Algebraic Invariants For Finite Group Actions I: Varieties

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To Sir Michael Atiyah On His Sixtieth Birthday

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Beweis: Das folgt offensichtlich aus den Definitionen und der vorhergehenden Proposition. *q.e.d.*

Lemma 1 *Sei $M \in \mathcal{D}_T(\lambda)$. Es gibt eine Multimenge $P(M) \subset \mathbf{h}^*$ so daß für alle $\varepsilon \in \mathbf{h}^* \cap SpecT$ der $\mathbf{g}$-Modul $M \otimes_T \mathbb{C}_\varepsilon$ eine Filtrierung mit Subquotienten $M(\mu + \varepsilon), \mu \in P(M)$ hat.*

Beweis: Für $M \in \mathbf{g} \otimes T\text{-}mod^e$ und $\nu \in \mathbf{h}^*$ definiere ich den $\mathbf{h}$-Gewichtsraum $M^\nu = \{v \in M | Xv = (X + \nu(X))v \forall X \in \mathbf{h}\}$, wo die erste Multiplikation mit $X \in \mathbf{h} \subset \mathbf{g}$ aufzufassen ist, die zweite jedoch mit $(X + \nu(X)) \in S$. Jedes $M \in \mathcal{D}_T(\lambda)$ zerfällt in Gewichtsräume, die über T lokal frei sind von endlichem Rang. Den Rest des Beweises überlasse ich dem Leser. *q.e.d.*

Sei speziell $R = S_{(0)}$ die Lokalisierung von S an der Stelle $0 \in \mathbf{h}^* \subset SpecT$. Es gibt genau einen einfachen R-Modul $\mathbb{C}_0 = \mathbb{C}$.

Proposition 4 *Seien $M, N \in \mathcal{D}_R(\lambda)$. Liefert $\phi : M \to N$ einen Isomorphismus auf der Nullfaser $M \otimes_R \mathbb{C} \to N \otimes_R \mathbb{C}$, so ist ϕ schon selbst ein Isomorphismus.*

Beweis: Man kann jedes Objekt von $\mathcal{D}_R(\lambda)$ in $\mathbf{h}$-Gewichtsräume zerlegen. Diese sind freie R-Moduln von endlichem Rang. Das Lemma von Nakayama beendet den Beweis. *q.e.d.*

Proposition 5 *Zu jedem $M \in \mathcal{D}_R(\lambda)$ gibt es eine Lokalisierung T von S nach einem Element und $\tilde{M} \in \mathcal{D}_T(\lambda)$, so daß $T \subset R$ und $\tilde{M} \otimes_T R \cong M$.*

Beweis: Die Projektoren einer Zerlegung von $E \otimes M_R(\lambda)$ in eine direkte Summe "leben" nach Proposition 2 auf einer offenen affinen Umgebung $U = SpecT$ von $0 \in \mathbf{h}^*$. *q.e.d.*

Wir können in diesem Zusammenhang auch Verschiebungsfunktoren einführen. Zunächst bemerken wir, daß $E \otimes M_T(\lambda) = E \otimes (\mathbf{U} \otimes_{\mathbf{b}} (\mathbb{C}_\lambda \otimes T)) = \mathbf{U} \otimes_{\mathbf{b}} (E \otimes \mathbb{C}_\lambda \otimes T)$ eine Kompositionsreihe mit Faktoren $M_T(\lambda + \nu)$ hat, wo ν die Gewichte von E mit Multiplizitäten durchläuft.

Weiter betrachten wir den Träger $Supp(M_R(\mu))$ von $M_R(\mu)$ in $Spec(Z \otimes R)$. Man erkennt, daß $Supp(M_R(\mu)) \cap Supp(M_R(\eta)) \neq 0 \Leftrightarrow \mathcal{W} \cdot \mu = \mathcal{W} \cdot \eta$. Für alle $M \in \mathcal{D}_R(\lambda)$ induziert die Zerlegung von $SuppM$ in Zusammenhangskomponenten eine Zerlegung von M in eine direkte Summe.

Introduction. The purpose of this paper and its sequel is to provide an algebraic framework for the cohomological study of finite group actions on spaces and chain complexes which are possibly infinite dimensional but with finitely generated total cohomology. In Part I, we introduce and study a new algebraic invariant for G–spaces and G–chain complexes (where G is a finite group) through the theory of varieties in modular representation theory. This invariant, called "the rank variety family" of a G–space (or a G–chain complex) is an algebraic substitute for the family of isotropy subgroups of a finite dimensional G–space. In Part II, we apply the theory of coherent algebraic sheaves to further study those aspects of the theory which involve the notion of an "orbit space". Some of the results of this paper have been announced in [6].

The traditional concepts and tools of finite transformation groups which have been successfully applied to study finite dimensional G–spaces (e.g. manifolds and CW complexes) are not sufficient for the cohomological study of infinite dimensional G–spaces and G–chain complexes. Here, tools and the language of abstract algebraic geometry provide us with a natural and suitable medium to enlarge the domain of study of transformation groups. Such generalizations are natural, and they arise in studying function spaces, homotopy actions, symmetries of varieties (e.g. defined over a field of positive characteristic) and modular representation theory to mention a few. We refer the reader to [1] [2] [5] [6] [7] [8] [9] [12] [19] for further motivation and discussion in this direction. We have also provided some applications to classical finite dimensional problems (Sections 6–8) in order to illustrate the usefulness and flexibility of this algebraic approach to the subject.

The main algebric tool in Part I comes from some recent achievements in modular representation theory, namely the theory of varieties developed by J. Carlson [23] [24] and Avrunin–Scott [17]. This theory is in turn inspired and built on the works of Borel [20], Atiyah–Segal [15], Quillen [40] [41], E. Dade [29], G. Avrunin [16], Alperin–Evens [3] and O. Kroll [38] among others. In the context of restricted Lie algebras, there has been a parallel theory developed by Friedlander–Parshall [31] and Janzen [36] [37].

We briefly mention the role of this theory in what follows. Let G be a finite group and p be a prime divisor of $|G|$, k an algebraically closed field of characteristic p, and M a finitely generated kG–module. For $E \cong (\mathbb{Z}_p)^n \subseteq G$, Dade [29] introduced certain well–behaved cyclic subgroups of the group algebra kE (called "shifted cyclic subgroups") which detect kE–projectivity of M. To measure the deviation from kE–projectivity of M, Jon Carlson [23] associated a homogeneous affine k–variety $V_E^r(M)$ which "parametrizes" those shifted cyclic subgroups of kE which do not act freely on M. There is also a cohomological support variety $V_E(M)$ associated to M. This was inspired by Quillen's work [40] [41] in transformation groups and group cohomology and is defined to be the support of the module $H^*(E;M)$ in the maximal spectrum Max H_E, where $H_E = \bigoplus_{i \geq 0} H^{2i}(E;k)$. For the trivial module k, there is a natural identification $V_E(k) = V_E^r(k)$. Carlson [23] showed that $V_E^r(M) \subseteq V_E(M)$ with $\dim V_E(M) = \dim V_E^r(M)$, and conjectured their equality. This conjecture was proved by Avrunin–Scott [17]. They used it in conjunction with Green's theory of vertices and sources [28] [32] to prove that for a general finite group G, and a kG–module M, a suitably defined cohomological support variety $V_G(M)$ possesses a Quillen stratification [17]. That is, $V_G(M)$ is obtained from gluing together the collection $V_E(M)$, where E ranges over p–elementary abelian subgroups of G. A similar stratification is demonstrated by S. Jackowski [35] for the equivariant cohomology with local coefficients, reproving Avrunin–Scott's stratification theorem along Quillen's original topological approach. See Section One for definitions and precise statements.

In our context, the above theory is interpreted and applied as follows. The "shifted subgroups of kG" enlarge the notion of "subgroups of G". A suitably defined rank variety $V_E^r(X)$ (or $V_E^r(X_*)$ for a kG–complex X_*) parametrizes the shifted cyclic subgroups of kG which (algebraically) behave like elements of E with non–empty fixed point sets. The cohomological support variety $V_E(X)$ using equivariant cohomology $H^*_G(X;k)$ (or its algebraic version $\mathbb{H}^*(G;X^*)$) is related to the rank variety in much the same way as for kG–modules: $V^r_E(X) \cong V_E(X)$. See Section Two. The theory is extended to arbitrary finite groups G via the above mentioned stratification theory. The interplay between the cohomological variety and the rank variety allows us to formulate and prove a suitable algebraic analogue for The Localization Theorem in equivariant cohomology of finite dimensional G–spaces. The topological form of this theorem is due to Borel [20], Quillen [40], and W.Y. Hsiang [34] in cohomology, and to Atiyah–Segal [15] in equivariant K–theory. See Section Three below for precise statements and a discussion. The most basic invariant of a variety is its (Krull) dimension, and this leads to the notion of complexity. In Section Three, Alperin's complexity theory [3] is adapted to this context, and it is used in Section Seven to apply the above–mentioned algebraic localization to topological circumstances.

The organization and contents of the paper are as follows. In Section One, we have collected some concepts and background material from the theory of varieties in modular representation theory, stratification theory, and more specialized aspects of transformation groups. Also, much of the notations and conventions are introduced. The heading of each subsection is intended to help the reader to choose only the needed definition or discussion. The basic notion of varieties are introduced in Section Two, and studied in Section Three via localization, complexity and the role of shifted subgroups. We found it useful (also for the sake of future reference) to include a comparison of the above varieties with their finite dimensional predecessors, namely the Quillen variety [40] and the support variety defined and studied by Jackowski [35]. The analogy between "the rank variety family" and the "family of p–elementary abelian isotropy subgroups" of a finite dimensional G–space (alluded to in the above) is made explicit in Section Five. The material of these sections are applied in Section Six to some problems regarding finite dimensionality and finite domination of spaces with finite fundamental group. Further applications in this direction appear in [7] [9] and [11]. Section Seven contains applications to a very old and classi–cal area of topological transformation groups. It contains an extension of the Borsuk–Ulam Theorem to infinite dimensional G–spaces and chain complexes, as well as a new and simple proof of a generalized form (for p–elementary abelian groups) in finite dimensions. Recall that the classical Borsuk–Ulam Theorem states that there are no cohomologically essential maps from the real porjective space $\mathbb{R}P^n$ for $m > n$. Considering S^m and S^n equipped with $\mathbb{Z}_2$–actions, another formulation of the Borsuk–Ulam theorem is that if there exists an equivariant map $f : S^m \longrightarrow S^n$ with $m > n$, then the fixed point set $(S^n)^{\mathbb{Z}_2} \neq \phi$. We have the following generalization to infinite dimensions. The statement $V_E^r(Y) = V_E^r(k)$ is the algebraic analogue of " $Y^E \neq \phi$ " when $\dim Y < \infty$.

<u>Theorem</u>. Let G be a finite group, and let X and Y be G–spaces (possibly infinite dimensional) with finitely generated homology, and let $f : X \longrightarrow Y$ be an equivariant map. Suppose for some $p \mid |G|$, $H_i(X,\mathbb{F}_p) = 0$ for $i \leq n$, and $H_i(Y,\mathbb{F}_p) = 0$ for $i > n$. Then $V_E^r(Y) = V_E^r(k)$ for all $E \in \mathscr{E}$.

The finite dimensional version has a more familiar statement:

<u>Corollary</u>. In the above theorem, assume that $G \cong (\mathbb{Z}_p)^m$ and $\dim(Y) < \infty$ in addition to the hypotheses on homologies of X and Y. Then there exists an equivariant map $f : X \longrightarrow Y$ if and only if $Y^G \neq \phi$.

Also, one may relate in this way the growth rate of the equivariant Betti numbers of Y to the existence of equivariant maps $f : X \longrightarrow Y$ as in the corollary below:

Corollary. Suppose that X and Y are G–spaces (possibly infinite dimensional) as in the above theorem. If the growth rate of dim $H_G^*(Y;\mathbb{F}_p)$ is less than the maximal rank of the elementary abelian subgroups of G , then there are no equivariant maps $f : X \longrightarrow Y$.

As a final application in this paper, we have considered an algebraic generalization of a conjecture of Conner and Floyd [26] [27] regarding the non–existence of periodic diffeomorphsms of odd prime power order of oriented closed connected n–manifolds with one single fixed point. This conjecture was proved first by Atiyah and Bott [13] (and hence known as the Atiyah–Bott theorem in literature) using their Lefschetz–type fixed point formula for elliptic complexes, and by Conner and Floyd [27] via equivariant cobordism theory methods. The generation of this conjecture to abelian p–group actions on smooth manifolds is due to W. Browder [21]. See Section 8 for further remarks and precise statements.

Further References and Acknowledgement. Since the completion of an earlier version of this work in 1985–86, there has been further developments which are related to this paper. A. Adem has extended independently the notion of complexity and also applied shifted cyclic subgroups to study group actions [1] [2], see also 3.7 (b) below. M. Özaydin has also obtained some generalizations of the Borsuk–Ulam Theorem for finite dimensional G–spaces (Section Seven) which will appear in his forthcoming paper. D. Puppe and V. Puppe have informed the author that they give different proofs of Corollary 7.4. There are also related works of A. Dold [47], Fadell–Husseini–Rabinowitz [48], Fadell–Husseini [49], and C.T. Yang [50] on generalizations of the Borsuk–Ulam Theorem among the extensive literature in this direction. Frank Quinn has also observed that the results of the author [5] lead to a finite domination criterion similar to Corollary 6.4.

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M. Özaydin for inspiring and informative conversations. Special thanks to the referee for helpful comments and corrections which has led to an improved exposition.

Section One: Preliminaries

1.1. Notation and Conventions. G denotes a finite group of order $|G|$, and p is a prime number dividing the order of G. $\mathbb{F}_p$ is the field with p–elements and $k = \overline{\mathbb{F}}_p$ is an algebraic closure. The cyclic group with p–elements is denoted by $\mathbb{Z}_p$, and a p–elementary belian group of rank n $E \cong (\mathbb{Z}_p)^n$ with generators $\{x_1,\dots,x_n\}$ is also denoted by $E = \langle x_1,\dots,x_n\rangle$. Then $rk(E) \overset{def}{=} rank(E) = n$, $rk_p(G) \overset{def}{=} \max\{n : (\mathbb{Z}_p)^n \subseteq G\}$, and $rk(G) = \max\{rk_p(G) : p \text{ divides } |G|\}$. The collection of p–elementary abelian subgroups of G is denoted by $\mathcal{E}$, or if emphasis on p is needed, by $\mathcal{E}_p$ or $\mathcal{E}_p(G)$. We may consider a category whose set of objects is $\mathcal{E}$ and whose morphisms are inclusions of subgroups and inner automorphisms of G. By a slight abuse of notation, this category is also denoted by $\mathcal{E}$ (or $\mathcal{E}_p(G)$ appropriately). Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{E} \longrightarrow \mathcal{C}$ be a functor. Then the inductive (or direct) limit of $\mathcal{F}$ is denoted by $\lim\operatorname{ind}_{E \in \mathcal{E}} \mathcal{F}(E)$. For a G–space X, X^G is the fixed point set of G, and $H^*_G(X)$ is the Borel–equivariant cohomology, cf. [34]. If C^* is a cochain complex with G–action, $\mathbb{H}^*(G;C^*)$ denotes the cohomolog of G with coefficients in C^* (cf. Brown [22] or Cartan–Eilenberg [25] who call it hypercohomology).

