]$ be an arbitrary geodesic joining x and y in $\mathbb{F}$. By Lemma 2.12, for any ϵ, $[x,y]\setminus(B(x,\epsilon)\cup B(y,\epsilon))$ is contained in $\mathrm{Sat}_0\{x_n,y_n\}\subset U$. Therefore $[x,y]$ is contained in the closure of U and its interior is contained in U. □

2.3 $\mathbb{R}$–tree quotients of tree-graded spaces

Notation: For every tree-graded space $(\mathbb{F},\mathcal{P})$ and every two points x,y in $\mathbb{F}$ let $\widetilde{\mathrm{dist}}(x,y)$ be $\mathrm{dist}(x,y)$ minus the sum of lengths of sub-arcs from $I([x,y])$.

Lemma 2.14. *1. The number $\widetilde{\mathrm{dist}}(x,y)$ is well defined (i.e. it does not depend on the choice of a geodesic $[x,y]$).*

2. The function $\widetilde{\mathrm{dist}}(x,y)$ is symmetric and satisfies the triangular inequality.

Proof. Statement 1 immediately follows from Corollary 2.8. Statement 2 follows from Lemma 2.7. □

Let us define the equivalence relation $\approx$ by

$$x\approx y \text{ if and only if } \widetilde{\mathrm{dist}}(x,y)=0.$$

Lemma 2.15. (1) *If $x\approx y$ then for every geodesic $[x,y]$, its saturation is contained in the same $\approx$–equivalence class as x and y. The same holds for every topological arc joining x and y. In particular, every piece intersecting an $\approx$–class is contained in it.*

(2) *The equivalence relation $\approx$ is closed.*

Proof. The statement in part (1) for geodesics follows immediately from the definition of $\approx$. Together with property (T_2') it then implies the same statement for topological arcs.

(2) Let $x=\lim x_n, y=\lim y_n$ where $x_n\approx y_n$, that is $\widetilde{\mathrm{dist}}(x_n,y_n)=0$. We need to show that $\widetilde{\mathrm{dist}}(x,y)=0$. For every $\epsilon>0$ consider x_n,y_n such that $\mathrm{dist}(x_n,x)<\epsilon, \mathrm{dist}(y_n,y)<\epsilon$. Since $\widetilde{\mathrm{dist}}(x_n,y_n)=0$, the union of non-trivial intersections of pieces with a geodesic $[x_n,y_n]$ is dense in the geodesic. By Lemma 2.12, the same is true for $[x,y]\setminus(B(x,\epsilon)\cup B(y,\epsilon))$. Hence $\mathrm{dist}(x,y)$ minus the sum of the lengths of sub-arcs from $I([x,y])$ cannot be bigger than 2ϵ. Thus $\widetilde{\mathrm{dist}}(x,y)<2\epsilon$ for every $\epsilon>0$, hence $\widetilde{\mathrm{dist}}(x,y)=0$. □

Remark 2.16. Note that $\approx$ is in general not the smallest equivalence relation satisfying the properties in Lemma 2.15. For example, consider the collection of closures of subintervals of the interval $[0,1]$ used in creating the Cantor set (the middle thirds). This collection together with the singletons not contained in any interval form a tree-graded structure $\mathcal{P}$ on the unit interval. It is easy to see that in this case $\approx$ has only one equivalence class. On the other hand the smallest equivalence relation with the properties listed in Lemma 2.15 has as equivalence classes all the pieces $\mathcal{P}$.

Lemma 2.17. *Every $\approx$–class is a connected union of pieces.*

Proof. It follows immediately from Lemma 2.15. □

Recall [DS] that for every point x in a tree-graded space $\mathbb{F}$, the *transversal tree at x* consists of all $y \in \mathbb{F}$ such that any geodesic $[x,y]$ has only trivial intersections with pieces. We proved in [DS] that transversal trees either coincide or do not intersect. Any two points in the same transversal tree are joined by a unique geodesic in $\mathbb{F}$. These geodesics are called *transversal geodesics* in $\mathbb{F}$. The following lemma immediately follows from the definition of $\approx$.

Lemma 2.18. *If $x \neq y$ are in the same transversal tree of $\mathbb{F}$ then* $\mathrm{dist}(x,y) = \widetilde{\mathrm{dist}}(x,y)$. *In particular, $x \not\approx y$. Thus every transversal tree projects into $\mathbb{F}/\approx$ isometrically.*

Let T be the quotient $\mathbb{F}/\approx$.

Lemma 2.19. *T is an $\mathbb{R}$–tree with respect to the metric induced by* $\widetilde{\mathrm{dist}}$. *Every geodesic in $\mathbb{F}$ projects onto a geodesic in T.*

Proof. We first show that T is a geodesic metric space. Consider two points $\bar{x}$ and $\bar{y}$ in T. We shall construct one geodesic $\mathfrak{g}_{\bar{x},\bar{y}}$ joining them. Let $x,y \in \mathbb{F}$ be representatives of $\bar{x}$ and $\bar{y}$, and consider a geodesic $[x,y]$. Let μ be the Lebesgue measure on $[x,y]$ (here the geodesic is identified with an interval). Define a new measure μ_0 on all Borel sets by

$$\mu_0(B) = \mu\left(B \setminus \bigcup_{\mathfrak{a} \in I([x,y])} \mathfrak{a}\right).$$

Note that μ_0 is absolutely continuous with respect to μ, hence there exists a measurable function $f : [x,y] \to \mathbb{R}$ such that

$$\mu_0(B) = \int_B f \, d\mu \, .$$

Let $D = \mathrm{dist}(x,y)$ and let $\delta = \widetilde{\mathrm{dist}}(\bar{x},\bar{y})$. For every $t \in [0,D]$ let x_t be the point on $[x,y]$ at distance t from x and let $[x,x_t]$ be the sub-arc of $[x,y]$ between x and x_t. The function $F : [0,D] \to [0,\delta]$, $F(t) = \int_{[x,x_t]} f \, d\mu$ is monotone non-decreasing and continuous. Define the map $\mathfrak{g}_{\bar{x},\bar{y}} : [0,\delta] \to T$ such that $\mathfrak{g}_{\bar{x},\bar{y}}(s)$ is the projection $\bar{x}_{t(s)}$, where $x_{t(s)}$ is a point in $F^{-1}(s)$.

Let $s < r$ be two numbers in $[0,\delta]$. In order to compute $\widetilde{\mathrm{dist}}(\mathfrak{g}_{\bar{x},\bar{y}}(s), \mathfrak{g}_{\bar{x},\bar{y}}(r))$ consider the sub-arc $\left[x_{t(s)}, x_{t(r)}\right]$ of $[x,y]$. We have the following equalities

$$\widetilde{\mathrm{dist}}(\mathfrak{g}_{\bar{x},\bar{y}}(s), \mathfrak{g}_{\bar{x},\bar{y}}(r)) = \mu_0\left(\left[x_{t(s)}, x_{t(r)}\right]\right) = \int_{\left[x_{t(s)}, x_{t(r)}\right]} f \, d\mu F\left(x_{t(r)}\right) - F\left(x_{t(s)}\right) = r - s \, .$$

Note that for every $m \in [x,y]$, its projection $\bar{m}$ onto T is in $\mathfrak{g}_{\bar{x},\bar{y}}$. Indeed, let $d = \widetilde{\mathrm{dist}}(\bar{x},\bar{m})$ and let $x_{t(d)}$ be chosen as before. Then $d = \int_{[x,m]} f\,d\mu = \int_{[x,x_{t(d)}]} f\,d\mu$, hence

$$\int_{[m,x_{t(d)}]} f\,d\mu = \widetilde{\mathrm{dist}}(\bar{m}\,,\,\bar{x}_{t(d)}) = 0$$

and $\mathfrak{g}_{\bar{x},\bar{y}}(d) = \bar{x}_{t(d)} = \bar{m}$.

Now we show that for every two points $\bar{x},\bar{y}$ there exists a unique geodesic joining them in T. Let $\mathfrak{p}$ be an arbitrary geodesic joining $\bar{x}$ and $\bar{y}$ in T. Let $\mathfrak{g}_{\bar{x},\bar{y}}$ be the geodesic constructed above between $\bar{x},\bar{y}$. Let $\bar{z}$ be an arbitrary point on $\mathfrak{p}$ and let z be a representative of $\bar{z}$ in $\mathbb{F}$. For two arbitrary geodesics $[x,z],[z,y]$ let z' be the farthest from z point in $[x,z]\cap[z,y]$. Then $[x,z']\cup[z',y]$ is a topological arc. Consider the maximal sub-arc $[a,b]$ in it containing z' and contained in a piece. Note that this sub-arc can be a point. Property (T_2') implies that $\{a,b\}\subset[x,y]$. By definition $\bar{a}=\bar{b}=\bar{z'}$, which is a point on $\mathfrak{g}_{\bar{x},\bar{y}}$. Consequently, $\widetilde{\mathrm{dist}}(\bar{x},\bar{z'})+\widetilde{\mathrm{dist}}(\bar{y},\bar{z'})=\widetilde{\mathrm{dist}}(\bar{x},\bar{y})$. On the other hand, since $z'\in[x,z]\cap[z,y]$, $\widetilde{\mathrm{dist}}(\bar{x},\bar{z'})\le\widetilde{\mathrm{dist}}(\bar{x},\bar{z})$ and $\widetilde{\mathrm{dist}}(\bar{y},\bar{z'})\le\widetilde{\mathrm{dist}}(\bar{y},\bar{z})$. It follows that the previous two inequalities are in fact equalities, and $\bar{z'}=\bar{z}$. $\square$

Lemma 2.20. *Let $\mathbb{F}$ be a tree-graded space.*

(1) An isometry ϕ of $\mathbb{F}$ permuting the pieces induces an isometry $\tilde{\phi}$ of the real tree T.

(2) For every non-trivial geodesic $\mathfrak{g}$ in T there exists a non-trivial geodesic $\mathfrak{p}$ in $\mathbb{F}$ such that its projection on T is $\mathfrak{g}$. Moreover the isometry $\tilde{\phi}$ fixes $\mathfrak{g}$ pointwise if and only if ϕ fixes $\mathrm{Cutp}\,(\mathfrak{p})$ *pointwise.*

In the particular case when $\mathfrak{g}$ is the projection of a geodesic $\mathfrak{g}_0$ in a transversal tree, $\mathfrak{p}$ can be taken equal to $\mathfrak{g}_0$.

Proof. (1) Since for every piece A and every geodesic $\mathfrak{g}$, the lengths of $\phi(A)\cap\phi(\mathfrak{g})$ and $A\cap\mathfrak{g}$ are the same, $\widetilde{\mathrm{dist}}(\mathfrak{g}_-,\mathfrak{g}_+)=\widetilde{\mathrm{dist}}(\phi(\mathfrak{g}_-),\phi(\mathfrak{g}_+))$.

(2) Consider the endpoints $a\neq b$ of $\mathfrak{g}$ in T. By Lemma 2.19 if $x,y\in\mathbb{F}$ are representatives of a and b respectively, and $[x,y]$ is a geodesic joining them, the latter projects onto $\mathfrak{g}$. The intersection of the $\approx$–equivalence class containing x with $[x,y]$ is closed and connected, by Lemma 2.15. Hence it is a geodesic segment $[x,x']\subsetneq[x,y]$. By replacing if necessary x by x', we may therefore assume that $[x,y]$ contains only one point in the $\approx$–equivalence class of x. A similar argument allows to assume that the intersection of $[x,y]$ and the $\approx$–class containing y is $\{y\}$. The same can then be said about $\phi[x,y]$ and its endpoints. We take $\mathfrak{p}=[x,y]$. Since the projection of $\mathrm{Cutp}\,(\mathfrak{p})$ is $\mathfrak{g}$, if ϕ fixes $\mathrm{Cutp}\,(\mathfrak{p})$ pointwise then $\tilde{\phi}$ fixes $\mathfrak{g}$ pointwise. Now we show that the converse also holds.

As $\tilde{\phi}(a)=a$ and $\tilde{\phi}(b)=b$, it follows that $\phi(x)\approx x$ and $\phi(y)\approx y$. Any geodesic $[x,\phi(x)]$ intersects $\phi(\mathfrak{p})$ only in $\phi(x)$, likewise any geodesic $[y,\phi(y)]$ intersects $\phi(\mathfrak{p})$ in $\phi(y)$. Then $\mathfrak{g}_1=[x,\phi(x)]\sqcup\phi(\mathfrak{p})\sqcup[\phi(y),y]$ is a topological arc and $\phi(x),\phi(y)$ belong to $\mathrm{Cutp}\,(\mathfrak{g}_1)$. It follows from (T_2') that the geodesic $\mathfrak{p}$ must contain $\phi(x),\phi(y)$. On the other hand $\mathrm{dist}(x,y)=\mathrm{dist}(\phi(x),\phi(y))$. Consequently $x=\phi(x)$ and $y=\phi(y)$. This and Remark 2.10 imply that $\mathrm{Cutp}\,(\phi(\mathfrak{p}))=\phi(\mathrm{Cutp}\,(\mathfrak{p}))=\mathrm{Cutp}\,(\mathfrak{p})$. Moreover, since ϕ is an isometry it must fix $\mathrm{Cutp}\,(\mathfrak{p})$ pointwise. $\square$

2.4 Nested tree-graded structures

If $\mathcal{P}$ and $\mathcal{P}'$ are two collections of subsets of a set $\mathbb{F}$, we write $\mathcal{P}\prec\mathcal{P}'$ if for every set $A\in\mathcal{P}$ there exists $A'\in\mathcal{P}'$ such that $A\subset A'$. The relation $\prec$ induces a partial order on the set of tree-graded structures of a tree-graded space $\mathbb{F}$ because of our convention that in a tree-graded structure, pieces cannot contain each other.

For every tree-graded space $(\mathbb{F}, \mathcal{P})$, consider the following equivalence relation $\sim$. Two points x and y are $\sim$–equivalent if one (hence any by Corollary 2.8) geodesic $[x, y]$ is inside the union of a finite number of pieces. This relation is transitive, by Lemma 2.7.

Let κ be an equivalence class for $\sim$. The following lemma is obvious.

Lemma 2.21. *For any two points x, y in κ, the saturation* $\mathrm{Sat}\,\{x, y\}$ *is contained in κ.*

Lemma 2.22. *The union of strict saturations of pairs of points in κ is equal to κ.*

Proof. By Lemma 2.7, the union of strict saturations of pairs of points from κ is contained in κ. On the other hand κ is the union of all geodesics with endpoints in κ. Since every geodesics is inside its strict saturation, we conclude that κ is inside the union of strict saturations of geodesics with endpoints in κ. □

Let $\mathcal{P}'$ be the collection of closures of $\sim$–equivalence classes. For every A in $\mathcal{P}'$ let $\kappa(A)$ be the $\sim$–equivalence class such that $A = \overline{\kappa(A)}$.

Lemma 2.23. $\mathbb{F}$ *is tree-graded with respect to $\mathcal{P}'$.*

Proof. By construction all pieces in $\mathcal{P}'$ are closed. By Lemmas 2.22 and 2.13, all pieces are geodesic subspaces.

Clearly every piece of $\mathcal{P}$ is inside a piece of $\mathcal{P}'$. Therefore pieces of $\mathcal{P}'$ cover $\mathbb{F}$ and (T_2) is satisfied.

Assume that the intersection of two sets A and B from $\mathcal{P}'$ contains two points a, b. The set A is the closure of a $\sim$–equivalence class $\kappa(A)$ and B is the closure of $\kappa(B)$.

By Lemma 2.13, the interior of a geodesic $[a, b]$ is contained in $\kappa(A) \cap \kappa(B)$, a contradiction (distinct equivalence classes do not intersect). □

Lemma 2.24. *Let A be a piece in $\mathcal{P}'$ which is the closure of an equivalence class κ of $\sim$, let $x \in \kappa$ and let $p \in A \setminus \kappa$. On every geodesic $[x, p]$, there exists an infinite sequence of pairwise distinct points $x_0 = x, x_1, x_2, ..., x_n, ...$ appearing in this order from x to p, such that x_n converges to p and $[x_n, x_{n+1}]$ is the intersection of $[x, p]$ with a piece in $\mathcal{P}$.*

Moreover all geodesics joining x and p contain this ordered countable set of points.

Proof. Let (y_n) be a sequence of points from κ converging to p. By the definition of $\sim$, any geodesic $[x, y_n]$ is covered by finitely many pieces from $\mathcal{P}$. By replacing, if necessary, y_n by the farthest point from it in $[x, y_n] \cap [y_n, p]$ one can assume that $[x, y_n]$ is contained in a topological arc joining x to p. Property (T_2') implies that all endpoints of intersections of $[x, y_n]$ with pieces of $\mathcal{P}$ are also on $[x, p]$. Let c_n be the nearest to y_n such a point. Then c_n also converges to p. Indeed, otherwise, all but finitely many c_n are equal to a point $c \in [x, p]$, and y_n must be in the same piece P_n as c. If all but finitely many pieces P_n intersect $[c, p]$ nontrivially, then these pieces are the same, and $p \in P_n$, a contradiction. If infinitely many pieces P_n intersect $[c, p]$ by a point (which must by c) then c is the projection of p onto P_n for infinitely many n's. Thus $\mathrm{dist}(y_n, p) \geq \mathrm{dist}(c, p)$ for infinitely many n's, therefore $c = p \in \kappa$, a contradiction.

We have, therefore, found a sequence $c_n \in \kappa \cap [x, p]$ of endpoints of non-trivial intersections of $\mathcal{P}$–pieces with $[x, p]$ that converges to p, as required. The "moreover" statement of the lemma follows from (T_2'). □

Lemma 2.25. *Let ψ be an isometry of $\mathbb{F}$ permuting pieces in $\mathcal{P}$ and such that $\psi(A) = B$, where A, B are pieces in $\mathcal{P}'$. Then $\psi(\kappa(A)) = \kappa(B)$.*

Proof. If $\kappa(A)$ is a singleton then the result is obvious. Thus we suppose that $\kappa(A)$ is not a singleton. Let x be an arbitrary point in $\kappa(A)$, let y be a point in $\kappa(A) \setminus \{x\}$ and let $\mathfrak{g}$ be a geodesic joining x with y. By the definition of $\sim$, this geodesic is covered by finitely many non-trivial intersections with pieces contained in $\kappa(A)$. Then $\psi(\mathfrak{g})$ is a geodesic joining $\psi(x)$ with $\psi(y)$, and it is also covered by finitely many non-trivial intersections with pieces from $\mathcal{P}$. Since $\psi(x)$ and $\psi(y)$ are two points in $B = \overline{\kappa(B)}$, by Lemma 2.13 the interior of $\psi(\mathfrak{g})$ is contained in $\kappa(B)$. It follows that all pieces intersecting $\psi(\mathfrak{g})$ non-trivially are contained in $\kappa(B)$, hence $\psi(\mathfrak{g}) \subset \kappa(B)$. In particular $\psi(x), \psi(y) \in \kappa(B)$. Since x was arbitrary in $\kappa(A)$, we conclude that $\psi(\kappa(A)) \subseteq \kappa(B)$. A similar argument applied to ψ^{-1} implies that $\psi(\kappa(B)) \subseteq \kappa(B)$. □

Lemma 2.26. *Let ψ be an isometry of $\mathbb{F}$ permuting pieces in $\mathcal{P}$ and let A be a piece in $\mathcal{P}'$ such that $\psi(A) = A$. Let x be an arbitrary point in $\kappa = \kappa(A)$, p a point in $A \setminus \kappa$ and $x_1, x_2, ...$ the sequence of points on $[x, p)$ converging to p as in Lemma 2.24. If $\psi(p) = p$ then ψ fixes all but finitely many of the x_i's. In particular ψ stabilizes all but finitely many pieces in $\mathcal{P}$ intersecting $[x, p]$ non-trivially.*

Proof. By Lemma 2.25, $\psi(x) \in \kappa$, and $\psi(x_0), \psi(x_1), ..., \psi(x_n), ...$ is a sequence of points on $[\psi(x), p]$ which converges to p and consists of endpoints of arcs from $I([\psi(x), p])$. A geodesic $\mathfrak{g} = [x, \psi(x)]$ is covered by finitely many pieces since $x \sim \psi(x)$. Let x' be the farthest from $\psi(x)$ point in $\mathfrak{g} \cap [\psi(x), p]$. It is different from p, and $[x, x'] \sqcup [x', p]$ is a topological arc. Property (T_2') implies that this arc contains all x_i, and since $[x, x']$ is covered by finitely many pieces, for some n_0, all x_n with $n \geq n_0$ are in $[x', p]$. By its definition, this sequence coincides with the intersection of the sequence $(\psi(x_n))$ with $[x', p]$. Thus we have that $x_n = \psi(x_{n+k})$ for all $n \geq n_0$ and some fixed $k \geq 0$. On the other hand, $\mathrm{dist}(x_n, p) = \mathrm{dist}(\psi(x_{n+k}), p) = \mathrm{dist}(x_{n+k}, p)$ implies that $k = 0$. Therefore ψ fixes all x_n with $n \geq n_0$. □

Lemma 2.27. *Consider a sequence $\mathcal{P}_1 \prec \mathcal{P}_2 \prec ...$ of tree-graded structures on $\mathbb{F}$ and an ascending sequence of pieces $A_1 \subseteq A_2 \subseteq ...$ where $A_i \in \mathcal{P}_i$. The closure $\widehat{A}$ of the union $\bigcup A_i$ contains together with any two points any geodesic joining them. Moreover the interior of such a geodesic is contained in $\bigcup A_i$ and a non-trivial sub-arc of it is contained in all but finitely many A_i.*

Proof. Let x, y be two arbitrary points in $\widehat{A}$ and let $\mathfrak{g}$ be an arbitrary geodesic with endpoints x, y. We have $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$ where $x_n, y_n \in \bigcup A_i$. Without loss of generality we may suppose that both x_n and y_n are in some piece A_{i_n}. Lemma 2.12 implies that $\mathfrak{g} \setminus \mathcal{N}_\epsilon(\{x, y\}) \subset A_{i_n}$ for n large enough. Thus the interior of $\mathfrak{g}$ is contained in $\bigcup A_i$ and $\mathfrak{g}$ is contained in $\widehat{A}$. □

Lemma 2.28. *For every sequence $\mathcal{P}_1 \prec \mathcal{P}_2 \prec ...$ of tree-graded structures on $\mathbb{F}$ there exists the smallest tree-graded structure $\bigcup \mathcal{P}_i$ such that $\mathcal{P}_j \prec \bigcup \mathcal{P}_i$ for every j.*

The pieces in $\bigcup \mathcal{P}_i$ are closures of unions of ascending sequences $A_1 \subseteq A_2 \subseteq ...$ where $A_i \in \mathcal{P}_i$.

Proof. Consider all possible sequences $A_1 \subseteq A_2 \subseteq ...$ where $A_i \in \mathcal{P}_i$. Let a collection $\widehat{\mathcal{P}}$ of subsets of $\mathbb{F}$ consist of closures of unions of all these sequences of sets. It is clear that $\mathcal{P}_j \prec \widehat{\mathcal{P}}$ for every j.

Let us prove that $\mathbb{F}$ is tree-graded with respect to $\widehat{\mathcal{P}}$. It is obvious that all pieces in $\widehat{\mathcal{P}}$ are closed and that (T_2) is satisfied.

The fact that every piece in $\widehat{\mathcal{P}}$ is a geodesic subspace follows from Lemma 2.27.

Let us prove (T_1). Let $A = \overline{\bigcup A_i}$, $B = \overline{\bigcup B_i}$ be two pieces in $\widehat{\mathcal{P}}$ and $x \neq y \in A \cap B$. Then every geodesic $[x,y]$ is in $A \cap B$. By Lemma 2.27, a non-trivial portion $[x', y']$ of a geodesic $[x,y]$ belongs to all but finitely many A_i and to all but finitely many B_i. Since $\mathcal{P}_i$ satisfy (T_1), $A_i = B_i$ for all but finitely many i. Hence $A = B$.

It remains to show that every tree-graded structure $\widetilde{\mathcal{P}} \succ \mathcal{P}_i$ for all i satisfies $\widetilde{\mathcal{P}} \succ \widehat{\mathcal{P}}$. Indeed, let $A \in \widehat{\mathcal{P}}$. Then $A = \overline{\bigcup A_i}$. If all A_i are points then A is a point and A is contained in a piece from $\widetilde{\mathcal{P}}$. Otherwise, all but finitely many A_i's are not points and by (T_1) applied to $\widetilde{\mathcal{P}}$, all A_i are in the same piece $\tilde{A} \in \widetilde{\mathcal{P}}$. Since $\tilde{A}$ is closed, $A \subseteq \tilde{A}$. □

The following lemma is obvious.

Lemma 2.29. *1. Let $(\mathbb{F}, \mathcal{P})$ be a tree-graded space and let ϕ be an isometry of $\mathbb{F}$ permuting pieces of $\mathcal{P}$. Then ϕ permutes pieces of $\mathcal{P}'$.*

2. If $(\mathbb{F}, \mathcal{P}_i)$ is a sequence of tree-graded structures on $\mathbb{F}$, $\mathcal{P}_i \prec \mathcal{P}_{i+1}$ and an isometry ϕ permutes pieces of each $\mathcal{P}_i$ then ϕ permutes pieces of $\bigcup \mathcal{P}_i$.

