Magic* in the spectra of the XXZ quantum chain with boundaries at $\Delta = 0$ and $\Delta = -1/2$.

Jan de Gier[1], Alexander Nichols[2], Pavel Pyatov[3], and Vladimir Rittenberg[2]

[1] *Department of Mathematics and Statistics, The University of Melbourne, VIC 3010, Australia*
[2] *Physikalisches Institut, Universität Bonn, Nussallee 12, Bonn, Germany*
[3] *Bogoliubov Laboratory of Theoretical Physics, JINR 141980 Dubna, Moscow Region, Russia & Max Planck Institute for Mathematics, Vivatsgasse 7, D-53111, Bonn, Germany*

June 6, 2005

Abstract

We show that from the spectra of the $U_q(sl(2))$ symmetric XXZ spin-1/2 finite quantum chain at $\Delta = -1/2$ ($q = e^{\pi i/3}$) one can obtain the spectra of certain XXZ quantum chains with diagonal and non-diagonal boundary conditions. Similar observations are made for $\Delta = 0$ ($q = e^{\pi i/2}$). In the finite-size scaling limit the relations among the various spectra are the result of identities satisfied by known character functions. For the finite chains the origin of the remarkable spectral identities can be found in the representation theory of one and two boundaries Temperley-Lieb algebras at exceptional points. Inspired by these observations we have discovered other spectral identities between chains with different boundary conditions.

1 Introduction

Finding the spectrum of the spin one-half XXZ quantum chain with non-diagonal boundaries is still an open problem. We think that this problem is of special interest because it brings together in a new way the Bethe Ansatz and the representation theory of some associative algebras.

The XXZ chain with very special diagonal boundary terms has an $U_q(sl(2))$ quantum group symmetry [1]. This chain is, from an algebraic point of view, simple as it can be written in terms of the generators of the Temperley-Lieb (TL) algebra.

The addition of a single boundary term to the $U_q(sl(2))$ chain can be described using the one-boundary Temperley-Lieb (1BTL) algebra [2–5]. In contrast to the TL algebra the 1BTL now depends on two parameters. At certain exceptional points (called "critical values" in [11]) this algebra becomes non-semisimple and possesses indecomposable representations. A third parameter, which is absent in the algebra, enters as a coefficient in the integrable Hamiltonian.

As shown in [6] this general one-boundary Hamiltonian has exactly the same spectrum as the XXZ Hamiltonian with purely diagonal boundary conditions [7]. It also has the same spectrum as a loop Hamiltonian defined on a 2^L dimensional space of link patterns [8].

All of these Hamiltonians can be written in terms of the 1BTL algebra using three different representations. These representations are equivalent except at the exceptional points of the 1BTL

*The art of producing illusions or tricks that fool or deceive an audience [Webster's Dictionary of American English].

[1] degier@ms.unimelb.edu.au
[2] pyatov@thsun1.jinr.ru
[3] nichols@th.physik.uni-bonn.de
[4] vladimir@th.physik.uni-bonn.de

algebra. At these exceptional points, although the spectra remain the same, degeneracies appear and the three Hamiltonians have different Jordan cell structure and therefore describe different physical problems [6].

Before proceeding we comment briefly on relations between various algebras we are considering here. The TL algebra is a quotient of the type A Hecke algebra. The 1BTL algebra contains a TL sub-algebra. In turn, it is a quotient algebra of type B Hecke algebra, which is a quotient of affine Hecke algebra (see [9, 10] and references therein).

The situation of non-diagonal boundary conditions at both ends of the chain is more complicated. In addition to the anisotropy parameter, one has five boundary parameters. As noticed in [16] the Hamiltonian can be written in terms of the generators of the two-boundary Temperley-Lieb algebra [8, 18]. This algebra depends on Δ and three boundary parameters only. The structure and representation theory of the two-boundary Temperley-Lieb (2BTL) algebra is essentially unknown.

In [12–16] the Bethe Ansatz equations for the case of non-diagonal boundaries was written only when the boundary parameters satisfy a particular constraint. The surprising fact is that this constraint involves *only* the parameters which enter the 2BTL algebra and not the coefficients in the Hamiltonian.

In this paper we shall make a conjecture for the location of the exceptional points of the 2BTL algebra and explain how this was obtained. At the exceptional points the algebra becomes non-semisimple and one finds representations which are indecomposable. These exceptional points are *exactly* the points at which Bethe Ansatz equations were written [12–16]. An explanation of this remarkable observation is still missing.

The main aim of this paper is to show some intriguing relations between the spectra of several Hamiltonians describing the XXZ quantum chain for a finite number of sites with various boundary terms for $\Delta = -1/2$ and $\Delta = 0$. Some insight into these relations can be gained from the TL, 1BTL and 2BTL algebras. We hope that the existence of relations of this kind may bring some new light into the unsolved problem of writing the Bethe Ansatz equations for arbitrary boundary terms.

We would like to mention that in the case of $\Delta = -1/2$, the Hamiltonians considered here have the same spectra (but necessarily not the same Jordan cell structures) as the Hamiltonians describing Raise and Peel stochastic models of fluctuating interfaces with different sources at the boundaries [20]. In these models the boundary parameters have a simple physical interpretation.

The paper is organized as follows. In Section 2 we define the TL, 1BTL and 2BTL algebras and give their representations in terms of XXZ quantum chains. We also give the relations which define the exceptional points. In sections 3 and 4 we give conjectures relating different spectra at some fixed values of the boundary parameters. The magic mentioned in the title of this paper is described here. These conjectures are based on exact diagonalizations at a low number of sites given in Appendix A and Appendix B. In section 5 we give proofs and generalizations of several of these conjectures using the Bethe Ansatz. In sections 6 and 7 we discuss the finite-size scaling limit. In this limit the conjectures for finite chains become identities between characters of the $c = 0$ and $c = -2$ conformal field theories. In section 8 we comment on the appearance of extra symmetries in finite chains for $\Delta = -1/2$. Conclusions and open questions are in section 9.