Throughout this paper, all G–spaces and G–chain complexes are required to have finitely generated total homology with appropriate coefficients. Chain and cochain complexes are often assumed to be connected, i.e. positively graded and their zero–degree homology is equal to the coefficient ring. Whenever needed, such complexes will be augmented and chain maps will be augmentation preserving of degree zero. Further notation and terminology will be introduced in the following paragraphs.

1.2. Equivariant Cohomology. Let $E_G \longrightarrow BG$ be the universal principal G–bundle, and let X be a G–space. The Borel construction $E_G \times_G X \xrightarrow{\pi} BG$ is the associated fibre bundle with fibre X . The notation $X_G \equiv E_G \times_G X$ is often used. For a commutative ring R with identity, the Borel equivariant cohomology with coefficients in R is $H_G^*(X;R) \overset{\text{def}}{=} H^*(E_G \times_G X;R)$. Using $\pi^* : H^*(G;R) \longrightarrow H^*(E_G \times_G X;R)$, we may regard $H_G^*(X;R)$ as a graded $H^*(G;R)$–module, where $H^*(G;R) \equiv H^*(BG;R)$. It is well–known that $H^*(G;R)$ is a finitely generated R–algebra and $H_G^*(X;R)$ is a finitely generated graded $H^*(G;R)$–module whenever $\bigoplus_{i=0}^{\infty} H^i(X;R)$ is finitely generated. Similar remarks apply to a pair of G–spaces (X,Y) .

<u>1.3. Equivariant Cohomology for Chain Complexes (Hypercohomology)</u>. Now assume that C^* is an RG–cochain complex and Q^* is an injective resolution of R over RG . In analogy with G–spaces, we form the double complex $Q^* \otimes_G C^*$ and define the cohomology of G with coefficients in C^* : $H^*(G;C^*) \overset{\text{def}}{=} H^*(Q^* \otimes_G C^*) \equiv$ cohomology of the associated total complex. In the terminology of Cartan–Eilenberg [25], $H^*(G;C^*)$ is called "the hypercohomology" of G with coefficients in C^* . See also Brown [22] for further details. We may take a projective resolution P_* of R over RG , and form the hypercohomology of an RG–chain complex C_* similarly: $H_*(G;C_*) \overset{\text{def}}{=} H_*(P_* \otimes C_*)$. These notions are the suitable algebraic analogues of equivariant homology and cohomology, and as such, they enjoy similar properties. Just as in the case of G–spaces, $H^*(G;C^*)$ is a module over $H^*(G;R)$. In fact, the usual spectral sequence $E_2^{i,j} = H^i(G;H^j(C^*))$ converging to $H^*(G;C^*)$ is the analogue of the Serre spectral sequence for the fibration $E_G \times_G X \longrightarrow BG$ with a similar construction. In our situation, $\bigoplus_i H^i(C^*)$ is finitely generated over R . Hence $H^*(G;C^*)$ is a finitely generated $H^*(G;R)$–module using the above spectral sequence. It is useful to remark that if M is a finitely generated RG–module, and we regard M as an RG–cochain complex $\mathcal{M}^*$ concentrated in one dimen-

sion only, then $H^*(\mathcal{M}) = M$ and $H^*(G; \mathcal{M}^*) \cong H^*(G;M)$ ($\equiv$ cohomology of G with coefficients in M) with a suitable shift in dimension. Thus, $H^*(G;C^*)$ is also a generalization of the usual group cohomology $H^*(G;M)$ with twisted coefficients. Part of our task will be to study this two–fold generalization of G–spaces and G–modules to G–chain complexes from both geometric and algebraic view–points.

1.4. Supports. Let A be a commutative ring with identity and let M be a finitely generated A–module. The set of maximal ideals of A , denoted by Max A , together with the Zariski topology is the subspace of closed points of Spec A . The support of M in Spec(A) is defined as usual: $\mathrm{supp}(M) = \{\, \mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$, and $\mathrm{Max\,supp}(M) \overset{\mathrm{def}}{=} \mathrm{Max}(A) \cap \mathrm{Supp}(M)$, where $M_{\mathfrak{p}}$ is the localization of M at the ideal $\mathfrak{p}$. Let $\mathrm{ann}_A(M) \subseteq A$ be the annihilating ideal of M . Then supp(M) and Max supp(M) are equivalently defined as "the varieties" determined by the ideal $\mathrm{ann}_A(M)$ in Spec(A) and respectively Max(A) . See Kunz [39] and Atiyah–Mac Donald [14]. Max supp(M) is often abbreviated to supp(M) as well.

Recall that "reduction by the radical" $A \twoheadrightarrow A_{red} \overset{\mathrm{def}}{=} A/\mathrm{Radical}$ induces a homeomorphism of topological spaces $\mathrm{Spec}(A_{red}) \xrightarrow{\cong} \mathrm{Spec}(A)$ and $\mathrm{Max}(A_{red}) \xrightarrow{\cong} \mathrm{Max}(A)$. This homomorphism respects the supports of modules as well, i.e. considering $M_{red} \overset{\mathrm{def}}{=} M \otimes_A A_{red}$ as an A_{red}–module, we have $\mathrm{supp}(M_{red}) \cong \mathrm{supp}(M)$ and $\mathrm{Max\,supp}(M_{red}) \cong \mathrm{Max\,supp}(M)$. These comments apply to our situation as follows. The ring $H^*(G;R)$ is graded commutative, and we will consider the strictly commutative ring $\bigoplus_{i=0}^{\infty} H^{2i}(G;R)$ when $\mathrm{char}(R) \neq 2$. Define H_G to be

$\bigoplus_{i\geq 0} H^{2i}(G;k)$ when $p = \mathrm{char}(k) = \mathrm{odd}$ and $H_G = \bigoplus H^i(G;k)$ if $p = 2$. Reduction modulo radical does not affect the supports, hence we may use the unreduced ring H_G instead of $(H_G)_{red} = H_G/\mathrm{Radical}$ which is commonly used in the literature (e.g. Quillen [40] or Benson [18]). In fact, even for $p = 2$ we may use $H_G^{ev} \overset{def}{=} \bigoplus_{i\geq 0} H^{2i}(G;k)$ instead of $\bigoplus_{i\geq 0} H^i(G;k)$ and obtian the same answers. However, we will not insist on this technicality and follow the established notation Avrunin–Scott [17], Jackowski [35], Quillen [40], Carlson [23] [24].

<u>1.5. The Cohomological Variety of Elementary Abelian p–Groups.</u> Let $E \cong (\mathbb{Z}_p)^n = \langle x_1,\dots,x_n\rangle$, and recall that $H^*(E;\mathbb{F}_p) \cong \Lambda(u_1,\dots,u_n) \otimes \mathbb{F}_p[t_1,\dots,t_n]$ for $p = \mathrm{odd}$, and $H^*(E;\mathbb{F}_2) \cong \mathbb{F}_2[t_1,\dots,t_n]$ otherwise (using the Künneth formula and the explicit computation of $H^*(\mathbb{Z}_p;\mathbb{F}_p)$). Here, $\Lambda(u_1,\dots,u_n)$ is the exterior algebra associated to the vector space $H^1(E;\mathbb{F}_p)$ whose basis $\{u_1,\dots,u_n\}$ is determined by the choice of generators $\{x_1,\dots,x_n\} \subset E$. Namely, u_i is dual to x_i when we consider E as an $\mathbb{F}_p$–vector space of dimension n with basis $\{x_1,\dots,x_n\}$ and identify $H^1(E;\mathbb{F}_p) \equiv \mathrm{Hom}_{\mathbb{Z}_p}(E,\mathbb{Z}_p)$. Moreover, the Bockstein $\beta: H^1(E;\mathbb{F}_p) \longrightarrow H^2(E;\mathbb{F}_p)$ is an isomorphism and $\beta(u_i) = t_i$. It follows that the radical of $H^*(E;\mathbb{F}_p)$ is generated by $H^1(E;\mathbb{F}_p)$ for $p = \mathrm{odd}$, and it is zero for $p = 2$. Thus changing rings from $\mathbb{F}_p$ to k yields $H_E^{ev}/\mathrm{Radical} \cong k[t_1,\dots,t_n]$ and the radical of H_E^{ev} is complementary to the subalgebra $k[t_1,\dots,t_n]$. Moreover, $\mathrm{Max}\, H_E = \mathrm{Max}(H_E)_{red}$ is isomorphic to the n–dimensional affine space k^n over k, and we may regard $k[t_1,\dots,t_n]$ as its coordinate ring.

Following Avrunin–Scott [17], we denote $\mathrm{Max}\, H_E$ by V_E, and proceed to describe V_E in terms of the group algebra kE. Let $0 \longrightarrow J \longrightarrow kE \overset{\epsilon}{\longrightarrow} k \longrightarrow 0$ be the augmentation sequence, and $0 \longrightarrow J^2 \longrightarrow J \overset{s}{\underset{\longrightarrow}{\leftarrow}} L \longrightarrow 0$ be a splitting of $J^2 \subset J$ as k–vector spaces where we will identify L and $s(L)$ so that $J = J^2 \oplus L$. It will be convenient to choose $L \equiv$ k–vector space generated by $\{x_1-1,\dots,x_n-1\}$ for $E = \langle x_1,\dots,x_n\rangle$. Since $H^1(E,k) \cong \mathrm{Hom}_k(J/J^2,k) \cong \mathrm{Hom}_k(L,k)$ and $\beta \otimes 1_k : H^1(E,k) \xrightarrow{\cong} H^2(E,k)$ is a natural isomorphism, we obtain a natural identification of affine k–spaces: $L \cong \mathrm{Max}(H_E)_{red} \cong \mathrm{Max}\, H_E = V_E$ for $p =$ odd and similarly for $p = 2$ where $(H_E)_{red} = H_E$.

1.6. Shifted Subgroups. Next, we discuss "shifted cyclic subgroups of kE", introduced by E. Dade [29] and subsequently used by J. Carlson [23] [24] in his theory of rank varieties. We will continue to follow Avrunin–Scott [17] in our exposition below, and refer the reader to [18] [23] and [24] for further developments and details. Corresponding to each vector $\alpha = (\alpha_1,\dots,\alpha_n) \in k^n = J/J^n$, we consider the element $u_\alpha = 1 + \sum_{i=1}^{n} \alpha_i(x_i-1) \in kE$ which is seen to be a unit of order p, so that $\langle u_\alpha\rangle \cong \mathbb{Z}_p$. The subgroup $\langle u_\alpha\rangle$ is called a "shifted cyclic subgroup", and although it is not in general a subgroup of E, it behaves very much like one. For example, kE is a free $k\langle u_\alpha\rangle$–algebra, and induction and restriction of representations, as well as Mackey's formula hold as in the case of genuine subgroups of E. Choosing m linearly independent vectors $\alpha^{(1)},\dots,\alpha^{(m)} \in k^n$, there is a corresponding subgroup $\langle u_{\alpha^{(1)}},\dots,u_{\alpha^{(m)}}\rangle$ which is isomorphic to $(\mathbb{Z}_p)^m$, and it is called a "shifted subgroup of rank m". Just as in the above kE is a free $k\langle u_{\alpha^{(1)}},\dots,u_{\alpha^{(m)}}\rangle$ – module. The choice of a vector $\alpha \in L$ leads to the inclusions $\rho_\alpha : k\langle u_\alpha\rangle \longrightarrow kE$, $\rho'_\alpha : k\{\alpha\} \longrightarrow L$ and projections $H^*(E;k) \xrightarrow{\pi_\alpha} H^*(\langle u_\alpha\rangle;k)$ and $H_E \xrightarrow{\pi'_\alpha} H_{\langle u_\alpha\rangle}$. Let $\tau_\alpha : \mathrm{Max}\, H_{\langle\alpha\rangle} \longrightarrow \mathrm{Max}\, H_E$ be induced by π'_α. Then the naturality of the above–mentioned isomorphisms and identifications yields the following commutative diagram:

$$\begin{array}{ccccccccc}
kE & \longleftarrow & L & \cong \mathrm{Max}(H_E)_{red}) & \cong \mathrm{Max}\, H_E & = V_E & \\
\rho_\alpha \uparrow & & \rho'_\alpha \uparrow & & & \uparrow \tau_\alpha & (*) \\
k<u_\alpha> & \longleftarrow & k\{\alpha\} & \cong \mathrm{Max}(H_{<u_\alpha>})_{red} & \cong \mathrm{Max}\, H_{<u_\alpha>} & = V_{<u_\alpha>} &
\end{array}$$

In analogy with our previous notation, we will use the notation $(H_{<u_\alpha>})_{red} = k[t_\alpha]$ to indicate this correspondence on the level of cohomology.