Lemma 2.30. *Let $\mathcal{P}_1 \prec \mathcal{P}_2 \prec \ldots$ be a sequence of tree-graded structures on a complete geodesic metric space $\mathbb{F}$.*

Let ψ be an isometry of $\mathbb{F}$ which permutes the pieces in $\mathcal{P}_i$ for any i, and let A, B be pieces in $\bigcup \mathcal{P}_i$ such that $\psi(A) = B$. Suppose that A and B are closures of unions $\bigcup A_i$ and respectively $\bigcup B_i$, with $A_i, B_i \in \mathcal{P}_i$. Then $\psi(A_i) = B_i$ for i large enough.

Proof. Suppose that $\bigcup A_i$ is not a singleton, otherwise the statement clearly holds. Let x be a point in $\bigcup A_i$. For some i large enough, x is in A_i together with a non-trivial geodesic $[x,y]$. The image of this geodesic $\psi([x,y])$ is in $\psi(A_i) \in \mathcal{P}_i$ and it is also in B. Lemma 2.27 implies that the interior of $\psi([x,y])$ is in $\bigcup B_i$ and that a non-trivial subarc $\mathfrak{g}$ of $\psi([x,y])$ is contained in some B_j.

Let $k = \max(i,j)$. Then $\psi(A_i) \subset \psi(A_k)$ and $B_j \subset B_k$. Hence $\psi(A_k)$ and B_k are two pieces in $\mathcal{P}_k$ which have in common $\mathfrak{g}$. Property (T_1) implies that $\psi(A_k) = B_k$ and the same holds for every $m \geq k$. □

2.5 Groups acting on $\mathbb{R}$–trees

An action of a group G on an $\mathbb{R}$–tree T is called *stable* if the set of stabilizers of arcs of T satisfies the ascending chain condition (ACC). An action is called *minimal* if T does not have a proper invariant subtree.

We begin by recalling two known result of Rips, Bestvina and Feighn, and Sela on non-trivial stable actions on trees.

Theorem 2.31 (Rips-Bestvina-Feighn [BF, Theorem 9.5]). *Let Λ be a finitely presented group with a non-trivial stable minimal action by isometries on an $\mathbb{R}$–tree T. Then one of the following two cases occurs:*

(1) Λ splits over an extension E-by-cyclic, where E is the stabilizer of a non-trivial arc of T;

(2) T is a line and Λ has a subgroup of index at most 2 that is the extension of the kernel of the action of Λ on T by a finitely generated free Abelian group.

Theorem 2.32 (Sela [Sel$_2$, Section 3]). *Let Λ be a finitely generated group with a non-trivial stable minimal action by isometries on an $\mathbb{R}$–tree T and assume that the stabilizers of all tripods in T are trivial. Then the conclusion of Theorem 2.31 holds.*

The following version of Theorem 2.32 is proved by V. Guirardel in [Gui]. Since the proof is still not published, we present this stronger version along with Theorem 2.32.

Definition 2.33. The *height* of an arc in an $\mathbb{R}$–tree with respect to the action of some group G on it is the maximal length of a decreasing chain of sub-arcs with distinct stabilizers. If the height of an arc is zero then it follows that all sub-arcs of it have the same stabilizer. In this case the arc is called *stable*.

The tree T is *of finite height* if any arc of it can be covered by finitely many arcs with finite height. If the action is minimal and G is finitely generated then this condition is equivalent to the fact that there exists a finite collection of arcs $\mathcal{I}$ of finite height such that any arc is covered by finitely many translates of arcs in $\mathcal{I}$ [Gui].

Theorem 2.34 (Guirardel [Gui]). *Let Λ be a finitely generated group and let T be a real tree on which Λ acts minimally and with finite height. Suppose that the stabilizer of any non-stable arc in T is finitely generated.*

Then one of the following three situations occurs:

(1) Λ splits over the stabilizer of a non-stable arc or over the stabilizer of a tripod;

(2) Λ splits over a virtually cyclic extension of the stabilizer of a stable arc;

(3) T is a line and Λ has a subgroup of index at most 2 that is the extension of the kernel of that action by a finitely generated free Abelian group.

Stability of the action on an $\mathbb{R}$–tree is a necessary condition and cannot be removed from Theorem 2.31 or 2.32 (see Dunwoody [Dun_2]). The next lemma shows that in some cases stability and finite height follow from the algebraic structure of stabilizers of arcs.

Lemma 2.35. *Let G be a finitely generated group acting on an $\mathbb{R}$–tree T with finite of size at most D tripod stabilizers, and (finite of size at most D)-by-Abelian arc stabilizers, for some constant D. Then*

(1) an arc with stabilizer of size $> (D+1)!$ is stable;

(2) every arc of T is of finite height (and so the action is of finite height and stable).

Proof. (1) Let H be the stabilizer of an arc $\mathfrak{g}$ in T, $|H| > (D+1)!$ and let $\mathfrak{g}_1$ be a sub-arc in $\mathfrak{g}$ with stabilizer $H_1 > H$. Then H_1 is an extension of a subgroup U of size at most D by an Abelian group, and the centralizer $C(U)$ of U in H_1 has index at most $D!$ since $H_1/C(U)$ is embedded into $\mathrm{Aut}(U)$. Therefore $|C(U) \cap H| > D$.

For every $h \in H_1 \setminus H$, hHh^{-1} fixes $h\mathfrak{g}$. Since $h \notin H$, $h\mathfrak{g} \neq \mathfrak{g}$ but $\mathfrak{g}_1 \subseteq \mathfrak{g} \cap h\mathfrak{g}$. Hence $\mathfrak{g} \cup h\mathfrak{g}$ contains at least one tripod. Thus the group $H \cap hHh^{-1}$, which stabilizes $\mathfrak{g} \cup h\mathfrak{g}$, is of size at most D. If $h \in U$ then $H \cap hHh^{-1}$ contains $C(U) \cap H$. This gives $U \leq H$. Since H_1/U is Abelian, H contains the derived subgroup of H_1. Hence H is normal in H_1. Therefore $H \cap hHh^{-1} = H$, and $|H| \leq D$, a contradiction.

(2) Indeed, from (1) it immediately follows that the height of every arc in T cannot exceed $(D+1)! + 1$. ∎

We need a pretree version of Levitt's theorem [Lev, Theorem 1].

Definition 2.36 (see [Bow], page 10). A *pretree* is a set equipped with a ternary *betweenness* relation xyz satisfying the following conditions:

- (PT0) $(\forall x, y)(\neg xyx)$.
- (PT1) $xzy \Leftrightarrow yzx$.
- (PT2) $(\forall x, y, z)(\neg(xyz \wedge xzy))$.
- (PT3) xzy and $z \neq w$ then $(xzw \vee yzw)$.

An *interval* in a pretree is a set $\lfloor x, y \rfloor$ composed of all z such that xzy holds.

Definition 2.37. An automorphism g of a pretree T is called *non-nesting* if for every interval I, $g \cdot I \subseteq I$ implies $g \cdot I = I$.

Theorem 2.38. *If a finitely presented group G admits a non-trivial non-nesting action by pretree automorphisms on an $\mathbb{R}$–tree T, then it admits a non-trivial isometric action on some complete $\mathbb{R}$–tree T' with the following properties:*

(1) the stabilizer of an arc in T' is also the stabilizer of an arc in T;

(2) the stabilizer of a tripod in T' is also the stabilizer of a tripod in T;

(3) if G stabilizes a line in T' then it stabilizes a line in T.

Proof. Property (1) was proved in [Lev] under the assumption that G acts on T by homeomorphisms. On the other hand, it is proved in Mayer and Oversteegen [MO] that for every action of a finitely generated group G on an $\mathbb{R}$–tree T by pretree automorphisms, one can modify the metric on T (preserving the pretree structure) so that G acts on T by homeomorphisms. Thus we can apply the results of [MO], and then Levitt's theorem.

Another way of proving property (1) is the following. It is easy to see that every pretree automorphism of an $\mathbb{R}$–tree preserves geodesic intervals, hence restricted to a finite subtree it becomes a homeomorphism. Since Levitt's proof only uses restrictions of the homeomorphisms to finite subtrees, it carries without any change.

Property (2) follows from property (1).

Property (3) easily follows from the proof of Levitt's theorem in [Lev]. $\square$

We are going to prove a version of Theorem 2.38 for finitely generated groups. We start with the following Lemma. For the definition of the notion of ω–limit used in it see Section 4.1.

Lemma 2.39. *Let $G = \langle S \rangle$ be an inductive limit of groups $G_n = \langle S \rangle$ and surjective homomorphisms $G_1 \to G_2 \to \ldots$ that are identical on S. For every n, let $(T_n, dist_n)$ be a complete $\mathbb{R}$–tree upon which G_n acts non-trivially by isometries. For every $x \in T_n$ let*

$$d_n(x) = \sup_{a \in S} \mathrm{dist}(ax, x),$$

and let x_n be a point in T_n such that $d_n = d_n(x_n) \leq \inf_{x \in T_n} d_n(x) + 1$. Let ω be any ultrafilter. Then

(1) G acts non-trivially by isometries on the ω–limit T of $(T_n, \mathrm{dist}_n/d_n, (x_n))$ by

$$(g_n) \lim^{\omega}(y_n) = \lim^{\omega}(g_n y_n)\ ;$$

(2) for every arc l in T with stabilizer $\mathrm{Stab}_G(l)$ there exists a sequence of arcs l_n in T_n such that $\lim^{\omega}(l_n) \subset l$ and such that any finitely generated subgroup K in $[\mathrm{Stab}_G(l), \mathrm{Stab}_G(l)]$ is inside the inductive limit of stabilizers $\mathrm{Stab}_{G_n}(l_n)$ for $n \in I_K \subseteq \mathbb{N}$, where $\omega(I_K) = 1$;

(3) *for every tripod abc in T there exists a sequence of tripods $\alpha_n\beta_n\gamma_n$ in T_n such that $\lim^\omega(\alpha_n\beta_n\gamma_n) \subset abc$ and such that the following holds. Any finitely generated subgroup L stabilizing abc in G is inside the inductive limit of stabilizers of $\alpha_n\beta_n\gamma_n$ in G_n for $n \in I_L \subseteq \mathbb{N}$, where $\omega(I_L) = 1$.*

Proof. (1) We need to prove that for every $g = (g_n) \in G$ and every $\lim^\omega(y_n) \in T$ the point $\lim^\omega(g_n y_n)$ is defined (i.e. the distance from it to (x_n) is not infinity), and that the action $(g_n)\lim^\omega(y_n) = \lim^\omega(g_n y_n)$ is well defined (i.e. it does not depend on the sequence representing g). The first statement follows from the choice of d_n and x_n, as $\mathrm{dist}(g_n y_n, x_n) \leq \mathrm{dist}(y_n, x_n) + |g|_S d_n$. The second statement follows from the fact that ω is a non-principal ultrafilter and so every set of natural numbers with finite complement has ω–measure 1.

(2) Since T is a tree and so any pair of points is connected by a unique arc, every arc l in T is the ω–limit of arcs $\ell^n \subseteq T_n$. Since the arc l has non-zero length, $|\ell^n| = O(d_n)$. Suppose that (g_n) fixes l. Then

$$\lim_\omega \frac{\mathrm{dist}(g_n\ell^n_-, \ell^n_-)}{d_n} = \lim_\omega \frac{\mathrm{dist}(g_n\ell^n_+, \ell^n_+)}{d_n} = 0\,. \tag{1}$$

Since T_n is a tree, (1) can only happen if ω-a.s. there exists a number $r_n = o(d_n)$ and a number $\epsilon_n = o(d_n)$ such that for every point $x_n \in \ell^n \setminus \mathcal{N}_{\epsilon_n}(\{\ell^n_-, \ell^n_+\})$,

$$\mathrm{dist}(g_n x_n, x_n) = r_n.$$

This immediately implies that for any fixed $\varepsilon > 0$, any two elements (g_n) and (h_n) from G stabilizing l, $[g_n, h_n]$ fixes the arc $l_n = \ell^n \setminus \mathcal{N}_{\varepsilon d_n}(\{\ell^n_-, \ell^n_+\})$ for $n \in I_{g,h} \subseteq \mathbb{N}$ where $\omega(I_{g,h}) = 1$. Hence $[g, h]$ is in the induction limit of stabilizers $\mathrm{Stab}_{G_n}(l_n)$, $n \in I_{g,h}$. This implies Part (2).

(3) The proof is similar to the proof of (2). A tripod abc in T is the ω–limit of tripods $a_n b_n c_n$ in T_n. If (g_n) fixes the tripod abc, then g_n must move the ends a_n, b_n, c_n of the tripod $a_n b_n c_n$ by distance $o(d_n)$ ω–a.s. This implies that g_n must fix the center mass m_n of $a_n b_n c_n$ ω–a.s. Therefore for any $\varepsilon > 0$, the element g_n must fix the tripod $\alpha_n\beta_n\gamma_n$ where $\alpha_n \in [a_n, m_n], \beta_n \in [b_n, m_n], \gamma_n \in [c_n, m_n]$, and $\mathrm{dist}(a_n, \alpha_n) = \mathrm{dist}(b_n, \beta_n) = \mathrm{dist}(c_n, \gamma_n) = \varepsilon d_n$ ω–a.s. ☐

Theorem 2.40. *If a finitely generated group G admits a non-trivial non-nesting action by pretree automorphisms on an $\mathbb{R}$–tree T, then it admits a non-trivial isometric action on some complete $\mathbb{R}$–tree T' and*

(1) *the derived subgroup of a stabilizer of an arc in T' is locally inside the stabilizer of an arc in T;*

(2) *the stabilizer of a tripod in T' is locally inside the stabilizer of a tripod in T.*

Proof. If G is finitely presented then we can apply Theorem 2.38. If G is not finitely presented, then we can represent G as the inductive limit of a sequence of finitely presented groups and surjective homomorphisms $G_1 \to G_2 \to ....$ Each G_n acts on T by pretree automorphisms, and we can apply Theorem 2.38 to this action. Hence each G_n acts by isometries on a complete $\mathbb{R}$–tree T_n and the stabilizers of arcs (tripods) of T_n in G_n are stabilizers of arcs (tripods) of T in G_n. By Lemma 2.39, G acts on a complete $\mathbb{R}$–tree T' by isometries and properties (1), (2), (3) of the lemma hold.

Let l be a non-trivial arc in T'. Consider the stabilizer $\mathcal{S}$ of l in G. Then by Lemma 2.39 any finitely generated subgroup of $[\mathcal{S}, \mathcal{S}]$ is inside the inductive limit of stabilizers $\mathcal{S}_n, n \in I \subseteq \mathbb{N}$, of arcs l^n in T_n, $\omega(I) = 1$. Each $\mathcal{S}_n$ stabilizes an arc ℓ^n in T. Notice that if $\mathcal{S}_n$ stabilizes a non-trivial arc in T then the image $\mathcal{S}'_n$ of $\mathcal{S}_n$ in G also stabilizes this arc. Hence any finitely generated subgroup of $[\mathcal{S},\mathcal{S}]$ is inside a stabilizer of a nontrivial arc in T. This proves (1).

Statement (2) is proved similar to (1). □

3 Groups acting on tree-graded spaces

Let G be a finitely generated group acting by isometries on a tree-graded space $(\mathbb{F},\mathcal{P})$. Let $S = S^{-1}$ be a finite generating set of G. Given a piece $B \in \mathcal{P}$ and a point $p \in B$ we denote by $\mathrm{Stab}(B)$ (resp. $\mathrm{Stab}(p)$ and $\mathrm{Stab}(B,p)$) the stabilizer of B (resp. stabilizer of p and the intersection $\mathrm{Stab}(B)\cap\mathrm{Stab}(p)$).

Let us fix the following notation:

- $\mathcal{C}_1(G)$ is the set of stabilizers of subsets of $\mathbb{F}$ all of whose finitely generated subgroups stabilize pairs of distinct pieces in $\mathcal{P}$.
- $\mathcal{C}_2(G)$ is the set of stabilizers of pairs of points of $\mathbb{F}$ not from the same piece.
- $\mathcal{C}_3(G)$ is the set of stabilizers of triples of points of $\mathbb{F}$ neither from the same piece nor from the same transversal geodesic.

By Lemma 2.20, the action of G on $\mathbb{F}$ induces an action by isometries of G on the $\mathbb{R}$–tree $T = \mathbb{F}/\!\approx$. The main result of this section is the following theorem.

Theorem 3.1. *Let G be a finitely generated group acting on a tree-graded space $(\mathbb{F},\mathcal{P})$. Suppose that the following hold:*

(i) *every isometry $g \in G$ permutes the pieces;*

(ii) *no piece in $\mathcal{P}$ is stabilized by the whole group G; likewise no point in $\mathbb{F}$ is fixed by the whole group G.*

Then one of the following four situations occurs:

(I) *the group G acts by isometries on the real tree $T = \mathbb{F}/\!\approx$ non-trivially, with stabilizers of non-trivial arcs in $\mathcal{C}_2(G)$, and with stabilizers of non-trivial tripods in $\mathcal{C}_3(G)$;*

(II) *there exists a point $x \in \mathbb{F}$ such that for any $g \in G$ any geodesic $[x, g\cdot x]$ is covered by finitely many pieces: in this case the group G acts non-trivially on a simplicial tree with stabilizers of vertices of the form $\mathrm{Stab}(B)$, $B \in \mathcal{P}$, or of the form $\mathrm{Stab}(p)$, $p \in \mathbb{F}$, and stabilizers of edges of the form $\mathrm{Stab}(B,p)$;*

(III) *the group G acts non-trivially on a simplicial tree with edge stabilizers from $\mathcal{C}_1(G)$;*

(IV) *the group G acts on a complete $\mathbb{R}$–tree by isometries, non-trivially, stabilizers of non-trivial arcs are locally inside $\mathcal{C}_1(G)$-by-Abelian subgroups, and stabilizers of tripods are locally inside subgroups in $\mathcal{C}_1(G)$; moreover if G is finitely presented then the stabilizers of non-trivial arcs are in $\mathcal{C}_1(G)$.*

Case A. Suppose that the action of G on $T = \mathbb{F}/\!\approx$ does not have a global fixed point. Then the action of G on T has all the other required properties from (I). Indeed, Lemma 2.20, (2), and Remark 2.10 imply that any stabilizer in G of a non-trivial arc of T coincides with the stabilizer of two distinct points in $\mathbb{F}$, not contained in the same piece, therefore it is an element of $\mathcal{C}_2(G)$. The same results imply that the stabilizer of a tripod in T coincides with the stabilizer of a triple of points in $\mathbb{F}$ projecting onto the vertices of the tripod in T. These three points are not in the same piece nor on the same transversal geodesic (otherwise their images in T would coincide or they would be on a geodesic).

Case B. Suppose that G fixes a point in T. Let $t \in T$ be this point. Let R be the $\approx$–equivalence class projecting onto t. By Lemma 2.17, R is tree-graded with respect to the pieces from $\mathcal{P}$ contained in R. Let $\mathcal{R}$ be this set of pieces.

Lemmas 2.23 and 2.28 allow us to define a transfinite sequence of tree-graded structures on R. Set $\mathcal{P}_0 = \mathcal{R}$, for every non-limit cardinal $\alpha + 1$, we define $\mathcal{P}_{\alpha+1}$ as $\mathcal{P}'_\alpha$, and for every limit cardinal α we set $\mathcal{P}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}_\beta$.

Definition 3.2. Let B be an arbitrary piece in $\mathcal{P}_\alpha$, $\alpha \geq 1$. Assume that α is not a limit cardinal, and let $\sim_{\alpha-1}$ be the $\sim$–equivalence relation defined in Section 2.4 corresponding to $(\mathbb{F}, \mathcal{P}_{\alpha-1})$. Then B is the closure of a $\sim_{\alpha-1}$–equivalence class. We denote this equivalence class by $\mathrm{Int}\,(B)$ and we call it the *interior* of B.

Assume that α is a limit cardinal. Then B is, by Lemma 2.28, the closure of an increasing union of pieces B_β from $\mathcal{P}_\beta$, $\beta < \alpha$. In this case we denote $\bigcup_{\beta<\alpha} B_\beta$ by $\mathrm{Int}\,(B)$ and also call it the *interior* of B.

Note that Lemma 2.21 in the first case and Corollary 2.8 in the second case imply that the interior of a piece in $\mathcal{P}_\alpha$, with $\alpha \geq 1$, is always convex.

Lemma 3.3. *If* $B \neq B' \in \mathcal{P}_\alpha$ *then* $\mathrm{Int}\,(B) \cap \mathrm{Int}\,(B') = \emptyset$.

Proof. This is obviously true if α is not a limit cardinal or if either B or B' is a singleton. Suppose that α is a limit cardinal, and that $B = \overline{\cup_{\beta<\alpha} B_\beta}$, $B' = \overline{\cup_{\beta<\alpha} B'_\beta}$ are not singletons. If there exists $p \in \mathrm{Int}\,(B) \cap \mathrm{Int}\,(B')$ then $p \in B_\beta \cap B'_\beta$ for some $\beta < \alpha$. Therefore B_β and B'_β are inside the same piece of $\mathcal{P}_{\beta+1}$. Moreover B_β and B'_β are not singletons. Property (T_1) of tree-graded spaces then implies that $B_\xi = B'_\xi$ for every $\xi > \beta$. Hence $B = B'$, a contradiction. □

It is obvious that the sequence $\mathcal{P}_\alpha$ stabilizes (for example when the cardinality of α is bigger than the cardinality of the set of collections of subsets of R). It is also clear that if $\mathcal{P}$ contains two pieces that intersect then $\mathcal{P}' \neq \mathcal{P}$. Hence for some α, $\mathcal{P}_\alpha$ consists of pairwise non-intersecting pieces. Then the equivalence classes of the relation $\sim_\alpha$ corresponding to $\mathcal{P}_\alpha$ are just the pieces from $\mathcal{P}_\alpha$. We shall assume that α is minimal with this property. By Lemma 2.29, we have an induced action of G on $\tilde{R} = R/\sim_\alpha$. Consider two cases.

Case B.1. Assume the group G fixes a point in $\tilde{R}$, that is G stabilizes a piece in $\mathcal{P}_\alpha$. Let $\delta \leq \alpha$ be the minimal cardinal such that G fixes a piece A in $\mathcal{P}_\delta$. By (ii), $\delta \geq 1$.

Pick a point $x \in \mathrm{Int}\, A$. By Lemmas 2.25 and 2.30, $G \cdot x \subset \mathrm{Int}\, A$.

Let us modify the set of generators S as follows. We know that there exists $s \in S$ such that $s \cdot x \neq x$. Let $s_1, ..., s_k$ be all elements of S such that $s_i \cdot x = x$. Then let us replace each s_i by ss_i, and s_i^{-1} by $s_i^{-1}s^{-1}$. The set S' of generators thus obtained is closed under taking inverses, and no element of S' fixes x. Without loss of generality we can assume that S itself satisfies this property.

Since $\mathrm{Int}\,(A)$ is convex, geodesics connecting pairs of points from $G \cdot x$ are in $\mathrm{Int}\,(A)$. Let us define an image of the Cayley graph $\mathrm{Cayley}(G, S)$ in $\mathrm{Int}\,(A)$. For every $s \in S$ choose a geodesic $\mathfrak{c}_s$ connecting x and $s \cdot x$ such that $s\mathfrak{c}_{s^{-1}} = \mathfrak{c}_s$. Now for every $g \in G$ let the image of the edge (g, gs) be $g \cdot \mathfrak{c}_s$. These geodesics will be also called *edges*. Note that by our assumption about S, none of the edges can be of length 0. This will be needed in the proof of Lemma 3.4

below. Thus $\mathrm{Int}\,(A)$ contains an image $\mathfrak{C}$ of the Cayley graph $\mathrm{Cayley}(G,S)$ such that all edges of $\mathrm{Cayley}(G,S)$ map to geodesic intervals in $\mathrm{Int}\,(A)$.

Lemma 3.4. *δ is not a limit cardinal.*

Proof. Suppose that δ is a limit cardinal. Then $\mathcal{P}_\delta = \cup_{\beta<\delta}\mathcal{P}_\beta$. Therefore $\mathrm{Int}\,A$ is an increasing union of pieces $A_\beta \in \mathcal{P}_\beta$, $\beta < \delta$. Then there exists $\beta < \delta$ such that $\bigcup_{s\in S}\mathfrak{c}_s \subset A_\beta$. By Lemma 2.29, for every $g \in G$ the union $\bigcup_{s\in S} g\cdot\mathfrak{c}_s$ belongs to the piece $g\cdot A_\beta \in \mathcal{P}_\beta$. Using induction on the length $|g|_S$, we prove that $g\cdot A_\beta = A_\beta$.

For $|g|_S = 0$ this is trivial. Suppose that the statement is true for $|g| = n$, consider an arbitrary element $|g|$ of length $n+1$. Then $g = g_1 s$, $|g_1|_S = n$. By induction the edge $g_1\mathfrak{c}_s$ is in A_β. On the other hand, $g_1\mathfrak{c}_s$ is the same geodesic as $g\mathfrak{c}_{s^{-1}}$ in $g\cdot A_\beta$. So the pieces A_β and $g\cdot A_\beta$ have a non-trivial arc in common, hence $A_\beta = g\cdot A_\beta$ by property (T_1) of tree-graded spaces.