2 Temperley Lieb algebras and XXZ chains with boundary terms

Following [6, 8, 16], we summarize the relations between the Temperley-Lieb algebra and its extensions and XXZ quantum chains with boundaries. As is going to be shown in detail in Section 3, these relations can help to explain part of the magic observed in the spectra of the quantum chains seen in Appendices A and B.

We start be defining the algebras. The Temperley-Lieb (TL) algebra is generated by the unity and the set of $L-1$ elements e_i, $i=1,\ldots,L-1$, subject to the relations:

$$\begin{aligned} e_i e_{i\pm1} e_i &= e_i \\ e_i e_j &= e_j e_i \qquad |i-j|>1 \\ e_i^2 &= (q+q^{-1})\, e_i \end{aligned} \tag{2.1}$$

with $q=\exp(\mathrm{i}\gamma)$. The one-boundary TL algebra (1BTL) (also called blob algebra [4]) is defined by the generators e_i and a new generator e_0. It has the following additional relations:

$$\begin{aligned} e_1 e_0 e_1 &= e_1 \\ e_0^2 &= \frac{\sin\omega_-}{\sin(\omega_-+\gamma)} e_0 \\ e_0 e_i &= e_i e_0 \qquad i>1 \end{aligned} \tag{2.2}$$

Notice that the new parameter ω_- is defined up to a multiple of π. It was shown by Martin and Saleur [4] that when for a given value of the bulk parameter γ, the boundary parameter ω_- takes one of the values:

$$\omega_- = k\gamma + \pi\mathbb{Z} \tag{2.3}$$

with k integer $|k|<L$, the algebra becomes non-semisimple and hence possesses indecomposable representations. We shall call the values of ω_- which satisfy (2.3), exceptional (in [11] they are called 'critical', a term which in physics might bring other associations). One should note that 1BTL can have exceptional points even when q is generic. However when we are in the exceptional cases (2.3) *and* q is a root of unity the indecomposable structure is much richer (called 'doubly critical' in [11]). The focus of this paper will be on such points.

The two-boundary TL algebra (2BTL) is defined by the generators e_0, e_i ($i=1,\ldots,L-1$), and a new generator e_L subject to the supplementary relations

$$\begin{aligned} e_{L-1} e_L e_{L-1} &= e_{L-1} \\ e_L^2 &= \frac{\sin\omega_+}{\sin(\omega_++\gamma)} e_L \\ e_L e_i &= e_i e_L \qquad i<L-1 \end{aligned} \tag{2.4}$$

and

$$I_L J_L I_L = b I_L \tag{2.5}$$

where

$$\begin{aligned} I_{2n} &= \prod_{x=0}^{n-1} e_{2x+1} \quad J_{2n} = \prod_{x=0}^{n} e_{2x} \\ I_{2n+1} &= \prod_{x=0}^{n} e_{2x} \quad J_{2n+1} = \prod_{x=0}^{n} e_{2x+1}. \end{aligned} \tag{2.6}$$

Notice that the boundary generators e_0 and e_L enter differently in the expressions of I_L and J_L for L even and odd. The 2BTL algebra has one bulk parameter γ and three boundary parameters $\omega_\pm$ and b. It is convenient to use instead of the parameter b another parameter θ defined by the relations:

$$b = \begin{cases} -\dfrac{\cos\theta+\cos(\gamma+\omega_-+\omega_+)}{2\sin(\gamma+\omega_-)\sin(\gamma+\omega_+)} & \text{for even } L \\ \dfrac{\cos\theta+\cos(\omega_--\omega_+)}{2\sin(\gamma+\omega_-)\sin(\gamma+\omega_+)} & \text{for odd } L. \end{cases} \tag{2.7}$$

Based on explicit analysis for small values of L, we conjecture that for a given value of the bulk parameter γ, the 2BTL algebra has exceptional points where the 2BTL is non-semisimple if the boundary parameters $\omega_\pm$ and θ satisfy the relations:

$$\begin{cases} \pm\theta = (2k+1)\gamma + \omega_- + \epsilon\,\omega_+ + \pi + 2\pi\mathbb{Z} & \text{for even } L, \\ \pm\theta = 2k\gamma + \omega_- + \epsilon\,\omega_+ + \pi + 2\pi\mathbb{Z} & \text{for odd } L, \end{cases} \tag{2.8}$$

where $\epsilon = \pm 1$ and k is an integer $|k| < L/2$. Our conjecture is based on the following facts.

For small values of L we have found zeros of the determinant of the Gram matrix also called the discriminant (see [17], p.112) in two representations of the 2BTL algebra. Namely, we checked the cases $L = 2, 3, 4$ in the loop representation of [16] and $L = 2, 3$ in the regular representation. The latter check, although more laborious, is a criterium of non-semisimplicity (see for example exercise 6, p.115 of [17]). The non-semisimplicity of the algebra implies the existence of indecomposable representations.

The algebras TL, 1BTL and 2BTL have a $2^L \times 2^L$ representation which can be given in terms of Pauli matrices:

$$e_i = \frac{1}{2}\left\{\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \cos\gamma\sigma_i^z\sigma_{i+1}^z + \cos\gamma + \mathrm{i}\sin\gamma\left(\sigma_i^z - \sigma_{i+1}^z\right)\right\} \tag{2.9}$$

for $i = 1, \ldots, L-1$, and

$$e_0 = -\frac{1}{2}\frac{1}{\sin(\omega_- + \gamma)}\left(\mathrm{i}\cos\omega_-\,\sigma_1^z + \sigma_1^x - \sin\omega_-\right), \tag{2.10}$$

$$e_L = -\frac{1}{2}\frac{1}{\sin(\omega_+ + \gamma)}\left(-\mathrm{i}\cos\omega_+\,\sigma_L^z + \cos\theta\,\sigma_L^x + \sin\theta\,\sigma_L^y - \sin\omega_+\right), \tag{2.11}$$

As explained in [6] in the case of the 1BTL, at the exceptional points, besides the representation given by (2.9)–(2.10) there are other nonequivalent, $2^L \times 2^L$ representations of the same algebra. The same is true for the 2BTL at the exceptional points given in (2.8).

