TESTING THE COHEN-MACAULAY PROPERTY UNDER BLOWING UP

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INTRODUCTION. Let X be an algebraic variety and let $X' \longrightarrow X$ be a blowing up of X with arbitrary center Y. In general, the Cohen-Macaulay properties of X and X' are totally unrelated: If X is Cohen-Macaulay and Y is permissible, X' need not be Cohen-Macaulay [15]; and if X is not Cohen-Macaulay, X' can be made Cohen-Macaulay by a suitable choice of Y [1], [2]. Replacing X by a local ring R and Y by an ideal I of R, we try to relate the Cohen-Macaulay property of R to the Cohen-Macaulay property of the Rees ring $R\overset{+}{e}(I,R) = \bigoplus_{n\geq 0} I^n \simeq R[It]$, and of $X' = \mathrm{Proj}(R\overset{+}{e}(I,R))$.

One line of thought is this: Given some ideal I of R, which may be thought of as a "testideal"; what can we say about blowing ups defined by other ideals J containing I? We restrict our investigations to a certain class of ideals I which we call equimultiple, and which are a common generalization of the two most important classical cases: 1) I is permissible (in the sense of Hironaka, e.g. the maximal ideal), 2) I is an ideal of the principal class. From the algebraic point of view, this class of ideals is characterized by the fact that $gr_I(R)$ has a homogeneous system of parameters, at least in the equidimensional case (see [11]). These properties of equimultiple ideals are essential in the proof of theorem 3.1.

In section 2 we describe the influence of the multiplicity $e(R/I)$ of R/I on the behaviour of $R\overset{+}{e}(M,R)$. In section 3 we compare the Rees rings of I and $I + \underline{x}R$, where $\underline{x}$ is a part of a system of parameters mod I . For this situation we prove a transitivity property for the Cohen-Macaulayness of the Rees rings (and the graded rings $\oplus I^n/I^{n+1}$), assuming that R itself is Cohen-Macaulay. This last assumption is necessary, as we show in theorem 3.8. This theorem and proposition 2.1 indicate that it will be somewhat complicated to construct examples of non-Cohen-Macaulay rings R with Cohen-Macaulay Rees rings $R\overset{+}{e}(I,R)$, at least if $\dim R \geq 3$. We give several examples for R Cohen-Macaulay as well as for R non-Cohen-Macaulay, in which the Cohen-Macaulay property of $R\overset{+}{e}(I,R)$ is tested for various ideals I . In the last section 4 we asked the same question as before in theorem 3.8 for the geometric blowing ups Proj $R\overset{+}{e}(I,R)$ and Proj $R\overset{+}{e}(J,R)$.

<u>1. NOTATIONS</u>. A) For any system $\underline{x} = \{x_1,\dots,x_r\}$ of parameters with respect to $I \subset R$ one has a numerical function $H^{(0)}(n) = e(\underline{x},I^n/I^{n+1})$, where $e(\ ,\ )$ denotes the multiplicity symbol of Wright and Northcott. We know by [7] that $H^{(1)}(n) = \sum_{i=0}^{n} H^{(0)}(i) = \sum_{P\in \mathrm{Assh}(R/I)} e(\underline{x};R/P)\cdot H^{(1)}[IR_P](n)$,

where $Assh(R/I) = \{P \in Ass(R/I) \ / \ \dim R/p = \dim R/I\}$ and $H^{(1)}[IR_p]$ is the usual Hilbert-Samuel function of the PR_p-primary ideal IR_p . For large n, $H^{(1)}(n)$ is a polynomial in n with rational coefficients. If d is the degree and a_d the highest coefficient of this polynomial, the number $e(\underline{x},I,R) := d!a_d$ is called the multiplicity of I with respect to $\underline{x}$. If $ht(I) = \dim R - \dim R/I$, then

$$e(\underline{x},I,R) = \sum_{P \in Assh(R/I)} e(\underline{x};R/P)\ e(IR_p) \quad ,$$

where $e(IR_p)$ is the Samuel multiplicity of IR_p .

B) Let I be a proper ideal in the local ring R . Then we define here the reduction exponent $r(I)$ of I as

$$r(I) = \inf\{n / \text{ there exists a minimal reduction } J \text{ of } I \text{ such that } I^n = JI^{n-1}\} .$$

C) I is said to be equimultiple, if $ht(I) = \ell(I)$. R is said to be normally Cohen-Macaulay along I if $depth(I^n/I^{n+1}) = \dim(R/I)$ for all $n \geq 0$. If $\dim R = \dim R/I + ht(I)$ then this condition implies equimultiplicity $ht(I) = \ell(I)$, s. [9].

D) An ideal I is said to be a complete intersection if it is generated by $ht(I)$ elements. I is said to be a generic complete intersection if IR_p is a complete intersection for all minimal primes P of I .

2. TESTIDEALS OF SMALL MULTIPLICITY

In general if $R\overset{+}{e}(I,R)$ is Cohen-Macaulay for some I then R need not be Cohen-Macaulay. For the case $I = M$ we know by [12] that depth $R \geq 2$ if $R\overset{+}{e}(M,R)$ is Cohen-Macaulay and $\dim R \geq 2$. So for $\dim R = 2$, R must be Cohen-Macaulay. This is no longer true for $\dim R \geq 3$ (see example 2.3). One result of this section (s. proposition 2.9) shows that by restricting the multiplicity of certain testideals the Cohen-Macaulay property of R follows from the same property of $R\overset{+}{e}(I,R)$. First we need a preliminary result.