1.7. Varieties for Modules Now suppose M is a finitely generated kE–module. Let $S \subset kE$ be a shifted subgroup of rank m corresponding to the m–dimensional linear subspace $k\{\alpha^{(1)},\ldots,\alpha^{(m)}\} = L_0 \subset L$ as in the above paragraph. "The cohomological support of M" is defined via $V_E(M) \overset{def}{=} \mathrm{Max\ supp}_{H_E}(H^*(E;M))$ and similarly $V_S(M) \overset{def}{=} \mathrm{Max\ supp}_{H_S}(H^*(S;M|S))$. Then the induced homomorphism $H_E \longrightarrow H_S$ induces the morphism $\tau_{S,E} : V_S \longrightarrow V_E$ such that $\tau_{S,E}(V_S(M)) = V_E(M)$. On the level of group algebra, J. Carlson [23] [24] has defined an algebraic analogue of support for M, which he calls "the rank variety of M" and denotes by $V^r_E(M)$. The definition of the rank variety appears at first to depend on the choice of L. Namely, "Carlson's rank variety of M with respect to L" is the set $\{\alpha \in L \mid M|_{<u_\alpha>}$ is not a free $k<u_\alpha>$–module$\} \cup \{0\} \subseteq L$ together with the induced Zariski topology of L. However, according to Carlson ([23] Lemma 6.4), if $a,b \in J \subset kE$ such taht $a-b \in J^2$, then $M|_{k<1+a>}$ is $k<1+a>$–free if and only if $M|_{k<1+b>}$ is $k<1+b>$–free. Therefore, the above subvariety of L is isomorphic to $V^r_E(M) \overset{def}{=} \{\alpha \in k^n = J/J^2 \mid M|_{<u_\alpha>}$ is not free or $\alpha = 0\}$, which is well–defined and does not depend on the choice of L. $V^r_E(M)$ is a homogeneous affine subvariety of J/J^2 (or of L under our choice of identification), see Carlson [23]. Since $<u_\alpha> \cong \mathbb{Z}_p$, $M|_{<u_\alpha>}$ is $k_{<u_\alpha>}$–free if and only if

$H^i(<u_\alpha>; M|_{<u_\alpha>}) = 0$ for all $i > 0$. It follows that $V_{<u_\alpha>}(M) \cong V_{<u_\alpha>}$ if and only if $V^r_{<u_\alpha>}(M|_{<u_\alpha>})$ is the one–dimensional subspace $k\{u_\alpha\} \subset L$. Accordingly, the commutative diagram (*) and the above discussion show that under the identification $L \cong V_E$, the subset $V^r_E(M) \subseteq L$ corresponds to a subset of $V_E(M) \subseteq V_E$. The inclusion $V^r_E(M) \subseteq V_E(M)$ (due to Carlson [23]) was proved by Avrunin–Scott [17] to be an equality, thus confirming Carlson's conjecture. Note that $V^r_L(k) = L$ and $V_E(k) = V_E$ for the trivial E–module k. We rephrase slightly Carlson's conjecture for future reference. See Avrunin–Scott [17], and Carlson [24] for a different proof.

<u>1.7.1. Theorem ([17] Theorem 1.1)</u> Let M be a finitely generated kE–module where $E = (\mathbb{Z}_p)^n$. Under the identification $V^r_E(k) \cong V_E(k)$, we have $V^r_E(M) \cong V_E(M)$. (Further, these identifications are natural with respect to the inclusion of shifted subgroups of kE in the above sense.) The following result of Avrunin–Scott [17] will be also useful in the sequel.

<u>1.7.2. Proposition ([17] Corollary 1.3)</u>. Let S be a shifted subgroup of kE, and let $\tau_{S,E} : V_S \longrightarrow V_E$ be the induced morphism as in the above. For any finitely generated kE–module M, $\tau^{-1}_{S,E}(V_E(M)) = V_S(M)$.

See Jackowski ([35] Lemma 2.3) also for a related result.

<u>1.8. ω–Stability</u>. Two kG–modules M_1 and M_2 are called projectively stably isomorphic if there are kG–projective modules P_1 and P_2 such that $P_1 \oplus M_1 \cong P_2 \oplus M_2$ (kG–isomorphism). Projective stable isomorphism is an equivalence relation and the equivalence class of M is denoted by <M>. We define an operator ω on the set of projective stable equivalence classes as follows. Let M represent <M>, and choose a kG–projective module P and a short exact sequence of kG–modules $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$.

Then the projective stable equivalence class of N is well–defined by Schanuel's Lemma [18], and we set $<N> = \omega(<M>)$. For each $n \in \mathbb{Z}$, we define ω^n inductively via $\omega^1 = \omega$ and $\omega^{n+1} = \omega \circ \omega^n$, where $<M> = \omega^{-1}<N>$ from the above sequence. To avoid excessive notation, we write $\omega^n(M)$ instead of $\omega^n(<M>)$. By a slight abuse of notation, we call two kG–modules M_1 and M_2 ω–stably equivalent, if for some integers $n,m \in \mathbb{Z}$, $\omega^n(M_1) = \omega^m(M_2)$. This is also an equivalence relation for kG–modules. If we consider finitely generated kG–modules, then we require that all the projective kG–modules in the above definitions be finitely generated.

1.8.1. Lemma. Suppose M_1 and M_2 are ω–stably eqivalent. Then $V_E^r(M_1) = V_E^r(M_2)$ and $V_E(M_1) = V_E(M_2)$ for all $E \in \mathscr{E}$.

1.9. Localization in Equivariant Cohomology. Let X be a finite dimensional paracompact G–space and consider $E \in \mathscr{E}$ as in 1.5 above. The simplest form of the localization theorem, originally due to Borel [20] states that $H_E^*(X;\mathbb{F}_p)$ is isomorphic to $H_E^*(X^E;\mathbb{F}_p)$ modulo $H^*(E;\mathbb{F}_p)$–torsion.

1.9.1 The Localization Theorem. Let $t_E \in H^*(E;\mathbb{F}_p)$ be the product of all non–nilpotent elements of $H^i(E;\mathbb{F}_p)$ where $i = 1$ if $p = 2$ and $i = 2$ if $p > 2$, and let $S_E = \{t^m : m \in \mathbb{Z}\}$. Then for any pair of G–spaces (X,Y) with $\dim(X-Y) < \infty$ and $X^E \subseteq Y$, we have $S_E^{-1}H_E^*(X;R) \cong S_E^{-1}H_E^*(Y;R)$ where $R = \mathbb{F}_p$ or k. In particular, if $\dim X < \infty$ then $X^E = \phi$ if and only if $S_E^{-1}H_E^*(X;R) = 0$.

See [20] [40], and for a generalized version [34]. This simple statement has far–reaching consequences, but it fails to be true if $\dim X = \infty$. See Section Three below for further discussion.

1.10. The Steenrod Algebra. We need a brief discussion of the role of the Steenrod algebra in this context. Let $\mathscr{A}$ denote the Steenrod algebra of reduced power operations mod p [44]. It is convenient to ignore the Bockstein when $p > 2$, so that $\mathscr{A}$ is generated by $\mathscr{P}^i$, $i \geq 1$, subject to the Adem relations. Let $\mathscr{P} : H^*(X;\mathbb{F}_p) \longrightarrow H^*(X;\mathbb{F}_p)[t]$ be the total operation $\mathscr{P}(x) = x + \mathscr{P}^1(x)t + \ldots + \mathscr{P}^i(x)t^i + \ldots$. Then $\mathscr{P}$ is a ring homomorphism by the Cartan formula. If k is any field of characteristic p, the operation of $\mathscr{P}^i$ and $\mathscr{P}$ are defined on $H^*(X;k)$ by extending the scalars. The action of $\mathscr{P}$ on $H_E \subseteq H^*(BE;k)$ is quite simple to describe. Let $H_E = k[x_1,\ldots,x_m]$. Then $\mathscr{P}(x_i) = x_i + x_i^p t^p$. The following theorem of Serre characterizes $\mathscr{A}$ – invariant ideals of H_G.

<u>1.10.1. Theorem</u> (Serre [43]). Let $I \subset K[x_1,\ldots,x_m]$ be a homogeneous ideal, where $k = \mathbb{F}_p$. Suppose the ideal I is invariant under the transformation $x_i \longrightarrow x_i + x_i^p$. Then I is generated by products of $\mathbb{F}_p$–rational (i.e. with $\mathbb{F}_p$–coefficients) linear polynomials.

In [40], Quillen applied 1.10.1 to identify the prime ideals in $H_G(X)(k)$ which are $\mathscr{A}$ – invariant. See 1.11 below.

<u>1.11. Stratification</u>. For a paracompact connected G–space X with $\dim X < \infty$, Quillen considered the commutative ring $H_G(X) \overset{\text{def}}{=} H_G^*(X;\mathbb{F}_p)/\text{Radical}$ and defined the cohomological variety of X, denoted by $H_G(X)(k)$ to be the set of ring homomorphisms $H_G(X) \longrightarrow k$ endowed with the Zariski topology. One of the main results of [40] is a precise description of $H_G(X)(k)$ as a stratified set obtained by "gluing" the strata $H_E(X)(k)$ in a precise manner. We describe briefly the combinatorially simpler case where X^E are connected for all $E \in \mathscr{E}$. Define $\mathscr{E}(X) = \{E \in \mathscr{E} \mid X^E \neq \phi\}$ and consider it as a full subcategory of $\mathscr{E}$. The stratification theorem of Quillen is conveniently incapsulated as the inductive limit in the category of affine varieties:

$H_G(X)(k) = \lim_{E \in \mathscr{E}(X)} \text{ind}\; H_E(X)(k)$. An analogue of Quillen's theorem is extended by S. Jackowski [35] to the case of equivariant cohomology with local coefficients.

Following Avrunin–Scott [17], the cohomological variety $V_G(M)$ is defined for a finitely generated kG–module M as the union of the supports of $H^*(G;M \otimes L)$ in $\text{Max}\, H_G$ where L ranges over the (finite) set of isomorphism classes of simple kG–modules. Thus, $V_G(k) = \text{Max}\, H_G$ and $V_G(M)$ is a subvariety of $V_G(k)$. In particular, for $E \in \mathscr{E}$, $V_E(M)$ is the subvariety of $\text{Max}\, H_E \cong k^{rk(E)}$ defined by the annihilating ideal $\text{ann}_{H_E}(H^*(E;M))$. Avrunin–Scott [17] proved that $V_G(M)$ has a stratification similar to Quillen's stratification of $H_G(X)(k)$. Accordingly, $V_G(M) \cong \lim_{E \in \mathscr{E}} \text{ind}\; V_E(M)$. See Jackowski [35] also for a "topological" proof of this theorem. J. Carlson [24] has a technically different definition using the $\text{Ext}^*_E(k,k)$–module $\text{Ext}^*_E(M,M)$, but the identification $\text{Ext}^*_E(k,k) \cong H^*(E;k)$ leads to the same answer. See also G. Avrunin [16] for an ideal–theoretic treatment.

For a kG–complex X^* , one defines the cohomological support variety in a manner similar to the case of modules.

<u>1.11.1. Definition</u>. $V_G(X^*)$ is the union of the supports of the H_G–module $\mathbb{H}^*(G;X^* \otimes_k L)$, where L ranges over all simple kG–modules, in $\text{Max}\, H_G$. For a kG–complex X_* , $V_G(X_*) \overset{def}{=} V_G(\text{Hom}(X_*,k))$. If $f : Z_* \longrightarrow Y_*$ is a kG–chain homomorphism of degree zero, and X_* is the mapping cone of f , then $V_G(f) \overset{def}{=} V_G(X_*)$. There is a stratification for $V_G(X^*)$ just as for the case of modules [17] and finite dimensional G–spaces [35].

<u>1.11.2. Proposition</u>. Let X^* be a kG–complex. Then $V_G(X^*) \cong \lim_{E \in \mathscr{E}} \text{ind}\; V_E(X^*)$. Simi-

larly, if $f : Z_* \longrightarrow Y_*$ is a degree zero chain map of kG–complexes, then $V_G(f) = \lim\limits_{E \in \mathcal{E}} \operatorname{ind} V_E(f)$. See Sections Two and Four.

Section Two. Varieties.

In this section, we define rank varieties for G–spaces and kG–chain complexes, and we study their basic properties which will be used in the following sections and the sequel to this paper.

The geometric motivation for the introduction of varieties as a basic object in studying finite transformation groups is as follows. If a torus $(S^1)^n$ acts, say smoothly, on a compact manifold X , then the collection of the isotropy subgroups of this action is in one–to–one correspondence with certain $\mathbb{Q}$–rational linear subspaces of the Lie algebra $T_1(S^1)^n \cong \mathbb{R}^n$. Such subspaces may be determined Lie theoretically from the Lie algebras of vector fields, and such a linearization procedure reduces the questions about isotropy subgroups of $(S^1)^n$ to the appropriate statements regarding linear subgroups of $T_1(S^1)^n$. For the group $E = (\mathbb{Z}_p)^n$, the rank variety $V_E^r(k) \cong k^n$ plays the role of the Lie algebra of $(S^1)^n$. Indeed, the spectrum of the local ring kE has a unique point which corresponds to the augmentation ideal J in $0 \longrightarrow J \longrightarrow kE \longrightarrow k \longrightarrow 0$. The Zariski cotangent space to Spec(kE) at the point J is the k–vector space J/J^2 , and the Zariski tangent space is $\mathrm{Hom}_k(J/J^2,k) \cong H^1(E;k)$. Furthermore, the k–linear subspaces of J/J^2 give rise to shifted subgroups of kE , in close analogy with the case of the torus. For instance, just as $\mathbb{Q}$–rational linear subspaces of the Lie lagebra of T correspond to closed subgroups of T , so do $\mathbb{F}_p$–linear subspaces of J/J^2 to subgroups of E itself. However, there are far more shifted subgroups of kE than subgroups of E . Therefore, in purely algebraic situations, we need to enlarge the notion of "subgroup" to include "shifted subgroups". It turns out that shifted subgroups play an important role in the geometric situations as well. Through a systematic exploitation of the concept of rank and support varieties which keep track of "distinguished subspaces" of J/J^2 for the problem at hand, we can play the same game with shifted subgroups as for ordinary isotropy subgroups in the familiar case of finite dimensional G–spaces.

Since the definition and properties of "rank" and "cohomological varieties" for G–spaces and G–chain complexes are quite similar, we use the notation X, Y , etc. for G–spaces and X_*, Y_* or X^*, Y^* etc. for chain and cochain complexes with G–action. Subject to the assumptions of Section One, G–spaces could be fairly general. In fact, we apply the results about RG–chain comlexes to G–spaces by choosing suitable RG–chain complexes associated to the G–spaces in question. For topological applications, the singular chains, or when appropriate, simplicial or cellular chains are often sufficient. For more general spaces, Čech cochain complex etc. may be used equally effectively. In particular, for algebro–geometric applications when the varieties (or more generally schemes) are considered equipped with Zariski, étale, or any of the other numerous non–Hausdorff topologies, the Čech complex (suitably defined) is used. This point of view is illustrated in Assadi [12] and will be investigated systematically elsewhere.

In Assadi [4], we introduced the notion of "free equivalence" for G–spaces and permutation G–complexes as a useful technical device. This equivalence relation is particularly suited for defining varieties and related applications as it is illustrated in Assadi [5] [6] [7] [8].

2.1. Definition. Two (connected) RG–chain complexes X_* and Y_* are called "freely equivalent" if there exists a (connected) RG–chain complex Z_* and injective chain homomorphisms $X_* \xrightarrow{f} Z_*$ and $Y_* \xrightarrow{f} Z_*$ such that Z_*/X_* and Z_*/Y_* are RG–free chain complexes with finitely generated total complexes over RG . Similarly, free equivalence of (connected) G–spaces X and Y is defined by requiring the existence of a (connected) G–space Z and equivariant embeddings $X \longrightarrow Z$ and $Y \longrightarrow Z$ such that Z–X and Y–X are free G–spaces and Z/X and Z/Y are compact.