This implies that $G\cdot x \subset A_\beta$, hence that $G\cdot A_\beta = A_\beta$, a contradiction with the minimality of δ. □

According to Lemma 3.4 and (ii), there exists $\delta - 1$. The collection $\mathcal{P}_\delta$ is thus equal to $\mathcal{P}'_{\delta-1}$. We have that $\mathfrak{C} \subset \mathrm{Int}\,(A)$, hence all edges $g\cdot\mathfrak{c}_s$ are covered by finitely many pieces in $\mathcal{P}_{\delta-1}$. On the other hand, no piece in $\mathcal{P}_{\delta-1}$ contains the whole graph $\mathfrak{C}$.

Consider the following graph Γ. The vertices of Γ are of two types. Vertices of the first type are the pieces in $\mathcal{P}_{\delta-1}$ intersecting non-trivially the edges of $\mathfrak{C}$. Vertices of the second type are the intersection points of the pieces representing vertices of the first type. We connect a vertex of the first type B with a vertex of the second type p if and only if $p \in B$. It is easy to show using Corollary 2.8 and Lemma 2.7 that Γ is a simplicial tree. The action of G on $\mathbb{F}$ induces a simplicial action of G on Γ (by Lemma 2.29). This action does not fix a point. Indeed, by the minimality of δ, no vertex of the first type is fixed by the whole G. The fact that no vertex of the second type is fixed follows from (ii). The group G cannot fix a midpoint of any edge either, because it would have to fix its endpoints.

Suppose that $\delta - 1 = 0$. Then $\mathcal{P}_{\delta-1} = \mathcal{R}$. Let K be the stabilizer of an edge (B,p). Then, by definition, K is of the form $\mathrm{Stab}(B,p)$, $B \in \mathcal{P}, p \in \mathbb{F}$. Note that in this case and by Lemma 2.7, any geodesic $[x, g\cdot x]$, $g \in G$, is covered by a finite number of pieces from $\mathcal{R}$. Thus case (II) of the theorem occurs.

Therefore we can assume that $\delta > 1$. Consider an edge (B,p) of Γ. Let $K \leq G$ be the stabilizer of this edge. We have that $p \in B$, $B = \overline{\mathrm{Int}\,(B)}$.

Lemma 3.5. *Suppose that p belongs to $\mathrm{Int}\,(B)$. Then the stabilizer of the edge (B,p) coincides with the stabilizer of the vertex p.*

Proof. We can assume that B is not a singleton. Let ξ be the smallest cardinal such that a $\mathcal{P}_\xi$–piece containing p is not a singleton. Then $\xi < \delta - 1$ since $\mathrm{Int}\,(B)$ is not a singleton either. Now suppose that $g \in G$ fixes $p \in \mathrm{Int}\,(B)$. Then g permutes the pieces of $\mathcal{P}_\xi$ containing p. The union of these pieces is inside a piece from $\mathcal{P}_{\xi+1} = \mathcal{P}'_\xi$. Note that $\xi + 1 \leq \delta - 1$. The element g stabilizes this union. Since different pieces of $\mathcal{P}_{\delta-1}$ intersect by at most a point, g must stabilize B. Thus we proved that $\mathrm{Stab}(B,p) = \mathrm{Stab}(p)$. □

Edges (p,B) of Γ satisfying the condition $p \in \mathrm{Int}\,(B)$ will be called *redundant*. Lemma 3.3 shows that for every vertex p of Γ there can be at most one redundant edge of the form (B,p).

Let Γ' be the simplicial tree obtained by collapsing the redundant edges of Γ (i.e. removing the interior of each redundant edge and identifying its ends). One can describe Γ' directly as follows. The vertices of Γ' are of two types. Vertices of the first type are pieces from $\mathcal{P}_{\delta-1}$

intersecting non-trivially the edges of $\mathfrak{C}$. Vertices of the second type are points $p = B_1 \cap B_2$ where B_1, B_2 are vertices of the first type, such that $p \notin \mathrm{Int}\,(B)$ for any vertex of the first type B. Vertices B and B' of the first type are connected if and only if $\mathrm{Int}\,(B) \cap B'$ or $B \cap \mathrm{Int}\,(B')$ is not empty. A vertex B of the first type is connected to the vertex p of the second type if $p \in B$.

According to Lemmas 2.25 and 2.30 the group G permutes redundant edges of Γ, hence it acts on Γ' by simplicial automorphisms.

Let us prove that the stabilizer K of any edge in Γ' is in $\mathcal{C}_1(G)$. If K stabilizes an edge of type (B, B'), i.e. it stabilizes two pieces $B, B' \in \mathcal{P}_{\delta-1}$ with the intersection point p in $\mathrm{Int}\,(B) \sqcup \mathrm{Int}\,(B')$, then K stabilizes p. Suppose that $p \in \mathrm{Int}\,(B')$. Then $K \subset \mathrm{Stab}(B, p)$, moreover by Lemma 3.5, $\mathrm{Stab}(B, p) \subset \mathrm{Stab}(B, B') = K$. Thus $K = \mathrm{Stab}(B, p)$.

Similarly, if (B, p) is an edge of Γ' where $B \in \mathcal{P}_{\delta-1}$, $p \in \mathbb{F}$, then $p \notin \mathrm{Int}\,(B)$, and $K = \mathrm{Stab}(B, p)$. Hence it is enough to prove the following statement.

Lemma 3.6. *For every $B \in \mathcal{P}_{\delta-1}$ and $p \in B \setminus \mathrm{Int}\,(B)$ the stabilizer K of the pair (B, p) is in $\mathcal{C}_1(G)$.*

Proof. Suppose first that $\delta - 1$ is a limit cardinal. Then $B = \overline{\bigcup_{\beta<\delta-1} B_\beta}$.

According to Lemma 2.30, for every $g \in K$ there exists $\beta(g)$ such that $gB_\beta = B_\beta$ for every $\beta > \beta(g)$.

Let K_1 be a finitely generated subgroup of K, and let $k_1, ..., k_m$ be a set of generators of K_1. There exists β_0 such that for every $\beta > \beta_0$, $k_i B_\beta = B_\beta$ for all i. Therefore K_1 also stabilizes the piece B_β, and it fixes the point p, hence it fixes the projection y of p onto B_β. Since p, y are in R, $p \approx y$. On the other hand $p \not\sim_\beta y$ (otherwise $p \in B_{\beta+1}$), hence $p \not\sim_0 y$ and $\mathrm{Sat}_0\,\{p, y\}$ contains at least two pieces. Then K_1 must also fix these pieces. Hence K_1 stabilizes a pair of distinct pieces.

Now suppose that $\xi = \delta - 2$ exists. Let again K_1 be a finitely generated subgroup of K and let S be a finite set generating K_1. Let y be a point in $\mathrm{Int}\,(B)$ and let $y_0, y_1, ..., y_n, ...$ be a sequence of points converging to p on the geodesic $[y, p]$ as in Lemma 2.24. For every $s \in S$, according to Lemma 2.26, there exists an n_s such that s fixes all y_n with $n \geq n_s$. It follows that the whole group K_1 fixes all y_n's for $n \geq m = \max_{s\in S} n_s$. Let $z = y_m$ and $z' = y_{m+2}$. They are by construction in the same $\approx$–class R, and $\mathrm{Sat}_0\,\{z, z'\}$ must contain at least two distinct pieces in $\mathcal{R}$. Since K_1 stabilizes $\{z, z'\}$, it must stabilize these two pieces, by Remark 2.10. We conclude that $K \in \mathcal{C}_1(G)$. □

Summarizing, we obtain the following proposition.

Proposition 3.7. *In Case B.1, either G acts non-trivially on a simplicial tree with edge stabilizers in $\mathcal{C}_1(G)$, or property (II) of Theorem 3.1 holds.*

Case B.2. Suppose G does not stabilize a point in $\tilde{R} = R/\sim_\alpha$.

Take a point p in R, and consider the corresponding image $\mathfrak{C}$ of the Cayley graph Cayley(G, S) as in Case B.1. Recall that it consists of vertices $g \cdot p$, $g \in G$, connected by edges $g \cdot \mathfrak{c}_s, s \in S,$, where $\mathfrak{c}_s$ is a geodesic joining p and sp, with the assumption that $s\mathfrak{c}_{s^{-1}} = \mathfrak{c}_s$.

Let us define a ternary *betweenness* relation on $\tilde{R}$ as follows. For every three points x, y, z in $\tilde{R}$ we set xyz if there exists a geodesic connecting a point from x with a point from z and containing a point from y. Recall that pieces from $\mathcal{P}_\alpha$ are pairwise disjoint, thus a point from y is neither in x nor in z.

Lemma 3.8. *For every $x, y, z \in \tilde{R}$, xyz if and only if any geodesic connecting a point in x with a point in z intersects y.*

Proof. It immediately follows from property (T_2'). $\square$

Lemma 3.9. *The set $\tilde{R}$ with the betweenness relation is a pretree in the sense of Definition 2.36.*

Proof. (PT0) $(\forall x, y)(\neg xyx)$. Indeed, suppose that a, b, c are points in x, y, x respectively and a geodesic $[a, c]$ contains b. Since x is convex (by Lemma 2.13), $[a, c]$ is inside x. Therefore x and y must intersect, a contradiction.

(PT1) $xzy \Leftrightarrow yzx$. That is obvious.

(PT2) $(\forall x, y, z)(\neg(xyz \wedge xzy))$. Indeed, suppose that $a, a' \in x, b, b' \in y, c, c' \in z$ and $[a, c]$ contains b while $[a', b']$ contains c'. Let b'' be the entry point of $[a', b']$ in y, and let b''' be the exit point of $[a, c]$ from y. Then the union $[c', b''] \sqcup [b'', b'''] \sqcup [b''', c]$ is an arc by [DS, Lemma 2.28]. By Property (T_2) we have that any geodesic connecting c' and c must pass through b''. Therefore we have that z is between y and y which contradicts (PT0).

(PT3) xzy and $z \neq w$ then $(xzw \vee yzw)$. Indeed, suppose that $\neg xzw$. Since xzy, we can find $a \in x, b \in y, c \in z$ such that $[a, b]$ contains c. Since $\neg xzw$, any geodesic connecting a and a point e in w does not intersect z. Let a' be the farthest from a intersection point of $[a, b]$ and $[a, e]$. Then the union $[e, a'] \sqcup [a', b]$ is an arc. Since it passes through z, every geodesic connecting b and e passes through z, and we have yzw. $\square$

For every two points x, y in the pretree $\tilde{R}$ the set $\{z \mid xzy\}$, denoted by $\lfloor x, y \rfloor$, is called the *interval* between x and y.

By $\rfloor x, y \lfloor$ we denote $\lfloor x, y \rfloor$ without x, y, and we call it *the open interval* between x and y.

Every interval $\lfloor x, y \rfloor$ has a natural order: $a < b$ if $a \in \lfloor x, b \rfloor$.

Lemma 3.10. *$\tilde{R}$ is a median pretree in the terminology of [Bow], i.e. for every three points $x, y, z \in \tilde{R}$ the intersection of intervals $\lfloor x, y \rfloor$, $\lfloor x, z \rfloor$ and $\lfloor y, z \rfloor$ is a singleton in $\tilde{R}$.*

Proof. Consider a geodesic triangle abc in R where $a \in x, b \in y, c \in z$. By Lemma 3.8, $\lfloor x, y \rfloor$ (resp. $\lfloor y, z \rfloor$, $\lfloor x, z \rfloor$) consist of all pieces intersecting the side $[a, b]$ (resp. $[b, c]$, $[a, c]$). Let a' be the farthest from a common point of $[a, b]$, $[a, c]$, and b', c' defined likewise. Then $a'b'c'$ is a simple geodesic triangle in R. By Property (T_2), $a'b'c'$ is contained in a piece. Since pieces of $\mathcal{P}_\alpha$ do not intersect, this piece is unique. It is the intersection of the intervals $\lfloor x, y \rfloor$, $\lfloor x, z \rfloor$ and $\lfloor y, z \rfloor$. $\square$

Notation: For every three points $x, y, z \in \tilde{R}$ we denote the unique common point of $\lfloor x, y \rfloor$, $\lfloor x, z \rfloor$ and $\lfloor y, z \rfloor$ by $m(xyz)$.

Definition 3.11. We call a subset U of a set V *dense* in V if between any two points x, y of V, there is a point from U distinct from x, y.

The following lemma is well known (see, for example, [H]).

Lemma 3.12. *A countable ordered set L is order isomorphic to the set of rational numbers from the open unit interval of the real line if and only if L is dense in itself and every element of L is between two other elements of L (i.e. there are no terminal elements in L).*

Lemma 3.13. *For every two points x, y in $\tilde{R}$ the interval $\lfloor x, y \rfloor$ in the pretree $\tilde{R}$ contains a dense subset that is order isomorphic to the set of rational numbers in the unit interval.*

Proof. Indeed, let $x, y \in \tilde{R}$. Let $\mathfrak{p}$ be a geodesic connecting a point $a \in x$ to a point $b \in y$. We assume that a, b are the exit and entry point of $\mathfrak{p}$ from x and to y respectively. Since different $\mathcal{P}_\alpha$–pieces do not intersect, and the transversal trees of R are trivial by Lemma 2.18, there are countably many $\mathcal{P}_\alpha$–pieces that intersect $\mathfrak{p}$ nontrivially, and between any two intersections of pieces with $\mathfrak{p}$, there is a non-trivial intersection of $\mathfrak{p}$ with a piece. By Lemma 3.12, the set of pieces that intersect $\mathfrak{p}$ non-trivially form an ordered set that is order isomorphic to $\mathbb{Q} \cap (0, 1)$. $\square$

Note that every $\mathbb{R}$–tree can be considered as a dense pretree with the natural betweenness relation: xyz if and only if y belongs to the geodesic $[x, z]$.

Proposition 3.14. *The pretree $\tilde{R}$ is isomorphic to an $\mathbb{R}$–tree Θ.*

The proof of Proposition 3.14 is done in several steps.

Definition 3.15. A subset S in a pretree R is called *connected* if for every two points s_1, s_2 in S, $\lfloor s_1, s_2 \rfloor$ is contained in S.

Lemma 3.16. *The pretree $\tilde{R}$ can be embedded into an $\mathbb{R}$–tree Θ such that the following properties hold:*

(R1) $\tilde{R}$ is dense in Θ in the sense of Definition 3.11;

(R2) every element in Θ is between two elements in $\tilde{R}$;

(R3) the topology induced by the metric topology of Θ on every interval $\lfloor x, y \rfloor$ of $\tilde{R}$ coincides with the subinterval topology on this interval.

Proof. Consider the set $\mathcal{Q}$ of triples (S, ϕ, Θ_S) where

(S1) S is a connected sub-pretree of $\tilde{R}$, i.e. for every $s_1, s_2 \in S$, $\lfloor s_1, s_2 \rfloor \subseteq S$;

(S2) Θ_S is an $\mathbb{R}$–tree, and ϕ is an embedding of pretrees with dense image;

(S3) the topology induced by the metric on Θ_S on any interval $\lfloor s_1, s_2 \rfloor$ of S coincides with the subinterval topology;

(S4) every element of Θ_S is between two elements of $\phi(S)$.

Note that the set $\mathcal{Q}$ is not empty because it contains the triple $(\{s\}, \iota, \{s\})$ where $s \in \tilde{R}$ and ι is the identity map.

Define a partial order $\prec$ on $\mathcal{Q}$ as follows. We say that (S, ϕ, Θ_S) is smaller than $(S', \phi', \Theta_{S'})$ if $S \subseteq S'$, $\Theta_S \subseteq \Theta_{S'}$, the metric of Θ_S is the restriction of the metric of $\Theta_{S'}$, and ϕ is the restriction of ϕ' onto S.

Let $(S_1, \phi_1, \Theta_{S_1}) \prec (S_2, \phi_2, \Theta_{S_2}) \prec ...$ be an increasing sequence of elements of $\mathcal{Q}$. Let $S = \bigcup S_i, \Theta_S = \bigcup \Theta_{S_i}$ and $\phi \colon S \to \Theta_S$ be defined in the natural way. It is obvious that $(S, \phi, \Theta_S) \succ (S_i, \phi_i, \Theta_{S_i})$ for every i and that (S, ϕ, Θ_S) satisfies (S1)-(S4). Thus $(\mathcal{Q}, \prec)$ satisfies the condition of the Zorn lemma.

Let (S, ϕ, Θ_S) be a maximal element of $\mathcal{Q}$. Let us show (by contradiction) that $S = \tilde{R}$. Suppose that $S \neq \tilde{R}$.

Consider the union $\cup S$ of $\mathcal{P}_\alpha$–pieces contained in S. Then $\cup S \neq R$. Note that $\cup S$ is connected. Moreover for any two points a, b in $\cup S$ the saturation $\mathrm{Sat}\,\{a, b\}$ is in $\cup S$. Indeed, consider a geodesic $\mathfrak{p}$ in R connecting a and b. Since R is convex, $\mathfrak{p}$ is covered by its intersections with $\mathcal{P}_\alpha$–pieces from R. By (S1), all these pieces are in S. Therefore $\mathrm{Sat}\,\mathfrak{p} \subseteq \cup S$.

Case 1. Suppose that $\cup S$ is not closed in R. Take p on the boundary $\partial(\cup S) \setminus \cup S$, and the unique $\mathcal{P}_\alpha$–piece P containing p. Let $S' = S \cup \{P\}$.

Consider a geodesic $\mathfrak{g}$ connecting a point $a \in \cup S$ with p. By Lemma 2.13, $\mathfrak{g} \setminus \{p\}$ is contained in $\cup S$. Let a_n, $n \geq 0$, be a sequence of elements of $\mathfrak{g} \setminus \{p\}$ converging to p such that $a_0 = a$ and $a_n \in [a, a_{n+1}]$. Let A_n be the $\mathcal{P}_\alpha$–piece containing a_n, $n = 0, 1, 2, \ldots.$ Then $\lfloor A_0, A_n \rfloor \subset \lfloor A_0, A_{n+1} \rfloor$ in the pretree $\tilde{R}$. Note that every piece in $\lfloor A, P \lfloor$ is inside one of $\lfloor A_0, A_n \rfloor$. Therefore every piece in $\tilde{R}$ between P and A is in S'. Hence S' is connected.

Consider the increasing union $U = \bigcup [\phi(A_0), \phi(A_n)]$ of geodesic intervals in the $\mathbb{R}$–tree Θ_S. Then U is either a geodesic ray or a half-open geodesic interval in Θ_S.

Suppose that U is a half-open geodesic interval in Θ_S. We define $\Theta_{S'}$ as the tree obtained from Θ_S by adding, if necessary, an endpoint t to the half-open geodesic interval and extending the metric in the natural way. Then consider $\phi_{S'}$ as the extension of ϕ to S' defined by $\phi_{S'}(P) = t$.

We show that $(S', \phi_{S'}, \Theta_{S'})$ satisfies the conditions (S1)-(S4). That would contradict the maximality of (S, ϕ_S, Θ_S). Property (S1) is already proved, (S3) and (S4) are obvious.

In order to show (S2) it suffices to prove that if $B, B' \in S$ are such that B' is between B and P, then $\phi_{S'}(B')$ is between $\phi_{S'}(B)$ and $\phi_{S'}(P)$. Let $b \in B$. Lemma 2.12 implies that for n large enough the piece B' intersects the geodesic $[b, a_n]$. Consequently $\phi_{S'}(B')$ is in the geodesic $[\phi_{S'}(B), \phi_{S'}(A_n)]$ for $n \geq n_0$. Thus, it is in $[\phi_{S'}(B), \phi_{S'}(P)]$.

Suppose that U is a geodesic ray in Θ_S. Then we redefine the metric on Θ_S so that the topology stays the same but the ray U becomes isometric to $[0, 1)$, for instance by taking the stereographic projection of $\mathbb{R}$ onto the unit circle and restricting it to $[0, \infty)$. Note that after this modification of the metric on the tree Θ_S properties (S1)-(S4) still hold. This and the previous case will end the argument.

Case 2. Now suppose that $\cup S$ is closed. Let $p \in R \setminus \cup S$.

For every point a in $\cup S$ a geodesic $\mathfrak{g}_a$ joining a with p contains a point p_a which is the nearest to p point in $\cup S$. If two geodesics $\mathfrak{g}_a, \mathfrak{g}_b$ are such that $p_a \neq p_b$ then it follows that p_a, p_b and a point c in $\mathfrak{g}_a \cap \mathfrak{g}_b$ are the vertices of a simple geodesic triangle. Then p_a, p_b and c are in the same $\mathcal{P}_\alpha$–piece, hence this piece is also in $\cup S$, and so is c. This contradicts the choice of p_a and p_b. Thus p_a is a point p' that does not depend on the point a or on the geodesic $\mathfrak{g}_a$, p' is in fact the nearest point projection of p onto $\cup S$.

Let P and P' be the $\mathcal{P}_\alpha$–pieces containing p and p', respectively. Let $\mathfrak{g}$ be a geodesic connecting p' and p, and let $\mathrm{Sat}\, \mathfrak{g}$ be its saturation. Let S' be the union of S and all pieces in $\mathrm{Sat}\, \mathfrak{g}$. Clearly S' is a connected sub-pretree of $\tilde{R}$. By Lemma 3.13, there exists a countable dense subset Y of the interval $\lfloor P', P \rfloor$, containing P and P', and order isomorphic to $\mathbb{Q} \cap [0, 1]$. Let ϕ' be an order isomorphism from Y onto $\mathbb{Q} \cap [0, 1]$, such that $\phi'(P') = 0$ and $\phi'(P) = 1$. The map ϕ' can be uniquely extended to an order-preserving map from $\lfloor P', P \rfloor$ to a dense subset of the interval $[0, 1]$ of the real line. We denote that extension by ϕ' also. The union $\Theta_{S'}$ of Θ_S and $[0, 1]$, with 0 identified to the point $\phi(P')$, is an $\mathbb{R}$–tree with the natural metric extending the metric on Θ_S and the interval metric on $[0, 1]$. The map $\phi_{S'}$ is defined as ϕ on S and as ϕ' on $\lfloor P', P \rfloor$. It is easy to see that $\phi_{S'}$ is an embedding of the pretree S' into $\Theta_{S'}$ and properties (S3) and (S4) are satisfied. Properties (S1) and (S2) have been established before. This contradicts the maximality of the triple (S, ϕ, Θ_S). $\square$

Remark 3.17. After completing the proof of Lemma 3.16, we discovered a paper by Bowditch and Crisp [BC] where for an arbitrary pretree $\tilde{P}$ satisfying some mild conditions an embedding into an $\mathbb{R}$–tree Φ is constructed, so that every automorphism of $\tilde{P}$ extends to an automorphism of the $\mathbb{R}$–tree Φ (considered as a pretree), and non-nesting automorphisms are extended to non-nesting automorphisms (for the definition of "non-nesting" see Definition 2.37)

Thus, it would suffice to prove that our pretree $\tilde{R}$ satisfies their conditions and then apply [BC]. Still we decided to include our original proof. The reason is that the embedding in [BC] is based on a relatively complicated construction from Bowditch [Bow], and our embedding is straightforward and the proof is much shorter. Note that our construction in Lemma 3.16 can shorten the arguments in [Bow] too. Note also that other embeddings of pretrees into $\mathbb{R}$–trees have been considered in Chiswell [C].

Conventions: In what follows, we identify $\tilde{R}$ with its image in Θ. Eventually we shall show, proving Proposition 3.14, that $\tilde{R} = \Theta$.

For two points $x, y \in \Theta$ we denote the geodesic joining them by $[x, y]$.

Lemma 3.18. *Every point p in $\Theta \setminus \tilde{R}$ is the unique intersection of a nested sequence of intervals $[x_n, y_n]$ with $x_n, y_n \in \tilde{R}$.*

Proof. Property (R2) of Lemma 3.16 implies that $p \in [x_0, y_0]$ with $x_0, y_0 \in \tilde{R}$. By Lemma 3.13, the interval $\lfloor x_0, y_0 \rfloor$ in $\tilde{R}$ contains a countable dense subset U. Let $U \cap [x_0, p] = \{u_0, u_1, ...\}$ and $U \cap [p, y_0] = \{v_0, v_1, ...\}$. Then $\{p\} = \bigcap [u_n, v_n]$. Indeed, suppose that $\bigcap [u_n, v_n]$ is an interval $[a, b]$ containing p. Property (R1) applied twice implies that there exist $u, v \in]a, p[\cap \tilde{R}$. Density of U implies that there exists $u_i \in]u, v[\subset]a, p[$. This yields a contradiction.