PROPOSITION 2.1: Let (R,M) be a local ring such that $R\overset{+}{e}(M,R)$ is Cohen-Macaulay. If $e(R) < \dim R$, then R is Cohen-Macaulay.

PROOF: Since $R\overset{+}{e}(M,R)$ is Cohen-Macaulay, R must be a Buchsbaum ring by [12], theorem 0.1. Therefore we know by [5] the following inequality

$$(*) \quad e(R) \geq 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} h^i(R) \quad ,$$

where $h^i(R)$ is the dimension of the cohomology module $H^i_M(R)$. Since depth $R \geq 2$ we get $h^0(R) = h^1(R) = 0$. Then the assumption $e(R) < \dim R$ implies also $h^i(R) = 0$ for $2 \leq i \leq d-1$.

<u>COROLLARY 2.2</u>: Let (R,M) be a local ring with $e(R) \lneq \dim R$. Then the following conditions are equivalent:

(i) $R\overset{+}{e}(M,R)$ is Cohen-Macaulay.

(ii) (R and) $gr_M R$ is Cohen-Macaulay.

<u>PROOF</u>: For $\dim R = 2$ the implication (i) $\Rightarrow$ (ii) is true without any assumption on $e(R)$.
If $\dim R \geq 3$, then (i) $\Rightarrow$ (ii) follows from proposition 2.1 and [11], theorem 4.8.
The implication (ii) $\Rightarrow$ (i) is true for $e(R) \leq \dim R$ by Corollary 5.4 in [11].

The following example 2.3 shows that for $e(R) = \dim R$ the equivalence of (i) and (ii) is not true in general.

<u>EXAMPLE 2.3</u>: $R = k[[X_1,X_2,X_3,Y_1,Y_2,Y_3]]/(X_1Y_1+X_2Y_2+X_3Y_3,(Y_1,Y_2,Y_3)^2)$, where k is a field, and X_i, Y_j are indeterminates. This ring is a non-Cohen-Macaulay Buchsbaum ring with $e(R) = \dim R = 3$, and $R\overset{+}{e}(M,R)$ is Cohen-Macaulay, see [20].

<u>REMARK 2.4</u>: a) If $e(R) = \dim R$ and $R\overset{+}{e}(M,R)$ is Cohen-Macaulay, then R is not too far from being Cohen-Macaulay. For if R is not Cohen-Macaulay, at most two cases are possible for $h^i = h^i(R)$:

case 1: $h^2 = 1$; $h^0 = h^1 = h^3 = \ldots = h^{d-1} = 0$

case 2: $h^{d-1} = 1$; $h^0 = h^1 = \ldots = h^{d-2} = 0$.

b) Assume that $\overset{+}{Re}(M,R)$ is Cohen-Macaulay again. Then we have:

a) If R is not Cohen-Macaulay then $e(R) \geq \dim R$ by proposition 2.1.

b) If R is a hypersurface (i.e. R is unmixed and $\operatorname{emdim} R \leq \dim R + 1$), then $e(R) \leq \dim R$ by [11], Cor. 5.5.

c) For any Cohen-Macaulay ring R the Cohen-Macaulayness of $\overset{+}{Re}(M,R)$ doesn't imply a special inequality between $e(R)$ and $\dim R$, as the following two examples show.

<u>EXAMPLE 2.5</u>: $R = k[[X^2,XY,Y^2,XZ,YZ,Z]]$, k a field, X,Y,Z indeterminates. R is a Cohen-Macaulay ring, see [11]. Since $(X^2,Y^2,Z)M = M^2$ we know [17], that $gr_M R$ is Cohen-Macaulay, hence $\overset{+}{Re}(M,R)$ is Cohen-Macaulay by [11], thm. 4.8. Furthermore we see that $e(R) = \operatorname{emdim} R - \dim R + 1 = 4$, i.e. $e(R) > \dim R$.

<u>EXAMPLE 2.6</u>: $R = k[[X]]/I_2(X)$, where $X = (X_{ij})$ is the 2×3 matrix of indeterminates X_{ij} over a field k and $I_2(X)$ is the ideal generated by the 2×2 minors of X. Then R is Cohen-Macaulay, $e(R) = 3 < \dim R = 4$, and $\operatorname{emdim} R = 6$. Therefore we have $e(R) = \operatorname{emdim} R - \dim R + 1$, i.e. $M^2 = (\underline{a})M$ [17], where $\underline{a}$ is a minimal reduction of M. The same argument as in example 2.5 shows that $\overset{+}{Re}(M,R)$ is Cohen-Macaulay.

To make use of testideals the following auxiliary result is needed.

<u>LEMMA 2.7</u>: Let (R,M) be a local ring. If I is an equimultiple ideal in R which is a generic complete intersection then $e(R/I) \geq e(R)$.