We define three different Hamiltonians which if we use the representation (2.9)–(2.11) have the same bulk terms with an anisotropy parameter $\Delta = -\cos\gamma$ but different boundaries:

$$H^{\mathrm{T}} = \sum_{i=1}^{L-1}(1 - e_i), \tag{2.12}$$

$$H^{1\mathrm{T}} = a_-(1 - e_0) + H^{\mathrm{T}}, \tag{2.13}$$

$$H^{2\mathrm{T}} = a_+(1 - e_L) + H^{1\mathrm{T}}. \tag{2.14}$$

It is convenient to parameterize the coefficients a_+ and a_- as follows:

$$a_\pm = \frac{2\sin\gamma\sin(\omega_\pm + \gamma)}{\cos\omega_\pm + \cos\delta_\pm}. \tag{2.15}$$

We have introduced in the definitions (2.12)–(2.14) of the Hamiltonians constant terms such that for $\gamma = \pi/3$, $\omega_\pm = -2\pi/3$ and $b = 1$, the three Hamiltonians describe stochastic processes and therefore their ground-state energies are equal to zero for any system size [20, 28].

The Hamiltonian H^{T} has special diagonal boundary conditions and is known to be $U_q(sl(2))$ symmetric [1]. The Hamiltonian $H^{1\mathrm{T}}$ has the most general boundary condition at one side of the chain but a fixed diagonal boundary condition at the other end of the chain:

$$H^{1T} = \frac{\sin\gamma}{\cos\omega_- + \cos\delta_-}\left(\mathrm{i}\cos\omega_-\sigma_1^z + \sigma_1^x - \sin\omega_-\right) + \frac{2\sin\gamma\sin(\omega_- + \gamma)}{\cos\omega_- + \cos\delta_-}$$
$$-\frac{1}{2}\left\{\sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \cos\gamma\sigma_i^z\sigma_{i+1}^z + \cos\gamma\right) + \mathrm{i}\sin\gamma\left(\sigma_1^z - \sigma_L^z\right)\right\} \tag{2.16}$$

In [6] it was shown that, remarkably, the Hamiltonian (2.13) has the same spectrum as the Hamiltonian:

$$H^{\mathrm{d}} = -\frac{1}{2}\left\{\sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \cos\gamma\sigma_i^z\sigma_{i+1}^z + \cos\gamma\right) + \sin\gamma\left[\tan\left(\frac{\omega_- + \delta_-}{2}\right)\sigma_1^z + \tan\left(\frac{\omega_- - \delta_-}{2}\right)\sigma_L^z + 2\frac{\sin\omega_- - 2\sin(\omega_- + \gamma)}{\cos\omega_- + \cos\delta_-}\right]\right\}. \quad (2.17)$$

H^{d} has diagonal boundary terms only and its spectrum was studied since a long time using the Bethe Ansatz [7]. The Hamiltonian H^{d} commutes with

$$S^z = \frac{1}{2}\sum_{i=1}^{L}\sigma_i^z. \quad (2.18)$$

and therefore for generic values of parameters one would not expect any degeneracies to occur. However at exceptional points (2.3) the 1BTL algebra gives rise to degeneracies in H^{1T}. As the spectrum of $H^{1\mathrm{T}}$ and H^{d} is identical this *implies* degeneracies in H^{d} at these points [6].

If one examines the coefficients of σ_1^z and σ_L^z in (2.17) there is no obvious difference between the parameters ω_- and δ_-. We know however that this is not the case in the Hamiltonian $H^{1\mathrm{T}}$: ω_- enters the algebra whereas δ_- does not. Some observations related to the properties of the spectrum of H^{d} can be explained [6] like the absence of δ_- in the finite-size scaling limit [26] and the fact that for finite chains, at the exceptional points, one observes degeneracies and some energy levels are δ_- independent and some are not [26].

Using (2.9)–(2.11), (2.14), and (2.15), $H^{2\mathrm{T}}$ has the following expression:

$$\begin{aligned} H^{2\mathrm{T}} = & -\frac{1}{2}\left\{\sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \cos\gamma\sigma_i^z\sigma_{i+1}^z - 2 + \cos\gamma\right)\right. \\ & - \frac{2\sin\gamma}{\cos\omega_+ + \cos\delta_+}\left(-\mathrm{i}(\cos\omega_+ - \cos\delta_+)\,\sigma_L^z + \cos\theta\,\sigma_L^x + \sin\theta\,\sigma_L^y - \sin\omega_+ + 2\sin(\omega_+ + \gamma)\right) \\ & \left. - \frac{2\sin\gamma}{\cos\omega_- + \cos\delta_-}\left(\mathrm{i}(\cos\omega_- - \cos\delta_-)\,\sigma_1^z + \sigma_1^x - \sin\omega_- + 2\sin(\omega_- + \gamma)\right)\right\}. \end{aligned} \quad (2.19)$$

One should keep track of the role of the parameters in (2.19). Only the parameters γ, $\omega_\pm$ and θ enter the 2BTL algebra not the $\delta_\pm$ parameters. If we take into account all the parameters, $H^{2\mathrm{T}}$ gives the most general XXZ model with boundaries.

Not much is known yet about the spectrum of $H^{2\mathrm{T}}$. One reason for this lack of understanding is the absence of Bethe Ansatz equations for generic values of the parameters. However, such equations were derived at a subset of the exceptional points, those with $\epsilon = 1$ [12–14,16] and those with $\epsilon = -1$ in the case of L odd [16]. It was further noticed that one needs two sets of Bethe Ansatz equations to describe the complete spectrum [15, 16].

In the next section, we are going to show that for $\gamma = \pi/3$ and $\gamma = \pi/2$, for certain boundary parameters the spectra of the Hamiltonian H^{T} for even and odd number of sites give all the energy levels observed in the Hamiltonians $H^{1\mathrm{T}}$ and $H^{2\mathrm{T}}$. In section 5 we shall prove generalizations of some of these conjectures.

3 Spectra with magic. The case $q = e^{i\pi/3}$

3.1 The open and one-boundary chains.

In this section we are going to take

$$e_j^2 = e_j, \quad (j = 0, 1, \ldots, L). \quad (3.1)$$

This implies $\gamma = \pi/3$ and $\omega_\pm = -2\pi/3$.