Now for every $n \in \mathbb{N}$ take x_n to be the first u_k at distance less than $1/n$ from p. Likewise we define y_n using v_k. Obviously the sequence of intervals $[x_n, y_n]$ is nested and its intersection is $\{p\}$. □

Proof of Proposition 3.14. It suffices to prove that $\tilde{R} = \Theta$, that is the embedding in Lemma 3.16 is surjective. Suppose that there exists a point p in $\Theta \setminus \tilde{R}$. By Lemma 3.18, p is the unique intersection of a nested sequence of intervals $[x_n, y_n]$ with $x_n, y_n \in \tilde{R}$. For every $n \in \mathbb{N}$ let $a_n \in x_n$ and $b_n \in y_n$ be the pair of points minimizing the distance in R. Since $x_n, y_n \in [x_{n-1}, y_{n-1}]$ it follows that x_n, y_n intersect any geodesic $[a_{n-1}, b_{n-1}]$, therefore by Lemma 2.2 the points a_n, b_n are contained in $[a_{n-1}, b_{n-1}]$. Thus one can produce a nested sequence of geodesics $[a_n, b_n]$ in R whose intersection is a geodesic $[a, b]$. Let x be the unique $\mathcal{P}_\alpha$–piece in $\tilde{R}$ containing a and let y be the unique piece containing b. The interval $\lfloor x, y \rfloor$ is contained in any $\lfloor x_n, y_n \rfloor$, hence in Θ the arc $[x, y]$ is contained in $\bigcap [x_n, y_n] \{p\}$. It follows that $x = y = p \in \tilde{R}$, a contradiction. □

Proposition 3.19. *The group G acts on the $\mathbb{R}$–tree Θ by non-nesting pretree automorphisms, non-trivially, and with stabilizers of non-trivial arcs in $\mathcal{C}_1(G)$.*

Proof. It suffices to prove that the action of G on $\tilde{R}$ satisfies all the required properties.

Step 1. The group G acts on the pretree $\tilde{R}$ by automorphisms. This follows from Lemma 3.8 and the fact that an isometry from G takes geodesics in R to geodesics in R and permutes $\mathcal{P}_\alpha$–pieces (by Lemma 2.29).

Step 2. The action of G on $\tilde{R}$ is non-nesting. Indeed, if $I = \lfloor a, b \rfloor$ is an interval of $\tilde{R}$, then by Lemma 3.8, $\lfloor a, b \rfloor$ consists of all $\mathcal{P}_\alpha$–pieces intersecting a certain geodesic $\mathfrak{g}$ in R. We can assume that $\mathfrak{g}$ is the shortest geodesic joining the pieces a and b. Recall that by Lemma 2.2 the endpoints of $\mathfrak{g}$ are uniquely defined, and that any geodesic joining a and b must contain them.

Suppose that $h \cdot I \subsetneq I$, $h \in G$. Note that $h \cdot I$ consist of all pieces intersecting $h \cdot \mathfrak{g}$, and that $h \cdot \mathfrak{g}$ is the shortest geodesic joining $h \cdot a$ and $h \cdot b$. Since $h \cdot I \subset I$, the pieces $h \cdot a$ and $h \cdot b$ must intersect $\mathfrak{g}$ therefore the endpoints $h \cdot \mathfrak{g}_-$ and $h \cdot \mathfrak{g}_+$ must be contained in $\mathfrak{g}$. On the other hand, $h \cdot I \neq I$ implies that either $h \cdot \mathfrak{g}_- \neq \mathfrak{g}_-$ or $h \cdot \mathfrak{g}_+ \neq \mathfrak{g}_+$. In particular dist $(h \cdot \mathfrak{g}_-, h \cdot \mathfrak{g}_+)$ is smaller than the length of $\mathfrak{g}$, a contradiction.

Step 3. The arc stabilizers of the action of G on $\tilde{R}$ are in $\mathcal{C}_1(G)$.

Indeed, let $[a, b]$ be a non-trivial interval in $\tilde{R}$ and let K be its stabilizer.

If $g \in G$ fixes a and b then in R it fixes the two points $x \in a$ and $y \in b$ such that $\mathrm{dist}(x, y) = \mathrm{dist}(a, b)$. According to the proof of Lemma 3.13, the strict saturation of $\{x, y\}$ in $(R, \mathcal{P}_\alpha)$ contains countably many disjoint pieces. Consequently the same is true for the strict saturation $\mathrm{Sat}_0\,\{x, y\}$ in $(R, \mathcal{R})$. Thus every element g fixing x, y must stabilize countably many distinct pieces in $\mathcal{R}$.

We conclude that K is the stabilizer of the set $x \cup y$ in $\mathbb{F}$, and that it stabilizes countably many distinct pieces in $\mathcal{R}$, hence K is in $\mathcal{C}_1(G)$. □

Theorem 2.38 in case when G is finitely presented and Theorem 2.40 in the general case, and Proposition 3.19 imply the following statement.

Proposition 3.20. *In Case B.2, the group G acts on an $\mathbb{R}$–tree non-trivially by isometries so that*

(a) the stabilizers of non-trivial arcs are locally inside $\mathcal{C}_1(G)$-by-Abelian subgroups of G, and stabilizers of tripods are locally inside subgroups in $\mathcal{C}_1(G)$;

(b) if G is finitely presented then the stabilizers of non-trivial arcs are in $\mathcal{C}_1(G)$.

This ends the proof of Theorem 3.1.

Proposition 3.21. *Let G be a finitely presented group satisfying the conditions of Theorem 3.1. Suppose that G stabilizes a (bi-infinite) line $\mathcal{T}$ in the $\mathbb{R}$–tree it acts upon according to the theorem. Then G stabilizes the strict saturation and the set of cut-points of a bi-infinite geodesic $\mathfrak{g}$ in $\mathbb{F}$ that is not contained in a piece of $\mathcal{P}$. In addition, any element from G fixes the line $\mathcal{T}$ pointwise if and only if it fixes the set of cut-points from* $\mathrm{Cutp}\,(\mathfrak{g})$ *pointwise.*

Proof. We shall consider the different cases in the proof of Theorem 3.1, which yield different real trees with a good action of G on them.

Case A. Suppose that $\mathcal{T}$ is a line in the $\mathbb{R}$–tree $T = \mathbb{F}/\!\approx$. First we construct a geodesic $\mathfrak{G}$ whose projection onto T is $\mathcal{T}$.

Let $\mathcal{T}_0$ be the maximal subinterval of $\mathcal{T}$ that can be obtained as the projection of a geodesic segment or ray from $\mathbb{F}$. Note that we may suppose that $\mathcal{T}_0$ is not a singleton. Assume that $\mathcal{T}_0 \neq \mathcal{T}$. Then by Lemma 2.19 it is a geodesic segment or ray (i.e. $\mathcal{T}_0$ cannot be an open or semi-open interval), and we can find a geodesic segment or ray $\mathfrak{g}_0$ in $\mathbb{F}$ projecting onto $\mathcal{T}_0$. Let x be an endpoint of $\mathfrak{g}_0$. The intersection of the $\approx$–equivalence class containing x with $\mathfrak{g}_0$ is closed and connected, by Lemma 2.15. It cannot contain the whole $\mathfrak{g}_0$, otherwise the projection of the latter would be a point. Hence it is a geodesic segment $[x', x]$. By replacing if necessary x by x', we may therefore assume that $\mathfrak{g}_0$ contains only one point in the $\approx$–equivalence class of x.

Let $\bar{x} \in \mathcal{T}_0$ be the projection of x in T. Then it is an endpoint of $\mathcal{T}_0$. Pick $\bar{y} \in \mathcal{T}_0$, $\bar{y} \neq \bar{x}$ and let $y \in \mathfrak{g}_0$ be a representative of it. Let $\bar{z} \in \mathcal{T} \setminus \mathcal{T}_0$ be such that $\bar{x} \in (\bar{y}, \bar{z})$ and let $z \in \mathbb{F} \setminus \mathfrak{g}_0$ be a representative of it. Then $\widetilde{\mathrm{dist}}(y, z) = \widetilde{\mathrm{dist}}(y, x) + \widetilde{\mathrm{dist}}(x, z)$.

Consider two geodesics $[y, x] \subset \mathfrak{g}_0$ and $[x, z]$ in $\mathbb{F}$. Assume that they have a point $x' \neq x$ in common. Then

$$\widetilde{\mathrm{dist}}(y, z) \leq \widetilde{\mathrm{dist}}(y, x') + \widetilde{\mathrm{dist}}(x', z) \leq \widetilde{\mathrm{dist}}(y, x) + \widetilde{\mathrm{dist}}(x, z) = \widetilde{\mathrm{dist}}(y, z)\,.$$

It follows that $\widetilde{\mathrm{dist}}(y, x) = \widetilde{\mathrm{dist}}(y, x')$, hence that $\widetilde{\mathrm{dist}}(x, x') = 0$. This contradicts the choice of x on $\mathfrak{g}_0$.

Therefore, $[y,x] \cap [x,z] = \{x\}$, and $\mathfrak{c} = [y,x] \sqcup [x,z]$ is a topological arc. By the choice of x, the maximal sub-arc in $\mathfrak{c}$ containing x and appearing as intersection with a piece is either $\{x\}$ or it has x as an endpoint.

Let now $[y,z]$ be a geodesic in $\mathbb{F}$. By (T_2') it has to contain x. Therefore $\mathrm{dist}(y,z) = \mathrm{dist}(y,x) + \mathrm{dist}(x,z)$.

We have thus obtained that for all $y \in \mathfrak{g}_0 \setminus \{x\}$, $\mathrm{dist}(y,z) = \mathrm{dist}(y,x) + \mathrm{dist}(x,z)$. One easily deduces from this that given a geodesic $[x,z]$, $\mathfrak{g}_0 \sqcup [x,z]$ is a geodesic segment or ray. It projects onto a set strictly larger than $\mathcal{T}_0$, contradicting its maximality.

We conclude that $\mathcal{T}$ is the projection of a bi-infinite geodesic $\mathfrak{g}$. We now show that G stabilizes the strict saturation and the set of cut-points of this geodesic.

Let $x, y \in \mathfrak{g}$ be two points with $\widetilde{\mathrm{dist}}(x,y) > 0$ and such that $[x,y] \subset \mathfrak{g}$ intersects the $\approx$–equivalence classes of x and y only in x and y respectively. In particular x and y belong to $\mathrm{Cutp}\, \mathfrak{g}$.

For every $g \in G$, the same can be said about $g \cdot [x,y]$ and its endpoints. On the other hand, $g \cdot [x,y]$ projects into $\mathcal{T}$. Thus there exist x' and y' in $\mathfrak{g}$ with $\widetilde{\mathrm{dist}}(x', g \cdot x) = 0$ and $\widetilde{\mathrm{dist}}(y', g \cdot y) = 0$. Since $\widetilde{\mathrm{dist}}(x', g \cdot x) = 0$, any geodesic $[x', g \cdot x]$ intersects $g \cdot [x,y]$ only in $g \cdot x$, likewise for a geodesic $[y', g \cdot y]$. Then $[x', g \cdot x] \sqcup g \cdot [x,y] \sqcup [g \cdot y, y']$ is a topological arc, and $g \cdot x, g \cdot y$ are endpoints of intersections of that arc with pieces. It follows that the geodesic $[x', y'] \subset \mathfrak{g}$ must contain $g \cdot x, g \cdot y$. This and Corollary 2.8 imply that $\mathrm{Sat}_0\, g \cdot [x,y] = g \cdot \mathrm{Sat}_0\, [x,y] \subset \mathrm{Sat}_0\, \mathfrak{g}$ and that $\mathrm{Cutp}\, g \cdot [x,y] = g \cdot \mathrm{Cutp}\, [x,y] \subset \mathrm{Cutp}\, \mathfrak{g}$.

Since there exists an increasing sequence of segments $[x_n, y_n]$ as above such that $\bigcup [x_n, y_n] = \mathfrak{g}$, the equalities $G \cdot \mathrm{Sat}_0\, \mathfrak{g} = \mathrm{Sat}_0\, \mathfrak{g}$ and $G \cdot \mathrm{Cutp}\, \mathfrak{g} = \mathrm{Cutp}\, \mathfrak{g}$ follow immediately.

Obviously if g fixes $\mathrm{Cutp}\, \mathfrak{g}$ pointwise then it fixes $\mathcal{T}$ pointwise, because the projection of $\mathrm{Cutp}\, \mathfrak{g}$ on T is $\mathcal{T}$. Let now g in G be such that $g \cdot \bar{t} = \bar{t}$ for every $\bar{t} \in \mathcal{T}$.

Let again $x, y \in \mathfrak{g}$ be two points with $\widetilde{\mathrm{dist}}(x,y) > 0$ and such that $[x,y] \subset \mathfrak{g}$ intersects the $\approx$–equivalence classes of x and y only in x and respectively y. With the argument above it follows that $[x,y]$ must contain $g \cdot x, g \cdot y$.

Since $\mathrm{dist}(x,y) = \mathrm{dist}(g \cdot x, g \cdot y)$ it follows that $g \cdot x = x$ and $g \cdot y = y$. Then $g \cdot \mathrm{Cutp}\, \{x,y\} = \mathrm{Cutp}\, \{x,y\}$. Moreover, since g is an isometry, it fixes $\mathrm{Cutp}\, \{x,y\}$ pointwise.

We now take an increasing sequence of segments $[x_n, y_n]$ as above such that $\bigcup [x_n, y_n] = \mathfrak{g}$. Then $\mathrm{Cutp}\, \mathfrak{g} = \bigcup \mathrm{Cutp}\, \{x_n, y_n\}$, hence g fixes every point in $\mathrm{Cutp}\, \mathfrak{g}$.

Case B.1. In this case $\mathcal{T}$ is a line in the simplicial tree Γ. Let $(B_n)_{n \in \mathbb{Z}}$ be the set of vertices of the first type in $\mathcal{T}$ enumerated in the order in which they appear in $\mathcal{T}$. Then each B_n is a piece in $\mathcal{P}_{\delta-1}$ intersecting $\mathfrak{C}$, and $B_n \cap B_{n+1}$ is a point p_n for all $n \in \mathbb{Z}$. Let $\mathfrak{c}_n \subset B_n$ be a geodesic joining p_{n-1} to p_n. By [DS, Lemma 2.28], $\mathfrak{g} = \bigcup_{n \in \mathbb{Z}} \mathfrak{c}_n$ is a geodesic line. The group G stabilizes the set of points $\{p_n \mid n \in \mathbb{Z}\}$ on $\mathfrak{g}$ hence it stabilizes $\mathrm{Sat}_0\, \mathfrak{g}$ and $\mathrm{Cutp}\, \mathfrak{g}$. An element $g \in G$ fixes $\mathcal{T}$ pointwise if and only if it fixes pointwise $\{p_n \mid n \in \mathbb{Z}\}$. Since $\mathrm{Cutp}\, \mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathrm{Cutp}\, \{p_{-n}, p_n\}$ it follows that $g \in G$ fixes pointwise $\{p_n \mid n \in \mathbb{Z}\}$ if and only if it fixes pointwise $\mathrm{Cutp}\, \mathfrak{g}$.

Case B.2. Suppose that the isometric action of G on the $\mathbb{R}$–tree T' from Theorem 2.38 stabilizes a line L. Then by Theorem 2.38, the action of G on Θ stabilizes a line as well. Let $\mathcal{T}$ be that line. It can also be seen that an element of G fixes L pointwise if and only if it fixes $\mathcal{T}$ pointwise by Lemma 3.16. Let $\mathcal{T}_{\tilde{R}}$ be the line in $\tilde{R}$ corresponding to the line $\mathcal{T}$ in Θ. Let $\lfloor l_n, r_n \rfloor$ be an increasing sequence of intervals in $\tilde{R}$ such that $\bigcup \lfloor l_n, r_n \rfloor = \mathcal{T}_{\tilde{R}}$. Let $x_n \in l_n$ and $y_n \in r_n$ be the unique pair of points minimizing the distance between the pieces l_n, r_n. The inclusion $\lfloor l_n, r_n \rfloor \subset \lfloor l_{n+1}, r_{n+1} \rfloor$ and Lemma 2.2 imply that x_n, y_n are contained in any geodesic joining x_{n+1} and y_{n+1}. It follows that if $\mathfrak{g}_n$ is an arbitrary geodesic of endpoints x_{n+1}, x_n, $\mathfrak{g}_n'$ is an arbitrary geodesic of endpoints y_n, y_{n+1} and $\mathfrak{g}_0$ is a geodesic joining x_1, y_1, then

$$\mathfrak{g} = \ldots \sqcup \mathfrak{g}_n \sqcup \ldots \sqcup \mathfrak{g}_1 \sqcup \mathfrak{g}_0 \sqcup \mathfrak{g}_1' \sqcup \ldots \sqcup \mathfrak{g}_n' \sqcup \ldots$$

is a geodesic line in $\mathbb{F}$. Since for every $g \in G$, $g \cdot \lfloor l_n, r_n \rfloor \subset \mathcal{T}_{\tilde{R}}$, it follows that $g \cdot l_n$ and $g \cdot r_n$ intersect $\mathfrak{g}$, hence by Lemma 2.2, $g \cdot x_n$ and $g \cdot y_n$ are contained in $\mathfrak{g}$. Thus $G \cdot \{x_n, y_n\} \subset \mathfrak{g}$ for any $n \in \mathbb{N}$. An argument as in Case A allows to deduce from this that G stabilizes $\mathrm{Sat}_0\, \mathfrak{g}$ and $\mathrm{Cutp}\, \mathfrak{g}$, and that $g \in G$ fixes $\mathcal{T}_{\tilde{R}}$ if and only if it fixes $\mathrm{Cutp}\, \mathfrak{g}$ pointwise. □

Remark 3.22. In the proof of Proposition 3.21, one can replace "finitely presented" by "finitely generated" in Cases A and B.1, since finite presentability is not used there.

In Case B.2, the finite presentability is used because we need to apply Theorem 2.38. Assume that a non-finitely presented group G stabilizes a bi-infinite line L' in the $\mathbb{R}$–tree T'. Since the group of isometries of the line is (torsion-free Abelian)-by-$\mathbb{Z}/2\mathbb{Z}$, G has an index at most 2 subgroup G_1 that is an extension of a subgroup K fixing L' pointwise by a finitely generated free Abelian group. By Theorem 2.40, the derived subgroup $[K, K]$ is locally inside the pointwise stabilizer of an arc in the pretree $\tilde{R}$ from the Case B.2 of the proof of Theorem 3.1. Then as in Case B.2 of the proof of Proposition 3.21, we can deduce that $[K, K]$ is locally inside a subgroup in $\mathcal{C}_1(G)$. Thus G_1 is inside a ($\mathcal{C}_1(G)$-by-Abelian)-by-(finitely generated free Abelian group).

Combining Theorems 3.1, 2.31 and Proposition 3.21, we obtain the following result.

Theorem 3.23 (see Theorem 1.3). *Let G be a finitely presented group acting by isometries on a tree-graded space $(\mathbb{F}, \mathcal{P})$ such that:*

- ***(i)*** *every isometry from G permutes the pieces;*
- ***(ii)*** *no piece in $\mathcal{P}$ is stabilized by the whole group G; likewise no point in $\mathbb{F}$ is fixed by the whole group G;*
- ***(iii)*** *the collection of subgroups $\mathcal{C}(G) = \mathcal{C}_1(G) \cup \mathcal{C}_2(G)$ satisfies the ascending chain condition.*

Then one of the following three cases occurs:

1. *G splits over a $\mathcal{C}(G)$-by-cyclic group;*
2. *G can be represented as the fundamental group of a graph of groups whose vertex groups are of the form $\mathrm{Stab}(B)$ or $\mathrm{Stab}(p)$ and edge groups are of the form $\mathrm{Stab}(B, p)$, $B \in \mathcal{P}, p \in \mathbb{F}$;*
3. *the group G has a $\mathcal{C}_1(G)$-by-(free Abelian) subgroup of index at most 2.*

The following statement is a version of Theorem 3.23 for finitely generated groups, using Theorem 2.32.

Theorem 3.24. *Let G be a finitely generated group acting by isometries on a tree-graded space $(\mathbb{F}, \mathcal{P})$ such that properties **(i)** and **(ii)** from Theorem 3.23 hold, and in addition*

- ***(iii)*** *the collection of subgroups $\mathcal{C}_2(G)$ satisfies the ascending chain condition and every subgroup in $\mathcal{C}_1(G) \cup \mathcal{C}_3(G)$ is trivial.*

Then one of the following three cases occurs:

1. *G splits over a $\mathcal{C}_2(G)$-by-cyclic group or over an Abelian-by-cyclic group;*
2. *same as case (2) in Theorem 3.23;*

(3) the group G has a metabelian subgroup of index at most 2.

Proof. The statement would follow from Theorems 3.1, 2.32, Proposition 3.21, and Remark 3.22 if we prove that in Case B.2 of the proof of Theorem 3.1, the action is stable. But in that case, by Proposition 3.20, the action is with Abelian arc stabilizers and with trivial tripod stabilizers, since all subgroups in $\mathcal{C}_1(G)$ are trivial by our assumption. Hence we can apply Lemma 2.35. $\square$

If instead of Theorem 2.32 we use Theorem 2.34, then by Theorem 3.1, Lemma 2.35 and Remark 3.22, we obtain the following version of our theorem:

Theorem 3.25 (Theorem 1.4). *Let G be a finitely generated group acting by isometries on a tree-graded space $(\mathbb{F}, \mathcal{P})$ such that properties **(i)** and **(ii)** from Theorem 3.23 hold, and in addition*

(iii) *the subgroups in $\mathcal{C}_2(G)$ are (finite of uniformly bounded size)-by-Abelian and the subgroups in $\mathcal{C}_1(G) \cup \mathcal{C}_3(G)$ have uniformly bounded cardinality.*

Then one of the following three cases occurs:

(1) G splits over a [(finite of uniformly bounded size)-by-Abelian]-by-(virtually cyclic) subgroup

(2) same as case (2) in Theorem 3.23;

(3) the group G has a subgroup of index at most 2 which is a [(finite of uniformly bounded size)-by Abelian]-by-(free Abelian) subgroup.

Remark 3.26. The proofs of Theorems 3.23, 3.24, 3.25 show that in case (2) of these theorems, the group splits as a non-trivial amalgamated product or HNN extension with vertex subgroup of the form $\mathrm{Stab}(B)$.

4 Applications: relatively hyperbolic groups

4.1 Asymptotic cones

Let I be an arbitrary countable set. Recall that a *non-principal ultrafilter* ω over I is a finitely additive measure on the class $\mathcal{P}(I)$ of subsets of I such that each subset has measure either 0 or 1 and all finite sets have measure 0. Since we only use non-principal ultrafilters, the word non-principal will be omitted throughout the paper.

If a statement $P(i)$ holds for all i from a set J such that $\omega(J) = 1$, then we say that $P(i)$ holds ω*–a.s.*.

Remark 4.1. The definition of an ultrafilter immediately implies the following. Let $P_1(i)$, $P_2(i) \ldots$, $P_m(i)$, $i \in I$, be statements such that for any $i \in I$ no two of them can be true simultaneously. If the disjunction of these statements holds ω–a.s. then there exists $k \in \{1, 2, \ldots, m\}$ such that ω–a.s. $P_k(i)$ holds and all $P_j(i)$ with $j \neq k$ do not hold.

Given a sequence of sets $(X_n)_{n \in I}$ and an ultrafilter ω, the *ultraproduct corresponding to* ω, $\Pi X_n/\omega$, consists of equivalence classes of sequences $(x_n)_{n \in I}$, $x_n \in X_n$, where two sequences (x_n) and (y_n) are identified if $x_n = y_n$ ω–a.s. The equivalence class of a sequence $x = (x_n)$ in $\Pi X_n/\omega$ is denoted either by x^ω or by $(x_n)^\omega$. In particular, if all X_n are equal to the same X, the ultraproduct is called the *ultrapower* of X and it is denoted by $\Pi X/\omega$.

If G_n, $n \geq 1$, are groups then $\Pi G_n/\omega$ is again a group with the multiplication $(x_n)^\omega (y_n)^\omega = (x_n y_n)^\omega$.

Lemma 4.2. *Let ω be an ultrafilter and let (X_i) be a sequence of sets of cardinality at most D ω–a.s. Then the ultraproduct $\Pi X_i/\omega$ contains at most D elements.*

Proof. Consider $X_i = \{x_i^1, ..., x_i^D\}$. If the cardinality of X_i is strictly less that D then the last element is repeated as many times as necessary. Let $x_\omega^j = (x_i^j)^\omega$ for $j = 1, 2, ..., D$. For every $y_\omega = (y_i)^\omega$ in $\Pi X_i/\omega$, there exists $j \in \{1, 2, ..., D\}$ such that ω–a.s. $y_i = x_i^j$, hence $y_\omega = x_\omega^j$. Thus $\Pi X_i/\omega = \{x_\omega^1, ..., x_\omega^D\}$. □

Remark 4.3. Lemma 4.2 is a simple corollary of the well known Los' theorem: if a first order property holds in ω–almost all X_i then it holds in the ultraproduct $\Pi X_i/\omega$: consider the formula $\exists x_1, ..., x_D \forall y (y = x_1 \vee \cdots \vee y = x_D)$.

For every sequence of points $(x_n)_{n \in I}$ in a topological space X, its ω*–limit* $\lim_\omega x_n$ is a point x in X such that every neighborhood U of x contains x_n ω–a.s.

- Suppose that the metric space X is Hausdorff. For every sequence (x_n) in X, if the ω–limit $\lim_\omega x_n$ exists, then it is unique.
- Every sequence of elements in a compact space has an ω–limit [Bou].