<u>PROOF</u>: The condition $ht(I) = \ell(I)$ implies by [8], [9] the equality $e(\underline{x},I,R) = e(I + \underline{x}R)$ for any system $\underline{x}$ of parameters of I . By assumption, IR_P is a parameter ideal for all minimal primes P of I . Therefore we have

$$e(\underline{x},I,R) = \sum_{P\in Min(I)} e(\underline{x},R/P)\cdot e(IR_P) \leq \sum_{P\in Min(I)} e(\underline{x},R/P)\cdot \ell(R_P/IR_P) \quad ,$$

where $Min(I)$ denotes the set of minimal primes of I . Hence we get: $e(\underline{x},I,R) \leq e(\underline{x},R/I)$.
Choosing a special system $\underline{x}$ of parameters for I which satisfies $e(\underline{x},R/I) = e(R/I)$ we have finally:

$$e(R) \leq e(I + \underline{x}R) = e(\underline{x},I,R) \leq e(R/I) \ .$$

<u>REMARK</u>: If in the lemma (R,M) is a Cohen-Macaulay ring with infinite residue field R/M , then I is a complete intersection already. This can be seen as follows:

Let $a_1,\dots,a_t$ be a minimal reduction of I with $t = ht(I)$. For $J := (a_1,\dots,a_t) \subset I$ we have JR_P is a minimal reduction of IR_P for all $P \in Min(I) = Min(J)$. By assumption IR_P is a complete intersection in R_P . Therefore, it has no proper minimal reduction by [14] § 4, thm. 4, hence $JR_P = IR_P$. Since J is an ideal of the principal class in a Cohen-Macaulay local ring, it is height-unmixed. So we have the following primary decompositions for I and J

$$I = Q_1 \cap \ldots \cap Q_n \cap Q_\ell$$
$$J = Q_1 \cap \ldots \cap Q_n$$

where the $Q_1,\ldots,Q_n$ are primary ideals associated to the $P_1,\ldots,P_n \in \mathrm{Min}(I)$ and Q_ℓ contains all embedded components of I. Hence we get $I = J$.

PROPOSITION 2.8: Let (R,M) be a local ring with a Cohen-Macaulay Rees ring $R\overset{+}{e}(M,R)$. Let I be an equimultiple ideal which is a generic complete intersection. If $e(R/I) < \dim R$, then R and $gr_M R$ are Cohen-Macaulay.

PROOF: Use lemma 2.7 and corollary 2.2.

A result similar to proposition 2.8 is the following one.

PROPOSITION 2.9: Let R be a local ring and let I be a complete intersection in R such that $R\overset{+}{e}(I,R)$ is Cohen-Macaulay and $e(R/I) = e(R)$. Then R is Cohen-Macaulay.

PROOF: 1) If $\dim R/I = 0$, we have $e(R) = e(R/I) = \ell(R/I)$, hence R is Cohen-Macaulay. [Here we don't use $R\overset{+}{e}(I,R)$ is Cohen-Macaulay.]

2) In the general case we may assume that R has an infinite residue field. Let $I = (y_1,\ldots,y_s)$ and let $x_1,\ldots,x_r$ be a system of parameters mod I such that

$\bar{x}_1,\dots,\bar{x}_r \in R/I$ form a minimal reduction of M/I in R/I. We put $\bar{R} = R/\underline{x}R$, $\underline{x} = x_1,\dots,x_r$.
$R\overset{+}{e}(I,R)$ Cohen-Macaulay implies that R is normally Cohen-Macaulay along I. Therefore $\underline{x}$ is a regular sequence on I^n/I^{n+1} for $n \geq 0$, hence on R too.
Note that $e(R/I) = e((\bar{x}_1,\dots,\bar{x}_r))$ since $(\bar{x}_1,\dots,\bar{x}_r)$ is a minimal reduction of M/I. Furthermore $e((\bar{x}_1,\dots\bar{x}_r)) = \ell(R/I+\underline{x}R) = e(R/I+\underline{x}R) \geq e(R/\underline{x}R) \geq e(R)$ since R/I is Cohen-Macaulay. Therefore $e(\bar{R}) = e(\bar{R}/I\bar{R})$, i.e. $\bar{R}$ is Cohen-Macaulay by step 1, hence R is Cohen-Macaulay.

3. TRANSITIVITY OF COHEN-MACAULAYNESS FOR REES RINGS

Now we assume that the given ring R is Cohen-Macaulay. Then we consider equimultiple ideals $J \subset I$ such that $I = J + \underline{x}R$, where $\underline{x}$ is part of a system of parameters mod J. For simplicity we are always working with an infinite residue field.

THEOREM 3.1: (Transitivity of Cohen-Macaulay property.) Let (R,M) be a local Cohen-Macaulay ring with infinite residue field. Let J be an equimultiple ideal of R, let $\underline{x} = (x_1,\dots,x_s)$ be a part of a system of parameters mod J and let $I = J + \underline{x}R$.

a) The following conditions are equivalent:

(i) $gr_J(R)$ is Cohen-Macaulay.

(ii) $gr_I(R)$ is Cohen-Macaulay, and $gr_{JR_P}(R_P)$ is Cohen-Macaulay for all $P \in Min(I)$.

b) If $ht(J) > 0$, the following conditions are equivalent:

(i) $\overset{+}{Re}(J,R)$ is Cohen-Macaulay.

(ii) $\overset{+}{Re}(I,R)$ is Cohen-Macaulay, and $\overset{+}{Re}(JR_P,R_P)$ is Cohen-Macaulay for all $P \in Min(I)$.

<u>PROOF</u>: a) Let $\underline{y}$ be a system of parameters mod I . Then $\underline{x} \cup \underline{y}$ is a system of parameters mod J .

(i) $\Rightarrow$ (ii) Clearly $gr_{JR_P}(R_P) \simeq gr_J(R) \otimes R_P$ is Cohen-Macaulay. By [11], Prop. 4.5, $gr_J(R)$ is Cohen-Macaulay if and only if $gr_{J+\underline{x}R+\underline{y}R}(R)$ is Cohen-Macaulay and R is normally Cohen-Macaulay along J . This implies that R is normally Cohen-Macaulay along I ([7], Satz 4.2, p. 132). Using $gr_{J+\underline{x}R+\underline{y}R}(R) = gr_{I+\underline{y}R}(R)$ we see that $gr_I(R)$ is Cohen-Macaulay (by [11], Prop. 4.5 again).