We start by considering the spectra of the Hamiltonian $H^{\rm T}$, defined in (2.12) with $\gamma = \pi/3$, which describes a stochastic process with open boundaries [28]. It follows that the ground-state energy is zero for any number of sites.

$H^{\rm T}$ is $U_q(sl(2))$ symmetric. Its spectrum can be computed using the Bethe Ansatz in the spin basis [7] or in the link pattern basis [6, 16]. The advantage of the latter basis, is that $H^{\rm T}$ has a block triangular form and that in order to compute the spectrum, one can disregard the "off-diagonal" blocks and be left with the "diagonal" ones only. To a given value of the spin S there are $2S+1$ identical diagonal blocks.

In Appendix A we present the characteristic polynomials which give the spectra of $H^{\rm T}$ for different sizes of the system and different values of S. We can notice that there are supplementary degeneracies which occur because $q = e^{i\pi/3}$ is a root of unity. These degeneracies are well understood [1]. The partition function for a system of size L and spin S and the total partition function are defined as follows:

$$Z^{\rm T}_{L,S}(z) = \sum_i z^{E_i} \tag{3.2}$$

$$Z^{\rm T}_{L}(z) = \sum_S (2S+1) Z^{\rm T}_{L,S}(z) \tag{3.3}$$

We next consider the Hamiltonian $H^{1\rm T}$ taking $\omega_- = -2\gamma = -2\pi/3$ in the 1BTL (2.2) and $a_- = 1$ (i.e. $\delta_- = \pi$). $H^{1\rm T}$ describes again a stochastic process [28]. It is helpful to use the spectral equivalence between $H^{1\rm T}$ and $H^{\rm d}$ (see (2.17)). Since S^z given by (2.18), commutes with $H^{\rm d}$, in Appendix A we give for different sizes L, the characteristic polynomials for different eigenvalues m of S^z. Notice that we did not need to use other polynomials than those used already for the $U_q(sl(2))$ symmetric chain. We denote by $Z^{1\rm T}_{L,m}(z)$ the partition function for a system of size L and charge m and the total partition function by:

$$Z^{1\rm T}_{L}(z) = \sum_{m\in\mathbb{Z}} Z^{1\rm T}_{L,m}(z) \tag{3.4}$$

Conjecture 1 *The following identities hold for finite chains:*

$$Z^{1\rm T}_{L,m} = \sum_{n\geq 0} \left\{ Z^{\rm T}_{L+1,(3/2+m+3n)} + Z^{\rm T}_{L,(m+3n)} \right\}, \quad \textit{for } m \geq 0 \ , \tag{3.5}$$

$$Z^{1\rm T}_{L,-m} = \sum_{n\geq 0} \left\{ Z^{\rm T}_{L+1,(1/2+m+3n)} + Z^{\rm T}_{L,(2+m+3n)} \right\}, \quad \textit{for } m \geq -1/2 \ . \tag{3.6}$$

This implies for the total partition functions:

$$Z^{1\rm T}_{L} = \frac{1}{3}(Z^{\rm T}_{L+1} + Z^{\rm T}_{L}) \ . \tag{3.7}$$

In (3.5) and (3.6) n spans all integer values such that the spin does not exceed the value $L/2$.

These identities were checked using data up to $L = 11$ (not included in Appendix A). In the finite-size scaling limit, as shown in (6.10) the identities amount to obtain the Gauss model from the $U_q(sl(2))$ symmetric partition functions [29].

How can we understand the identities (3.5)–(3.7)? We first notice that for $\gamma = \pi/3$, $\omega_- = -2\pi/3$ the 1BTL algebra is at an exceptional point (see (2.3)). Then, since $e_0^2 = e_0$ the 1BTL algebra has a quotient $e_0 = 1$. If we take $e_0 = 1$ in the one-boundary Hamiltonian (2.13) it becomes the Temperley-Lieb Hamiltonian (2.12). This may explain why one part of the spectrum of $H^{1\rm T}$ with L sites comes from the spectrum of $H^{\rm T}$ with L sites. On the other hand, if e_0 is subject to the supplementary condition $e_0 e_1 e_0 = e_0$, $H^{1\rm T}$ becomes $H^{\rm T}$ with $L+1$ sites. This may explain why another part of the spectrum of H^{1T} with L sites comes from the spectrum of $H^{\rm T}$

with $L+1$ sites[1]. This suggests how one might generalize Conjecture 1 to the case $a_- \neq 1$ (i.e. $\delta_- \neq \pi$) keeping $\omega_- = -2\gamma = -2\pi/3$ unchanged. Let $Z_L^{1\mathrm{T}}(\delta_-, z)$ be the partition function of H^{1T} in this case.

We also consider a new Hamiltonian (describing again a stochastic process [28]):

$$H^{\mathrm{T}}(\delta_-) = a_-(1-\tilde{e}_0) + H^{\mathrm{T}}, \tag{3.8}$$

where $\tilde{e}_0$ and e_i ($i = 1, \cdots, L-1$) are generators of a TL algebra with L generators. Obviously, $H^{\mathrm{T}}(\delta_-)$ is $U_q(sl(2))$ symmetric. Let $Z_L^{\mathrm{T}}(\delta_-, z)$ be the total partition function given by the spectrum of $H^{\mathrm{T}}(\delta_-)$.

Conjecture 2 *The following identity holds for finite chains:*

$$Z_L^{1\mathrm{T}}(\delta_-, z) = \frac{1}{3}\left(Z_L^{\mathrm{T}}(z) + Z_{L+1}^{\mathrm{T}}(\delta_-, z)\right). \tag{3.9}$$

This identity was checked up to $L = 6$. There are several implications of this conjecture. Firstly, if L is even and assuming that all levels except the ground-state of $H^{\mathrm{T}}(\delta_-)$ depend on δ_-, out of the 2^L levels, $p+1$ levels–including the ground-state–will be δ_- independent and $2p$ will be δ_- dependent. Here $p = (2^L - 1)/3$. If L is odd, and $p = (2^{L-1} - 1)/3$, then out of the 2^{L+1} levels, $1 + 2p$ are δ_- independent and $4p + 1$ are δ_- dependent. In principle, this result can also be obtained using the methods developed in [6]. The observation that in the spectrum of H^{d} there are levels independent of δ_- is known for a long time [26, 27], the fact that this unusual behaviour of the energy levels is related to exceptional points of the 1BTL algebra is new (see also [6]). Moreover, barring accidental coincidences of energy levels occurring for special values of a_-, the degeneracies seen in the H^{d} chain can be obtained from the known degeneracies of the $U_q(sl(2))$ symmetric chains H^{T} (δ_-) (they are δ_- independent).