Definition 4.4 (ω–limit of metric spaces). Let (X_n, dist_n), $n \in I$, be a sequence of metric spaces and let ω be an ultrafilter over I. Consider the ultraproduct $\Pi X_n/\omega$. For every two points $x = (x_n)^\omega, y = (y_n)^\omega$ in $\Pi X_n/\omega$ let

$$D(x, y) = \lim_\omega \mathrm{dist}_n(x_n, y_n) .$$

Consider an *observation point* $e = (e_n)^\omega$ in $\Pi X_n/\omega$ and define $\Pi_e X_n/\omega$ to be the subset of $\Pi X_n/\omega$ consisting of elements which are finite distance from e with respect to D. The function D is a pseudo-metric on $\Pi_e X_n/\omega$, that is, it satisfies the triangle inequality and the property $D(x, x) = 0$, but for some $x \neq y$ the number $D(x, y)$ can be 0.

The ω*–limit* $\lim^\omega (X_n)_e$ *of the metric spaces* (X_n, dist_n) *relative to the observation point* e is the metric space obtained from $\Pi_e X_n/\omega$ by identifying all pairs of points x, y with $D(x, y) = 0$. The equivalence class of a sequence (x_n) in $\lim^\omega (X_n)_e$ is denoted by $\lim^\omega (x_n)$.

Note that if $e, e' \in \Pi X_n/\omega$ and $D(e, e') < \infty$ then $\lim^\omega (X_n)_e = \lim^\omega (X_n)_{e'}$.

Definition 4.5 (asymptotic cone). Let (X, dist) be a metric space, ω be an ultrafilter over a set I, $e = (e_n)^\omega$ be an observation point. Consider a sequence of numbers $d = (d_n)_{n \in I}$ called *scaling constants* satisfying $\lim_\omega d_n = \infty$.

The ω–limit $\lim^\omega \left(X, \frac{1}{d_n}\mathrm{dist}\right)_e$ is called an *asymptotic cone of* X. It is denoted by $\mathrm{Con}^\omega(X; e, d)$.

Note that if X is a group G endowed with a word metric then $\Pi_1 G/\omega$ is a subgroup of the ultrapower of G.

Definition 4.6. For a sequence $(A_n), n \in I$, of subsets of (X, dist) we denote by $\lim^\omega (A_n)$ the subset of $\mathrm{Con}^\omega(X; e, d)$ that consists of all the elements $\lim^\omega (x_n)$ such that $x_n \in A_n$ ω–a.s. Notice that if $\lim_\omega \frac{\mathrm{dist}(e_n, A_n)}{d_n} = \infty$ then the set $\lim^\omega (A_n)$ is empty.

Properties of asymptotic cones:

1. Any asymptotic cone of a metric space is a complete metric space [VDW]. The same proof gives that $\lim^\omega (A_n)$ is always a closed subset of the asymptotic cone $\mathrm{Con}^\omega(X; e, d)$.

2. Let G be a finitely generated group endowed with a word metric. The group $\Pi_1 G/\omega$ acts on $\mathrm{Con}^\omega(G;1,d)$ transitively by isometries:

$$(g_n)^\omega \lim^\omega (x_n) = \lim^\omega (g_n x_n)\,.$$

Given an arbitrary sequence of observation points x, the group $x^\omega(\Pi_1 G/\omega)(x^\omega)^{-1}$ acts transitively by isometries on the asymptotic cone $\mathrm{Con}^\omega(G;x,d)$. In particular, every asymptotic cone of G is homogeneous.

More generally if a group G acts by isometries on a metric space (X,dist) and there exists a bounded subset $B\subset X$ such that $X=G\cdot B$, then all asymptotic cones of X are homogeneous metric spaces.

Convention: When we consider an asymptotic cone of a finitely generated group, unless otherwise stated, we shall assume that the observation point e is $(1)^\omega$.

4.2 Asymptotically tree-graded metric spaces

Definition 4.7 (asymptotically tree-graded spaces). Let (X,dist) be a metric space and let $\mathcal{A}=\{A_i \mid i\in I\}$ be a collection of subsets of $\mathbb{F}$. In every asymptotic cone $\mathrm{Con}^\omega(\mathbb{F};e,d)$, we consider the collection of subsets

$$\mathcal{A}_\omega=\left\{\lim^\omega (A_{i_n}) \mid (i_n)^\omega\in \Pi I/\omega \text{ such that the sequence } \left(\frac{\mathrm{dist}(e_n,A_{i_n})}{d_n}\right) \text{ is bounded}\right\}.$$

We say that X is *asymptotically tree-graded with respect to* $\mathcal{A}$ if every asymptotic cone $\mathrm{Con}^\omega(X;e,d)$ is tree-graded with respect to $\mathcal{A}_\omega$.

This notion is a generalization, in the setting of metric spaces, of the usual notion of (strongly) relatively hyperbolic group.

There is no need to vary the ultrafilter in Definition 4.7: if a space is tree-graded with respect to a collection of subsets for one ultrafilter, it is tree-graded for any other with respect to the same collection of subsets [DS, Corollary 4.30].

We need the following facts from [DS].

Lemma 4.8 (Theorem 4.1, Remark 4.2, (2), in [DS]). *Let (X,dist) be a geodesic metric space and let $\mathcal{A}=\{A_i \mid i\in I\}$ be a collection of subsets of X. If the metric space X is asymptotically tree-graded with respect to $\mathcal{A}$ then the following properties are satisfied:*

(α_1) *for every $\delta>0$ the diameters of the intersections $\mathcal{N}_\delta(A_i)\cap\mathcal{N}_\delta(A_j)$ are uniformly bounded for all $i\neq j$;*

(α_2) *for every $L\geq 1$, $C\geq 0$, and $\theta\in\left[0,\frac{1}{2}\right)$ there exists $M>0$ such that for every (L,C)–quasi-geodesic $\mathfrak{q}$ defined on $[0,\ell]$ and every $A\in\mathcal{A}$ such that $\mathfrak{q}(0),\mathfrak{q}(\ell)\in\mathcal{N}_{\theta\ell/L}(A)$ we have $\mathfrak{q}([0,\ell])\cap\mathcal{N}_M(A)\neq\emptyset$.*

Notation: Let $M(L,C)$ be the maximum between the constant defined in [DS, Definition 4.20] and the constant given by (α_2) for $\theta=\frac{1}{3}$.

Definition 4.9. Let $\mathfrak{q}$ be an (L,C)–quasi-geodesic in X. The *saturation of* $\mathfrak{g}$, denoted by $\mathrm{Sat}\,(\mathfrak{g})$, is the union of $\mathfrak{g}$ and all the sets from $\mathcal{A}$ whose $M(L,C)$–tubular neighborhood crosses $\mathfrak{g}$.

Lemma 4.10 (Lemma 4.15 in [DS]). *Let X be a geodesic metric space which is asymptotically tree-graded with respect to a collection of subsets $\mathcal{A}$. For every $L \geq 1$ and $C \geq 0$, there exists $t \geq 1$ such that for every $d \geq 1$ and for every $A \in \mathcal{A}$, every (L,C)–quasi-geodesic joining two points in $\mathcal{N}_d(A)$ is contained in $\mathcal{N}_{td}(A)$.*

Lemma 4.11. *Let X be a metric space asymptotically tree-graded with respect to a collection of subsets $\mathcal{A}$. For every $L \geq 1$, $C \geq 0$, $M \geq M(L,C)$ and $\delta > 0$, there exists $D_0 > 0$ such that the following holds. Let $A \in \mathcal{A}$ and let $\mathfrak{q}_i : [0, \ell_i] \to X$, $i = 1, 2$, be two (L,C)–quasi-geodesics with the two respective start points a_i in $\mathcal{N}_M(A)$, and such that the diameter of $\mathfrak{q}_i \cap \overline{\mathcal{N}}_M(A)$ does not exceed δ for $i = 1, 2$.*

If $\operatorname{dist}(a_1, a_2) \geq D_0$ then $\mathfrak{q}_1 \sqcup [a_1, a_2] \sqcup \mathfrak{q}_2$ is an $(L + C + 1, C_1)$–quasi-geodesic, where $C_1 = C_1(D_0, \delta, C)$.

Proof. According to [DS, Lemma 4.19], $\mathfrak{q}_1 \sqcup [a_1, a_2]$ and $[a_1, a_2] \sqcup \mathfrak{q}_2$ are (L_1, C_1)–quasi-geodesics. It remains to prove that for every $t \in [0, \ell_1]$ and $s \in [0, \ell_2]$,

$$\operatorname{dist}(\mathfrak{q}_1(t), \mathfrak{q}_2(s)) + O(1) \geq \frac{1}{L}(t + s + \operatorname{dist}(a_1, a_2)) .$$

Lemma 4.28 and Corollary 8.14 in [DS] imply that a geodesic $[\mathfrak{q}_1(t), \mathfrak{q}_2(s)]$ contains two points a_1' and a_2' at distance $\varkappa$ of a_1 and a_2, respectively, with $\varkappa = \varkappa(X, \mathcal{A})$. Then $\operatorname{dist}(\mathfrak{q}_1(t), \mathfrak{q}_2(s)) \geq \operatorname{dist}(\mathfrak{q}_1(t), a_1) + \operatorname{dist}(a_1, a_2) + \operatorname{dist}(a_2, \mathfrak{q}_2(s)) - 4\varkappa \geq \frac{1}{L}(t + s) - 2C + \operatorname{dist}(a_1, a_2) - 4\varkappa$. □

Lemma 4.12 (Lemmas 4.26 and 4.28 in [DS]). *Let $\bigcup_{i=1}^m \mathfrak{q}_i$ be a polygonal line composed of (L,C)–quasi-geodesics.*

1. ***(uniform property** (α_2) **for saturations of polygonal lines)** For every $\lambda \geq 1$, $\kappa \geq 0$ and $\theta \in \left[0, \frac{1}{2}\right)$ there exists R such that for every (λ, κ)–quasi-geodesic $\mathfrak{c} : [0, \ell] \to X$ joining two points in $\mathcal{N}_{\theta\ell/\lambda}\left(\bigcup_{i=1}^m \operatorname{Sat}(\mathfrak{q}_i)\right)$, we have $\mathfrak{c}([0, \ell]) \cap \mathcal{N}_R(\bigcup_{i=1}^m \operatorname{Sat}(\mathfrak{q}_i)) \neq \emptyset$ (in particular, the constant R does not depend on $\mathfrak{q}_i$, only on m).*

2. ***(uniform quasi-convexity of saturations of polygonal lines)** For every $\lambda \geq 1$, $\kappa \geq 0$, there exists τ such that for every $R \geq 1$, the union of saturations $\bigcup_{i=1}^m \operatorname{Sat}(\mathfrak{q}_i)$ has the property that every (λ, κ)–quasi-geodesic $\mathfrak{c}$ joining two points in its R–tubular neighborhood is entirely contained in its τR–tubular neighborhood.*

3. *For every $\delta > 0$ and every $A \in \mathcal{A}$ such that $A \not\subset \bigcup_{i=1}^m \operatorname{Sat}(\mathfrak{q}_i)$, the intersection $\mathcal{N}_\delta(A) \cap \mathcal{N}_\delta\left(\bigcup_{i=1}^m \operatorname{Sat}(\mathfrak{q}_i)\right)$ has diameter bounded uniformly in $A, \mathfrak{q}_1, \dots, \mathfrak{q}_m$.*

4. *For every $R > 0$ and $\delta > 0$ there exists $\varkappa > 0$ such that if $A, B \in \mathcal{A}, A \cup B \subset \bigcup_{i=1}^m \operatorname{Sat}(\mathfrak{q}_i), A \neq B$, the following holds. Let $a \in \mathcal{N}_R(A)$ and $b \in \mathcal{N}_R(B)$ be two points that can be joined by a quasi-geodesic $\mathfrak{p}$ such that $\mathfrak{p} \cap \mathcal{N}_R(A)$ and $\mathfrak{p} \cap \mathcal{N}_R(B)$ has diameter at most δ. Then $\{a, b\} \subset \mathcal{N}_\varkappa\left(\bigcup_{i=1}^m \mathfrak{q}_i\right)$.*

Lemma 4.13 (Corollary 8.14 in [DS]). *For every $L \geq 1$, $C \geq 0$, $M \geq M(L,C)$ and $\delta > 0$ there exists $D_1 > 0$ such that the following holds. Let $A \in \mathcal{A}$ and let $\mathfrak{q}_i : [0, \ell_i] \to X$, $i = 1, 2$, be two (L,C)–quasi-geodesics with one common endpoint b and the other two respective start points $a_i \in \mathcal{N}_M(A)$, such that the diameter of $\mathfrak{q}_i \cap \overline{\mathcal{N}}_M(A)$ does not exceed δ. Then $\operatorname{dist}(a_1, a_2) \leq D_1$.*

In what follows we consider the image of a quasi-geodesic $\mathfrak{q} : [0, \ell] \to X$ endowed with the order from $[0, \ell]$.

Given a subset $A \in X$ intersecting $\mathfrak{q}[0,\ell]$ we call *entrance point* of $\mathfrak{q}$ into A the image $\mathfrak{q}(t)$ of the smallest number $t \in [0,\ell]$ such that $\mathfrak{q}(t) \in A$ and $\mathfrak{q}(t-1)$ or $\mathfrak{q}(0)$ if $0 \leq t < 1$ is in the complementary of A.

We call *exit point* of $\mathfrak{q}$ from A the image $\mathfrak{q}(s)$ of the largest number $s \in [0,\ell]$ such that $\mathfrak{q}(s) \in A$ and $\mathfrak{q}(s+1)$ or $\mathfrak{q}(\ell)$ if $\ell \geq s > \ell - 1$ is in the complementary of A.

Lemma 4.14. *Let $\bigcup_{i=1}^m \mathfrak{q}_i$ be a polygonal line composed of (L,C)–quasi-geodesics. Let R be the constant given by Lemma 4.12, (1), for $\lambda \geq 1$, $\kappa \geq 0$ and $\theta = \frac{1}{3}$. For every $R \leq R_1 < R_2$ and every (λ,κ) –quasi-geodesic $\mathfrak{q} : [0,\ell] \to X$ with $\mathfrak{q}(0)$ in $\mathcal{N}_1 = \mathcal{N}_{R_1}\left(\bigcup_{i=1}^m \mathrm{Sat}\,(\mathfrak{q}_i)\right)$ and $\mathfrak{q}(\ell)$ outside $\mathcal{N}_2 = \mathcal{N}_{R_2}\left(\bigcup_{i=1}^m \mathrm{Sat}\,(\mathfrak{q}_i)\right)$ the exit point $\mathfrak{q}(t_1)$ from $\mathcal{N}_1$ and the exit point $\mathfrak{q}(t_2)$ from $\mathcal{N}_2$ satisfy $t_2 - t_1 \leq 3\lambda(R_2 + \lambda + \kappa)$. In particular the two exit points are at uniformly bounded distance.*

Proof. If on the contrary $t_2 - t_1 > 3\lambda(R_2 + \lambda + \kappa)$, this together with Lemma 4.12, (1), applied to $\mathfrak{q}|_{[t_1+1,t_2]}$ would contradict the fact that $\mathfrak{q}(t_1)$ is an exit point from $\mathcal{N}_1$. $\square$

Lemma 4.15. *The statement in Lemma 4.13 holds with A replaced by the saturation of a third (L,C)–quasi-geodesic $\mathfrak{p}_0$ and $M \geq \max(M(L,C),R)$, where R is the constant given in Lemma 4.12, (1), for $(\lambda,\kappa) = (L,C)$, $\theta = \frac{1}{3}$ and $m = 1$.*

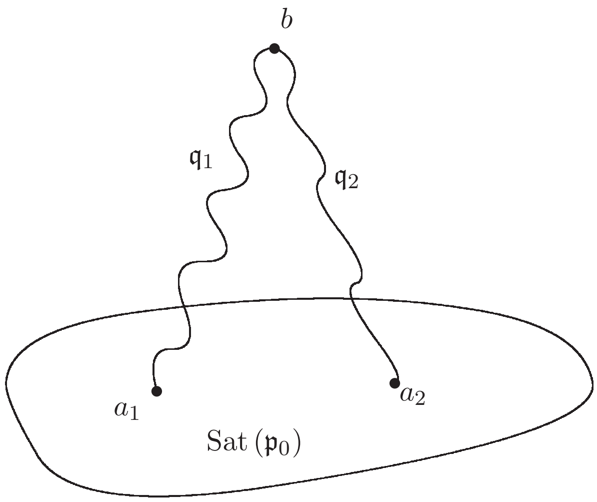

Figure 1: Lemma 4.15.

Proof. We have $a_i \in \mathcal{N}_M\,(\mathrm{Sat}\,\mathfrak{p}_0)$. There are three cases.

Case 1. Both a_i are in $\mathcal{N}_M\,(\mathfrak{p}_0)$. Let a_i' be a point in $\mathfrak{p}_0$ such that $\mathrm{dist}(a_i, a_i') \leq M$. Let $\mathfrak{p} : [0,\ell] \to X$ be a sub-quasi-geodesic of $\mathfrak{p}_0$ of endpoints a_1' and a_2'. Note that the number ℓ is of the order of $\mathrm{dist}(a_1,a_2)$.

By Lemma 4.12, (2), $\mathfrak{p}[0,\ell] \subset \mathcal{N}_\tau(\mathrm{Sat}\,(\mathfrak{q}_1)) \cup \mathcal{N}_\tau(\mathrm{Sat}\,(\mathfrak{q}_2))$. Let $m = \mathfrak{p}(\ell/2)$. If $m \in \mathcal{N}_\tau(\mathfrak{q}_1) \cup \mathcal{N}_\tau(\mathfrak{q}_2)$ and ℓ is large enough this and Lemma 4.12, (1), contradict the hypothesis that the diameter of $\mathfrak{q}_i \cap \overline{\mathcal{N}}_M(\mathrm{Sat}\,(\mathfrak{p}))$ is at most δ.

Assume that $m \in \mathcal{N}_\tau(A)$ with $A \subset \mathrm{Sat}\,(\mathfrak{q}_1)$. Note that the entrance point e_1 of $\mathfrak{p}$ in $\overline{\mathcal{N}}_\tau(A)$ and the entrance point e_2 of $\mathfrak{q}_1$ in $\mathcal{N}_M(A)$ are at distance $O(1)$, by Lemmas 4.13 and 4.14. If the distance from e_2 to a_1 is too large then Lemma 4.12, (1), allows us to find a point in

$\mathfrak{q}_1 \cap \mathcal{N}_R(\mathrm{Sat}\,(\mathfrak{p}))$ at distance $\gg \delta$ from a_1. Thus the diameter of $\{a_1, e_1, e_2\}$ is $O(1)$. If ℓ is large enough then $\mathrm{dist}(m, e_1)$ is larger than any constant fixed in advance, and by Lemma 4.12, (3), $A \subset \mathrm{Sat}\,(\mathfrak{p})$. It follows that if e_2' is the exit point of $\mathfrak{q}_1$ from $\mathcal{N}_M(A)$, then it is at distance at most δ from a_1. Therefore $\mathrm{dist}(e_2', m)$ is also larger than any constant fixed in advance.

Let e_1' be the exit point of $\mathfrak{p}$ from $\overline{\mathcal{N}}_\tau(A)$. We have that m is between e_1 and e_1', hence we can assume that $\mathrm{dist}(e_2', e_1')$ is large enough. The sub-quasi-geodesic of $\mathfrak{q}_1$ between b and e_2', together with $[e_2', e_1']$ and the sub-quasi-geodesic of $\mathfrak{p}$ between e_1' and a_2 compose a quasi-geodesic, by Lemma 4.11. According to Lemma 4.12, (2), this quasi-geodesic is contained in $\mathcal{N}_{\tau'}(\mathrm{Sat}\,(\mathfrak{q}_2))$. In particular $\mathcal{N}_\tau(A)$ intersects $\mathcal{N}_{\tau'}(\mathrm{Sat}\,(\mathfrak{q}_2))$ in a set of diameter at least $\mathrm{dist}(e_1', e_2')$, so by Lemma 4.12, (3), it is contained in $\mathrm{Sat}\,(\mathfrak{q}_2)$. The intersection of $\mathfrak{q}_2$ with $\mathcal{N}_M(A)$ is contained in its intersection with $\mathcal{N}_M(\mathrm{Sat}\,(\mathfrak{p}_0))$, therefore it is in $B(a_2, \delta)$. We have thus obtained that both a_1 and a_2 are at distance $O(1)$ from the entrance points of $\mathfrak{q}_1$ and respectively $\mathfrak{q}_2$ in $\mathcal{N}_M(A)$. We may then use Lemma 4.13.

Case 2. Assume that $a_1 \in \mathcal{N}_M(A)$ for some $A \subset \mathrm{Sat}\,(\mathfrak{p}_0)$ while $a_2 \in \mathcal{N}_M(\mathfrak{p}_0)$. Let a_2' be a point in $\mathfrak{p}_0 \cap B(a_2, M)$. Assume that $\mathcal{N}_M(A)$ intersects the sub-quasi-geodesic $\mathfrak{p}_1$ of $\mathfrak{p}_0$ between $\mathfrak{p}_0(0)$ and a_2'. Let e be the exit point of $\mathfrak{p}_1$ from $\overline{\mathcal{N}}_M(A)$, and let $\mathfrak{p}$ be the sub-quasi-geodesic of $\mathfrak{p}_1$ between e and a_2'.

The union $\mathfrak{q}_1' = \mathfrak{q}_1 \sqcup [a_1, e]$ is a quasi-geodesic. Indeed, if $\mathrm{dist}(a_1, e)$ is large enough, this follows from Lemma 4.11 while if $\mathrm{dist}(a_1, e) \leq D_0$ it is obvious. Also if the intersection of $\mathfrak{q}_1'$ with $\mathrm{Sat}\,(\mathfrak{p})$ contains a point too far from e, then A has a large intersection with $\mathrm{Sat}\,(\mathfrak{p})$, hence it is contained in it, and this contradicts the choice of $\mathfrak{p}$. Thus, the intersection $\mathfrak{q}_1' \cap \mathrm{Sat}\,(\mathfrak{p})$ is at distance $O(1)$ from e. We may then apply Step 1 to $\mathfrak{q}_1'$ and $\mathfrak{q}_2$ and deduce that $\mathrm{dist}(e, a_2)$ is $O(1)$. Then we apply Lemma 4.13 to $\mathfrak{q}_1, \mathfrak{q}_2$ and A, and deduce that $\mathrm{dist}(a_1, a_2)$ is $O(1)$.

Case 3. Assume that $a_i \in \mathcal{N}_M(A_i)$ for some $A_i \subset \mathrm{Sat}\,(\mathfrak{p}_0)$, $i = 1, 2$. Without loss of generality we may suppose that A_2 intersects $\mathfrak{p}_0$ between its exit point from $\overline{\mathcal{N}}_M(A_1)$ and the end of $\mathfrak{p}_0$. Let e be the exit point of $\mathfrak{p}_0$ from $\overline{\mathcal{N}}_M(A_1)$ and let $\mathfrak{p}$ be the sub-quasi-geodesic of $\mathfrak{p}_0$ between e and its end. As above, $\mathfrak{q}_1' = \mathfrak{q}_1 \sqcup [a_1, e]$ is a quasi-geodesic with the property that the intersection $\mathfrak{q}_1' \cap \mathrm{Sat}\,(\mathfrak{p})$ is at distance $O(1)$ from e. Step 2 for $\mathfrak{q}_1'$, $\mathfrak{q}_2$ and $\mathrm{Sat}\,(\mathfrak{p})$ implies that $\mathrm{dist}(e, a_2)$ is $O(1)$. Lemma 4.13 applied to $\mathfrak{q}_1, \mathfrak{q}_2$ and A_1 implies that $\mathrm{dist}(a_1, a_2)$ is $O(1)$. □

Lemma 4.16. *The statement in Lemma 4.11 holds with A replaced by the saturation of a third (L, C)–quasi-geodesic $\mathfrak{p}_0$ and $M \geq \max(M(L, C), R)$, where R is the constant given in Lemma 4.12, (1), for $(\lambda, \kappa) = (L, C)$, $\theta = \frac{1}{3}$ and $m = 1$.*

Proof. **Step 1.** We prove that $\widetilde{\mathfrak{q}}_1 = \mathfrak{q}_1 \sqcup [a_1, a_2]$ is a quasi-geodesic (the same argument implies that $\widetilde{\mathfrak{q}}_2 = [a_1, a_2] \sqcup \mathfrak{q}_2$ is a quasi-geodesic). Let $t \in [0, \ell_1]$ and let $x \in [a_1, a_2]$. Consider a geodesic $[\mathfrak{q}_1(t), x]$. Its entrance point in $\overline{\mathcal{N}}_M(\mathrm{Sat}\,(\mathfrak{p}))$ is at distance $O(1)$ from a_1, by Lemma 4.15. It follows that $\mathrm{dist}(\mathfrak{q}_1(t), x) = t + \mathrm{dist}(a_1, x) + O(1)$.

Step 2. We now prove the statement in the Lemma. Let $t \in [0, \ell_1]$ and $s \in [0, \ell_2]$. Let $\mathfrak{g}$ be a geodesic joining $\mathfrak{q}_1(t)$ and $\mathfrak{q}_2(s)$. By Lemma 4.12, $\widetilde{\mathfrak{q}}_1$ is contained in the τ-tubular neighborhood of $\mathrm{Sat}\,(\mathfrak{q}_2) \cup \mathrm{Sat}\,(\mathfrak{g})$. In particular the point a_1 is in the same tubular neighborhood.