2.2. Remark. The hypotheses of finite generation and compactness in the above definition are for convenience, since in the applications below this will suffice. More generally, we may require only the finite generation of the total cohomology together with finite dimensionality of Z_*/X_* , Z/X , etc. and modify the following arguments accordingly. The following lemma is elementary.

2.3. Lemma. (a) Free equivalence is an equivalence relation. (b) In the category of G–CW complexes, if two G–CW complexes are freely equivalent, then their cellular RG–chain complexes are freely equivalent as RG–complexes. ∎

2.4. Proposition. Suppose X^* and Y^* are kG–complexes which are freely equivalent. For each $E \in \mathcal{E}$, let $S_E \subset H_E$ be a non–empty multiplicatively closed subset. Then there exist isomorphisms $\varphi_E : S_E^{-1}\mathbb{H}^*(E;X^*) \longrightarrow S^{-1}\mathbb{H}^*(E;Y^*)$. If the collection $\{S_E : E \in \mathcal{E}\}$ is chosen compatible with the morphisms of $\mathcal{E}$, (i.e. $E \longmapsto S_E$ is a contravariant functor of $\mathcal{E}$), then $\varphi \equiv \{\varphi_E \mid E \in \mathcal{E}\}$ describes a natural transformation between the functors $E \longmapsto S_E^{-1}\mathbb{H}^*(E;X^*)$ and $E \longmapsto S_E^{-1}\mathbb{H}^*(E;Y^*)$.

Proof: Without loss of generality, we may assume that S_E has no nilpotent elements. Further, the assertion of the proposition reduces to the case where we have a surjective kG–chain map $f : X^* \longrightarrow Y^*$ such that $\mathrm{Ker}(f)$ is a finitely generated bounded free kG–complex. The short exact sequence $0 \longrightarrow \mathrm{Ker}(f) \longrightarrow X^* \xrightarrow{f} Y^* \longrightarrow 0$ yields a long exact sequence in hypercohomology

$$\cdots\cdots \longrightarrow \mathbb{H}^i(E;\mathrm{Ker}(f)) \longrightarrow \mathbb{H}^i(E;X^*) \xrightarrow{f_E^*} \mathbb{H}^i(E;Y^*) \longrightarrow \cdots\cdots$$

as in the case of group cohomology (see Cartan–Eilenberg [25] or Brown [22]). Since localization is an exact functor, the first assertion of the proposition follows from the claim that $S_E^{-1}\mathbb{H}^*(E;\mathrm{Ker}(f)) = 0$. Setting $\varphi_E \overset{\mathrm{def}}{=} S_E^{-1}f_E^*$, the second assertion is easily verified due to functoriality of the above argument. Thus the following lemma finishes the proof of the proposition. ■

2.5. Lemma. Suppose that C^* is a bounded free kG–complex, and $E \in \mathcal{E}$ and S_E are as in the above proposition. Then $S_E^{-1}\mathbb{H}^*(E;C^*) = 0$.

Proof: First assume that H^*–length $(C^*) \overset{\mathrm{def}}{=} \{n : H^n(C^*) \neq 0\} = 1$. Consider the spectral sequence $H^*(E;H^*(C^*)) \Rightarrow \mathcal{G}_r(\mathbb{H}^*(E;C^*))$ in which the E_2^{**} , and hence all the E_i^{**} are H_E–modules. This spectral sequence degenerates when H^*–length $(C^*) = 1$, so that $\mathbb{H}^*(E;C^*) \cong H^*(E;H^*(C^*))$ with a suitable shift in the dimension. Moreover, it is easy to see that in this case $H^*(C^*)$ is also a free kE–module since C^* is bounded and kG–free (e.g. by splitting C^* into short exact sequences). Hence $S_E^{-1}\mathbb{H}^*(E;C^*) \cong S_E^{-1}H^*(E;H^*(C^*)) = 0$. Next, assume that H^*–length $(C^*) = m+1$. Then we can "kill the first non–vanishing cohomology" of C^* and obtain a short exact sequence of kG–complexes $0 \longrightarrow C^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0$ in which H^*–length $(B^*) = n$ and H^*–length $(A^*) = 1$. The long exact sequence of hypercohomology, localization, and induction on H^*–length (C^*) completes the proof of the lemma. ■

2.6. Corollary. Suppose X^* and Y^* are freely equivalent kG–complexes. Then:

(a) for each $E \in \mathcal{E}$, the support varieties $V_E(X^*) = V_E(Y^*)$.

(b) $V_G(X^*) = V_G(Y^*)$.

Proof: By definition, $V_E(X^*)$ is the support of the H_E–module $\mathbb{H}^*(E;X^*)$. (a) follows from Proposition 2.4 and elementary considerations about supports, (see Section One 1.4). (b) follows from (a), the functoriality assertion in Proposition 2.4, and the stratification theorem:

$$V_G(X^*) = \lim_{E \in \mathscr{E}} \text{ind}\, V_E(X^*) = \lim_{E \in \mathscr{E}} \text{ind}\, V_E(Y^*) = V_G(Y^*) \ .$$

See 1.11. ■

Just as in the case of modules, the next step is to describe cohomological support varieties in terms of G itself (or rather kG).

2.7. Proposition. Let X^* be a kG–complex. There exists a kG–chain complex $\hat{X}^*$ freely equivalent to X^* such that $H^*(\hat{X}^*)$ is concentrated in one degree, say d . Further, the rank varieties $V_E^r(H^d(\hat{X}^*))$ are well–defined for all $E \in \mathscr{E}$ and they depend only on X^* and not on the choice of $\hat{X}^*$. In fact, $V_E^r(H^d(\hat{X}^*)) = V_E(X^*)$ under the identification $V_E^r(k) = V_E$ of Section One 1.5.

Proof: The existence of $\hat{X}^*$ follows from the familiar procedure of "killing homology" inductively as in the case of G–spaces. (See Assadi [5]). Namely, let n be the smallest integer such that $H^n(X^*) \neq 0$. Consider a free kG–module F , and let F^* be the cochain complex with $F^i = 0$ for $i \neq n$ and $F^n = F$. We choose F and an injective kG–homomorphism $f : H^n(X^*) \longrightarrow F$. As usual, there is a cochain map $\varphi : X^* \longrightarrow F^*$ of degree zero such that the induced homomorphism $\varphi^* : H^n(X^*) \longrightarrow H^n(F^*) = F$ is the above f . The mapping cone of φ , say Y^* , is a kG–complex such that $H_i(Y^*) = 0$ for $i \leq n$. The repetition of this procedure yields a kG–complex $\hat{X}^*$ as desired above. To see that $V_E^r(H^d(\hat{X}^*))$ does not depend on the choice of $\hat{X}^*$, we may proceed directly as in Assadi [5] and show that $H^d(\hat{X}^*)$ and $H^{d'}(\hat{X}'^*)$ are ω–stably equivalent for two choices of $\hat{X}^*$ and $\hat{X}'^*$ satisfying the assertion of the proposition (cf. 1.8). Since the rank variety of a module does not change under the operation ω (see 1.8.1), the result follows. Alternatively, we may proceed as follows. According to Corollary 2.6, $V_E(X^*) = V_E(\hat{X}^*)$. But $\mathbb{H}^*(E;\hat{X}^*) \cong H^*(E;H^d(\hat{X}^*))$ since the hypercohomology spectral sequence $H^*(E;H^d(\hat{X}^*)) \Rightarrow \mathbb{H}^*(E;\hat{X}^*)$ degenerates. Thus $V_E(\hat{X}^*) = V_E(H^d(\hat{X}^*))$. By Avrunin–Scott's affirmative answer to the Carlson Conjecture [17], $V_E(H^d(\hat{X}^*)) = V_E^r(H^d(\hat{X}^*))$ (see 1.7.1.). Therefore, $V_E^r(H^d(\hat{X}^*)) = V_E(X^*)$ and $V_E^r(H^d(\hat{X}^*))$ does not depend on the choice of $\hat{X}^*$. ∎

2.8. Definition. Let X^* be a kG–complex, and let $\hat{X}^*$ be a kG–complex freely equivalent to X^* as in Proposition 2.7. For each $E \in \mathcal{E}$, the E–rank variety of X^* is defined via $V_E^r(X^*) \overset{def}{=} V_E^r(H^d(\hat{X}^*))$. For a chain complex X_*, $V_E^r(X_*) \equiv V_E^r(X^*)$. For $f : Z_* \longrightarrow Y_*$ a kG–chain map of degree zero with mapping cone X_*, $V_E^r(f) \overset{def}{=} V_E^r(X_*)$. The collection $V\mathcal{E}(X^*) \overset{def}{=} \{V_E^r(X^*) : E \in \mathcal{E}\}$ is called the rank variety family of X^*.

2.9. Remark. Similarly, for paracompact G–spaces we define the rank variety using their Čech or singular cochain complexes, e.g. For a G–map $f : Z \longrightarrow Y$ between paracompact G–spaces, $V_E^r(f)$ is defined using the reduced cochain complex of the mapping cone of f. For a based G–space X with $x_0 \in X^G$, $V_E^r(X,x_0) = V_E^r(C_*(X)/C_*(\{x_0\}))$. In Assadi [5], the notation $V_E^r(X)$ was used instead of $V_E^r(X,x_0)$ above. Since in [5] all G–spaces are based, with this slight change of notation all other definitions agree and yield the same results.

In Sections Three and Four we study further properties of varieties.

Section Three. Localization and Complexity.

Despite its simplicity, the Localization Theorem 1.9.1 plays an important role in cohomology theory of transformation groups of finite dimensional (or compact) spaces. As pointed out in the Introduction, this theorem fails for infinite dimensional G–spaces. The purpose of this section is to introduce localization for G–spaces and G–chain complexes of arbitrary dimension in a similar spirit. The notion of complexity allows one to find suitable multiplicatively closed subsets with respect to which localization will take place. The main idea is to enlarge the notion of "subgroups of G " to "shifted subgroups of G ". It turns out that this larger supply of subgroups in conjunction with the family of rank varieties $V\mathscr{E}(X^*)$ (see 2.8 and 2.9 above) suggests a useful localization of hypercohomology.

Recall that the identification $V_E^r(k) = V_E \equiv V_E(k)$ of 1.5 establishes a correspondence between shifted cylic subgroups of KE (corresponding to vectors in L) and elements of Max $H_E = V_E(k)$ (see 1.6). We will not consider the zero vector in $V_E^r(k)$ or $V_E(k)$ which corresponds to the degenerate case of the trivial subgroup $\{1\} \subset kE$. Let $\{\alpha^{(1)},\ldots,\alpha^{(m)}\}$ be a set of linearly independent vectors in $V_E^r(k)$, $E \in \mathscr{E}$, and let $S = S_1 \times \ldots \times S_m$ be the corresponding shifted subgroup of kE . As in 1.5, let $t_i \in H^\epsilon(S;k)$ be the polynomial generators of H_S corresponding to $\alpha(i)$, where $\epsilon = 1$ or 2 according to $p = 2$ or $p > 2$. Let L_m , respectively T_m , be the k–linear span of $\{\alpha^{(1)},\ldots,\alpha^{(m)}\}$ in $V_E^r(k)$, respectively in $H^\epsilon(S;k)$. For any non–empty subset $\Sigma \subset T_m - \{0\}$, we denote the localization with respect to the multiplicity closed subset generated by Σ via $(\ldots\ldots)[\Sigma^{-1}]$.

3.1. Proposition. Let X^* be a kG–complex, $E \in \mathscr{E}$, and S, Σ, L_m etc. as in the above. Assume that $L_m \cap V_E^r(X^*) = 0$. Then $\mathbb{H}^*(S;X^*)[\Sigma^{-1}] = 0$.

Proof: Unwinding the definitions, the above statement follows from the equality

$V_E^r(X) = V_E(X^*)$. Let $\hat{X}^*$ be freely equivalent to X^* , satisfying the conditions of Proposition 2.7. Then $L \cap V_E^r(H^*(\hat{X}^*)) = 0$, which implies that $H^*(\hat{X}^*)$ is kS–free using Carlson's version of Dade's Lemma (cf. 1.7 above). Therefore $H^*(S;\hat{H}^*(\hat{X}^*))[\Sigma^{-1}] = 0$. But $\mathbb{H}^*(S;X^*)[\Sigma^{-1}] \cong \mathbb{H}^*(S;\hat{X}^*)[\Sigma^{-1}]$ by Proposition 2.4. (or rather its proof) and $\mathbb{H}^*(S;\hat{X}^*) \cong H^*(S;H^*(\hat{X}^*))$ by the hypercohomology spectral sequence (cf. 1.3). ■

<u>3.2. Corollary</u>. Keep the notation of Proposition 3.1. Assume that $f: Z^* \longrightarrow Y^*$ is a kG–chain map such that $L_m \cap V_E^r(f) = 0$. Then $\mathbb{H}^*(S;Z^*)[\Sigma^{-1}] \longrightarrow \mathbb{H}^*(S;Y^*)[\Sigma^{-1}]$ is an isomorphism. ■

The above corollary generalizes the localization for the equivariant cohomology of a pair of paracompact finite dimensional G–spaces (Z,Y) in which Y is a closed subspace of Z containing the fixed point set Z^E .

To find such shifted subgroups $S \subset kG$, we consider the growth rate of the equivariant Betti numbers of a g–space or G–complex as follows. For a finitely generated kG–module M , J. Alperin introduced the notion of "complexity", denoted by $cx_G(M)$, cf. Alperin–Evens [3]. Consider a minimal projective kG–resolution:

$$\cdots\cdots \longrightarrow P_n \longrightarrow \cdots\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \ .$$

Then $cx_G(M) \overset{\text{def}}{=} \min\left\{ s \mid \lim_{n\to\infty} \frac{\dim P_n}{n^s} = 0 \right\}$. This is the same as the growth rate of $\dim_k H^j(G;M)$ as $j \longrightarrow \infty$. In terms of the family $\mathcal{E}$,
$cx_G(M) = \max\{cx_E(M\,|\,kE) : E \in \mathcal{E}\}$, see [3]. J. Carlson proved [24]
$cx_E = \dim V_E(M) = \dim V_E(M) = \dim V_E^r(M)$. These motivate the following:

<u>3.3. Definition</u>. Let X^* be a kG–complex. Then the complexity of X^* is defined as the

growth rate $\min\left\{ s \mid \lim_{n\to\infty} \dim_k \frac{\mathbb{H}^n(G;X^*)}{n^s} = 0 \right\}$ and it is denoted by $cx_G(X^*)$. Similarly, for a G–pair of spaces (Z,Y) or a kG–cochain homomorphism $f : X^* \longrightarrow Y^*$ we define their kG–complexity via the growth rate of the corresponding equivariant cohomology $H^*_G(Z,Y;k)$ or hypercohomology of the mapping cone. Similar definitions are made for chain complexes, using the dual cochain complex.