Another interesting consequence of the Conjecture 2 is that it gives the finite-size scaling limit of the spectrum of the Hamiltonian (3.8) which, to our knowledge is unknown. Since in the finite-size scaling limit $Z_L^{1\mathrm{T}}(\delta_-, z)$ is δ_- independent [26], from Conjecture 2 so is $Z_L^{\mathrm{T}}(\delta_-, z)$.

It is convenient to divide the spectrum of $H^{1\mathrm{T}}$ for $\delta_- = \pi$ and L even into two groups according to the value of m. We denote these two groups by by 0_{e} and $1/3_{\mathrm{e}}$. The group 0_{e} contains all the states in the sectors $m = 3k$ and $m = 3k+2$ where k is an integer. The group $1/3_{\mathrm{e}}$ contains the states with $m = 3k+1$. For L odd, the two groups denoted by 0_{o} and $1/3_{\mathrm{o}}$ contain the same states with m replaced by $\tilde{m}$ where $\tilde{m} = 1/2 - m$. In Appendix A for each value of system size L the two groups are separated. The partition functions given by the energy levels of the four groups are denoted by $Z_L^{0_{\mathrm{e,o}}}(z)$, $Z_L^{1/3_{\mathrm{e,o}}}(z)$. These partition functions are going to be used when we consider the two-boundary case and in section 6.

3.2 The two-boundary chain

We turn now to the Hamiltonian $H^{2\mathrm{T}}$ (see (2.14) or (2.19)). We take $\omega_+ = \omega_- = -2\pi/3$ and $a_- = a_+ = 1$ ($\delta_- = \delta_+ = \pi$). We did not consider other values of a_- and a_+

In order to fix the Hamiltonian $H^{2\mathrm{T}}$ we have to specify the values of b in the 2BTL. Firstly, we take $b = 1$. This choice makes the Hamiltonian $H^{2\mathrm{T}}$ describe a stochastic process [20] and therefore the ground-state energy is equal to zero for any number of sites. Let us observe that for $b = 1$ the angle θ in the quantum chain is different for L odd and L even. It is equal to $\pm 2\pi/3$ for L even and to $\pm\pi/3$ for L odd.

The spectra of the quantum chain for different sizes of the system are shown in Appendix A. A closer look at the characteristic polynomials shows that the energy levels which appear for L sites, are contained in the one-boundary chains of size L and $L+1$ and hence in the open chains of size L, $L+1$ and $L+2$. This is not entirely surprising since we are at exceptional points of the 2BTL algebra (see (2.8)). As in the case of the one-boundary chain one can take quotients in the

[1] The existence of these quotients also explains relations between properties of stationary states of various raise and peel models [20].

algebra $e_0 = e_L = 1$ or conversely to promote a boundary generator of the 2BTL to a generator of the 1BTL algebra. We first separate the states in $0_{\rm e}$ and $0_{\rm o}$ into two groups

$$\begin{aligned} Z_L^{0_{\rm e}} &= Z_L^{{\rm e},I} + Z_L^{{\rm e},II} \quad (3.10)\\ Z_L^{0_{\rm o}} &= Z_L^{{\rm o},I} + Z_L^{{\rm o},II} \quad (3.11)\end{aligned}$$

where

$$Z_L^{{\rm e},I} = \sum_{m\in\mathbb{Z}} Z^{1T}_{L,3m}, \qquad Z_L^{{\rm e},II} = \sum_{m\in\mathbb{Z}} Z^{1T}_{L,3m+2}, \quad (3.12)$$

$$Z_L^{{\rm o},I} = \sum_{m\in\mathbb{Z}} Z^{1T}_{L,3m+1/2}, \qquad Z_L^{{\rm o},II} = \sum_{m\in\mathbb{Z}} Z^{1T}_{L,3m+3/2}. \quad (3.13)$$

The data presented in Appendix A, suggest the following conjecture:

Conjecture 3 *The total partition function $Z_L^{b=1}(z)$ for the Hamiltonian $H^{2\rm T}$ for $b = 1$ and L sites can be written in terms of partition functions of the one-boundary chain:*

$$\begin{aligned} Z^{b=1}_{L=2r}(z) &= Z^{{\rm e},I}_{L=2r}(z) + Z^{{\rm o},II}_{L=2r+1}(z) = Z^{{\rm e},II}_{L=2r}(z) + Z^{{\rm o},I}_{L=2r+1}(z), \quad (3.14)\\ Z^{b=1}_{L=2r-1}(z) &= Z^{{\rm o},I}_{L=2r-1}(z) + Z^{{\rm e},II}_{L=2r}(z) = Z^{{\rm o},II}_{L=2r-1}(z) + Z^{{\rm e},I}_{L=2r}(z). \quad (3.15)\end{aligned}$$

This conjecture was checked up to $L = 8$.

We take now $b = 0$. This implies taking in the quantum chain (2.19) $\theta = 0$ for L even and $\theta = \pm\pi$ for L odd. The value $b = 0$ might look "natural", but as we are going to notice shortly, there is more to this choice.