(ii) $\Rightarrow$ (i) By [7], Satz 4.2, p. 132 R is normally Cohen-Macaulay along J , and $gr_{J+\underline{x}R+\underline{y}R}(R) = gr_{I+\underline{y}R}(R)$ is Cohen-Macaulay, so $gr_J(R)$ is Cohen-Macaulay.

b) By [11], thm.4.8, we know that $\overset{+}{Re}(J,R)$ is Cohen-Macaulay if and only if $gr_J(R)$ is Cohen-Macaulay and $r(J) \le ht(J)$.

(i) $\Rightarrow$ (ii) Obviously we have $r(I) \le r(J) \le ht(J) \le ht(I)$, and also $r(JR_P) \le r(J) \le ht(J) = ht(JR_P)$. Therefore the assertion follows from a), (i) $\Rightarrow$ (ii) .

(ii) $\Rightarrow$ (i) By a) and [11], thm.4.8, we have to show that $r(J) \le ht(J)$. Equivalently, taking any minimal reduction J' of J and putting $t = ht(J)$, we have to show that $J^t \subset J'$ (compare [11], thm. 4.8). Note that R/J' is

Cohen-Macaulay, and therefore $\mathrm{Ass}(R/J') = \mathrm{Min}(J)$. So we are reduced to prove that $J^t R_Q \subset J'R_Q$ for all $Q \in \mathrm{Min}(J)$. Now if $Q \in \mathrm{Min}(J)$, we claim that $Q \subset P$ for some $P \in \mathrm{Min}(I)$. Otherwise we would have $Q \not\subset \bigcup_{P \in \mathrm{Min}(I)} P$, and therefore Q would contain an element y which is a non-zerodivisor mod I . Since R/J is Cohen-Macaulay, any non-zerodivisor mod I is also a non-zerodivisor mod J , which gives a contradiction to $Q \in \mathrm{Min}(J)$. Now given $P \in \mathrm{Min}(I)$ such that $Q \subset P$, we know from assumption (ii) that $J^t R_P \subset J'R_P$, and a forteriori $J^t R_Q \subset J'R_Q$, which completes the proof.

A class of examples is given by the following corollary.

COROLLARY 3.2: Let (R,M) be a Cohen-Macaulay ring and let P be an ideal in R such that R/P is regular and $e(R) = e(R_P)$ i.e. $\mathrm{ht}(P) = \ell(P)$ by [8]. If $\overset{+}{\mathrm{Re}}(P,R)$ is Cohen-Macaulay then $\overset{+}{\mathrm{Re}}(QR_Q,R_Q)$ is Cohen-Macaulay for all prime ideals $Q \supset P$; in particular $\overset{+}{\mathrm{Re}}(M,R)$ is Cohen-Macaulay.

Assume that $\overset{+}{\mathrm{Re}}(P,R)$ is Cohen-Macaulay for some equimultiple ideal P such that R/P is regular. In order to apply Corollary 3.2 to conclude that $\overset{+}{\mathrm{Re}}(M,R)$ is Cohen-Macaulay, we need to show that R is Cohen-Macaulay. Some results in this direction are given in the next two propositions.

PROPOSITION 3.3: Let P be an equimultiple ideal in (R,M) such that $\overset{+}{\mathrm{Re}}(P,R)$ is Cohen-Macaulay. If R/P is regular and $\mathrm{ht}(P) \leq 2$ then R and $\overset{+}{\mathrm{Re}}(M,R)$ are Cohen-Macaulay.

<u>PROOF</u>: <u>1.case</u>: $ht(P) = 1$. Then P is generated by one element f , s. [10], proposition 1.5. This implies R is regular, since $M = fR + (x_1,\dots,x_{d-1})R$, where $x_1,\dots,x_{d-1}$ form a regular system of parameters mod P .

<u>2.case</u>: $ht(P) = 2$. By assumption we have $M = P + \underline{x}R$, where $\underline{x} = (x_1,\dots,x_r)$ is a system of parameters mod P . Since $R\overset{+}{e}(P,R)$ is Cohen-Macaulay and $ht(P) = \ell(P)$, R must be normally Cohen-Macaulay along P, s. [10]. Therefore $\underline{x}$ is a regular sequence on P^n/P^{n+1} for $n \geq 0$, hence on R too. Moreover putting $\overline{R} = R/\underline{x}R$ and $\overline{M} = M/\underline{x}R$, we know that $R\overset{+}{e}(\overline{M},\overline{R}) \cong R\overset{+}{e}(P,R)/\underline{x}\,R\overset{+}{e}(P,R)$ is Cohen-Macaulay, i.e. depth $\overline{R} \geq 2 = \dim \overline{R}$, so $\overline{R}$ and R must be Cohen-Macaulay. Then $R\overset{+}{e}(M,R)$ is Cohen-Macaulay by theorem 3.1.

<u>PROPOSITION 3.4</u>: Let $P \neq M$ be an equimultiple ideal in (R,M) with $ht(P) \geq 2$. Assume that

(i) $R\overset{+}{e}(P,R)$ is Cohen-Macaulay

(ii) R/P is regular

(iii) $e(R) = 2$.

Then R and $R\overset{+}{e}(M,R)$ are Cohen-Macaulay.

<u>PROOF</u>: We may assume by [8] that $M = P + \underline{x}R$, where $\underline{x} = (x_1,\dots,x_r)$ is a sequence of superficial elements with $e(R/\underline{x}R) = e(R) = 2$, $r = \dim R/P$.