Step 2.a. Assume that a_1 is in the τ-tubular neighborhood of $\mathrm{Sat}\,(\mathfrak{q}_2)$. If a_1 is at distance less than τ from $\mathfrak{q}_2$, then by the hypothesis on $\mathfrak{q}_2$ and Lemma 4.14, a_1 is at uniformly bounded distance from a_2. Thus if D_0 is large enough this case cannot occur.

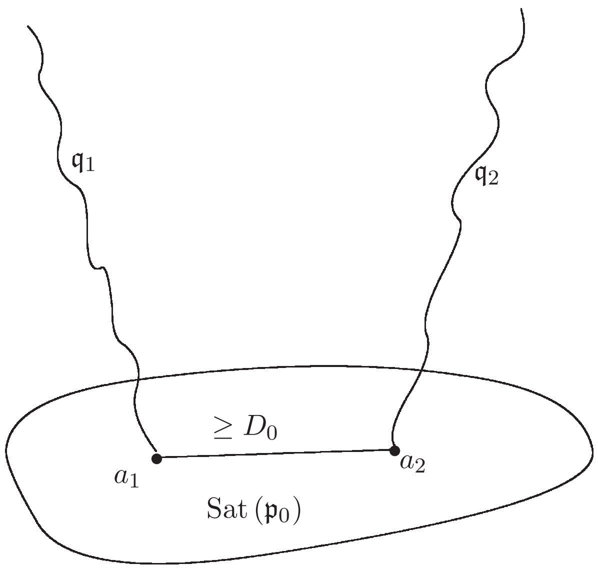

Figure 2: Lemma 4.16.

Suppose that a_1 is in the τ-tubular neighborhood of $A \subset \text{Sat}\,(\mathfrak{q}_2)$. Then the portion of $\widetilde{\mathfrak{q}}_2$ between a_1 and a point in $\mathfrak{q}_2 \cap \mathcal{N}_M(A)$ is contained in $\mathcal{N}_{\tau'}(A)$, hence the same holds for $[a_1, a_2]$. If D_0 is large enough this implies that $A \subset \text{Sat}\,(\mathfrak{p}_0)$.

We may conclude in this case that $\mathfrak{q}_1 \sqcup [a_1, a_2] \sqcup \mathfrak{q}_2$ is a quasi-geodesic by Lemma 4.11.

Step 2.b. Suppose that a_1 is in $\mathcal{N}_\tau(A)$ for some $A \subset \text{Sat}\,(\mathfrak{g})$. By Lemma 4.13 the exit point of $\mathfrak{g}$ from $\mathcal{N}_M(A)$ is at distance $O(1)$ from a point x in $\widetilde{\mathfrak{q}}_2$. If x is not on $[a_1, a_2]$ then $x \in \mathfrak{q}_2$, and it follows that a part of $\widetilde{\mathfrak{q}}_2$ containing $[a_1, a_2]$ is in some $\mathcal{N}_{\tau'}(A)$, whence $A \subset \text{Sat}\,(\mathfrak{p}_0)$ if D_0 is large enough. Lemma 4.11 then implies that $\mathfrak{q}_1 \sqcup [a_1, a_2] \sqcup \mathfrak{q}_2$ is a quasi-geodesic.

Suppose then that $x \in [a_1, a_2]$. Then $\text{dist}(\mathfrak{q}_1(t), \mathfrak{q}_2(s)) = \text{dist}(\mathfrak{q}_1(t), x) + \text{dist}(x, \mathfrak{q}_2(s)) + O(1)$. Since $\widetilde{\mathfrak{q}}_1$ is a quasi-geodesic, it follows that $\text{dist}(\mathfrak{q}_1(t), x) \geq \frac{1}{L_1}(t + \text{dist}(a_1, x)) - C_1$ for some $L_1 \geq 1$ and $C_1 \geq 0$. Also $\widetilde{\mathfrak{q}}_2$ is a quasi-geodesic, therefore $\text{dist}(\mathfrak{q}_2(s), x) \geq \frac{1}{L_1}(s + \text{dist}(a_2, x)) - C_1$. We conclude that

$$\text{dist}(\mathfrak{q}_1(t), \mathfrak{q}_2(s)) + O(1) \geq \frac{1}{L_1}(t + s + \text{dist}(a_1, a_2))\,.$$

Now assume that a_1 is at distance at most τ from a point in $\mathfrak{g}$. Then the argument above can be repeated with $x = a_1$.

□

4.3 Relatively hyperbolic groups

The following result of the authors and Denis Osin [DS] shows that Cayley graphs of relatively hyperbolic groups are asymptotically tree-graded metric spaces.

Theorem 4.17 (Theorem 8.5 and Appendix of [DS]). *A group G is relatively hyperbolic with respect to its finitely generated subgroups $H_1, ..., H_m$ if and only if every asymptotic cone of G is tree-graded with respect to the collection of ω–limits of sequences of cosets $\gamma_n H_i$ ($\gamma_n \in G, i = 1, ..., m$).*

From now on we fix an infinite finitely generated group G that is relatively hyperbolic with respect to a finite collection of finitely generated peripheral subgroups $\mathcal{H} = \{H_1, ..., H_m\}$, $G \neq H_i$. Let $\mathcal{G}$ be the set of all left cosets of H_i.

Definition 4.18. Recall that H_i are called *peripheral subgroups* of G, subgroups of conjugates of H_i are called *parabolic subgroups* of G, conjugates of H_i are called *maximal parabolic subgroups.*

Note that a maximal parabolic subgroup $\gamma H_i \gamma^{-1}$ is the stabilizer of a left coset γH_i.

Lemma 4.19. *Let $\mathcal{K} = \mathrm{Con}^\omega(G; x, d)$ be an asymptotic cone of G and let $\lim^\omega(\gamma_n H)$ be an ultralimit, where $\gamma_n \in G$ and H is a peripheral subgroup such that $\lim^\omega(\gamma_n H)$ is a non-empty subset of $\mathcal{K}$. The stabilizer in $x^\omega(\Pi_1 G/\omega)(x^\omega)^{-1}$ of $\lim^\omega(\gamma_n H)$ is inside $\Pi\left(\gamma_n H \gamma_n^{-1}\right)/\omega$.*

Proof. Let $\mathcal{S}$ be the stabilizer in $x^\omega(\Pi_1 G/\omega)(x^\omega)^{-1}$ of $\lim^\omega(\gamma_n H)$. For every $(g_n)^\omega \in \mathcal{S}$, $\lim^\omega(g_n\gamma_n H) = \lim^\omega(\gamma_n H)$, therefore ω–a.s. $g_n\gamma_n H = \gamma_n H$, and $g_n \in \gamma_n H \gamma_n^{-1}$. □

Lemma 4.20. *Let H, H' be peripheral subgroups of G, and let $g \in G$ be such that $gH' \neq H$. Then the subgroup $H \cap gH'g^{-1}$ is a conjugate of a subgroup inside the ball $B(1, R)$ for some uniform constant R. In particular, the size of this subgroup is uniformly bounded.*

Proof. Let $a \in H, b \in gH'$ be such that $\mathrm{dist}(a, b) = \mathrm{dist}(H, gH')$. In particular, if $\mathrm{dist}(H, gH') = 0$ then one can take $a = b$.

Let $x \in H \cap gH'g^{-1}$. Then $\mathrm{dist}(xa, xb) = \mathrm{dist}(H, gH')$. By Lemma 4.11, if $\mathrm{dist}(a, xa) > D_0$ then the union $[b, a] \sqcup [a, xa] \sqcup [xa, xb]$ is a $(2, C_1)$–quasi-geodesic (a geodesic if $a = b$). By Lemma 4.10, this quasi-geodesic is in the t-neighborhood of gH' for some uniform constant t. Hence $[a, xa] \subseteq \mathcal{N}_t(H) \cap \mathcal{N}_t(gH')$. By (α_1) the distance $\mathrm{dist}(a, xa)$ is uniformly bounded. The same is obviously true in the case when $\mathrm{dist}(a, xa) \leq D_0$. Hence $a^{-1}(H \cap gH'g^{-1})a$ is in a ball of uniformly bounded radius. □

Lemma 4.21. *Let K be a subgroup of G. If K contains a central element of infinite order, or if K does not contain free non-Abelian subgroups, then either K is virtually cyclic or K is a parabolic subgroup.*

Proof. Suppose that K has a central element z of infinite order. If no power of z is in a parabolic subgroup then the normalizer of $\langle z \rangle$ is virtually cyclic by [Os], so K is virtually cyclic. If z^n is inside a maximal parabolic subgroup H then for every $k \in K$, kHk^{-1} intersects H in an infinite set, since the intersection contains $\langle z^n \rangle$ and z is of infinite order. Hence $kH = H$ by Lemma 4.20, and $k \in H$. Thus $K \subseteq H$.

Suppose that K contains no free non-Abelian subgroups. By [Tu] any subgroup of a relatively hyperbolic group either has a free non-Abelian subgroup or it is elementary (that is, either virtually cyclic or parabolic). In particular K is elementary. □

In the case of non-parabolic subgroups the following can be said.

Lemma 4.22. *There exist a constant R and a positive integer m_0 depending on $G, H_1, ..., H_m$, such that for any finite subset S of G one of the following three cases occurs:*

- *S is contained in a parabolic subgroup of G;*
- *S is conjugate to a subset inside the ball $B(1, R)$;*
- *$S^{m_0} = \{s_1...s_k\ ;\ s_i \in S \cup S^{-1},\ k \leq m_0\}$ contains a hyperbolic element.*

In particular, every finite non-parabolic subgroup of G is conjugate to a subgroup inside $B(1, R)$.

Proof. The argument in the proof of [Xie, Lemma 5.3], relying on a result in [Kou], proves in fact the statement of the lemma. □

Lemma 4.23. *(1) If each H_i is hyperbolic relative to $H_{i,j}$ ($j = 1, ..., s_i$) then G is hyperbolic relative to $H_{i,j}$, ($i = 1, ..., n, j = 1, ..., s_i$).*

(2) Each peripheral subgroup H_i is undistorted in G.

(3) If H is an undistorted subgroup of G then it is hyperbolic with respect to finitely many parabolic subgroups of H.

(4) Suppose that none of the peripheral subgroups H_i is hyperbolic relative to proper subgroups. Then every automorphism of G permutes maximal parabolic subgroups.

Proof. Part (1) is proved in [DS, Corollary 1.14]. Part (2) immediately follows from Lemma 4.10 and Theorem 4.17. Part (3) is proved in [DS, Theorem 1.8].

To prove part (4) let ψ be an automorphism of G. Suppose that the image by ψ of a maximal parabolic subgroup gH_ig^{-1} is not parabolic. It follows that $\psi(H_i)$ is likewise non-parabolic. Since H_i is undistorted by part (2), the subgroup $\psi(H_i)$ is undistorted as well, and we can apply part (3) and conclude that $\psi(H_i)$ is relatively hyperbolic with respect to some parabolic subgroups of it. It follows that H_i is also relatively hyperbolic with respect to some proper subgroups of it, a contradiction.

Thus for every maximal parabolic subgroup $H = gH_ig^{-1}$, $\psi(H)$ is parabolic, hence contained in some maximal parabolic subgroup H'. The same argument applied to H' and ψ^{-1} implies that $H < \psi^{-1}(H') < H''$, where H'' is maximal parabolic. Since H is not finite (otherwise it would be relatively hyperbolic with respect to the trivial subgroup), it follows by Lemma 4.20 that $H = H''$, thus $\psi(H) = H'$. □

The following lemma shows how a maximal parabolic subgroup acts outside the left coset that it stabilizes.

Lemma 4.24. *Let g be an element in a maximal parabolic subgroup $\gamma H\gamma^{-1}$ and let x be a point in $G \setminus \gamma H$. Let x_1 be a nearest point projection of x onto γH. Then there exists a uniform constant C such that one of the following two situations occurs:*

(1) $\mathrm{dist}(x_1, gx_1) \leq C$;

(2) $\mathrm{dist}(x, gx) \geq \mathrm{dist}(x, \gamma H) + \frac{1}{2}\mathrm{dist}(x_1, gx_1) - C$.

Proof. Consider the constant D_0 provided by Lemma 4.11 for $(L, C) = (1, 0)$ and $M = \delta = M(1, 0)$. If $\mathrm{dist}(x_1, g\,x_1) \geq D_0$ then $[x, x_1] \sqcup [x_1, g\,x_1] \sqcup [g\,x_1, g\,x]$ is a $(2, C_1)$-quasi-geodesic, where C_1 is a uniform constant. This implies that $\mathrm{dist}(x, g\,x) \geq \mathrm{dist}(x, x_1) + \frac{1}{2}\mathrm{dist}(x_1, g\,x_1) - C_1$. □

Lemma 4.25. *Let $\mathcal{K} = \mathrm{Con}^\omega(G; x, d)$ be an asymptotic cone of G. Then any subgroup $\mathcal{S} < x^\omega(\Pi_1 G/\omega)(x^\omega)^{-1}$ which stabilizes a piece in $\mathcal{K}$ and a point outside the piece, is conjugate to a subgroup in $\Pi B(1, R)/\omega$ for some universal constant $R = R(G)$. In particular, its size is bounded by a universal constant $D = D(G)$.*

Proof. Let $\lim^\omega(\delta_i H)$ and $b = \lim^\omega(b_i)$ be the piece and respectively the point outside the piece, fixed by $\mathcal{S}$. Let $g = (g_i)^\omega$ be an element in $\mathcal{S}$. Note that since $g \cdot \lim^\omega(\delta_i H) = \lim^\omega(\delta_i H)$, we have $g_i \in \delta_i H \delta_i^{-1}$ ω–a.s.

The distance from b to $\lim^\omega(\delta_i H)$ is positive. Therefore the distance from b_i to $\delta_i H$ is at least $O(d_i)$. The distance from b_i to $g_i b_i$ is $o(d_i)$ since $g \cdot b = b$. Let c_i be a nearest point projection of b_i onto $\delta_i H$. By Lemma 4.24, $c_i^{-1} g_i c_i$ is inside a ball of radius C ω–a.s. Hence we can take $R(G) = C$. Let D be the number of elements in the ball of radius C in G. The set $\mathcal{S}$ cannot contain more than D elements, by Lemma 4.2. $\square$

Lemma 4.26. *Let $\mathcal{K} = \mathrm{Con}^\omega(G; x, d)$ be an asymptotic cone of G. Then any subgroup $\mathcal{S} < x^\omega(\Pi_1 G/\omega)(x^\omega)^{-1}$ which fixes pointwise a non-degenerate tripod in a transversal tree of $\mathcal{K}$ is a conjugate of a subgroup in $\Pi B(1, R)/\omega$ for some universal constant $R = R(G)$. In particular, the size of $\mathcal{S}$ is bounded by a universal constant $D = D(G)$.*

Proof. **Step 1.** Let $u = \lim^\omega(u_i)$, $v = \lim^\omega(v_i)$ and $w = \lim^\omega(w_i)$ be the endpoints of a tripod in a transversal tree. All the statements about sequences (a_i) below are to be understood as holding for ω–almost every i.

Let R be the constant given by Lemma 4.12, (1), for $m = 1$, $(\lambda, \kappa) = (L, C) = (1, 0)$ and $\theta = \frac{1}{3}$. Consider a point π_i in $\overline{\mathcal{N}}_R(\mathrm{Sat}([u_i, v_i]))$ minimizing the distance to w_i. Then $\pi = \lim^\omega(\pi_i) \in \mathrm{Sat}([u, v])$ and $\mathrm{dist}(w, \pi) \le \mathrm{dist}(w, [u, v])$. If $\pi \notin [u, v]$ then $\pi \in A$ for some piece A intersecting $[u, v]$ in a point p. Since $\pi \neq p$, Lemma 2.28 in [DS] implies that $[w, \pi] \sqcup [\pi, p] \sqcup [p, u]$ is a geodesic. This contradicts the fact that u and w are in the same transversal tree. Hence $\pi \in [u, v]$ and it is the center of the tripod (uvw). In particular it follows that the distances from π_i to u_i, v_i and w_i respectively are of order d_i.

Let $g = (g_i)^\omega$ be an element in $\mathcal{S}$.

Notation: In the sequel, for any $a = \lim^\omega(a_i) \in \mathcal{K}$ we use the notations a_i' to denote $g_i a_i$ and respectively a' to denote $\lim^\omega(a_i') = g \cdot a$.

Note that if a is in the tripod (uvw) then $a' = a$, that is $\mathrm{dist}(a_i, a_i') = o(d_i)$.

With the above notation π_i' is a point in $\overline{\mathcal{N}}_R(\mathrm{Sat}([u_i', v_i']))$ minimizing the distance to w_i', and its distances to u_i', v_i' and respectively w_i' are of order d_i.

It suffices to show that $\mathrm{dist}(\pi_i, \pi_i') = O(1)$. Indeed, if this is true, then $(\pi_i)^{-1} g_i \pi_i$ is in a ball around 1 of uniformly bounded radius. Thus, in what follows we prove that $\mathrm{dist}(\pi_i, \pi_i') = O(1)$.

Let $\widehat{\pi}_i$ be a point in $\overline{\mathcal{N}}_R(\mathrm{Sat}([u_i', v_i']))$ minimizing the distance to w_i. Arguing as before we obtain that $\lim^\omega(\widehat{\pi}_i)$ is the center of the tripod (uvw), therefore the distances from $\widehat{\pi}_i$ to u_i, v_i, w_i and to their images by g_i are of order d_i.

According to Lemma 4.16 if $\mathrm{dist}(\widehat{\pi}_i, \pi_i') \ge D_0$ then $[w_i, \widehat{\pi}_i] \sqcup [\widehat{\pi}_i, \pi_i'] \sqcup [\pi_i', w_i']$ is a quasi-geodesic. This contradicts the fact that $\mathrm{dist}(w_i, w_i') = o(d_i)$, while $\mathrm{dist}(w_i, \widehat{\pi}_i)$ and $\mathrm{dist}(\pi_i', w_i')$ are of order d_i. We conclude that $\mathrm{dist}(\widehat{\pi}_i, \pi_i') \le D_0$.

Thus, in order to finish the argument, we need to prove that $\mathrm{dist}(\pi_i, \widehat{\pi}_i) = O(1)$.

Step 2. For every $\varepsilon > 0$ take $\bar{u}_i, \bar{v}_i \in [u_i, v_i]$, at distance εd_i from u_i and v_i respectively. By Lemma 4.12, (1), there exists a point in $[u_i, \bar{u}_i]$ contained in $\mathcal{N}_R(\mathrm{Sat}\,[u_i', v_i'])$ and a similar point in $[\bar{v}_i, v_i]$. Lemma 4.12, (2), implies that $[\bar{u}_i, \bar{v}_i] \subset \mathcal{N}_{\tau R}(\mathrm{Sat}\,[u_i', v_i'])$.

Take y_i an arbitrary point on $[\bar{u}_i, \bar{v}_i]$ and assume that $y_i \notin \mathcal{N}_{\tau R}([u_i', v_i'])$. Then $y_i \in \mathcal{N}_{\tau R}(A_i)$, with $A_i \subset \mathrm{Sat}([u_i', v_i'])$. Let $\bar{e}_i, \bar{f}_i$ be the entry and respectively exit point of $[\bar{u}_i, \bar{v}_i]$ in $\overline{\mathcal{N}}_{\tau R}(A_i)$. Likewise let e_i', f_i' be the entry and respectively exit point of $[u_i', v_i']$ in $\overline{\mathcal{N}}_M(A_i)$. Lemmas 4.14 and 4.11 imply that if $\mathrm{dist}(\bar{e}_i, e_i') \ge D_0$ then $[u_i, \bar{e}_i] \sqcup [\bar{e}_i, e_i'] \sqcup [e_i', u_i']$ is a quasi-geodesic. This contradicts the fact that $\mathrm{dist}(u_i, u_i') = o(d_i)$ while $\mathrm{dist}(u_i, \bar{e}_i)$ is of order d_i. Thus $\mathrm{dist}(\bar{e}_i, e_i') \leq D_0$. Arguing similarly we obtain that $\mathrm{dist}(\bar{f}_i, f_i') \leq D_0$.

If $\mathrm{dist}(\bar{e}_i, \bar{f}_i)$ is large enough then by Lemma 4.12, (3), $A_i \subset \mathrm{Sat}\,[u_i, v_i]$, hence $A_i \subset \mathrm{Sat}\,[u_i, v_i] \cap \mathrm{Sat}\,[u_i', v_i']$. If not it follows that y_i is also at distance $O(1)$ from $\{e_i', f_i'\}$.

We have thus obtained that every $y_i \in [u_i, v_i]$ such that $\mathrm{dist}(u_i, y_i)$ and $\mathrm{dist}(y_i, v_i)$ is of order d_i is either at distance $O(1)$ from $[u_i', v_i']$ or it is contained in $\mathcal{N}_R(A_i)$ for some $A_i \subset \mathrm{Sat}\,[u_i, v_i] \cap \mathrm{Sat}\,[u_i', v_i']$, y_i at large distance from the entrance and exit point of $[u_i, v_i]$ in $\overline{\mathcal{N}}_M(A_i)$.

A similar statement can be formulated for points $y_i' \in [u_i', v_i']$ and $[u_i, v_i]$.

Step 3. Suppose that $\pi_i \in \overline{\mathcal{N}}_R([u_i, v_i])$. Likewise $\pi_i' \in \overline{\mathcal{N}}_R([u_i', v_i'])$. The argument in Step 2 implies that $\pi_i' \in \mathcal{N}_{\tau R+R}\,(\mathrm{Sat}\,[u_i, v_i])$. Hence $\widehat{\pi}_i$ is in $\overline{\mathcal{N}}_{R+D_0}([u_i', v_i']) \cap \overline{\mathcal{N}}_{(\tau+1)R+D_0}(\mathrm{Sat}\,[u_i, v_i])$. Lemmas 4.15 and 4.14 imply that the entrance point $\widetilde{\pi}_i$ of $[w_i, \widehat{\pi}_i]$ in $\overline{\mathcal{N}}_{(\tau+1)R+D_0}(\mathrm{Sat}\,[u_i, v_i])$ and π_i are at distance $O(1)$. Then $\mathrm{dist}(\widetilde{\pi}_i, [u_i, v_i]) = O(1)$. Also by Step 2 we can deduce that $\widetilde{\pi}_i$ is at distance $O(1)$ from $\mathrm{Sat}\,([u_i', v_i'])$. By Lemma 4.12, (1), $\widetilde{\pi}_i$ is at a distance from $\widehat{\pi}_i$ which is at most thrice the distance from $\mathrm{Sat}\,([u_i', v_i'])$, otherwise $[\widetilde{\pi}_i, \widehat{\pi}_i]$ would intersect $\mathcal{N}_R(\mathrm{Sat}\,([u_i', v_i']))$, a contradiction. Thus $\mathrm{dist}(\widetilde{\pi}_i, \widehat{\pi}_i) = O(1)$ and $\mathrm{dist}(\widetilde{\pi}_i, \pi_i) = O(1)$, which finishes the argument.

Step 4. Suppose that $\pi_i \in \overline{\mathcal{N}}_R(A_i)$ with A_i a left coset in $\mathrm{Sat}\,([u_i, v_i])$, and that π_i is at large distance from $[u_i, v_i]$. Then also $\pi_i' \in \overline{\mathcal{N}}_R(A_i')$ with A_i' a left coset in $\mathrm{Sat}\,([u_i', v_i'])$ and π_i' is at large distance from $[u_i', v_i']$.

Let e_i, f_i be the entrance and, respectively, the exit points of $[u_i, v_i]$ in $\overline{\mathcal{N}}_M(A_i)$. By the argument in Step 2, e_i is at distance $O(1)$ from a point y_i in $[u_i', v_i']$ or it is contained in $\mathcal{N}_M(B_i)$ for some $B_i \subset \mathrm{Sat}\,([u_i, v_i]) \cap \mathrm{Sat}\,([u_i', v_i'])$, far from the extremities of $[u_i, v_i] \cap \mathcal{N}_M(B_i)$. Without loss of generality we may suppose that either $y_i \in [f_i', v_i']$, or $\mathcal{N}_M(B_i)$ intersects $[f_i', v_i']$. In the second case let y_i be the entrance point of $[f_i', v_i']$ into $\overline{\mathcal{N}}_M(B_i)$.

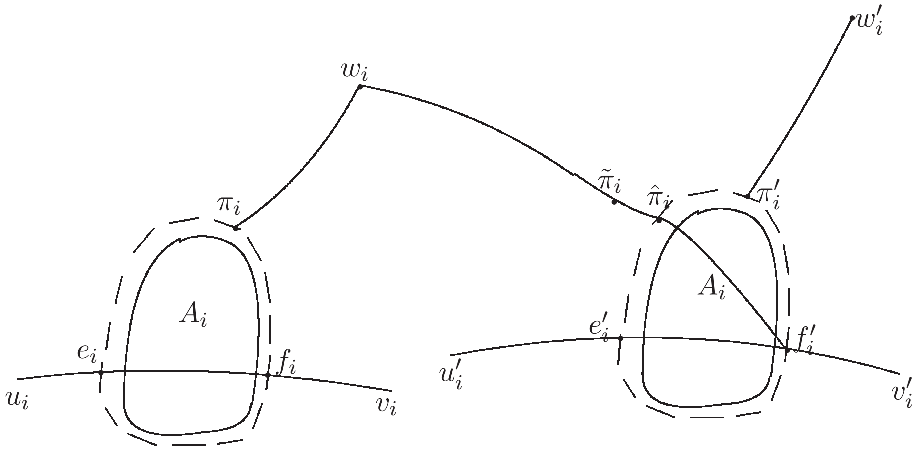

Figure 3: Step 4 in Lemma 4.26.