The spectra for $b = 0$ and different lattice sizes are shown in Appendix A. Keeping the notation introduced at the end of the previous section:

$$Z^{1/3_{\rm e}}_{L=2r} = \sum_{m\in\mathbb{Z}} Z^{1\rm T}_{L,3m+1}, \qquad Z^{1/3_{\rm o}}_{L=2r-1} = \sum_{m\in\mathbb{Z}} Z^{1\rm T}_{L,3m-1/2}, \quad (3.16)$$

one observes that the following conjecture is compatible with the data:

Conjecture 4 *The total partition function $Z_L^{b=0}(z)$ for the Hamiltonian $H^{2\rm T}$ for $b = 0$ and L sites can be written in terms of partition functions of the one-boundary chain:*

$$\begin{aligned} Z^{b=0}_{L=2r}(z) &= Z^{1/3_{\rm e}}_{L=2r}(z) + Z^{1/3_{\rm o}}_{L=2r+1}(z), \quad (3.17)\\ Z^{b=0}_{L=2r-1}(z) &= Z^{1/3_{\rm o}}_{L=2r-1}(z) + Z^{1/3_{\rm e}}_{L=2r}(z). \quad (3.18)\end{aligned}$$

This conjecture was checked up to $L = 8$. In section 5 we shall prove generalizations of conjectures 3 and 4 relating spectra of H^{1T} and H^{2T} for arbitrary values of ω_- and δ_- keeping $(\pm\theta - \omega_-)$ fixed. These are always exceptional points of the 2BTL algebra (2.8).

We have shown that for the one and two-boundary chains with proper boundary conditions, the energy levels can all be found in the open chain. Should we look for other values of b? Probably not since for $\omega_\pm = -2\gamma$, $\gamma = \pi/3$, one can see from (2.8) that there are no other exceptional points.

4 Spectra with magic. The case $q = e^{i\pi/2}$.

4.1 The open and one-boundary chain.

The case $q = e^{i\pi/2}$ is special since one can find the spectrum and the wavefunctions of the XX model with the most general boundary conditions without using the Bethe Ansatz (see [30] and [31] and references therein).

We are going to show, that similar to the case $q = e^{i\pi/3}$, magic exists and can be again related to the TL algebra and its extensions. Throughout this section we are going to take

$$e_j^2 = 0, \quad (j = 0, 1, \ldots, L). \tag{4.1}$$

This implies $\gamma = \pi/2$, $\omega_- = \omega_+ = -\pi$. (We have fixed the parameters of the TL and 1BTL algebra and three out of the four parameters of the 2BTL algebra).

We start with the $U_q(sl(2))$ symmetric Hamiltonian (2.12). The characteristic polynomials for different sizes and spin sectors S are given in Appendix B.

Next, we consider $H^{1\mathrm{T}}$, defined in (2.13), in which we take $a_- = 1$ ($\delta_- = \pi$). Using the spectral equivalence of $H^{1\mathrm{T}}$ and H^{d} (see (2.17)) in Appendix B we give the characteristic polynomials for different values of m (the eigenvalues of S_z). We have to keep in mind that $\omega_- = -\pi$ is an exceptional value of the 1BTL algebra (see (2.3)).

One notices the following two identities:

$$Z^{1\mathrm{T}}_{L,m}(z) = Z^{1\mathrm{T}}_{L,-m}(z), \tag{4.2}$$

and

$$Z^{1\mathrm{T}}_{L,m}(z) = \sum_{n\geq 0} Z^{\mathrm{T}}_{L+1,1/2+m+2n}(z) \quad (m \geq -1/2). \tag{4.3}$$

from which it follows that:

$$\sum_{n\geq 0} Z^{\mathrm{T}}_{L,2n}(z) = \sum_{n\geq 0} Z^{\mathrm{T}}_{L,2n+1}(z), \qquad \text{for } L \text{ even} \tag{4.4}$$

Here $Z^{\mathrm{T}}_{L,S}$ is the partition function in the spin sector S for a system size L of H^{T} and $Z^{1\mathrm{T}}_{L,m}$ is the partition function in the sector m for a system of size L of $H^{1\mathrm{T}}$. The relations (4.4)–(4.3) can be proven for any system size [32].

¿From (4.3) one can show that

$$Z^{1\mathrm{T}}_{L}(z) = \frac{1}{2} Z^{\mathrm{T}}_{L+1}(z). \tag{4.5}$$

where $Z^{1\mathrm{T}}_{L}(z)$ is the total partition function of the open chain in which one takes into account the multiplicity of each spin S sector, and $Z^{\mathrm{T}}_{L}(z)$ is the total partition function for the one-boundary chain obtained by summing over the values of m ($-L/2 \leq m \leq L/2$).

We want to check if, as observed for the case $q = e^{i\pi/3}$, one cannot extend the relation (4.5) to the case when $a_- \neq 1$ (i.e. $\delta_- \neq \pi$) in (2.13).

We consider the $U_q(sl(2))$ invariant Hamiltonian (3.8) and compare it's spectrum with that of $H^{1\mathrm{T}}$, this brings us to a new conjecture

Conjecture 5 *The following identity holds for finite chains:*

$$Z^{1\mathrm{T}}_{L}(\delta_-, z) = \frac{1}{2} Z^{\mathrm{T}}_{L+1}(\delta_-, z) \tag{4.6}$$

where $Z^{\mathrm{T}}_{L+1}(\delta_-, z)$ is the total partition function for $H^{\mathrm{T}}(\delta_-)$ with L sites and $Z^{1\mathrm{T}}_{L}(\delta_-, z)$ is the total partition function for $H^{1\mathrm{T}}$ for L sites.

This conjecture was checked up to $L = 5$ sites. The existence of this relation should again be related to the fact that one is at an exceptional point of the 1BTL algebra.

Before proceeding, let us pause for a moment and look again at the relation (4.5) in order to illustrate how little the spectra tell us about the physical problem. H^{T} for q a root of unity has, as is well known, indecomposable Jordan cells, H^{d} is Hermitian and therefore is fully diagonalizable. Moreover, if one adds a dummy site to the L-site Hamiltonian H^{d} (a zero fermionic mode) the spectrum of the Hamiltonian in the larger vector space is precisely the one of H^{T} with $L+1$ sites. On the other hand, $H^{1\mathrm{T}}$ although it has the same spectrum as H^{d}, has again Jordan cell structures [6]. These observations are relevant since, as we are going to see, various characters of $c = -2$ conformal field theory are going to show up in the finite-size scaling limit (see Section 7) and the observation concerning different theories having the same partition functions for the finite chains should imply different conformal field theories related to the same $c = -2$ characters expressions.