Putting $\overline{R} = R/\underline{x}R$ and $\overline{M} = M/\underline{x}R$ as in the proof of proposition 3.3., we see again that $R\overset{+}{e}(\overline{M},\overline{R})$ is Cohen-Macaulay. Hence $\overline{R}$ is a Buchsbaum ring of multiplicity 2, which satisfies the Serre condition S_2. Using the inequality (*) in section 2, we get $h^i(\overline{R}) = 0$ for $i \neq \dim \overline{R}$. Therefore $\overline{R}$ and R are Cohen-Macaulay rings, proving that $R\overset{+}{e}(M,R)$ is Cohen-Macaulay by theorem 3.1.

<u>PROPOSITION 3.5</u>: Let (R,M) be a Buchsbaum ring of dimension $d \geqq 3$ with an algebraically closed residue field k. Let $P \neq M$ be an equimultiple prime ideal in R such that

(i) $R\overset{+}{e}(P,R)$ is Cohen Macaulay

(ii) $P^* = gr_M(P,R)$ [1] is prime .

If $e(R) = 3$ then R and $R\overset{+}{e}(M,R)$ are Cohen-Macaulay rings.

<u>PROOF</u>: Condition (i) tells us that $\operatorname{depth} R \geqq \dim R/P + 1$ by [10], proposition 1.5. Therefore R satisfies Serre's condition S_2. The high point of proof is to show that R is Cohen-Macaulay. For that we use the sharp relation (see [5])

$$(**) \qquad e(R) = 1 + \ell(M/J) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} h^i(R) \quad ,$$

[1] The ideal of the initialforms of P with respect to M.

where $J = \sum_{i=1}^{d} (x_1,\ldots,\check{x}_i,\ldots,x_d) : x_i$ and $(x_1,\ldots,x_d)$ a minimal reduction of M .

If we assume that R is not-Cohen-Macaulay then the equality (**) tells us that

(1) $d = 3$ and

(2) $\ell(M/J) = 0$,

since $e(R) = 3$ and $h^0(R) = h^1(R) = 0$, $h^2(R) = 1$. From (2) we conclude by [4] that $r(M) = 2$ and that $gr_M R$ is Buchsbaum. Moreover by Ikeda [20] we know - up to isomorphisms - exactly this graded ring, namely

$$gr_M R = k[X_1,X_2,X_3,Y_1,Y_2,Y_3]/(X_1Y_1+X_2Y_2+X_3Y_3,(Y_1,Y_2,Y_3)^2) \quad .$$

From (1) we get $ht(P) \leq 2$. Clearly $ht(P) \neq 1$ if R is not-Cohen-Macaulay, s. [11], proposition 4.11, i.e. $ht(P) = \ell(P) = 2$. Since $G = gr_M R$ is Buchsbaum, we have $ht(P^*) = \dim(G) - \dim(G/P^*) = 2$.

Now, putting $y_i = \overline{Y}_i \in G$, $i = 1,2,3$, we get:

$$Q := P^*/\underline{y}G \subset G/\underline{y}G = k[X_1,X_2,X_3] \quad .$$

Since P^* is prime and $ht(P^*) = 2$, Q corresponds to a closed point in $\operatorname{Proj} k[X_1,X_2,X_3]$. We may assume that $X_3 \notin Q$. Since k is algebraically closed, we must have $Q = (X_1-\alpha X_3,\ X_2-\beta X_3)$ for some $\alpha,\beta \in k$. Hence $G/P^* \cong k[Z]$, where Z is an indeterminate over k, i.e. R/P is regular. But this property cannot occur together with $R\overset{+}{e}(P,R)$ is Cohen-Macaulay and $ht(P) = \ell(P) = 2$ for a non-Cohen-Macaulay ring R, by proposition 3.3. Therefore R must be Cohen-Macaulay under the assumptions of proposition 3.5. But then we know by [17] that $gr_M R$ is Cohen-Macaulay, since $e(R) = 3$. Moreover we get that $R\overset{+}{e}(M,R)$ is Cohen-Macaulay by [11], Corollary 5.4. This completes the proof.

REMARK 3.6: R/P regular implies P^* prime.

QUESTION 3.7: Is the statement of proposition 3.5 true without the restriction on the multiplicity $e(R)$?

THEOREM 3.8: Let (R,M) be a local ring, J an equimultiple ideal of R, $\underline{x} = \{x_1,\dots,x_s\}$ part of a system of parameters mod J and $I = J + \underline{x}R$. Assume that $s > 0$ and that $R\overset{+}{e}(J,R)$ and $R\overset{+}{e}(I,R)$ are Cohen-Macaulay. Then R is Cohen-Macaulay.

PROOF: Since $ht(J) = \ell(J)$ and $R\overset{+}{e}(J,R)$ is Cohen-Macaulay we know that R normally Cohen-Macaulay along J. Therefore $\underline{x}R \cap J^i = (\underline{x})\cdot J^i$ for $i \geq 1$, implying $\underline{x}R \cap I^i = (\underline{x})\cdot I^{i-1}$.

We write: $G_I = gr_I R$, $G_J = gr_J R$;

$$G_I^{(0)} = G_I \ ; \quad G_I^{(j)} = G_I/(x_1^*,\dots,x_j^*) \ , \ 1 \leq j \leq s \ ,$$

where x_j^* is the initialform of x_j with respect to I, and $G_I(-1)$ is the shifting of G_I by -1.