Since $\mathrm{dist}(\widehat{\pi}_i, \pi_i') \leq D_0$, we have $\widehat{\pi}_i \in \overline{\mathcal{N}}_{R+D_0}(A_i')$ and that $\widehat{\pi}_i$ is at distance $> O(1)$ from $[u_i', v_i']$. In particular $\mathrm{dist}(\widehat{\pi}_i, f_i') > O(1)$. Lemma 4.16 implies that $\mathfrak{q}_i = [w_i, \widehat{\pi}_i] \sqcup [\widehat{\pi}_i, f_i']$ is a quasi-geodesic. By Step 2, f_i' is at distance $O(1)$ from $\mathrm{Sat}\,([u_i, v_i])$. Let $D_1 \geq R$ be such that $f_i' \in \mathcal{N}_{D_1}\,(\mathrm{Sat}\,([u_i, v_i]))$. Lemmas 4.15 and 4.14 imply that the entrance point $\widetilde{\pi}_i$ of $\mathfrak{q}_i$ in $\overline{\mathcal{N}}_{D_1}(\mathrm{Sat}\,([u_i, v_i]))$ and π_i are at distance $O(1)$. In particular $\widetilde{\pi}_i$ is at distance $O(1)$ from A_i and at $> O(1)$ distance from $[u_i, v_i]$.

Step 4.a. Assume that $\widetilde{\pi}_i \in [\widehat{\pi}_i, f'_i]$. Then $\widetilde{\pi}_i \subset \mathcal{N}_{\tau R}(A'_i)$. If $\mathrm{dist}(\widetilde{\pi}_i, [e'_i, f'_i])$ is large, as both $\widetilde{\pi}_i$ and $[e'_i, f'_i]$ are at distance $O(1)$ from both A'_i and $\mathrm{Sat}\,([u_i, v_i])$ it would follow that $A'_i \subset \mathrm{Sat}\,([u_i, v_i])$. This would contradict the fact that $\widetilde{\pi}_i$ is the entrance point of $\mathfrak{q}_i$ in $\overline{\mathcal{N}}_{D_1}(\mathrm{Sat}\,([u_i, v_i]))$.

Hence $\mathrm{dist}\,(\widetilde{\pi}_i, [e'_i, f'_i])$ is $O(1)$. Moreover, the fact that A'_i cannot be in $\mathrm{Sat}\,([u_i, v_i])$ implies that $\mathrm{dist}(e'_i, f'_i)$ is $O(1)$. Hence $\mathrm{dist}(e'_i, \widetilde{\pi}_i) = O(1)$, and since $\mathrm{dist}(\widetilde{\pi}_i, \pi_i)$ is likewise $O(1)$, we deduce that $\mathrm{dist}(\pi_i, e'_i)$ is $O(1)$.

By Step 2, either e'_i is at distance $O(1)$ from $[u_i, v_i]$ or it is in $\mathcal{N}_M(B_i) \cap [u'_i, v'_i]$, far from the extremities of the intersection, where $B_i \subset \mathrm{Sat}\,([u_i, v_i]) \cap \mathrm{Sat}\,([u'_i, v'_i])$. In the first case, π_i also is at distance $O(1)$ from $[u_i, v_i]$. We are then back in the case of Step 3, with a constant possibly larger than R, and we can argue similarly.

In the second case, we have that π_i is at distance $O(1)$ from B_i. If $B_i \neq A_i$ then by Lemma 4.12, (4), π_i is at distance $O(1)$ from $[u_i, v_i]$ and we argue as previously. Assume therefore that $B_i = A_i$. By isometry $\mathrm{dist}(e_i, f_i)$ is also $O(1)$. On the other hand, by the argument in Step 2, e_i is at distance $O(1)$ from the entrance point of $[u'_i, v'_i]$ into $\overline{\mathcal{N}}_M(B_i)$ at the same for f_i and the exit point. It follows that $\mathrm{diam}\overline{\mathcal{N}}_M(B_i) \cap [u'_i, v'_i]$ is $O(1)$ and that $\mathrm{dist}(e_i, e'_i)$ is $O(1)$. Hence $\mathrm{dist}(\pi_i, e_i) = O(1)$ and we are back again to Step 3.

Step 4.b. Assume that $\widetilde{\pi}_i \in [w_i, \widehat{\pi}_i]$. If $\mathcal{N}_M(A_i) \cap [u_i, v_i]$ is too large, by Step 2 and Lemma 4.12, (4), it follows that $A_i \subset \mathrm{Sat}\,([u'_i, v'_i])$. Together with the fact that $\widetilde{\pi}_i$ is in a tubular neighborhood of A_i, with the choice of $\widehat{\pi}_i$ and with Lemma 4.14, this implies that $\mathrm{dist}(\widetilde{\pi}_i, \widehat{\pi}_i) = O(1)$. Hence we may assume in what follows that $\mathcal{N}_M(A_i) \cap [u_i, v_i]$ has small diameter, and same for its image.

Let D_2 be the maximum between $R + D_0$ and $\mathrm{dist}(\widetilde{\pi}_i, A_i)$. Let $[z_i, z'_i]$ be either the sub-arc of $[\widetilde{\pi}_i, \widehat{\pi}_i]$ of extremities the exit point from $\overline{\mathcal{N}}_{D_2}(A_i)$ and the entry point into $\overline{\mathcal{N}}_{D_2}(A'_i)$ or a degenerate segment composed of one point in $[\widetilde{\pi}_i, \widehat{\pi}_i] \cap \overline{\mathcal{N}}_{D_2}(A_i) \cap \overline{\mathcal{N}}_{D_2}(A'_i)$.

Lemma 4.12, (4), applied to the polygonal line $\mathfrak{l} = [e_i, y_i] \cup [y_i, f'_i]$, to the left cosets A_i and A'_i and the points z_i, z'_i implies that $\{z_i, z'_i\}$ is in $\mathcal{N}_{\varkappa}(\mathfrak{l})$.

By Lemma 4.14, $\mathrm{dist}(z'_i, \widehat{\pi}_i) = O(1)$, which implies that $\widehat{\pi}_i$ is at distance $O(1)$ from $[e_i, y_i] \cup [y_i, f'_i]$.

If $\mathrm{dist}(e_i, y_i) = O(1)$ then it follows that $\widehat{\pi}_i$ is at distance $O(1)$ from $[y_i, f'_i]$. This implies that π'_i is at distance $O(1)$ from $[u'_i, v'_i]$, and we are back in the case of Step 3.

The other possibility is that $\mathrm{dist}(e_i, y_i)$ is large, which by Step 2, corresponds to the case when e_i is in $\mathcal{N}_M(B'_i) \cap [u_i, v_i]$, far from the extremities of this intersection, where $B'_i \subset \mathrm{Sat}\,([u_i, v_i]) \cap \mathrm{Sat}\,([u'_i, v'_i])$.

Assume that $A'_i \neq B'_i$. Then one can apply Lemma 4.11 twice and deduce that $[e_i, y_i] \sqcup [y_i, f'_i] \sqcup [f'_i, \widehat{\pi}_i]$ is a quasi-geodesic. Therefore $\mathrm{dist}(\widehat{\pi}_i, [e_i, y_i] \cup [y_i, f'_i]) = O(1)$ implies that $\mathrm{dist}(\widehat{\pi}_i, f'_i)$ is $O(1)$. But then it follows that $\mathrm{dist}(\pi'_i, [u'_i, v'_i])$ is $O(1)$, and we can again argue as in Step 3.

Assume that $A'_i = B'_i$. By the argument in Step 2, e'_i and f'_i are at distance $O(1)$ from the entrance and exit points of $[u_i, v_i]$ into $\overline{\mathcal{N}}_M(A'_i)$. By the argument in the beginning of the current step, $\mathrm{diam}\overline{\mathcal{N}}_M(A'_i) \cap [u'_i, v'_i]$ is $O(1)$. It follows that $\mathrm{diam}\overline{\mathcal{N}}_M(A'_i) \cap [u_i, v_i]$ is $O(1)$, in particular e_i is at distance $O(1)$ from e'_i. We can then repeat the argument done above in the case when $\mathrm{dist}(e_i, y_i) = O(1)$, with y_i replaced by e'_i, deduce that $\widehat{\pi}_i$ is at distance $O(1)$ from $[y_i, f'_i]$, hence that π'_i is at distance $O(1)$ from $[u'_i, v'_i]$, and get back to Step 3. $\square$

Corollary 4.27. *Let $\mathcal{K} = \mathrm{Con}^\omega(G; x, d)$ be an asymptotic cone of G. There exists a constant $R = R(G)$ such that the following holds. Any subgroup $\mathcal{S} < x^\omega(\Pi_1 G/\omega)(x^\omega)^{-1}$ which fixes three points not in the same piece nor on the same transversal geodesic in $\mathcal{K}$ is conjugate to a subgroup in $\Pi B(1, R)/\omega$.*

Proof. Let u, v, w be the three points. If any of the strict saturations $\mathrm{Sat}_0\{a, b\}$ with $a \neq b$, $a, b \in \{u, v, w\}$, contains a piece then $\mathcal{S}$ stabilizes the piece and fixes a point outside it, and we may apply Lemma 4.25.

If not, then the three points are in the same transversal tree, as vertices of a tripod. We apply Lemma 4.26 in this case. □

4.4 Homomorphisms into relatively hyperbolic groups

The following observation of Bestvina and Paulin is well known (see for instance [Be$_2$]).

Lemma 4.28. *Let Λ and Γ be two finitely generated groups, let $S = S^{-1}$ be a finite set generating Λ and let* dist *be a word metric on Γ. Given $\phi_n : \Lambda \to \Gamma$ an infinite sequence of homomorphisms, one can associate to it a sequence of positive integers defined by*

$$d_n = \inf_{x \in \Gamma} \sup_{a \in S} \mathrm{dist}(\phi_n(a)x, x)\,. \tag{2}$$

If (ϕ_n) are pairwise non-conjugate in Γ then $\lim_{n \to \infty} d_n = \infty$.

Remark 4.29. For every $n \in \mathbb{N}$, $d_n = \mathrm{dist}(\phi_n(a_n)x_n, x_n)$ for some $x_n \in \Gamma$ and $a_n \in S$.

Let $\Lambda = \langle S \rangle$, (ϕ_n) and (d_n) be as in Lemma 4.28, with $\Gamma = G$.

Consider an arbitrary ultrafilter ω. According to Remarks 4.29 and 4.1, there exists $a \in S$ and $x_n \in G$ such that $d_n = \mathrm{dist}(\phi_n(a)x_n, x_n)$ ω–a.s.

Lemma 4.30. *Under the assumptions of Lemma 4.28, the group Λ acts on the asymptotic cone $\mathcal{K}_\omega = \mathrm{Con}^\omega(G; (x_n), (d_n))$ by isometries, without a global fixed point, as follows:*

$$g \cdot \lim^\omega (x_n) = \lim^\omega (\phi_n(g)x_n)\,. \tag{3}$$

This defines a homomorphism ϕ_ω from Λ to the group $x^\omega(\Pi_1\Gamma/\omega)(x^\omega)^{-1}$ of isometries of $\mathcal{K}_\omega$.

The action in Lemma 4.30 of a group Λ on the asymptotic cone $\mathcal{K}_\omega$ (which is tree-graded with respect to limits of sequences of cosets from $\mathcal{G}$ by Theorem 4.17) satisfies the hypotheses in Theorem 3.23 if one more condition on (ϕ_n) holds.

Definition 4.31. A homomorphism $\phi : \Lambda \to G$ is called *parabolic* if its image is a parabolic group.

Proposition 4.32. *Suppose that a finitely generated group $\Lambda = \langle S \rangle$ has infinitely many non-parabolic homomorphisms ϕ_n into a relatively hyperbolic group G, which are pairwise non-conjugate in G.*

Then the action of Λ on an asymptotic cone $\mathcal{K}_\omega$ of G defined by (ϕ_n) as in (3) satisfies the properties (i),(ii) and (iii) of Theorem 3.23.

The proof is done in several steps.

As before (see the notation before Theorem 3.1), we use the following notation:

- $\mathcal{C}_1(\Lambda, \omega)$ is the set of stabilizers in Λ of proper subsets in $\mathcal{K}_\omega$ such that all their finitely generated subgroups stabilize pairs of pieces in $\mathcal{K}_\omega$;
- $\mathcal{C}_2(\Lambda, \omega)$ is the set of stabilizers in Λ of pairs of points in $\mathcal{K}_\omega$ that are not in the same piece;

Lemma 4.33. *Let K be a subgroup in Λ such that all its finitely generated subgroups stabilize pairs of pieces. Then $\phi_\omega(K)$ is a conjugate of a subgroup in the finite set $\Pi B(1,R)/\omega$ for some uniform constant R.*

Proof. Take an increasing sequence

$$K_1 \subset K_2 \subset ... \subset K_i \subset ... \tag{4}$$

of finitely generated subgroups of K such that $K = \bigcup K_i$. By hypothesis each K_i stabilizes two distinct pieces, that is two different nonempty limits $\lim^\omega (g_n H)$ and $\lim^\omega (h_n H')$. By Lemma 4.19, for every $k \in K_i$ ω–a.s. $\phi_n(k) \in g_n H g_n^{-1} \cap h_n H' h_n^{-1}$. In particular, given a finite generating set S_i of K_i, ω–a.s. $\phi_n(S_i) \subset g_n H g_n^{-1} \cap h_n H' h_n^{-1}$. Then also $\phi_n(K_i)$ is contained in $g_n H g_n^{-1} \cap h_n H' h_n^{-1}$ ω–a.s. By Lemma 4.20, there exists an element $a_n(i)$ in $g_n H$ such that $\phi_n(K_i)$ is conjugate by $a_n(i)$ to a subgroup inside $B(1,R)$, for some uniform constant $R > 0$, ω–a.s. Since there are finitely many subgroups inside $B(1,R)$, Remark 4.1 implies that there exists a finite subgroup U_i in $B(1,R)$ such that $\phi_n(K_i) = a_n(i)^{-1} U_i a_n(i)$, ω–a.s.

Again because there are finitely many subgroups inside $B(1,R)$, there exists a subgroup U inside $B(1,R)$ and a subsequence $i_0 < i_1 < i_2 < ...$ such that $U_{i_m} = U$ for every $m \geq 0$. In particular, since $\phi_n(K_{i_0}) < \phi_n(K_{i_m})$ for every $m > 0$ and both groups have the same cardinal as U ω–a.s., it follows that $\phi_n(K_{i_0}) = \phi_n(K_{i_m})$ ω–a.s. Hence for every $j \geq i_0$, $\phi_n(K_j)\phi_n(K_{i_0})$ ω–a.s. Consequently, given $a_n = a_n(i_0)$, the group $\phi_n(K_j)$ is equal to $a_n^{-1} U a_n$, ω–a.s.

Now an arbitrary element k in K is contained in some K_j with $j \geq i_0$, hence $\phi_n(k) \in \phi_n(K_j)$ ω–a.s. Then a_n conjugates $\phi_n(k)$ to an element of U ω–a.s. Hence $(a_n)^\omega$ conjugates $\phi_\omega(k)$ to an element in the finite set $\Pi U/\omega$. $\square$

Lemma 4.34. *Let $\mathfrak{g}$ be a non-trivial geodesic segment in a transversal tree of $\mathcal{K}_\omega$, and let L be the pointwise stabilizer of $\mathfrak{g}$ in Λ. Then up to conjugacy, $\phi_\omega(L)$ is an extension of a subgroup from $\Pi B(1,R)/\omega$ by an Abelian group.*

Proof. **Step 1.** Let γ be an element in L. We associate with it a sequence of translation numbers.

The geodesic $\mathfrak{g}$ is the ω–limit of geodesics $\mathfrak{g}_n$ of length $O(d_n)$. Let a_n and b_n be the initial and terminal points of $\mathfrak{g}_n$. The fact that γ fixes $\mathfrak{g}$ means that the left action of $\phi_n(\gamma)$ on $\mathfrak{g}_n$ must move it within $o(d_n)$ distance from itself.

Note that $\mathfrak{g}_n$ intersects each coset from $\mathcal{G}$ by a sub-geodesic of length $o(d_n)$ ω–a.s.

Let $\mathrm{Sat}\,(\mathfrak{g}_n)$ be the saturation of $\mathfrak{g}_n$ in the sense of Definition 4.9.

Let m_n be the midpoint of $\mathfrak{g}_n$. Let A_n be a coset from $\mathcal{G}$ whose M–tubular neighborhood contains m_n and such that the length ℓ_n of the intersection of $\mathcal{N}_M(A_n)$ with $\mathfrak{g}_n$ is maximal possible. Here $M = M(1,0)$.

If $\ell_n \geq \ell$, where $\ell = \ell(D)$ is a constant to be defined later (see Step 2), then take m_n' to be the entry point of $\mathfrak{g}_n$ into $\mathcal{N}_M(A_n)$. We know that $\mathrm{dist}(m_n', m_n) = o(d_n)$. If $\ell_n \leq \ell$, take $m_n' = m_n$.

Let $m_n(\gamma)$ be a projection of $\phi_n(\gamma)m_n'$ onto $\mathfrak{g}_n$.

We define the *n-th translation number* $\lambda_n(\gamma)$ as

$$\lambda_n(\gamma) = (-1)^\epsilon \mathrm{dist}(m_n', \phi_n(\gamma)m_n')$$

where $\epsilon = 0$ if $\mathrm{dist}(b_n, m_n(\gamma)) \leq \mathrm{dist}(b_n, m_n')$ and $\epsilon = 1$ otherwise.

Step 2. We are going to show that λ_n satisfies the quasi-homomorphism condition:

$$\forall \gamma\,,\, \zeta \text{ in } \Lambda\,, \quad \omega\text{-a.s.} \quad |\lambda_n(\gamma\zeta) - (\lambda_n(\gamma) + \lambda_n(\zeta))| \leq \Delta\,, \tag{5}$$

where Δ is a universal constant.

Consider the geodesic $\phi_n(\gamma)\mathfrak{g}_n$. By our assumption, $\mathrm{dist}(\phi_n(\gamma)a_n, a_n)$ and $\mathrm{dist}(\phi_n(\gamma)b_n, b_n)$ are $o(d_n)$. Let $\mathfrak{p}_n$ be the middle third of $\mathfrak{g}_n$. Let a_n' and b_n' be the initial and terminal points of $\mathfrak{p}_n$. Then by [DS, Lemmas 4.24 and 4.25], $\phi_n(\gamma)\mathfrak{p}_n$ is in the D–tubular neighborhood of $\mathrm{Sat}\,(\mathfrak{g}_n)$ for a uniform constant $D \geq M$.

According to [DS, Lemma 4.22], for every $t > 0$, every geodesic $\mathfrak{c}$ and every $A \in \mathcal{G}$ either the diameter of the intersection $\mathcal{N}_t(A) \cap \mathcal{N}_t(\mathrm{Sat}\,(\mathfrak{c}))$ is uniformly bounded by a constant $\ell(t)$ or $A \subset \mathrm{Sat}\,(\mathfrak{c})$.

Since $\mathfrak{p}_n$ is a geodesic, the intersection between $\mathcal{N}_M(\phi_n(\gamma)A_n)$ and the D–tubular neighborhood of $\mathrm{Sat}\,(\mathfrak{p}_n)$ has diameter at least ℓ_n.

Step 2.a Suppose that $\ell_n \geq \ell(D)$. Then $\phi_n(\gamma)A_n \subset \mathrm{Sat}\,(\mathfrak{p}_n)$.

The endpoint $\phi_n(\gamma)a_n'$ is also in the D–tubular neighborhood of $\mathrm{Sat}\,(\mathfrak{p}_n)$.

Suppose that $\phi_n(\gamma)a_n'$ is in the D–tubular neighborhood of a coset $B_n \subseteq \mathrm{Sat}\,(\mathfrak{g}_n)$. Since the length of the intersection of $\mathfrak{g}_n$ with $\mathcal{N}_D(B_n)$ is $o(d_n)$, $B_n \neq A_n$ ω–a.s. Then by [DS, Lemma 4.28], $\mathrm{dist}(\phi_n(\gamma)m_n', m_n(\gamma))$ is uniformly bounded ω–a.s.

Suppose now that $\phi_n(\gamma)a_n'$ is in the D–tubular neighborhood of $\mathfrak{g}_n$. Then by [DS, Lemma 8.13], again, $\mathrm{dist}(\phi_n(\gamma)m_n', m_n(\gamma))$ is uniformly bounded ω–a.s.

Step 2.b Now suppose that $\ell_n < \ell(D)$. Then $m_n' = m_n$. Since $\phi_n(\gamma)m_n$ is in the D–tubular neighborhood of $\mathrm{Sat}\,(\mathfrak{g}_n)$, $\phi_n(\gamma)m_n$ is either in $\mathcal{N}_D(\mathfrak{g}_n)$ or in $\mathcal{N}_D(A_n')$ where $A_n' \subset \mathrm{Sat}\,(\mathfrak{g}_n)$. In the first case $\mathrm{dist}(\phi_n(\gamma)m_n, m_n(\gamma))$ is uniformly bounded ω–a.s.

Suppose therefore that $\phi_n(\gamma)m_n$ is in $\mathcal{N}_D(A_n')$ where $A_n' \subset \mathrm{Sat}\,(\mathfrak{g}_n)$.

Suppose moreover that $\phi_n(\gamma)\mathfrak{g}_n$ does not intersect $\mathcal{N}_M(A_n')$. Then the length of the intersection of $\phi_n(\gamma)\mathfrak{g}_n$ with $\mathcal{N}_D(A_n')$ is at most $3D+1$. Otherwise property (α_2) and the choice of M in Definition 4.9 would imply that $\phi_n(\gamma)\mathfrak{g}_n$ intersects $\mathcal{N}_M(A_n')$. In particular $\phi_n(\gamma)m_n$ is at distance at most $3D+1$ from the entry point of $\phi_n(\gamma)\mathfrak{g}_n$ into $\mathcal{N}_D(A_n')$. An argument as in Step 2.a implies that this entry point is at uniformly bounded distance from $\mathfrak{g}_n$, hence the same holds for $\phi_n(\gamma)m_n$.

Suppose that $\phi_n(\gamma)\mathfrak{g}_n$ intersects $\mathcal{N}_M(A_n')$. Let c_n be the entry point of $\phi_n(\gamma)\mathfrak{g}_n$ into $\mathcal{N}_M(A_n')$. If $\phi_n(\gamma)m_n$ is not in $\mathcal{N}_M(A_n')$, then its distance to c_n is at most $3D+1$, otherwise one obtains a contradiction with the fact that c_n is the entry point into $\mathcal{N}_M(A_n')$.

Suppose that $\phi_n(\gamma)m_n \in \mathcal{N}_M(A_n')$. Note that the intersection of $\phi_n(\gamma)^{-1}\mathcal{N}_M(A_n')$ with $\mathfrak{g}_n$ contains m_n and has the same length as the intersection of $\mathcal{N}_M(A_n')$ and $\phi_n(\gamma)\mathfrak{g}_n$. Therefore these lengths are smaller than $\ell(D)$. In particular $\mathrm{dist}(\phi_n(\gamma)m_n, c_n) \leq \ell(D)$.

An argument as in Step 2.a gives that c_n is at uniformly bounded distance from $\mathfrak{g}_n$. Therefore this is also true for $\phi_n(\gamma)m_n$.

We conclude that $\mathrm{dist}(\phi_n(\gamma)m_n', m_n(\gamma))$ is uniformly bounded ω–a.s.

Thus in all cases, for some constant D'',

$$\mathrm{dist}(\phi_n(\gamma)m_n', m_n(\gamma)) < D'' \quad \omega\text{-a.s.} \tag{6}$$

Step 2.c Now we are ready to prove (5). For simplicity, assume that $\lambda_n(\gamma), \lambda_n(\zeta) \geq 0$ (the other cases are similar). All the equalities in the proof below are true ω–a.s. By (6), $\mathrm{dist}(m_n', m_n(\gamma)) = \lambda_n(\gamma) + O(1)$, $\mathrm{dist}(m_n', m_n(\gamma\zeta)) = \lambda_n(\gamma\zeta) + O(1)$, $\mathrm{dist}(\phi_n(\gamma)m_n', \phi_n(\gamma)m_n(\zeta)) = \lambda_n(\zeta) + O(1)$. The last equality implies that $\mathrm{dist}(m_n(\gamma), m_n(\gamma\zeta)) = \lambda_n(\zeta) + O(1)$. Combining these equalities together, we obtain (5).

Step 3. Let γ, ζ be two elements of L, $[\gamma, \zeta] = \gamma\zeta\gamma^{-1}\zeta^{-1}$ be their commutator. Then by (5), $\lambda_n(\phi_n([\gamma, \zeta])) = O(1)$ ω–a.s. Therefore

$$\mathrm{dist}(\phi_n([\gamma, \zeta])m_n', m_n') = O(1) \ \ \omega\text{–a.s.}$$

Therefore $|(m'_n)^{-1}\phi_n([\gamma,\zeta])m'_n| = O(1)$ ω–a.s. Hence up to conjugacy $\phi_n([\gamma,\zeta])$ is in the ball $B(1,R)$ ω–a.s. for some R.