4.2 The two-boundary chain.

We consider the Hamiltonian $H^{2\mathrm{T}}$ (see (2.14) or (2.19)) in which we take only $a_- = a_+ = 1$ ($\delta_- = \delta_+ = \pi$). We employ the same strategy as in the case of $q = e^{i\pi/3}$ and we are going to get a surprise.

We first take $b = 1$ in the 2BTL algebra (see (2.5)). This choice is quite "natural" if one has in mind the link pattern representation of the 2BTL, not discussed here (see [16]). For L even, (the angle θ in (2.7) becomes complex) the spectra of $H^{2\mathrm{T}}$ have nothing to do with those of H^{T} or $H^{1\mathrm{T}}$ (they are not shown in Appendix B). The situation is entirely different if L is odd. In this case, using (2.7) one finds $\theta = 0$, the Hamiltonian is Hermitian and has a very simple form:

$$H = \frac{1}{2}\left(\sum_{i=1}^{L-1} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_1^x + \sigma_L^x\right). \tag{4.7}$$

Before proceeding let us mention a known curious fact [33]. For L odd only, the operator

$$Y = \frac{1}{8}\sum_{i=1}^{L-1}\left[\left(1 + (-)^j\right)\sigma_i^x \sigma_{i+1}^y - \left(1 - (-)^j\right)\sigma_i^y \sigma_{i+1}^x\right] + \frac{1}{4}\left(\sigma_1^y - \sigma_L^y\right), \tag{4.8}$$

commutes with H given by (4.7). The fact that in the presence of two boundaries, quantum chains with even and odd number of sites behave so differently, is a surprise. One possible origin of the surprise can be found in (2.8). If we take $\gamma = \pi/2$, and $\omega_- = \omega_+ = -\pi$ in the equations, for L odd, one finds two solutions: $\theta = 0$ and $\theta = \pm\pi$ whereas for L even only one solution: $\theta = \pm\pi/2$. The latter two solutions correspond to the case $b = 0$ which will be discussed soon.

The characteristic polynomials which give the spectrum of H in (4.7) are shown in Appendix B. Comparing the characteristic polynomials for a system of size $L = 2r-1$ with the characteristic polynomials for the one-boundary case and size $L = 2r$, one is led to the following conjecture:

Conjecture 6 *The total partition function for the Hamiltonian $H^{2\mathrm{T}}$, $\theta = 0$ and $L = 2r - 1$ sites is related to the charge even sector ($m = 2n$) of $H^{1\mathrm{T}}$ with $L = 2r$ sites:*

$$Z_{L=2r-1}^{b=1}(z) = \sum_{n\in\mathbb{Z}} Z_{L=2r,2n}^{1\mathrm{T}}(z). \tag{4.9}$$

We consider $b = 0$ in the 2BTL algebra. We start with L odd. This corresponds to $\theta = \pm\pi$. The characteristic polynomials are shown in Appendix B, their expressions suggest the next conjecture:

Conjecture 7 *The total partition function for the Hamiltonian $H^{2\mathrm{T}}$, $\theta = \pm\pi$ and $L = 2r - 1$ sites is related to the charge odd sector ($m = 2n + 1$) of $H^{1\mathrm{T}}$ with $L = 2r$ sites:*

$$Z_{L=2r-1}^{b=0}(z) = \sum_{n\in\mathbb{Z}} Z_{L=2r,2n+1}^{1\mathrm{T}}(z). \tag{4.10}$$

We consider again $b = 0$, with L even. This corresponds to $\theta = \pm\pi/2$. The characteristic polynomials given in Appendix B, suggest the conjecture:

Conjecture 8 *The total partition function for the Hamiltonian $H^{2\mathrm{T}}$, $\theta = \pm\pi/2$ and $L = 2r$ sites is related to $1/2$ of the partition function of $H^{1\mathrm{T}}$ with $L = 2r + 1$ sites:*

$$Z_{L=2r}^{b=0}(z) = \frac{1}{2}\sum_{m\in\mathbb{Z}+1/2} Z_{L=2r+1,m}^{1\mathrm{T}}(z). \tag{4.11}$$

These three conjectures were checked up to $L=8$. It is interesting to mention that the spectra of the three quantum chains $H^{2\mathrm{T}}$ (4.9), (4.10), and (4.11) exhaust the spectra of $H^{1\mathrm{T}}$. This is another argument for not expecting more exceptional points of the 2BTL algebra for $\gamma=\pi/2, \omega_\pm=-\pi$. As will be discussed in Section 7, all the conjectures made above are valid in the finite-size scaling limit.

In the next section we shall prove generalizations of conjectures 6, 7 and 8 relating spectra of H^{1T} and H^{2T} for arbitrary values of ω_- and δ_- keeping $(\pm\theta-\omega_-)$ fixed. These are always exceptional points of the 2BTL algebra (2.8).

5 Spectral equivalences from the Bethe Ansatz

The Bethe Ansatz equations for the spectrum of $H^{2\mathrm{T}}$ have been written down only at exceptional points of the 2BTL [12–14, 16]. The cases in this paper are precisely of this kind, and we will use the Bethe Ansatz solution to prove conjectures 3, 4 and 6-8. In fact, we will prove a generalization of these conjectures.

Let us start by briefly stating the results for $H^{2\mathrm{T}}$ at the exceptional point (2.8) with $\epsilon=1$ [12–14]. To simplify the presentation we recall from (2.15) that

$$a_\pm = \frac{2\sin\gamma\sin(\gamma+\omega_\pm)}{\cos\omega_\pm+\cos\delta_\pm}, \tag{5.12}$$

and we will also use the definition

$$s_\pm = \frac{\sin\omega_\pm}{\sin(\gamma+\omega_\pm)}. \tag{5.13}$$

The eigenvalues of $H^{2\mathrm{T}}$ split into two groups, $E_1^{2\mathrm{T}}(k)$ and $E_2^{2\mathrm{T}}(k)$, which can be written as

$$E_1^{2T}(k) = a_-(1-s_-)+a_+(1-s_+)+L-1-\sum_{j=1}^{\lfloor (L-1)/2\rfloor-k}\frac{2\sin^2\gamma}{\cos 2u_i-\cos\gamma}, \tag{5.14}$$