For future reference, we list some of the consequences of the above definitions following from the known properties of complexities of G–modules.

<u>3.4. Proposition</u>. (a) If X^* is a kG–chain complex concentrated in one dimension only, say d, then the Alperin complexity of X^d and $cx_G(X^*)$ agree.

(b) If X^* and Y^* are freely equivalent, then $cx_G(X^*) = cx_G(Y^*)$.

(c) Let X^* be an arbitary kG–chain complex and $\hat{X}^*$ be freely equivalent to X^* with $H^i(\hat{X}^*) = 0$ unless $i = d$. Then $cx_G(X^*) = cx_G(\hat{X}^*)$ = Alperin complexity of $H^d(\hat{X}^*)$.

(d) $cx_G(X^*) = \max\{cx_E(X^*) : E \in \mathscr{E}\}$, and similarly for all other complexities.

(e) $cx_G(X^*) = \dim V_G(X^*) = \max\{\dim V^r_E(X^*) \mid E \in \mathscr{E}\}$

<u>Proof</u>: (a) A finite shift in dimensions of the cohomology (i.e. iterated suspension or desuspension) does not affect growth rate, and $\mathbb{H}^*(G;X^*) \cong H^*(G;X^d)$ possibly with a dimension shift.

(b) It suffices to consider a freely equivalent pair of kG–complexes $X^* \subseteq Y^*$. A comparison of the hypercohomology spectral sequences (see 1.3) shows that $\mathbb{H}^i(G;X^*) \cong \mathbb{H}^i(G;Y^*)$ for sufficiently large i.

(c) With a possible shift in dimension, $\mathbb{H}^*(G;X^*) \cong H^*(G;H^d(\hat{X}^*))$. Moreover, $\dim_k P^n$ and $\dim_k H^n(P^*)$ have the same growth rate as $n \longrightarrow \infty$ for any projective resolution P_* of $H^d(\hat{X}^*)$ of finite type, and in particular the minimal resolution. Together with (a) and (b), the conclusion follows.

(d) Follows from (c) and Alperin–Evens theorem [3] mentioned above.

(e) Follows from (d) and Proposition 2.7. above. ■

3.5. Remark. Note that $cx_G(X^*) \leq rk_p(G)$ and $cx_G(k) = rk_p(G)$.

3.6. Corollary. Let X^* be a kG–complex such that $cx_G(X^*) = s$. There exists a shifted elementary abelian subgroup $S \subset kG$ of rank m such that $m+s = rk_p(G)$ and $\mathbb{H}^*(S;X^*)[\Sigma^{-1}] = 0$, where $\Sigma \subset H_S$ is as in Proposition 3.1 above.

Proof: By 3.4 (e), $\dim V_E^r(X^*) \leq s$ for all $E \in \mathscr{E}$, and there exists $E_0 \in \mathscr{E}$ with $rk(E_0) = s+m$. We may assume $m > 0$, and choose a linear subspace $L \subset V_{E_0}^r(k) \cong k^{m+1}$ such that $\dim L = m$ and $L \cap V_{E_0}^r(X^*) = \{0\}$ by a general position argument. The shifted subgroup S corresponding to L and the multiplicative subset Σ chosen in Proposition 3.1 above yield the desired conclusion. ■

3.7. Remarks. (a) The above corollary should be compared to the theorem of O. Kroll [38] who proves by delicate algebraic arguments that if the complexity of a kE–module M is s, then for a shifted subgroup $S \subset kE$ of rank $rk(E)-s$, the restricted module $M\,|\,kS$ is kS–free.

(b) The notion of complexity for finite G–CW complexes and a related "isotropy variety" has been defined and studied independently by A. Adem [1] [2] in a different context, thus generalizing Alperin's notion of complexity of kG–modules. Adem has applied these, as well as O. Kroll's theorem to fixed point theory of G–CW complexes and other problems.

(c) The growth rate of equivariant Betti numbers of finite dimensional paracompact

G–spaces has been studied by Quillen [40] where he proves the analogues of Proposition 3.5 (d) and (e) in terms of his varieties (i.e. the spectrum of $H_G(X)$). See the following section for further relationship to Quillen's results.

Section Four. Comparison of Varieties.

In this section, we compare Quillen's varieties [40] and the varieties under consideration in this paper. In particular, we point out how varieties attached to infinite dimensional G–spaces provide an abstraction of the geometric properties of the finite dimensional case in this context. This leads to a basic conjecture whose understanding is intimately related to a deeper study of the cohomological aspects of infinite dimensional G–spaces . We provide some evidence for the truth of the conjecture.

Recall from 1.11 that to a finite dimensional G–space X , Quillen [40] has attached the variety of the geometric points of the equivariant cohomology ring $H_G(X)$, which is denoted by $H_G(X)(k)$ for the variety of k–valued points. Let $H_G(X;k) \overset{\text{def}}{=} H_G^*(X;k)/\text{Radical}$ be the associated affine k–algebra , and identify $H_G(X)(k)$ with the maximal spectrum $\text{Max}\, H_G(X;k)$. The projection $\pi_G : X_G \longrightarrow BG$ induces a k–algebra homomorphism $(H_G)_{red} \longrightarrow H_G(X;k)$, and consequently a morphism ${}^a\pi_G : H_G(X)(k) \longrightarrow \text{Max}(H_G)_{red}$ of varieties. (See the comments in 1.4). The following describes the relationship between Quillen's variety and the support variety. The connectivity hypothesis below is technical only, in order to avoid a lengthy discussion of the details of Quillen's stratification theory, (see 1.11 above). The formulation and similar proof of the general case is left to the interested reader. See Atiyah–Mac Donald [14] or Kunz [39] for details and definitions from commutative algebra.

4.1. Proposition: Let X be a paracompact G–space with $\dim X < \infty$. Then:

(a) For each $E \in \mathscr{E}$ such that $X^E \neq \phi$, the projection $\pi_E : X_E \longrightarrow B_E$ induces a morphism ${}^a\pi_E : H_E(X)(k) \longrightarrow V_E(X)$ which is surjective and finite.

(b) Suppose that for each maximal $E \in \mathscr{E}$, X^E is connected (or empty). Then there is a morphism $H_G(X)(k) \longrightarrow V_G(X)$ induced by $\{\pi_E \mid E \in \mathscr{E}\}$ which is surjective and finite.

(c) $\dim H_G(X)(k) = \dim V_G(X)$.

Proof: Suppose $X^E \neq \phi$. Then $\pi_E^* : H^*(E;k) \longrightarrow H_E^*(X;k)$ is injective. Since $H^*(X;k)$ is a finite dimensional k–vector space, $H_E^*(X;k)$ is a finitely generated k–algebra, and a finitely generated $H^*(E;k)$–module. It follows that the k–homomorphism $\pi_E : (H_E)_{red} \longrightarrow H_E(X;k)$ yields an integral extension of Noetherian k–algebras so that the induced morphism ${}^a\pi_E : H_E(X)(k) \longrightarrow \mathrm{Max}(H_E)_{red}$ is surjective and finite (cf. Kunz [39] pp. 44–48). Since $V_E(X) \equiv \mathrm{Max}(H_E)_{red}$ in this case, (a) follows. To see (b), observe that for a finite dimensional G–space X, Quillen [40] and Jackowski [35] describe the stratifications of $H_G(X)(k)$ and the support variety $V_G(X)$ in terms of the corresponding locally closed subsets $H_E(X)(k)$ and $V_E(X)$. Recall from 1.11 $\mathscr{E}(X) = \{E \in \mathscr{E} : X^E \neq \phi\}$ and $V_G(X) = \underset{E \in \mathscr{E}(X)}{\mathrm{lim\,ind}}\, V_E(X)$ (Jackowski [35]). Quillen's index category, denoted by $\mathscr{A}(G,X)$ in [40], depends on the path components of X^E and is somewhat more elaborate. The connectivity hypothesis in (b) above implies that $\mathscr{A}(G,X) = \mathscr{E}(X)$ in this case. Thus $H_G(X)(k) = \underset{E \in \mathscr{E}(X)}{\mathrm{lim\,ind}}\, H_E(X)(k)$. By naturality of the morphism π_E in (a) above, we have $\underset{E \in \mathscr{E}(X)}{\mathrm{lim\,ind}}\, \pi_E : H_E(X)(k) \longrightarrow V_G(X)$ which is surjective and finite since each π_E is surjective and finite. (c) follows from (b). □

It is clear from the definitions that we may still consider $H_G(X)(k)$ even if $\dim X = \infty$. According to our standing hypothesis $\dim_k H^*(X;k) < \infty$, and this suffices to have a finitely generated k–algebra $H_G(X;k)$ which is also a finitely generated $(H_G)_{red}$–module. However, the stratification theorem of Quillen [40] for $H_G(X)(k)$

and Jackowski [35] for $V_G(X)$ uses the localization theorem and other finite dimensional features of a G–space X. As far as $V_G(X)$ is concerned, we may proceed as follows. Replace X by a suitable kG–cochain complex X^* and choose a kG–cochain complex $\hat{X}^*$ freely equivalent to X^* for which $H^*(\hat{X}^*) = H^d(\hat{X}^*)$ as in Proposition 2.7. Thus $V_G(X) = V_G(X^*) = V_G(\hat{X}^*)$ and for each $E \in \mathcal{E}$, $V_E^r(H^d(\hat{X}^*)) = V_E(X^*)$. At this point, we may apply Avrunin–Scott's stratification theorem [17] (see 1.11 above) to obtain a generalization of Jackowski's theorem [35] below:

4.2. Proposition. Let X be an arbitrary paracompact G–space. Then
$V_G(X) = \lim\limits_{\substack{\longrightarrow \\ E \in \mathcal{E}}} \text{ind}\ V_E(X)$. □

As for $H_G(X)(k)$, we may well expect a generalization of Proposition 4.1 above to hold. Indeed, let us formulate the following:

4.3. Conjecture:

(1) Let X be a paracompact G–space of arbitrary dimension. Then $H_G(X)(k) = \lim\limits_{E \in \mathcal{E}} \text{ind}\ H_E(X)(k)$ where $H_E(X)(k)$ are locally closed subspaces. Further, each $H_E(X)(k)$ is a finite disjoint union of irreducible pieces of the form V_E^+/Q, where V_E^+ is the complement of a union of suitable $\mathbb{F}_p$–rational linear subspaces of V_E on which an appropriate subgroup $Q \subseteq N_G(E)/C_G(E)$ acts freely. Here $N_G(E)$ = normalizer and $C_G(E)$ = centralizer of E in G.

(2) There is a finite surjective morphism $H_G(X)(k) \longrightarrow V_G(X)$ which is induced by ${}^a\pi_E : H_E(X)(k) \longrightarrow V_E(X)$ in the following Proposition 4.4.

4.4. Proposition. Let X be an arbitrary paracompact G–space. Then for each $E \in \mathscr{E}$, the induced morphism ${}^a\pi_E : H_E(X)(k) \longrightarrow V_E(X)$ is surjective and finite. Furthermore, if G is an abelian p–group, then the above Conjecture 4.3 (2) holds.

Proof: Suppose G is an abelian p–group and $E \subseteq G$ is its (unique) maximal p–elementary abelian subgroup. Then the restriction $\rho_E : E \longrightarrow G$ induces an isomorphism $\rho_E^* : H^*(G;k) \longrightarrow H^*(E;k)$. Moreover, $N_G(E') = G = C_G(E')$ for each $E' \in \mathscr{E}$, and the category of elementary abelian subgroups of E and G coincide. Therefore, $V_G = V_E$, and $V_E(X) = V_G(X)$ by Proposition 4.2 above. From the commutative diagram:

$$\begin{array}{ccc} H_G(X;k) & \longrightarrow & H_E(X;k) \\ \pi_G^* \uparrow & & \uparrow \pi_E^* \\ (H_G)_{red} & \xrightarrow{\cong} & (H_E)_{red} \end{array}$$

it follows that Conjecture 4.3 above for G reduces to the case G = p–elementary abelian which we will consider next.

Let $I_E \subseteq (H_E)_{red}$ be the annihilating ideal of $H_E(X;k)$ as an $(H_E)_{red}$–module. $H_E(X;k)$ is a finitely generated $(H_E)_{red}$–module, hence also finitely generated as a module over $(H_E)_{red}/I_E$. Therefore, $H_E(X;k)$ is an integral extension of $(H_E)_{red}/I_E$, so that ${}^a\pi_E : \operatorname{Spec} H_E(X;k) \longrightarrow \operatorname{Spec}(H_E)_{red}/I_E$ is a finite and surjective morphism. Since $\operatorname{Max}(H_E)_{red}/I_E = V_E(X)$, the restriction of ${}^a\pi_E$ to the subspace of closed points yields a finite surjective morphism ${}^a\pi_E : H_E(X)(k) \longrightarrow V_E(X)$ as desired. □

<u>4.5. Remark</u>. As observed by Quillen ([40] Section 12) I_E is invariant under the Steenrod algebra See 1.10 above. Thus, the variety $V_E(X)$ defined by I_E is a union of $\mathbb{F}_p$–rational linear subspaces corresponding to suitable subgroups $E' \subseteq E$, by a theorem of Serre (cf. Theorem 1.10.1 above and compare with [40] 12.2). Using this identification, $V_E(X) = \cup V_{E'}(X)$, abusing the notation slightly, where the union is over subgroups $E' \subseteq E$ for which $V_{E'}(X) = V_{E'}$. This shows that $V_E(X)$ is stratified by pieces which are described in Conjecture 4.3 (1). It is reasonable to expect that the localization theorem of Section Three and the above results could be used to prove the above conjecture. The proof of the above conjecture is closely connected to the problem of determining when a given free infinite dimensional G–space is of the form $E_G \times K$ for a finitely dominated G–space K, cf. Assadi [5] [9] [10].

Section Five. Varieties and Isotropy Subgroups.

In this section we continue the comparison of finite dimensional and infinite dimensional G–spaces through the more familiar notion of isotropy subgroups. The rank variety family $V^r \mathcal{E}(X)$ of 2.8 is a reasonable substitute for the family of isotropy subgroups of a finite dimensional G–space X . The following two propositions make this point explicit. Fix a prime $p \mid |G|$, and consider $\mathcal{E}(X) = \{E \in : X^E \neq \phi\}$. $\mathcal{E}(X)$ is a good homological invariant for finite dimensional G–spaces, and it includes the family of isotropy subgroups of X if $G \in \mathcal{E}$.