Then we consider the exact sequence

$$(1)\quad 0 \longrightarrow G_I^{(j)}(-1) \xrightarrow{\cdot x^*_{j+1}} G_I^{(j)} \longrightarrow G_I^{(j+1)} \longrightarrow 0 \quad .$$

Now set $G_1 = G_I^{(s)}$ and $G_2 = G_J/\underline{x}G_J$. Denote by M_J and M_I the unique maximal homogeneous ideals of $R\overset{+}{e}(J,R)$ and $R\overset{+}{e}(I,R)$ respectively. Then we get from (1) the long exact sequence for the local cohomology:

$$(2)\quad \ldots \longrightarrow H^{i-1}_{M_I}(G_1) \longrightarrow H^i_{M_I}(G_I^{(s-1)})(-1) \xrightarrow{\delta} H^i_{M_I}(G_I^{(s-1)}) \longrightarrow \ldots \quad ,$$

where δ is defined by multiplying with x^*_s .

Now $G_1 \simeq G_2$ over S : $R\overset{+}{e}(J,R)/\underline{x}R\overset{+}{e}(J,R) \simeq R\overset{+}{e}(I,R)/(\underline{x},\underline{x}t) \simeq$ $\simeq R[It]/\oplus(\underline{x}R \cap I^n)t^n$. Since $\underline{x}R\overset{+}{e}(J,R)$ is a regular sequence on $R\overset{+}{e}(J,R)$, S is Cohen-Macaulay, hence by [10] , proposition 1.5:

$$(3)\quad H^{i-1}_{M_I}(G_1)_n \simeq H^{i-1}_{M_J}(G_2)_n = 0 \quad \text{for} \quad n \geq 0 \ , \ i \leq d-s \quad .$$

This implies that

$$H^i_{M_I}(G_I^{(s-1)})_{n-1} \xrightarrow{\delta} H^i_{M_I}(G_I^{(s-1)})_n$$

is injective. Therefore we get $\underset{n\geq -1}{\oplus} H^i_{M_I}(G_I^{s-1})_n = 0$.

By induction on j we see that

$$H^i_{M_I}(G_I^{(s-j)})_n = 0 \quad \text{for} \quad n \geq -j \ , \ i \leq d-r+j \ , \ 0 \leq j \leq r \ .$$

For $j = s$ and $i \leq d - 1$ this implies in particular:

$$0 = H^i_{M_I}(G_I)_{-1} = H^i_M(R) \quad ,$$

since $ht(I) = \ell(I)$ and $R\overset{+}{e}(I,R)$ is Cohen-Macaulay, see [10], proposition 1.5. This completes the proof.

REMARK 3.9: If $J = (0)$ in the above theorem, we have a similar conclusion as above, replacing the assumption on $R\overset{+}{e}(J,R)$ by the assumption that I is generated by a regular sequence. For we know from $R\overset{+}{e}(I,R)$ Cohen-Macaulay that R/I is Cohen-Macaulay, hence the same is true for R .

EXAMPLE 3.10: (Compare [3]): $R = k[[s^2,s^3,st,t]]$, s,t indeterminates, is a non-Cohen-Macaulay Buchsbaum ring. We consider $J = (s^2)R$ and $I = (s^2,t)R$. Since s^2,t form a system of parameters in a Buchsbaum domain of dimension 2 we know by [19] that $R\overset{+}{e}(I,R)$ is Cohen-Macaulay. Hence $R\overset{+}{e}(J,R)$ cannot be Cohen-Macaulay by theorem 3.8. [Compare also [11], proposition 4.11].

At the end of this section we consider again the ring of example 2.3. We want to test the structure of $R\overset{+}{e}(I,R)$ for various ideals I :

$$R = k[[X_1,X_2,X_3,Y_1,Y_2,Y_3]]/(X_1Y_1+X_2Y_2+X_3Y_3,(Y_1,Y_2,Y_3)^2)$$
$$\simeq k[[x_1,x_2,x_3,y_1,y_2,y_3]] \quad .$$

We consider these ideals:

$M = (x_1,x_2,x_3,y_1,y_2,y_3) \supset P_1 = (x_2,x_3,y_1,y_2,y_3) \supset P_2 = (x_3,y_1,y_2,y_3)$,

$P_3 = (y_1,y_2,y_3)$,

$Q_3 = (x_1,x_2,x_3) \supset Q_2 = (x_1,x_2) \supset Q_1 = (x_1)$

The following can be said about the Rees rings:

a) Since R is not-Cohen-Macaulay $\overset{+}{Re}(P_1,R)$ and $\overset{+}{Re}(P_2,R)$ are not-Cohen-Macaulay by theorem 3.8.

b) $\overset{+}{Re}(P_3,R) \simeq R \oplus P_3 t$ is finitely generated. Since R is not-Cohen-Macaulay, $\overset{+}{Re}(P_3,R)$ cannot be Cohen-Macaulay.

c) $\overset{+}{Re}(Q_2,R)$ is not-Cohen-Macaulay. Otherwise R would be normally Cohen-Macaulay along Q_2 by [10], i.e. in particular R/Q_2 would be Cohen-Macaulay.

d) $\overset{+}{Re}(Q_3,R)$ is not-Cohen-Macaulay. Otherwise we must have $H^i_M(R) = 0$ for $i \neq 1,d$ by [18], theorem 3.1, since Q_3 is a parameter ideal in R . Hence we would have $H^2_M(R) = 0$, but this is a contradiction in our case.

e) $\overset{+}{Re}(Q_1,R)$ is not Cohen-Macaulay. Otherwise R would be Cohen-Macaulay by [11], 4.11.