This implies that up to conjugacy the set of commutators of $\phi_\omega(L)$ is contained in the set $\Pi B(1,R)/\omega$ which is of bounded cardinality by Lemma 4.2. Therefore every finitely generated subgroup $L_1 \leq \phi_\omega(L)$ has conjugacy classes of bounded size, i.e. L_1 is an FC-group [N]. By [N], the set of all torsion elements of L_1 is finite, and the derived subgroup of L_1 is finite and is generated, up to conjugacy, by a subset of $\Pi B(1,R)/\omega$ for some R. There exists only finite number of finite subgroups generated by subsets of $\Pi B(1,R)/\omega$. Since elements of $\Pi B(1,R)$ are sequences $(g_n)^\omega$ that are constants ω–a.s., every finite subgroup generated by a subset of $\Pi B(1,R)/\omega$ is inside $\Pi B(1,R')/\omega$ for some $R' > R$. Hence the derived subgroup of $\phi_\omega(L)$ is conjugate to a subgroup of $\Pi B(1,R')/\omega$. □

Lemma 4.35. *Let $L \in \mathcal{C}_2(\Lambda,\omega)$. Then $\phi_\omega(L)$ is inside a conjugate of an extension of a subgroup in $\Pi B(1,R)/\omega$ by an Abelian group.*

Let x, y be two points in $\mathcal{K}_\omega$ that are not in the same piece and let L be the stabilizer in Λ of the pair (x, y).

Case 1. Suppose that $\mathrm{Sat}_0\,\{x,y\}$ contains a piece A. Then the stabilizer of x, y coincides with the stabilizer of $A \cup \{x,y\}$. Either x or y is not in A. Lemma 4.25 implies that $\phi_\omega(L)$ is conjugated to a subgroup in $\Pi B(1,O(1))/\omega$.

Case 2. Suppose now that $\mathrm{Sat}_0\,\{x,y\}$ contains no piece. Then x, y are contained in the same transversal tree, and so they are joined by a unique geodesic $\mathfrak{g}$ in that transversal tree and the stabilizer of x, y coincides with the stabilizer of $\mathfrak{g}$. It remains to use Lemma 4.34.

Proof of Proposition 4.32. **(i)** Obviously property (i) in Theorem 3.23 is satisfied: the action of Λ permutes pieces of $\mathcal{K}_\omega$.

(ii) According to Lemma 4.30 there is no point in $\mathcal{K}_\omega$ fixed by the whole Λ.

If $\Lambda \cdot A = A$ for some piece $A \in \mathcal{P}$ then, by Lemma 4.19, $\phi_\omega(\Lambda) \subset \Pi\left(g_n H g_n^{-1}\right)/\omega$ for some sequence (g_n) in G and some peripheral subgroup $H \in \mathcal{H}$. In particular ω–a.s. $\phi_n(S) \subset g_n H g_n^{-1}$, hence the image of ϕ_n is a parabolic group. This contradicts the hypothesis that ϕ_n are non-parabolic homomorphisms.

(iii) We prove (iii) in two steps.

(iii.a) Let us prove that $\mathcal{C}_1(\Lambda,\omega)$ satisfies ACC.

Let

$$K_1 \subset K_2 \subset ... \subset K_i \subset ... \tag{7}$$

be an increasing sequence of subgroups from $\mathcal{C}_1(\Lambda,\omega)$. By Lemma 4.33, for every $i \in \mathbb{N}$, the cardinality $\mathrm{card}\,\phi_\omega(K_i)$ is bounded by a constant D. The sequence

$$\mathrm{card}\,\phi_\omega(K_1) \leq \mathrm{card}\,\phi_\omega(K_2) \leq ... \leq \mathrm{card}\,\phi_\omega(K_i) \leq ...$$

must stabilize. Thus we may assume that all $\mathrm{card}\,\phi_\omega(K_i)$ are the same. It follows that for all $i > 1$, $\phi_\omega(K_i) = \phi_\omega(K_1)$.

Now since each K_i is in $\mathcal{C}_1(\Lambda,\omega)$, it is the stabilizer of some proper subset $\mathcal{M}_i$ in $\mathcal{K}$. The equality $\phi_\omega(K_i) = \phi_\omega(K_1)$ implies that for every $k_i \in K_i$ there exists $k_1 \in K_1$ such that $\phi_n(k_i) = \phi_n(k_1)$ ω–a.s. In particular k_i also stabilizes $\mathcal{M}_1$, thus $K_i \subset \mathrm{Stab}(\mathcal{M}_1) = K_1$. We obtain that $K_i = K_1$ for every $i > 1$.

(iii.b) Let us prove that $\mathcal{C}_2(\Lambda,\omega)$ satisfies ACC. Let L be an arbitrary subgroup in $\mathcal{C}_2(\Lambda,\omega)$, that is L is the stabilizer of two points x, y in $\mathcal{K}_\omega$ not in the same piece.

If $\mathrm{Sat}_0 \{x, y\}$ contains a piece then $\phi_\omega(L)$ is conjugate to a subgroup in $\Pi B(1, R)/\omega$, by Lemma 4.25. In this case ACC is proved with an argument as in (iii.a).

If $\mathrm{Sat}_0 \{x, y\}$ contains no piece, then x and y are the endpoints of a transversal geodesic $\mathfrak{p}$. Recall that the action of Λ on $\mathcal{K}_\omega$ induces an action of $\phi_\omega(\Lambda)$ on the $\mathbb{R}$-tree $T = \mathcal{K}_\omega/ \approx$. Let $\mathfrak{g}$ be the projection of $\mathfrak{p}$ onto T. By Lemma 2.20, $\phi_\omega(\Lambda)$ is the stabilizer of $\mathfrak{g}$. Note that the action of $\phi_\omega(\Lambda)$ on T has finite of bounded size tripod stabilizers, by Lemma 2.20 and Corollary 4.27, and it has (finite of bounded size)-by-Abelian arc stabilizers, by Lemma 4.34. It follows by Lemma 2.35 that for every ascending sequence of subgroups from $\mathcal{C}_2(\Lambda, \omega)$

$$L_1 \subset L_2 \subset \ldots \subset L_i \subset \ldots \tag{8}$$

the ascending sequence of images

$$\phi_\omega(L_1) \subset \phi_\omega(L_2) \subset \ldots \subset \phi_\omega(L_i) \subset \ldots \tag{9}$$

must stabilize, because it is a sequence of arc stabilizers for the action of $\phi_\omega(\Lambda)$ on T. Assume that $\phi_\omega(L_i) = \phi_\omega(L_1)$, for every $i > 1$. Since L_1 is defined as the stabilizer of two points x_1, y_1 in $\mathcal{K}_\omega$, not in the same piece, it follows that every element l in L_i must also stabilize x_1, y_1, as $\phi_\omega(l) \in \phi_\omega(L_1)$. Hence $L_i \subset L_1$, therefore $L_i = L_1$. □

Proposition 4.36. *If, in addition to the assumptions of Proposition 4.32, the kernel of the homomorphism ϕ_ω is finite, then the action of Λ on $\mathcal{K}_\omega$ satisfies property (iii) of Theorem 3.25.*

Proof. This follows immediately from Lemmas 4.33 and 4.35, and Corollary 4.27. □

Here are the main applications of Theorem 3.25 together with Propositions 4.32 and 4.36 to relatively hyperbolic groups. We assume that Theorem 2.34 is correct (recall that the proof is not published yet). If one does not want to assume that, one can add the assumption that the group Λ in these applications is either torsion-free or finitely presented and use Theorem 2.32 or 2.31, respectively.

First we show that a "generic" finitely generated group has only finitely many non-parabolic homomorphisms into a given relatively hyperbolic group G, up to conjugacy by elements of G.

As an immediate consequence of Theorem 3.23 and Proposition 4.32 we obtain the following statement. Recall that a group Λ satisfies *property FA* of Serre [Ser] if every action of Λ on a simplicial tree has a global fixed point.

Corollary 4.37. *If a finitely generated group Λ satisfies property FA then for every relatively hyperbolic group G there are only finitely many pairwise non-conjugate non-parabolic homomorphisms $\Lambda \to G$.*

If Λ is an arbitrary group, we can still obtain a lot of information about its homomorphisms into relatively hyperbolic groups.

Definition 4.38. Let G be a relatively hyperbolic group, and let $K < A$ be two subgroups of G. The subgroup K is called *locally parabolic in A* if for every finitely generated subgroup K_1 of K there exists an embedding $\phi \colon A \to G$ such that $\phi(K_1)$ is parabolic in G.

Remarks 4.39. 1. We do not know any examples where K is locally parabolic in $A \leq G$ but no embedding $A \to G$ maps K into a parabolic subgroup of G.

2. If K is finitely generated and locally parabolic in A then (obviously) some embedding $A \to G$ maps K inside a parabolic subgroup of G.

3. By Lemma 4.21, if parabolic subgroups do not contain non-Abelian free subgroups then one can drop the "finitely generated" assumption in 2.

Theorem 4.40. *Let Λ be a finitely generated subgroup in G which is neither virtually cyclic nor parabolic. Assume moreover that Λ does not split over any subgroup K of it such that K is virtually cyclic or locally parabolic in Λ. Then the number of conjugacy classes in G of injective non-parabolic homomorphisms $\Lambda \to G$ is finite.*

Proof. Suppose, by contradiction, that there exists a sequence of injective non-parabolic homomorphisms $\phi_n \colon \Lambda \to G$ pairwise non-conjugate in G.

By Proposition 4.32, the sequence (ϕ_n) defines an action of Λ on an asymptotic cone $\mathcal{K}_\omega$ of G satisfying properties (i), (ii) and (iii) of Theorem 3.23. Moreover the homomorphism ϕ_ω is injective, hence by Proposition 4.36 the assumptions of Theorem 3.25 also hold for the action of Λ on $\mathcal{K}_\omega$. Consequently one of the cases (1), (2) or (3) of Theorem 3.25 must occur for Λ.

Suppose that (1) holds. Then Λ splits over a (finite of uniformly bounded size)-by-Abelian-by-(virtually cyclic) subgroup K. By Lemma 4.21, the subgroup K is either virtually cyclic or it is a parabolic subgroup of G. This contradicts the assumption that Λ does not split over a virtually cyclic or locally parabolic subgroup.

Suppose (2) holds. Then Λ splits over the pre-image K by ϕ_ω of a stabilizer $\mathrm{Stab}(B, p)$ where $B \in \mathcal{P}, p \in B$. By Lemma 4.19, $\mathrm{Stab}(B, p)$ is inside the ultraproduct $\Pi_{n\in\mathbb{N}} P_n/\omega$ of some maximal parabolic subgroups $P_n < G$. This means that for every element $k \in K$ ω–almost all ϕ_n map k to P_n. Therefore K is locally parabolic in Λ, a contradiction.

If (3) holds then by Lemma 4.21 the group Λ is either virtually cyclic or parabolic. This again contradicts the choice of Λ. □

Theorem 4.40 immediately implies the following corollary. For every subgroup $\Lambda < G$ let $N_G(\Lambda)$ (resp. $C_G(\Lambda)$) be the normalizer (resp. the centralizer) of Λ in G. Clearly there exists a natural embedding ε of $N_G(\Lambda)/C_G(\Lambda)$ into the group of automorphisms $\mathrm{Aut}(\Lambda)$.

Corollary 4.41. *Suppose that $\Lambda \leq G$ is neither virtually cyclic nor parabolic, and that it does not split over a locally parabolic or virtually cyclic subgroup. Then $\varepsilon(N_G(\Lambda)/C_G(\Lambda))$ has finite index in* $\mathrm{Aut}(\Lambda)$. *In particular, if* $\mathrm{Out}(\Lambda)$ *is infinite then Λ has infinite index in its normalizer.*

Proof. Indeed, every automorphism of Λ is an embedding of Λ into G. □

Lemma 4.42. *Suppose that the peripheral subgroups of G are not relatively hyperbolic with respect to proper subgroups.*

Suppose that $\mathrm{Out}(G)$ *is infinite, and for some sequence (ϕ_n) of coset representatives of* $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ *define a non-trivial action of G on an asymptotic cone $\mathcal{K}_\omega$ of G, as in Lemma 4.30. Then*

(1) the stabilizers of pieces of $\mathcal{K}_\omega$ in G are either conjugates of subgroups in a fixed ball $B(1, R)$ or maximal parabolic subgroups;

(2) the stabilizer of a point $y = \lim^\omega(y_n)$ of $\mathcal{K}_\omega$ is the subgroup

$$G_y = \left\{ g \in G \mid \lim{}^\omega \left(\frac{|y_n^{-1}\phi_n(g)y_n|}{d_n} \right) = 0 \right\} .$$

Proof. (1) Let L be a stabilizer of a piece $\lim^\omega(\gamma_n H)$ of $\mathcal{K}_\omega$, for some $\gamma_n \in G$ and a peripheral subgroup H. By Lemma 4.19, $\phi_\omega(L)$ is inside $\Pi\left(\gamma_n H \gamma_n^{-1}\right)/\omega$. This means that for every $a \in L$, $\phi_n(a)$ is in $\gamma_n H \gamma_n^{-1}$ ω–a.s. Hence $a \in P_n = \phi_n^{-1}(\gamma_n H \gamma_n^{-1})$ ω–a.s. Since peripheral subgroups are not relatively hyperbolic with respect to proper subgroups, by Lemma 4.23, (4), the groups $P_n = \phi_n^{-1}(\gamma_n H \gamma_n^{-1})$ are maximal parabolic.

Suppose that L contains $N+1$ distinct elements, where N is the cardinal of the ball $B(1,R)$ appearing in Lemma 4.20. Let $a_1, ..., a_{N+1}$ be these elements. There exists $I \subset \mathbb{N}$ with $\omega(I) = 1$ and such that $\{a_1, ..., a_{N+1}\} \subset P_n$ for $n \in I$. Lemma 4.20 implies that for any $n \in I$ the group P_n coincides with a fixed maximal parabolic group P.

Suppose that L has at most N elements but that it is not conjugate to a subgroup in $B(1,R)$. A similar argument allows to conclude that there exists $I \subset \mathbb{N}$ with $\omega(I) = 1$ such that for all $n \in I$, P_n coincides with some fixed P.

Thus $L < P$. On the other hand, P clearly stabilizes the piece $\lim^{\omega}(\gamma_n H)$. Hence $L = P$. This implies (1).

Statement (2) can be obtained by a direct computation. □

Theorem 4.43. *Suppose that the peripheral subgroups of G are not relatively hyperbolic with respect to proper subgroups. If $\mathrm{Out}(G)$ is infinite then one of the followings cases occurs:*

(1) G splits over a virtually cyclic subgroup;

(2) G splits over a parabolic [(finite of uniformly bounded size)-by-Abelian]-by-(virtually cyclic) subgroup;

(3) G can be represented as a fundamental group of a graph of groups such that each vertex group is either maximal parabolic or of the form G_y in Lemma 4.42, (2), and the edge groups are parabolic; thus G splits as an amalgamated product or an HNN extension with a maximal parabolic subgroup H as a vertex group and a proper subgroup of H as an edge group.

Proof. Given a sequence (ϕ_n) of automorphisms as in Lemma 4.42, consider the corresponding action of G on an asymptotic cone $\mathcal{K}_\omega$ of G.

Conditions (i) and (ii) of Theorem 3.25 obviously hold, condition (iii) holds by Lemmas 4.33 and 4.35, and Corollary 4.27. Thus G is either in Case (1) or in Case (2) of Theorem 3.25. Case (3) cannot occur since G contains a non-Abelian free subgroup by Lemma 4.21.

Suppose that Case (1) of Theorem 3.25 occurs. By Lemma 4.21, a (finite of uniformly bounded size)-by-Abelian-by-(virtually cyclic) subgroup of G is either virtually cyclic, and then Case (1) in the present theorem holds, or it is parabolic and Case (2) of the theorem holds.

If Case (2) of Theorem 3.25 occurs then by Lemma 4.42 we have Case (3) of this theorem. □

The next theorem describes co-Hopfian relatively hyperbolic groups. The following lemma answers a question in [BS].

Lemma 4.44. *For any monomorphism $\phi\colon G \to G$ such that $\phi^k(G)$ is not parabolic for any k, let Z_k be the (finite) centralizer of $\phi^k(G)$. Then the increasing union Z of Z_k is finite.*

Proof. Suppose that Z is infinite. It is clear that Z is locally finite. Hence Z is a parabolic subgroup by Lemma 4.21, since it does not contain free non-Abelian subgroups. Let R be the constant from Lemma 4.20. For some $k >> 1$, Z_k contains more elements than the ball $B(1,R)$. Note that conjugation by any element $g \in \phi^k(G)$ fixes elements of Z_k. Let H be the maximal parabolic subgroup containing Z. Then $gHg^{-1} \cap H \geq Z_k$. By Lemma 4.20, $g \in H$. Hence H contains $\phi^k(G)$, a contradiction. □

Theorem 4.45. *Suppose that G is not co-Hopfian. Let ϕ be an injective but not surjective homomorphism $G \to G$. Then one of the following holds:*

- *$\phi^k(G)$ is parabolic for some k;*

- *G splits over a parabolic or virtually cyclic subgroup.*

Proof. By Lemma 4.44 and [RS, Theorem 3.1], we can assume that there are infinitely many pairwise non-conjugate powers of ϕ. Then G acts on an asymptotic cone of it $\mathcal{K}_\omega$. The rest of the proof is similar to the proof of Theorem 4.43 and it is left to the reader. $\square$

References

[AN] S. A. Adeleke, P. M. Neumann. Relations related to betweenness: their structure and automorphisms, Memoirs of Amer. Math. Soc. **131** (1998), no. 623.

[Ali] E. Alibegović. Makanin-Razborov diagrams for limit groups. preprint, arXiv: math.GR/0406581.

[Alo] J. M. Alonso, et al. Notes on word hyperbolic groups. Edited by H. Short. Group theory from a geometrical viewpoint (Trieste, 1990), 3–63, World Sci. Publishing, River Edge, NJ, 1991.

[B] J. Behrstock, Asymptotic geometry of the mapping class group and Teichmüller space, preprint, arXiv:math.GT/0502367.

[BDM] J. Behrstock, C. Druţu, L. Mosher. Thick metric spaces, relative hyperbolicity and quasi-isometric rigidity, preprint, arXiv:math.GT/0512592.

[BS] I. Belegradek, A. Szczepański. Endomorphisms of relatively hyperbolic groups. preprint, arXiv:math.GR/0501321.

[Be$_1$] M. Bestvina. Degenerations of the hyperbolic space. Duke Math. J. 56 (1988), 143–161.

[Be$_2$] M. Bestvina. $\mathbb{R}$–trees in topology, geometry, and group theory. Handbook of geometric topology, 55–91, North-Holland, Amsterdam, 2002.

[BF] M. Bestvina, M. Feighn. Stable actions of groups on real trees. Invent. Math. **121** (1995), 287–321.

[Bou] N. Bourbaki. *Topologie générale*, quatrième édition, Hermann, Paris, 1965.

[Bo] O. V. Bogopolsky. Arboreal decomposabiliy of the group of automorphisms of a free group. Algebra i Logika, 26 (1987), no. 2, 131-149.

[Bow] B. Bowditch. Treelike structures arising from continua and convergence groups. Memoirs Amer. Math. Soc. 662 (1999).

[BC] B. Bowditch, J. Crisp. Archimedean actions on median pretrees. Math. Proc. Cambridge Philos. Soc. 130 (2001), no. 3, 383–400.

[C] I. M. Chiswell. Generalised trees and Λ–trees. In Combinatorial and geometric group theory, London Math. Soc. Lecture Notes Series, no. 204, ed. A. J. Duncan, N. D. Gilbert, J. Howie (Cambridge University Press, 1995), pp. 43–55.

[CV] Marc Culler, Karen Vogtmann. A group-theoretic criterion for property FA. Proc. Amer. Math. Soc. 124 (1996), no. 3, 677–683.

[Dah] F. Dahmani. Accidental Parabolics and Relatively hyperbolic Groups, to appear in Isr. J. Math., 2006; preprint arXiv:math.GR/0210421.

[DP] T. Delzant, and L. Potyagailo. Endomorphisms of Kleinian groups. Geom. Funct. Anal. 13 (2003), no. 2, 396–436.

[DS] C. Druţu, M. Sapir. Tree-graded spaces and asymptotic cones of groups. With an appendix by Denis Osin and Mark Sapir. Topology 44 (2005), no. 5, 959–1058.

[Dun_1] M.J. Dunwoody. An inaccessible group. Geometric group theory, Vol. 1 (Sussex, 1991), 75-78, London Math. Soc. Lecture Note Ser., 181, Cambridge Univ. Press, Cambridge, 1993.

[Dun_2] M. J. Dunwoody. A small unstable action on a tree. Math. Res. Lett. 6 (1999), no. 5-6, 697–710.

[Fa] B. Farb. Relatively hyperbolic groups. Geom. Funct. Analysis **8** (1998), 810–840.

[Gr] M. Gromov. Asymptotic Invariants of Infinite Groups. Geometric Group Theory(vol. 2), G. A. Niblo, M. A. Roller (eds), Proc. of the Symposium held in Sussex, LMS Lecture Notes Series 181, Cambridge University Press 1991.

[Gro_1] D. Groves. Limit groups for relatively hyperbolic groups, I: The basic tools, preprint, arXiv:math.GR/0412492

[Gro_2] D. Groves. Limits of (certain) CAT(0) groups, II: The Hopf property and the shortening argument. preprint, arXiv:math.GR/0408080.

[Gro_3] D. Groves. Limit groups for relatively hyperbolic groups, II: Makanin-Razborov diagrams. Geometry and Topology 9 (2005), 2319–2358.

[Gui] V. Guirardel. preprint 2005.

[H] F. Hausdorff. Set theory. Second edition. Translated from the German by John R. Aumann et al Chelsea Publishing Co., New York 1962.

[K] M. Kapovich. Hyperbolic manifolds and discrete groups. Progress in Mathematics, 183. Birkhäuser Boston, Inc., Boston, MA, 2001. xxvi+467 pp. ISBN: 0-8176-3904-7

[KKL] M. Kapovich, B. Kleiner, B. Leeb, Quasi-isometries and the de Rham decomposition. Topology 37 (1998), no. 6, 1193–1211.

[KlL] B. Kleiner, B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. Publ. Math. IHES 86 (1997), 115–197.

[Kou] M. Koubi. Croissance uniforme dans les groupes hyperboliques. Ann. Inst. Fourier 48 ((1998), no.5 , 1441–1453.

[Lev] G. Levitt. Non-nesting actions on real trees. Bull. London Math.Soc. 30 (1998), 46–54.

[MO] J. C. Mayer and L. G. Oversteegen. A topological characterization of $\mathbb{R}$–trees. Trans. Amer. Math. Soc. 320 (1990), 395–415.

[N] B.H. Neumann. Groups with finite classes of conjugate elements. Proc. London Math. Soc. 1 (1951), 178–187.

[OP] K. Ohshika, L. Potyagailo. Self-embeddings of Kleinian groups. Ann. Sci. École Norm. Sup. 31 (1998), 329–343.

[Os] D.V. Osin. Relatively hyperbolic groups: Intrinsic geometry, algebraic properties and algorithmic problems. to appear in Memoirs of Amer. Math. Soc.; preprint arXiv:math.GR/0404040.

[Pau] F. Paulin. Topologie de Gromov équivariante, structures hyperboliques et arbres réels. Invent. Math. 94 (1988), 53–80.

[RS] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I, Geom. Funct. Anal. 4 (1994), no. 3, 337–371.

[Sel_1] Z. Sela. Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II., Geom. Funct. Anal. 7 (1997), no. 3, 561–593.

[Sel_2] Z. Sela. Acylindrical accessibility for groups. Invent. Math. 129 (1997), no. 3, 527–565.

[Sel_3] Z. Sela. Endomorphisms of hyperbolic groups. I. The Hopf property. Topology 38 (1999), no. 2, 301–321.

[Sel_4] Z. Sela. Diophantine geometry over groups. I. Makanin-Razborov diagrams. Publ. Math. Inst. Hautes tudes Sci. No. 93 (2001), 31–105.

[Ser] J. P. Serre. Trees. Springer, New York, 1980.

[Th] W. Thurston. The geometry and topology of 3-manifolds. Princeton University Press, 1978.

[Ti] J. Tits. A "theorem of Lie-Kolchin" for trees. In: "Contributions to algebra: a collection of papers dedicated to Ellis Kolchin", eds. H. Bass et al, Academic Press 1977, 377–388.

[Tu] P. Tukia. Convergence groups and Gromov's metric hyperbolic spaces. New Zealand J. Math. **23** (1994), 157–187.

[VDW] L. van den Dries, A. J. Wilkie. On Gromov's theorem concerning groups of polynomial growth and elementary logic. J. of Algebra **89** (1984), 349–374.

[Xie] X. Xie. Growth of relatively hyperbolic groups, preprint, arXiv:math.GR/0504191.

Cornelia Druţu
Department of Mathematics
University of Lille 1
and UMR CNRS 8524
Cornelia.Drutu@math.univ-lille1.fr

Mark V. Sapir
Department of Mathematics
Vanderbilt University
m.sapir@vanderbilt.edu