The following elementary proposition formulates some properties of $\mathcal{E}(X)$ which generalize to similar ones for its algebraic abstraction $V^r \mathcal{E}(X)$.

5.1. Proposition: Let X and Y be finite dimensional connected G–spaces. Then the following hold:

(i) If $f : X \longrightarrow Y$ is equivariant, then $\mathcal{E}(X) \subseteq \mathcal{E}(Y)$.

(ii) If f in (i) above induces an isomorphism $f_* : H_*(X;\mathbb{F}_p) \longrightarrow H_*(Y;\mathbb{F}_p)$, then $\mathcal{E}(X) = \mathcal{E}(Y)$. In particular, if $f \times id : X \times E_G \longrightarrow Y \times E_G$ is a G–homotopy equivalence, then $\mathcal{E}(X) = \mathcal{E}(Y)$.

(iii) $A \in \mathcal{E}(X)$ if and only if in the Borel construction $E_A \times_A X \xrightarrow{\pi} BA$, $\pi^* : H_A \longrightarrow H_A (X;\mathbb{F}_p)$ is injective.

(iv) If X is a Moore space and $|A| = p$, then in Tate cohomology $\hat{H}^*(A;\overline{H}_*(X) = 0)$ implies that $A \in \mathcal{E}(X)$ (and similarly for mod p Moore spaces).

(v) $E \in \mathcal{E}(X)$ if and only if $S_E^{-1}H_E^*(X;\mathbb{F}_p) \neq \phi$.

The proofs (ii)–(v) follow from the Localization Theorem 1.9.1.

Similarly, for a kG–chain complex X_* let $V^r\mathscr{E}(X_*) = \{V_E^r(X_*) \mid E \in \mathscr{E}\}$. The following Proposition makes the analogy between $V^r\mathscr{E}(X_*)$ and $\mathscr{E}(X)$ quite explicit:

<u>5.2 Proposition</u>: Let X_* and Y_* be connected augmented kG–complexes. Then the following hold:

(i) If $f : X_* \longrightarrow Y_*$ is a kG–map of augmented complexes, then $V^r\mathscr{E}(X_*) \subseteq V^r\mathscr{E}(Y_*)$.

(ii) If f in (i) above induces a homology isomorphism, then $V^r\mathscr{E}(X_*) = V^r\mathscr{E}(Y_*)$.

(iii) $V_E^r(k) \in V^r\mathscr{E}(X_*)$ if and only if $\pi^* : (H_E)_{red} \longrightarrow H^*(E;X^*)$ is injective.

(iv) Suppose X_* has only one non–trivial reduced homology group, and $A \subset kE$ is a shifted cyclic subgroup for some $E \in \mathscr{E}$. Then $\hat{H}^*(A;\overline{H}^*(X^*)) = 0$ implies that $\rho_A(V_A^r(k)) \subset V_E^r(X_*)$.

(v) $V_E^r(k) \in V^r\mathscr{E}(X_*)$ if and only if $S_E^{-1}H^*(E;X^*) \neq 0$.

(vi) If X is a G–space, $V_E^r(X)$ is a union of $\mathbb{F}_p$–rational linear subspaces of $V_E^r(k)$. In particular, if $\dim X < \infty$, there is a one–to–one correspondence between $\mathscr{E}(X)$ and $\{E \in \mathscr{E} \mid V_E^r(X) = V_E^r(k)\}$.

<u>Proof</u>: Suppose $\lambda \in V_E^r(X)$ for some $E \in \mathscr{E}$, and let $S \subset kE \subseteq kG$ be the corresponding shifted cyclic subgroup, and $t \in H^\varepsilon(S;k)$, $t \neq 0$ be the corresponding element, where $\varepsilon = 1$ if $p = 2$ and $\varepsilon = 2$ otherwise. See 1.5 and 1.6. We have the following commutative diagram.

$$\begin{array}{ccc} H^*(S;Y^*) & \xrightarrow{f^*} & H^*(S;X^*) \\ \beta \nwarrow & & \nearrow \alpha \\ & H^*(S;k) & \end{array}$$

Localization with respect to t yields a corresponding commutative diagram:

$$\begin{array}{ccc} H^*(S;Y^*)\left[\frac{1}{t}\right] & \xrightarrow{\bar{f}} & H^*(S;X^*)\left[\frac{1}{t}\right] \\ \bar{\beta} \nwarrow & & \nearrow \bar{\alpha} \\ & H^*(S;k)\left[\frac{1}{t}\right] & \end{array}$$

Since $\lambda \in V_E^r(X)$, $\rho_S^{-1}(V_E^r(X)) \cong V_S^r(k) = V_S(k)$.

Hence $\bar{\alpha}$ is an injection which implies that $\bar{\beta}$ is an injection. Thus $V_S^r(Y) = k = V_S(Y)$ and consequently $V_E^r(X_*) \subseteq V_E^r(Y_*)$. (ii)–(v) follow from similar considerations, using Proposition 2.7 $V_E(X^*) = V_E^r(X^*)$ and the localization in 3.1. (vi) follows from Remark 4.5. □

Section Six. Applications to Finite Dimensionality and Finite Domination Problems.

In this section we apply the theory developed in the previous sections to provide local–to–global criteria for finite dimensionality (up to homotopy) and finite domination of CW complexes. Such finiteness problems arise naturally and often in geometric topology involving non–simply connected spaces. The above results allow one to reduce the problem involving an arbitrary finite group to a similar one involving cyclic group of prime order for which algebraic and homological computations are considerably simpler. See Assadi [7] [8] [9] [11] for further applications of these ideas.

Let X be a CW complex. Then X is called finitely dominated if there exists a finite CW complex K and a map $f : K \longrightarrow X$ which has a right homotopy inverse $r : X \longrightarrow K$. A finitely dominated complex is homotopy equivalent to a finite dimensional complex with finitely generated homology. This concept has an analogue in the category chain complexes over an arbitrary ring Λ with a similar definition. Wall has provided in [45] [46] algebraic conditions for finite dimensionality and finite domination of a CW complex X. Let $\Lambda = \mathbb{Z}\,\pi_1(X)$ and $C_* \overset{\text{def}}{=} C_*(X;\mathbb{Z}\,\pi_1(X)) \equiv$ cellular chain complex of the universal covering space of X regarded as a Λ – complex. Clearly C_* is a free connected Λ – complex. According to Wall, X is homotopy equivalent to a finite dimensional CW complex if and only if C_* is chain homotopy equivalent to a projective finite dimensional Λ – chain complex. This reduces the problem of finite dimensionality for spaces to a similar one for chain complexes. If X is of finite type, i.e. every finite dimensional skeleton of X is a finite complex, then "finite dimensional" in the above may be replaced by "finitely dominated". Quite generally, a projective positive chain complex (P_*,d_*) is chain homotopy equivalent to an n–dimensional complex if and only if $H_i(P_*) = 0$ for $i > n$ and Image $(d : P_{n+1} \longrightarrow P_n)$ is a projective module, (see Wall [46] Theorem 6). If P_* is of finite type, then the analoguous statement holds for finitely dominated projective complexes.

<u>6.1. Proposition</u>. Let X_* be a connected projective $\mathbb{Z}G$–chain complex with finitely generated total homology. Then X_* is chain homotopy equivalent to a finite dimensional $\mathbb{Z}G$–free chain complex if and only if $X_* \otimes \overline{\mathbb{F}}_p$ is $\overline{\mathbb{F}}_p[E]$ – chain homotopy equivalent to a finite dimensional $\overline{\mathbb{F}}_p[E]$ – chain complex for all $E \in \mathcal{E}_p(G)$ and all $p \mid |G|$.

<u>Proof</u>: Let $M = \mathrm{Ker}(d_n : X_n \longrightarrow X_{n-1})$ for a sufficiently large n . From Wall's criterion, one sees that it suffices to prove that M is $\mathbb{Z}G$–projective. According to Chouinard's theorem (cf. [18]), M is $\mathbb{Z}G$–projective if and only if $M \mid \mathbb{Z}E$ is $\mathbb{Z}E$–projecitve for all $E \in \mathcal{E}_p(G)$ and all $p \mid |G|$. It is easy to see that the $\mathbb{Z}$–free $\mathbb{Z}E$–module M is $\mathbb{Z}E$–projective if and only if $M \otimes \mathbb{F}_p$ is $\mathbb{F}_p[E]$–free . The change of fields from $\mathbb{F}_p$ to $\overline{\mathbb{F}}_p$ and vice versa, sends free $\mathbb{F}_p[E]$–modules to $\overline{\mathbb{F}}_p[E]$–modules . The passage from $\mathbb{Z}G$–projective chain complexes to $\mathbb{Z}G$–free complexes in the finite dimensional case is a standard application of the Eilenberg trick. (Note that for the finitely dominated case, this step is measured by the finiteness obstruction in $\tilde{K}_0(\mathbb{Z}G)$) . □

Having reduced the problem of finite domination involving a finite group G to a similar question for a p–elementary abelian group E , we proceed further to give a criterion in terms of cyclic groups of order p . First we mention a useful observation.

It is easy to see that finite domination is preserved under free equivalence. We record the following generalization for future reference.

6.2. Proposition. Let $R = \mathbb{Z}$, $\mathbb{F}_p$, or k, and consider a short exact sequence of projective RG–complexes $X_* \xrightarrow{\varphi} Y_* \longrightarrow Z_*$. The finite domination of any two complexes implies that the third is finitely dominated.

Proof: Using Proposition 6.1 above, it suffices to consider the case $R = k$ and restrict to $E \in \mathscr{E}_p(G)$. Assume that X_* and Z_* are finitely dominated. Choose n large enough so that $H_i(X_*)$, $H_i(Y_*)$, and $H_i(Z_*)$ vanish for all $i \geq n$. Denote the truncated chain complex $0 \longrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \longrightarrow \ldots \longrightarrow 0$ by $X_*[n]$ and the complementary complex by $\tilde{X}_*$ such that the sequence

$0 \longrightarrow X_*[n] \longrightarrow X_* \longrightarrow \tilde{X}_* \longrightarrow 0$ is exact. Use a similar notation for Y_* and Z_*. Observe that $\mathbb{H}^*(E;\tilde{X}^*) \cong H^*(E;H^*(\tilde{X}^*))$ and $H^*(\tilde{X}^*) \cong \ker \partial_n$ by our choice of n. In the ladder of E–hypercohomology long exact sequences:

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathbb{H}^i(E;\tilde{X}^*) & \longrightarrow & \mathbb{H}^i(E;X^*) & \longrightarrow & \mathbb{H}^i(E;X^*[n]) & \longrightarrow \ldots \\
& \uparrow \tilde{\varphi}^* & & \uparrow \varphi^* & & \uparrow \varphi[n]^* & \\
\ldots \longrightarrow & \mathbb{H}^i(E;\tilde{Y}^*) & \longrightarrow & \mathbb{H}^i(E;Y^*) & \longrightarrow & \mathbb{H}^i(E;Y^*[n]) & \longrightarrow \ldots
\end{array}$$

the localized homomorphisms $S_E^{-1}\varphi^*$ and $S_E^{-1}\varphi[n]^*$ are isomorphisms, where S_E is the multiplicatively closed subset of H_E as in Proposition 2.4. This follows from the hypothesis that Z_* is a finitely dominated kG–projective complex, Lemma 2.5, and the invariance of hypercohomology under chain homotopy equivalence. By the Five–Lemma, $S_E^{-1}\tilde{\varphi}^*$ is an isomorphism. Let ∂' be the boundary operator of Y^* then $S_E^{-1}H^*(E;\text{Ker } \partial') \cong S_E^{-1}\mathbb{H}^*(E;\tilde{Y}^*) \cong S_E^{-1}\mathbb{H}^*(E;\tilde{X}^*) \cong S_E^{-1}H^*(E;\text{Ker } \partial) = 0$ since X_* is finitely dominated by hypothesis. S_E and E are arbitrary, this implies that

Ker ∂' is cohomologically trivial (in the sense of Tate cohomology, cf. [22] [25]). For the ring kE , cohomologically trivial modules are projective ([22]), so that Ker ∂' is kE–projective and Wall's criterion implies that Y^* is finitely dominated. The other cases are similar. □

6.3. Corollary. Let X_* be a connected kG–chain complex with finitely generated total homology. Then X_* is finitely dominated by a kG–projective complex if and only if $V_E^r(X_*) = 0$ for all $E \in \mathscr{E}$, or equivalently, $V_G(X_*) = 0$. Thus, X_* is kG–finitely dominated if and only if $X_* | kS$ is kS–finitely dominated for each shifted cyclic subgroup of G .

Proof. As before, we may restrict attention to an arbitrary $E \in \mathscr{E}$. Since free equivalence preserves finite domination, we may replace X_* by $\hat{X}_*$ as in Proposition 2.7 with homology concentrated in one dimension, say d , only. For any $n > d$, $\mathrm{Ker}(\partial : X_n \longrightarrow X_{n-1})$ is stably equivalent to $H_d(\hat{X}_*)$, cf. 1.8. Thus $V_E^r(X_*) = V_E^r(H_d(\hat{X}_*)) = V_E^r(\mathrm{Ker}\ \partial)$ vanishes if and only if Ker ∂ is kE–free (by Carlson's version of Dade's Lemma, cf. [23]). Now Wall [46] applies to yield the desired claim. □

6.4. Corollary. Suppose C_* is the cellular chain complex of the universal covering space of the CW complex X with $\pi_1(X) = G$. Then X is finitely dominated if and only if for each prime $p \mid |G|$ and each $C \subseteq G$ with $|C| = p$, $C_* \otimes \mathbb{F}_p$ is finitely dominated as an $\mathbb{F}_p[C]$–complex .

Proof: By the above Corollary, we must show that finite domination of $C_* \otimes \mathbb{F}_p$ over the ring $\mathbb{F}_p[C]$ as indicated in the above statement implies that $V_E^r(X_*) = 0$ for all $E \in \mathscr{E}$. The reverse implication is trivial. But $V_E^r(X_*) = V_E(X_*) = V_E(X)$, and

$V_E(X)$ is a union of $\mathbb{F}_p$–rational linear subspaces of V_E by Serre's theorem [43], cf. 1.10 and Remark 4.5 above. By the discussion in 1.5 and 1.6, cyclic subgroups of E correspond to non–zero $\mathbb{F}_p$–rational vectors of V_E (and the corresponding group algebras to $\mathbb{F}_p$–rational lines of V_E). Since $V_C(X_*) = 0$ and $V_E(X_*) \cap V_C = V_C(X_*) = 0$ (see 1.7.2), $V_E(X_*)$ contains no non–zero $\mathbb{F}_p$–rational point. Hence $V_E^I(X) = V_E(X) = 0$ as claimed. □

To illustrate an application to geometric topology, consider the problem of constructing free G–actions on a closed highly–connected manifold W^{2n}. As explained in Assadi [5] [7] methods of homotopy theory (under suitable hypotheses) lead one to the situation where we have an infinite dimensional CW complex X with $\pi_1(X) = G$, and the universal covering space of X, say $\tilde{X}$, is homotopy equivalent to W^{2n}. In order to replace X with a closed manifold homotopy equivalent to X, we need to decide that X is finitely dominated. The following criterion reduces this decision procedure to a familiar question in algebra. Notice that the structure of finitely generated $\mathbb{Z}$–free $\mathbb{Z}[\mathbb{Z}_p]$–modules is completely understood and it is easy to describe. But the same question for all but a very small number of finite groups is considered a hopeless problem, cf. Curtis–Reiner [28] and Benson [18]. For related applications, cf. Assadi [7] [8].