<u>REMARK 3.11</u>: Ikeda [20] has recently shown that the ideal $I = (x_1,x_2,y_3) \subset R$ has a Cohen-Macaulay-Rees ring. Hence the Rees ring of $J_1 = (x_1,x_2,x_3,y_3)$ or $J_2 = (x_i,y_3)$, $i = 1$ or 2 cannot be Cohen-Macaulay by theorem 3.8.

QUESTION 3.12: Relate S_n of $\overset{+}{Re}(J,R)$ and $\overset{+}{Re}(I,R)$, where $I = J + \underline{x}R$ as in the theorem 3.8, to S_{n-1} of R.

4. THE GEOMETRIC BLOWING UP

If we replace the Rees rings $\overset{+}{Re}(I,R)$ and $\overset{+}{Re}(J,R)$ in theorem 3.8 by the Proj's of these rings then the corresponding question becomes more difficile. The exact question is as follows:

"Let (R,M) be a local ring of depth $R \gneq 0$. Let J be an equimultiple ideal and let $I = J + \underline{x}R$, where $\underline{x}$ is part of a system of parameters mod J. Assume that

(i) $\mathrm{Proj}(\oplus I^n)$ is Cohen-Macaulay and

(ii) $\mathrm{Proj}(\oplus J^n)$ is Cohen-Macaulay.

Is R a Cohen-Macaulay ring?"

In theorem 4.5 we will give a partial answer to this question. Before formulating this result we want to remark that the assumptions depth $R \gneq 0$ and J is equimultiple are necessary:

EXAMPLE 4.1: Let (S,N) be a regular local ring with residue field k. Consider the ring

$$R = S[X]/(X^2, NX) \simeq S \oplus k \quad ,$$

where X is an indeterminate. Then $H^0_M(R) \simeq k$, hence $R/H^0_M(R)$ is Cohen-Macaulay.

Now take $I = (a_1,\dots,a_d)$ and $J = (a_1,\dots,a_i)$, $2 \leq i < d$, where $a_1,\dots,a_d$ is a system of parameters in R. Since $H^0_M(R) \subset \ker(R \longrightarrow R\left[\frac{K}{a_j}\right] \subset R_{a_j})$ for $K = I$ and $K = J$, we have $\mathrm{Proj}(\oplus I^n) \simeq \mathrm{Proj}(\oplus \bar{I}^n)$ and $\mathrm{Proj}(\oplus J^n) \simeq \mathrm{Proj}(\oplus \bar{J}^n)$, where $\bar{I}$ and $\bar{J}$ are the images of I and J in $R/H^0_M R)$.

But $\mathrm{Proj}(\oplus \bar{I}^n)$ and $\mathrm{Proj}(\oplus \bar{J}^n)$ are Cohen-Macaulay, since $\bar{I}$ and $\bar{J}$ are formed by a regular sequence in $R/H^0_M(R)$.

<u>EXAMPLE 4.2</u>: $R = k[[s^2,s^3,st,t]]$. We take $J = (s^2,s^3,st)$ and $I = (s^2,s^3,st,t)$. Now J is not an equimultiple ideal in R, since (s^2,st) is a minimal reduction of J, i.e. $\mathrm{ht}(J) = 1$ and $\ell(J) = 2$.

Since $\mathrm{Proj}(\oplus J^n) = \mathrm{Spec}\, R_1 \cup \mathrm{Spec}\, R_2$, where $R_1 = R[s,\frac{t}{s}]$ and $R_2 = R[t,\frac{s}{t}]$, we see that $\mathrm{Proj}(\oplus J^n)$ is isomorphic to the blowing up of the plane at the origin, hence Cohen-Macaulay. Furthermore $\mathrm{Proj}(\oplus I^n)$ is Cohen-Macaulay since R is a (non-Cohen-Macaulay) Buchsbaum ring of multiplicity 2 [3].

[Note that $R^+_e(I,R) = \bigoplus_{n\geq 0} I^n$ is not-Cohen-Macaulay, otherwise depth R would be 2, hence R would be Cohen-Macaulay.]

Now we are going to specialize I to a complete intersection. We denote the blowing up $\mathrm{Proj}(\oplus I^n)$ of R with center I by $B\ell_I(R)$. First we need an auxiliary result.

<u>LEMMA 4.3</u>: Let I be a complete intersection in the local ring (R,M), and let R_1 be a local ring obtained by blowing up R with center I. If R_1 corresponds to a closed point of $B\ell_I(R)$, then $\dim R_1 = \dim R$.

<u>PROOF</u>: We note that in general we have $\dim R_1 \leq \dim R$ without any assumption on I. (This can be shown by using [13], 14.c for the irreducible components of Spec R.)

<u>Case $ht(I) = \dim R$</u>, i.e. I is generated by a system of parameters $a_1,\ldots,a_d$ of R. We may assume $R_1 = \left[\frac{a_2}{a_1},\ldots,\frac{a_d}{a_1}\right]_N$ for some maximal ideal N of $R' = R\left[\frac{a_2}{a_1},\ldots,\frac{a_d}{a_1}\right]$. Now, by the analytic independence of systems of parameters, we have $R'/MR' \simeq R/M[T_2,\ldots,T_d]$, showing that every maximal ideal in R'/MR' has height $d-1$. Since R_1/a_1R_1 is, up to nilpotent elements, a localization of R'/MR' at a maximal ideal, we conclude that $\dim R_1/a_1R_1 = d-1$, and therefore $\dim R_1 = d$.