6.5. Theorem. Let W^{2n} be an $(n-1)$–connected finite Poincré complex $n \geq 2$. Suppose that X is a CW complex whose universal covering space $\tilde{X}$ is homotopy equivalent to W and $\pi_1(X) = G$. The necessary and sufficient conditions for X to be finitely dominated are as follows. For all primes $p \mid |G|$ and all subgroups $C \subseteq G$ with $|C| = P$, the following statements are satisfied:

(I) the spectral sequence of the Borel construction $E_C \times_C \tilde{X} \longrightarrow BC$ does not collapse.

(II) $\dim_{\mathbb{F}_p} H^{\varepsilon}(C;H^n(\tilde{X})) \geq 2$, where $\varepsilon = 1$ if n is even and $\varepsilon = 2$ if n is odd.

6.6. Remark: Condition (II) above may be replaced by the following more explicit condition: (II') $H_n(\tilde{X}) \mid \mathbb{Z}C$ is projectively stably isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ (with trivial action) for n = odd and $I \oplus I$ for n = even , where I is the augmentation ideal of $\mathbb{Z}[C]$.

Proof: The proof is similar to the proof of Assadi [8] Theorem 6.1, where the case of $n = 2$ was treated, with some modifications of details. Therefore, we will only indicate the outline of the proof incorporating such changes. The necessity of condition (I) and (II) or (II') is just as in [8] Theorem 6.1, and one may appeal to Assadi [12] (Theorem 3.1 and Example 3.5) for details in the case n = odd or $n > 2$. By Corollary 6.4 above, it suffices to prove that (I) and (II) above imply that $\tilde{X}/C$ is finitely dominated for each $C \subsetneq G$, $|C| = p$.

Next, we choose Y freely equivalent to $\tilde{X}$ by adding free orbits of cells of dimension $i \leq 2n$ such that Y is $(2n-1)$–connected . Since free equivalence preserves finite domination, it suffices to prove that Y is finitely dominated. Just as in [8] Lemma 6.3, one proves that the latter is satisfied if and ony if $H^i_C(Y) = 0$ for $i \geq 2n+1$. Furthermore, $H^i_C(Y) = 0$ for $i \geq 2n+1$ if and only if $H^i_C(X) = 0$ for $i \geq 2n+1$. One verifies these conditions explicitly using the Cartan–Leray–Serre spectral sequence of $X = E_C \times_C \tilde{X} \longrightarrow BC$. The details of these calculations are carried out in [8] Lemmas 6.4, 6.5, and 6.6 for $n = 2$. However, the general case follows by similar arguments, since $H^i(C;\mathbb{Z}) \cong H^{i+1}(C;I)$ for $i > 0$ and these H_C–modules are periodic. □

Now we can apply Wall's obstruction theory [45] [46] to decide when a finitely dominated X as in Theorem 6.5 above (provided (I) and (II) are verified) is homotopy equivalent to a finite Poincaré complex. For example:

6.7. Corollary. Suppose X is as in Theorem 6.5 and conditions (I) and (II) of 6.5 are satisfied. Then there exists a well–defined obstruction $\theta(X) \in \tilde{K}_0(\mathbb{Z}G)$ such that X is homotopy equivalent to a finite Poincaré complex if and only if $\theta(X) = 0$.

Proof: This is a consequence of the above discussion and the following remark about Poincaré duality. If X is finitely dominated and $\tilde{X}$ satisfies Poincaré duality, then X satisfies Poincaré duality. See Quinn [42] and Gottlieb [33] for details. □

It is possible to be more specific about the obstruction $\theta(X)$. For example:

6.8. Corollary. Suppose X is as in Theorem 6.5, and it satisfies the conditions (I) and (II') of 6.5 and 6.6. Assume that G is cyclic and using (II') write $H_n(\tilde{X}) \cong L \oplus L \oplus P$ where L is $\mathbb{Z}$ or I and P is $\mathbb{Z}G$–projective . Then X is homotopy equivalent to a finite Poincaré complex if and only if P is $\mathbb{Z}G$–free . □

6.9. Remarks.

(a) Notice that $|G|$ need not be a prime, and (II') is still satisfied for this case. See [12] (Example 3.5 (2) and Theorem 3.1).

(b) For n = odd , according to [12] Corollary 3.2, and since $H_{2n}(G;\mathbb{Z}) = 0$, we have a $\mathbb{Z}G$–isomorphism $H_n(\tilde{X}) \cong \omega^{n+1}(\mathbb{Z}) \oplus \omega^{-n-1}(\mathbb{Z})$ (cf. 1.8). Therefore,

$\tilde{H}_n(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus P$ where P is $\mathbb{Z}G$–projective due to periodicity of $H^*(G;\mathbb{Z})$. The finiteness obstruction for X is precisely $\pm[H_{2n}(Y)]$, where Y is the G–space constructed in the proof of Theorem 6.5 above.

Section Seven. Equivariant Maps and Algebraic Analogues of the Borsuk–Ulam Theorem.

As pointed out in Section Five, for a (possibly infinite dimensional) G–space X or a kG–chain complex X_* and each $E \in \mathscr{E}$, the statement $V_E^r(X) = V_E^r(k)$ and $V_E^r(X_*) = V_E^r(k)$ are the algebraic analogues of the geometric statement $X^E \neq \phi$ for the case $\dim X < \infty$. Furthermore, the existence of equivariant maps between G–spaces X and Y is related to fixed point theory of infinite dimensional G–spaces as follows. Let $\mathrm{Map}(X,Y)$ be the mapping space together with the usual action: for $g \in G$, $f \in \mathrm{Map}(X,Y)$, $(g,f) \longmapsto f^g$ with $f^g(x) = g \cdot f(g^{-1}x)$. Then the fixed point set $\mathrm{Map}(X,Y)^G$ consists of G–equivariant maps denoted by $\mathrm{Map}_G(X,Y)$. For $\dim Y < \infty$ and $X = E_G$, $\mathrm{Map}_G(E_G,Y) \neq \phi$ if and only if $Y^G \neq \phi$ for $G \cong (\mathbb{Z}_p)^n$ by the Localization Theorem 1.9.1 (see also W.Y. Hsiang [34]). The theorems of this section may be interpreted as generalizations of these results, and at the same time, algebraic versions of the classical Borsuk–Ulam Theorem. See also the Introduction.

7.1. Theorem: Suppose $f : X \longrightarrow Y$ is a G–map between connected (possibly infinite dimensional) G–spaces such that X is n–connected and $\bar{H}_i(Y) = 0$ for $i > n$. Then for any $p \mid |G|$, $k = \bar{\mathbb{F}}_p$, and the corresponding kG–varieties, one has $V_G(Y) = V_G(k)$ and $V_E^r(Y) = V_E^r(k)$ for all $E \in \mathscr{E}$. Similarly, if X_* and Y_* are connected augmented kG–chain complexes satisfying the above connectivity assumptions, and $f : X_* \longrightarrow Y_*$ is an augmentation preserving kG–chain homomorphism, then $V_E^r(Y_*) = V_E^r(k)$ for all $E \in \mathcal{E}$ and $V_G(Y_*) = V_G(k)$.

Proof: It suffices to fix a prime $p \mid |G|$ and prove the theorem on the level of kG–complexes. Consider the kE–cochain complexes and the corresponding kG–homomorphism $f^* : Y^* \longrightarrow X^*$. Suppose $V_E^r(Y^*) \neq V_E^r(k)$ so that for some $\lambda \in V_E^r(k)$, $\lambda \notin V_E^r(Y^*)$. Let $S \subset kE$ be a shifted cyclic subgroup corresponding to λ , so that λ belongs to the image of $\rho_S : V_S(k) \longrightarrow V_E(k)$. We need the following:

7.2. Lemma: Suppose that X^* is a connected kG–complex such that $H^j(X^*)$ is finitely generated for all j and vanishes for $j > n$. Suppose $\lambda \notin V_E^r(X^*)$ and λ corresponds to a shifted cyclic subgroup $S \subset kE$. Then $H^j(S;k) \longrightarrow H^j(S;X^*)$ is trivial for $j > n$.

Proof: By taking a resolution of X^* using kE–free chain complexes having a single non–zero group, we may embed X^* in a cochain complex $\hat{X}^*$ freely equivalent to X^* such that $H^0(X^*) \cong k$ and $H^i(\hat{X}^*) = 0$ for $i \neq n,0$. (Cf. Proposition 2.7). Then in the S–hypercohomology spectral sequence we have only two rows and the only relevant differentials to consider are $d_n^j : H^j(S;H^n(\hat{X}^*)) \longrightarrow H^{n+j+1}(S;k) \cong k$. Since $\hat{X}^*$ is freely equivalent to X^* , $V_E(\hat{X}^*) = V_E(X^*)$. Thus $\lambda \notin V_E^0(\hat{X}^*)$. This shows that $d_n^0 : H^0(S;H^n(\hat{X}^*)) \longrightarrow H^{n+1}(S;k) \cong k$ is surjective. Otherwise, $d_n^j \equiv 0$ for all $j \geq 0$ (using the multiplicative properties of the spectral sequence and the fact that all E_2–terms are modules over $H^*(S;k)$ which has periodic cohomology). This, in turn, implies that $H^*(S;k) \longrightarrow H^*(S;\hat{X}^*)$ is injective, and consequently $V_S(\hat{X}^*) \cong V_S(k)$. The latter implies that $\lambda \in V_E^0(\hat{X}^*)$, which contradicts our assumption. (Compare with 5.2.) The surjectivity of d_n^j yields the following calculation:

$$\mathbb{H}^j(S;\hat{X}^*) = \begin{cases} H^j(S;k) & \text{for } j < n \\ H^n(S;k) \oplus \operatorname{Ker} d_n^0 & \\ \operatorname{Ker} d_n^j & \text{for } j > n \end{cases} .$$

It follows that $H^j(S;k) \longrightarrow \mathbb{H}^j(S;\hat{X}^*)$ is trivial for $j > n$. Since $H^j(S;\hat{X}^*) \cong H^j(S;X^*)$ for $j > n$, the desired conclusion follows. □

By the above Lemma, $H^{n+1}(S;k) \xrightarrow{\alpha} \mathbb{H}^{n+1}(S;Y^*)$ is the zero homomorphism. Let $\phi : \mathbb{H}^{n+1}(S;Y^*) \longrightarrow \mathbb{H}^{n+1}(S;X^*)$ be the homomorphism in hypercohomology induced by f^*. Then we have a commutative diagram:

$$\begin{array}{ccc} & & \mathbb{H}^{n+1}(S;Y^*) \\ & \overset{\alpha}{\nearrow} & \\ k \cong H^{n+1}(S;k) & & \Big\downarrow \phi \\ & \underset{\beta}{\searrow} & \\ & & \mathbb{H}^{n+1}(S;X^*) \end{array}$$

which shows that $\beta \equiv 0$. On the other hand, since X^* is n–connected, β is injective, and therefore $\beta \not\equiv 0$. This contradiction establishes the theorem. □

Let X and Y be G–spaces as in the above Theorem.

<u>7.3. Corollary</u>: Suppose that $cx_G(Y) < \min\{cx_G(X), rk_p(G)\}$. Then there are no equivariant maps $f : X \longrightarrow Y$.

Proof. Since $cx_G(X) \leq rk_p(G)$, we have $cx_G(X) < rk_p(G)$. According to the proof of Corollary 3.6, there exists a $\lambda \in V_G^r(k)$ such that $\lambda \notin V_G^r(Y)$. The Corollary now follows as in the proof of 7.2. □

The following special case may be regarded as a generalization of the Borsuk–Ulam theorem.

7.4. Theorem: Suppose X and Y are G–spaces such that $\dim Y < \infty$ and $H_j(Y;\mathbb{F}_p) = 0$ for $j > n$ and $H_j(X;\mathbb{F}_p) = 0$ for $j \leq n$. Assume that G is p–elementary abelian. Then there esists an equivariant map $f : X \longrightarrow Y$ if and only if $Y^G \neq \phi$.

Proof: If $Y^G \neq \phi$, then such an f clearly exists. If $Y^G = \phi$, then $V_G(Y) \neq V_G(k)$ since G is p–elementary abelian. Now apply Theorem 7.1. □

Section Eight. An Algebraic Analogue of a Conjecture of Conner and Floyd.

In this section, we illustrate another application of the above invariants to formulate and prove a generalization of a conjecture of Conner and Floyd. To motivate the statement, assume first that X is a finite dimensional G–CW complex and Y is a G–invariant subcomplex. Let $C_*(-)$ denote the cellular chains, so that $C_*(X)$ etc. may be considered as permutation complexes. Then, the statement $(X-Y)^G \neq \phi$ for $G = (\mathbb{Z}_p)^m$, is the same as $V_G^r(C_*(X)/C_*(Y)) = V_G^r(k)$ by the results of Section Five. Therefore, if X_* is a kG–chain complex, and Y_* is a kG–subcomplex, then the statement $V_G^r(X_*/Y_*) = V_G^r(k)$ is an algebraic substitute for the geometric statement $(X-Y)^G \neq \phi$. Moreover, the condition $V_G^r(X_*/Y_*) = V_G^r(k)$ depends only on the free equivalence class of X_*/Y_*, and it is meaningful even if X_* and Y_* are infinite dimensional. Further, since $V_G^r(X_*/Y_*) = V_G(X_*/Y_*)$, we may assume that X_* and Y_* are free kG–complexes without loss of generality. In this way, we may formulate algebraic analogues of the well–known fixed–point theorems as in Section Seven and in the following.

Using their cobordism theoretic methods, Conner and Floyd proved in [26] that if $G = (\mathbb{Z}_2)^m$ acts smoothly on a connected positive dimensional closed smooth manifold X, then X^G cannot consist of one point only. They conjectured that a similar result should hold if $G = \mathbb{Z}_{p^r}$, p is odd, and X is orientable. In their classical paper [13], Atiyah and Bott applied their version of Lefschetz fixed point formula for elliptic complexes to prove the Conner–Floyd conjecture: If $f : X \longrightarrow X$ is a periodic diffeomorphism of the compact oriented Riemannian positive dimensional manifold of period p^r, where p is an odd prime, then the fixed–point set of f cannot consist of one point only. Conner and Floyd also proved their conjecture using cobordism arguments [27]. When G is an abelian p–group, p = odd prime, Browder in [21] proved by different techniques that the conclusion of the Conner–Floyd conjecture holds in this case for smooth G–action on orientable closed manifolds. According to Ewing–Stong [30] the smoothness of the G–action and the validity of the Conner–Floyd conjecture for $G = \mathbb{Z}_{p^r}$ implies a similar result for all abelian p–groups. On the other hand, if G is not an abelian p–group (p = odd prime), smooth actions with one fixed point exist as demonstrated in Browder [21] and Ewing–Stong [30]. In the following, we assume that $G = (\mathbb{Z}_p)^m$ and formulate an analogue of the Conner–Floyd conjecture which uses a weak version of Poincaré duality on the level of chain complexes. The connectivity assumption in Theorem 8.1 is not necessary, but the proof will be more involved.

Let C_* be a connected kG–chain complex such that $C_i = 0$ for $i < 0$ and $H_0(C_*) \cong k$. We say that C_* satisfies Poincaré duality in formal dimension d, if there exists kG–homomorphisms $h_i : C_{d-i} \longrightarrow C^i$ which induce a chain homotopy equivalence between C^* and C_* (with the indicated dimension shift). For example, the cellular chain complex of a Poincaré duality complex of formal dimension d, or the singular chain complex of a closed oriented topological manifold (with coefficients in k) are chain complexes with Poincaré duality. If $d = 2n$ and $H_i(C_*) = 0$ for $0 < i < n$, then $H_j(C_*) = 0$ for $n < j < 2n$ and the isomorphism $h_* : H_n(C_*) \longrightarrow H^n(C^*) \cong \mathrm{Hom}(H_n(C_*),k)$ allows us to define the pairing $\eta : H^n(C^*) \otimes H^n(C^*) \longrightarrow H^{2n}(C^*) \cong k$ via $\alpha \otimes \beta \longmapsto \langle \alpha, h_*^{-1}(\beta) \rangle [C^*]$ where $[C^*] \in H^{2n}(C^*) \cong k$ is a fixed fundamental class (i.e. a generator) and $\langle\ ,\ \rangle$ is the evaluation (Kronecker pairing). As usual, η is a non–degenerate (± 1)–symmetric pairing, and it is this structure of $H^*(C^*)$ which we will use to prove the following version of the Conner–Floyd conjecture.

8.1. Theorem. Assume that p is an odd prime, and $G = (\mathbb{Z}_p)^m$. Suppose X_* is an $(n-1)$–connected kG–chain complex (possibly infinite dimensional) with Poincaré duality in formal dimension $2n > 0$. Let $Y_* \subset X_*$ be an acyclic subcomplex, i.e. $H_*(Y_*) \cong k$. Then $V_G^r(X_*,Y_*) = V_G^r(k)$.

Proof: To prove the assertion of the theorem, it suffices to show that for each $\lambda \in V_G^r(k)$ and each shifted cyclic subgroup $S \cong \mathbb{Z}_p$ of kG the corresponding kS–chain complexes X_* and Y_* satisfy $V_S^r(X_*,Y_*) \neq 0$, i.e. $\lambda \in V_G^r(X_*,Y_*)$. Thus, we need to prove 8.1 for the special case $G = \mathbb{Z}_p$, as in the following lemma:

8.2. Lemma. Assume that $G = \mathbb{Z}_p$, and (X_*,Y_*) satisfy the hypotheses of 8.1 above. Then $V_G^r(X_*,Y_*) \neq 0$.

Proof of 8.2.: We show that the assumption $V_G^r(X_*,Y_*) = 0$ leads to a contradiction. Let $C^* \overset{\text{def}}{\equiv} \mathrm{Hom}_k(X_*/Y_*,k)$ be the dual cochain complex. By Proposition 2.7 $V_G(C^*) = V_G^r(C^*) = 0$ so that $\mathbb{H}^*(G;C^*)\left[\frac{1}{t}\right] = 0$, i.e. the polynomial generator $t \in H^2(G;k)$ acts nilpotently on $\mathbb{H}^*(G;C^*)$. We need to prove a few lemmas:

8.3. Lemma. $\mathbb{H}^*(G;X^*) \cong \mathbb{H}^*(G;C^*) \oplus \mathbb{H}^*(G;Y^*)$.

Proof of 8.3.: Let $j : Y_* \longrightarrow X_*$ be the inclusion, and $j^{\#} : X^* \longrightarrow Y^*$ be the corresponding surjection on the associated cochain complexes, and $j_G^* : \mathbb{H}^*(G;X^*) \longrightarrow \mathbb{H}^*(G;Y^*)$ the induced homomorphism on hypercohomologies. Consider the following commutative diagram:

$$\begin{array}{ccc}
H^*(G;X^*) & \underset{\overset{\tau}{\longleftarrow - - -}}{\xrightarrow{\;j_G^*\;}} & H^*(G;Y^*) \\
\alpha \uparrow & & \cong \uparrow \beta \\
H^*(G;H^0(X^*)) & \xrightarrow[u]{\cong} & H^*(G,H^0(Y^*)) \\
& \cong \nwarrow \qquad \nearrow \cong & \\
& H^*(G;k) &
\end{array}$$

where α and β are edge–homomorphisms in the hypercohomology spectral sequence. Define $\tau = \alpha \circ u^{-1} \circ \beta^{-1}$. Then $j_G^* \circ \tau =$ identity, and τ gives a splitting in the long exact sequence of hypercohomology associated to the short exact sequence $0 \longrightarrow C^* \longrightarrow X^* \xrightarrow{j^\#} Y^* \longrightarrow 0$ of kG–cochain complexes. This proves the lemma. □

<u>8.4. Lemma</u>. Let $J \subset H^*(G;k)$ be the ideal of nilpotent elements, and $R \equiv H^*(G;k)/J \cong k[t]$ the reduced commutative k–algebra with the quotient field $K \cong k(t)$. For any graded $H^*(G;k)$–module M^*, let $M^*_{red} \overset{def}{\equiv} M^*/JM^*$ be the associated (reduced) R–module. Then $\dim_K H^*(G;X^*)_{red} \otimes_R K = 1$.

<u>Proof</u>: Since $H^*(G;C^*)\left[\frac{1}{t}\right] = 0$ by the above discussion, the localized homomorphism $j_G^*\left[\frac{1}{t}\right]$ induces an isomorphism $H^*(G;X^*)\left[\frac{1}{t}\right] \cong H^*(G;Y^*)\left[\frac{1}{t}\right] \cong H^*(G;k)\left[\frac{1}{t}\right]$ (by acyclicity of Y^*). Reduction modulo J and localization commute, so that $(H^*(G;X^*)_{red})\left[\frac{1}{t}\right] \cong (H^*(G;Y^*)_{red})\left[\frac{1}{t}\right] \cong (H^*(G;k)_{red})\left[\frac{1}{t}\right] \cong k[t,t^{-1}]$. Therefore, $H^*(G;X^*)_{red} \otimes_R K \cong (H^*(G;X^*)_{red})\left[\frac{1}{t}\right] \otimes_{k[t,t^{-1}]} K \cong K$ as desired. □

<u>8.5. Lemma</u>. If $H^i(X^*)$ is kG–projective for all $0 < i < 2n$, then $V_G(X^*,Y^*) \neq 0$.

Proof: In the hypercohomology spectral sequence $E_2^{i,j} = H^i(G;H^j(C^*)) \Rightarrow \mathbb{H}^*(G;C^*)$, the only non–vanishing terms are $E_2^{*,n} = H^*(G;H^n(X^*)) \cong H^*(G;k)$. Hence $\mathbb{H}^*(G;C^*) \cong H^{*-n}(G;k)$ as $H^*(G;k)$–modules, and the annihilating ideal of $\mathbb{H}^*(G;C^*)$ consists of nilpotent elements. □

8.6. Lemma. Consider the homomorphism
$\gamma : \bigoplus_{i,j} H^i(G;H^n(C^*)) \otimes H^j(G;H^n(C^*)) \longrightarrow H^{i+j}(G;k)$ which is induced by the duality pairing $\mu : H^n(C^*) \otimes H^n(C^*) \longrightarrow k$ and cup product in group cohomology. Then γ is surjective.

Proof: By the previous lemma, we may assume that $H^n(C^*)$ is not kG–projective . Let M_1 be a non–projective indecomposable direct summand. The duality isomorphism $D : H^n(C^*) \xrightarrow{\cong} \mathrm{Hom}(H^n(C^*),k)$ gives rise to an indecomposable direct summand $M_2 \subseteq H^n(C^*)$ such that $D(M_1) = \mathrm{Hom}(M_2,k)$ using the Krull–Schmidt–Azumaya theorem. If $M_1 \neq M_2$, then choose $M = M_1 \oplus M_2$. Otherwise, let $M = M_1$. Then $M \subseteq H^n(C^*)$ and the restriction $\mu | M \otimes M$ yields the duality isomorphism $D|M : M \xrightarrow{\cong} \mathrm{Hom}(M,k)$. This allows us to identify $\mu|M \otimes M : M \otimes M \longrightarrow k$ with the trace homomorphism $\tau : \mathrm{End}_k(M) \longrightarrow k$ where for any $f \in \mathrm{End}_k(M) \cong M \otimes \mathrm{Hom}_k(M,k) \cong M \otimes M$, $\tau(f) = \mathrm{trace}(f)$ as a k–linear homomorphism. Since G acts on $H^n(C^*)$ by isometrics, τ is a kG–isomorphism . On the other hand, $\dim M_1 \equiv 0 \bmod p$, since M_1 is indecomposable and non–projective and $G = \mathbb{Z}_p$. Since p is odd, $\dim M = 2\dim M_1 \equiv 0 \bmod p$ as well. Therefore, α admits a right inverse over kG , $\rho : k \longrightarrow \mathrm{End}_k(M)$ given by $\rho(1) = (1/\dim M)(\mathrm{id})$, where $\mathrm{id} : M \longrightarrow M$ is the identity and G acts trivially on k . This shows that $M \otimes M \cong k \oplus \ker(\tau)$, and the induce homomorphism $H^i(G;M \otimes M) \longrightarrow H^i(G;k)$ is split surjective. On the other hand, using the periodicity of group cohomology for

$G = \mathbb{Z}_p$ in the form $\hat{H}^*(G;M) \cong \hat{H}^*(G;k) \otimes [\hat{H}^0(G;M) \oplus H^1(G;M)]$ and the surjectivity $\oplus \hat{H}^i(G;M) \otimes \hat{H}^j(G;M) \longrightarrow \hat{H}^{i+j}(G;M \otimes M)$ of the cup product in cohomology for small values $0 \leq i + j \leq 2$ by direct inspection, we conclude that the cup product $\eta : \bigoplus_{i,j} H^i(G;M) \otimes H^j(G;M) \longrightarrow H^{i+j}(G;M \otimes M)$ is also surjective onto the factor $H^{i+j}(G;k)$ in the decomposition $M \otimes M = k \oplus \operatorname{Ker} \tau$. Therefore the composition $\gamma = \tau_* \circ \eta$ on $\bigoplus_{i,j} H^i(G;M) \otimes H^j(G;M)$ is surjective as claimed. □

<u>Completion of the proof of 8.2.</u> In the hyper–cohomology spectral sequence $E_2^{i,j} = H^i(G;H^j(C^*)) \Rightarrow H^*(G;C^*)$ all terms are modules over $H^*(G;k)$, and the differentials are $H^*(G;k)$–linear (cf. 1.3 and its references). Here, there are only two non–trivial rows corresponding to $H^j(C^*)$ for $j = n$ and $2n$. $E_2^{*,2n} \cong H^*(G;k)$ and $E_2^{*,n} \neq 0$ by Lemma 8.5 and our assumption $V_G^r(X^*,Y^*) = 0$ in the beginning of the proof of 8.2 above. The Poincaré duality of X^* allows us to define the pairing $E_2^{i,n} \otimes E_2^{j,n} \longrightarrow E_2^{i+j,2n}$. Since for G–modules M_1 and M_2 , in the pairing $C^i(G;M_1) \otimes C^j(G;M_2) \longrightarrow C^{i+j}(G;M_1 \otimes M_2)$ the diagonal approximation of $C^*(G) = C^*(BG)$ is used, we have a "Leibniz rule" in $C^*(G;-)$ of the form $d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes \beta$, see Brown [22] and [25]. In the hypercohomology spectral sequence set–up, we have a pairing between (a priori different) three spectral sequences and the corresponding differentials satisfy a similar Leibniz rule for the same reason: $E_r(M_*') \otimes E_r(M_*'') \longrightarrow E_r(M_*' \otimes M_*'')$ and $d_r(b' \otimes b'') = d_r(b') \otimes b'' \pm b \otimes d_r(b')$, (cf. [25]). We use this structure to show that the first possibly non–trivial differential $d_{n+1} : E_{n+1}^{*,2n} \longrightarrow E_{n+1}^{*+n,n}$ also vanishes on the polynomial generators $t^r \in H^{2r}(G;H^{2n}(C^*)) = E_{n+1}^{2r,2n}$. Consider $\gamma : H^i(G;H^n(C^*)) \otimes H^j(G;H^n(C^*)) \longrightarrow H^{i+j}(G;k) \cong E_{n+1}^{i+j,2n}$ which is surjective by Lemma 8.6, and let $\gamma(b' \otimes b'') = t$. Then

$d_{n+1}(t) = d_{n+1}(\gamma(b' \otimes b'')) = \gamma[d_{n+1}(b') \otimes b'' \pm b' \otimes d_{n+1}(b'')] = 0$ since

$d_{n+1}(b') = 0 = d_{n+1}(b'')$ for degree reasons. Similarly, $d_{n+1}(t^r) = 0$ for all $r \geq 1$ as claimed. This in turn implies that $(E_\infty^{*,2n})_{red} \neq 0$, and consequently $H^*(G;C^*)_{red} \supseteq k[t]$. Together with Lemma 8.3, we conclude $\dim_K H^*(G;X^*)_{red} \otimes_R K \geq 2$. But this contradicts Lemma 8.4, and this contradiction shows that Lemma 8.2 is true. This completes the proof of the Theorem. □

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