<u>GENERAL CASE</u>: If $I = (a_1,\ldots,a_s)$, $s = ht(I) \leq \dim R$, we extend $a_1,\ldots,a_s$ to a system of parameters $a_1,\ldots,a_d$ of R and we put $I' = (a_1,\ldots,a_d)R$. We may assume that $IR_1 = a_1R_1$. Let $R'' = R\left[\frac{a_2}{a_1},\ldots,\frac{a_d}{a_1}\right] = R\left[\frac{a_2}{a_1},\ldots,\frac{a_s}{a_1}\right]\left[\frac{a_{s+1}}{a_1},\ldots,\frac{a_d}{a_1}\right]$ and assume that $R_1 = R\left[\frac{a_2}{a_1},\ldots,\frac{a_s}{a_1}\right]_N$. Put $N'' = NR'' + \left(\frac{a_{s+1}}{a_1},\ldots,\frac{a_d}{a_1}\right)R''$ and $R_2 = R''_{N''}$. Then R_2 corresponds to a closed point of $B\ell_{I'}(R)$ and therefore $\dim R_2 = \dim R$ by the special case above. On the other hand R_2 is obtained by blowing up

$(a_1, a_{s+1}, \ldots, a_d)$ in R_1, and therefore $\dim R_2 \leq \dim R_1 \leq \dim R$, which concludes the proof.

<u>REMARK 4.4</u>: Using Ratliff's well developed theory of quasi-unmixed rings one can show that the statement of the lemma is true for any equimultiple ideal in a quasi-unmixed local ring.

<u>THEOREM 4.5</u>: Let (R,M) be a Buchsbaum local ring with depth $R > 0$. If $B\ell_I(R)$ is Cohen-Macaulay for a complete intersection I of R such that $2 \leq \mathrm{ht}(I) < d = \dim R$, then R is Cohen-Macaulay.

<u>PROOF</u>: Let $s = \mathrm{ht}(I)$ and let $a_1, \ldots, a_d$ be a system of parameters of R such that $I = (a_1, \ldots, a_s)R$. We put

$$R' = R\left[\frac{a_2}{a_1}, \ldots, \frac{a_s}{a_1}\right] \quad ,$$

$$N = MR' + \left(\frac{a_2}{a_1}, \ldots, \frac{a_s}{a_1}\right) R' \quad ,$$

$$R_1 = R'_N \quad .$$

Since $\dim R_1 = d$ by the lemma, we see that $a_1, \frac{a_2}{a_1}, \ldots, \frac{a_s}{a_1}, a_{s+1}, \ldots, a_d$ is a system of parameters of R_1. Using the Buchsbaum property of R it is not difficult to see that

$$R' \Big/ \left(a_1, \frac{a_2}{a_1}, \ldots, \frac{a_s}{a_1}\right) R' \simeq R/K$$

where $K = a_1R + ((a_2,\dots,a_s) : a_1)_R$, [6], and therefore also

$$R_1 \Big/ \left(a_1, \frac{a_2}{a_1}, \dots, \frac{a_s}{a_1}\right) R_1 \simeq R/K \quad .$$

Since R_1 was assumed to be Cohen-Macaulay and $a_1, \frac{a_2}{a_1}, \dots, \frac{a_s}{a_1}$ is part of a system of parameters of R_1 , we see that R/K is Cohen-Macaulay. We consider the following exact sequence

(1) $$0 \to K/I \to R/I \to R/K \to 0 \quad .$$

Using again the Buchsbaum property of R one obtains

(2) $$K/I \simeq \frac{(a_2,\dots,a_s):M}{(a_2,\dots,a_s)} \simeq H^0_M(R/(a_2,\dots,a_s)) \quad ,$$

i.e. K/I is a vector space over R/M . From the sequence (1) and from the fact that R/K is Cohen-Macaulay we get:

(3) $$h^j(R/I) = h^j(K/I) \quad \text{for} \quad 0 \leq j < d-s \quad ,$$

and since $\dim K/I = 0$

(4) $$h^j(R/I) = 0 \quad \text{for} \quad 0 < j < d-s \quad .$$

From $h^j(R/xR) = h^j(R)+h^{j+1}(R)$ (see [5], p. 494) we conclude by induction

(5) $$h^j(R/I) = \sum_{r=0}^{s-1} \binom{s}{r} h^{j+r}(R) \quad \text{for} \quad 0 \leq j < d-s \quad ,$$

and similarly, together with (2), we have

(6) $$h^0(K/I) = \sum_{r=0}^{s-1} \binom{s-1}{r} h^r(R) \quad .$$

Putting $j = 0$ in (3) and (5) and comparing with (6) we obtain

$$h^j(R) = 0 \quad \text{for} \quad 0 < j \leq s \quad .$$

On the other hand, comparing (4) and (5) we also have

$$h^j(R) = 0 \quad \text{for} \quad s < j < d \quad .$$

Finally $h^0(R) = 0$ since depth $R > 0$, and this completes the proof of the theorem.

<u>REMARK 4.6</u>: Since depth $R > 0$ and R is Buchsbaum in the theorem 4.5, the ring $R/H^0_M(R) \simeq R$ is Buchsbaum. Hence $B\ell_H(R) = \mathrm{Proj}(\oplus H^n)$ is Cohen-Macaulay for $H = (a_1,\ldots,a_d)$ by [6]; i.e. theorem 4.5 is indeed a special case of our question at the beginning of this section (for the pair of ideals $I \subset H$).

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