$$E_2^{2T}(k) = a_-+a_++L-1-\sum_{j=1}^{\lfloor L/2\rfloor+k}\frac{2\sin^2\gamma}{\cos 2v_i-\cos\gamma}, \tag{5.15}$$

for all $k\in\mathbb{Z}$ satisfying (2.8). The complex numbers u_i and v_i are solutions of the equations

$$z(u_i)^{2L} = \frac{K_-(u_i-\omega_-)K_+(u_i-\omega_+)}{K_-(-u_i-\omega_-)K_+(-u_i-\omega_+)}\prod_{\substack{j=1\\ j\neq i}}^{\lfloor (L-1)/2\rfloor-k}\frac{S(u_i,u_j)}{S(-u_i,u_j)}, \tag{5.16}$$

$$z(v_i)^{2L} = \frac{K_-(v_i)K_+(v_i)}{K_-(-v_i)K_+(-v_i)}\prod_{\substack{j=1\\ j\neq i}}^{\lfloor L/2\rfloor+k}\frac{S(v_i,v_j)}{S(-v_i,v_j)}, \tag{5.17}$$

where we have used the following functions,

$$z(u) = \frac{\sin(\gamma/2+u_i)}{\sin(\gamma/2-u_i)}, \qquad S(u,v)=\cos 2v-\cos(2\gamma+2u), \tag{5.18}$$

$$K_\pm(u) = \cos\delta_\pm+\cos(\gamma+\omega_\pm+2u). \tag{5.19}$$

We are going to compare these solutions to the Bethe Ansatz for $H^{1\mathrm{T}}$ [16]. As described in [6], the eigenvalues $E^{1\mathrm{T}}$ of $H^{1\mathrm{T}}$ are the same as those of H^{d} [7] and can therefore be grouped into sectors labelled by the eigenvalues m of S_z. We give the Bethe Ansatz equations for $H^{1\mathrm{T}}$ explicitly, to emphasize their similarity with (5.14)–(5.17),

$$E^{1T}(m) = a_-(1-s_-)+L-1-\sum_{j=1}^{L/2-m}\frac{2\sin^2\gamma}{\cos 2\mu_i-\cos\gamma}, \tag{5.20}$$

and

$$E^{1T}(m) = a_- + L - 1 - \sum_{j=1}^{L/2+m} \frac{2\sin^2\gamma}{\cos 2\nu_i - \cos\gamma}. \tag{5.21}$$

Here, the complex numbers μ_i and ν_i satisfy the equations

$$z(\mu_i)^{2L} = \frac{K_-(\mu_i - \omega_-)}{K_-(-\mu_i - \omega_-)} \prod_{\substack{j=1\\ j\neq i}}^{L/2-m} \frac{S(\mu_i,\mu_j)}{S(-\mu_i,\mu_j)}, \tag{5.22}$$

$$z(\nu_i)^{2L} = \frac{K_-(\nu_i)}{K_-(-\nu_i)} \prod_{\substack{j=1\\ j\neq i}}^{L/2+m} \frac{S(\nu_i,\nu_j)}{S(-\nu_i,\nu_j)}. \tag{5.23}$$

We will now describe some connections between the spectra of $H^{2\mathrm{T}}$ and that of $H^{1\mathrm{T}}$. In particular, we set $a_+ = 1$, $\omega_+ = -2\gamma$ (i.e. $\delta_+ = \pi$) and take several values for γ and b, see (2.7) and (2.8).

5.1 $\gamma = \pi/3$

We take $\delta_+ = \pi$ and $\omega_+ = -2\pi/3$ and find

$$\frac{K_+(u + 2\pi/3)}{K_+(-u + 2\pi/3)} = 1, \qquad \frac{K_+(u)}{K_+(-u)} = \left(\frac{\sin(\gamma/2 - u)}{\sin(\gamma/2 + u)}\right)^2 = z(u)^{-2}, \tag{5.24}$$

This will allow us to identify (5.16) and (5.17) with (5.22) and (5.23) either with the same system size L or with L replaced by $L+1$. We will consider this correspondence in detail for $b = 1$ and $b = 0$.

- $b = 1$, L even

  ¿From (2.7) we find that $b = 1$ implies $\theta = \pm\omega_-$, and it follows from (2.8) that $k \equiv 2 \bmod 3$. The eigenvalue (5.14) and the Bethe Ansatz equations (5.16) are identical to in (5.20) and (5.22) by identifying $m = k + 1 \equiv 0 \bmod 3$. Similarly, the eigenvalue (5.15) and the Bethe Ansatz equations (5.17) are identical to (5.21) and (5.23) with L replaced by $L + 1$ and $m = k - \frac{1}{2} \equiv \frac{3}{2} \bmod 3$.

  Let $Z_L^{b=1}(\omega_-, \delta_-, z)$ be the total partition function of the Hamiltonian $H^{2\mathrm{T}}$ with $b = 1$, $\omega_+ = -2\gamma = -2\pi/3$, $a_+ = 1$ (i.e. $\delta_+ = \pi$), and arbitrary values of δ_- and ω_- in the left boundary term. Denote by $Z_{L=2r}^{\mathrm{e},I(\text{or } II)}(\omega_-, \delta_-, z)$, $Z_{L=2r+1}^{\mathrm{o},I(\text{or } II)}(\omega_-, \delta_-, z)$, $Z_{L=2r}^{1/3_\mathrm{e}}(\omega_-, \delta_-, z)$ and $Z_{L=2r+1}^{1/3_\mathrm{o}}(\omega_-, \delta_-, z)$ the partition functions defined as in eqs.(3.12), (3.13), (3.16) (see also discussion at the end of section 3.1) for the Hamiltonian $H^{1\mathrm{T}}$ with arbitrary values of the left boundary parameters ω_- and δ_-

  ¿From the Bethe Ansatz equations one therefore obtains:

$$Z_{L=2r}^{b=1}(\omega_-, \delta_-, z) = Z_{L=2r}^{\mathrm{e},I}(\omega_-, \delta_-, z) + Z_{L=2r+1}^{\mathrm{o},II}(\omega_-, \delta_-, z), \tag{5.25}$$

  This is a generalization of the first equality (3.14) in conjecture 3.

- $b = 1$, L odd

  ¿From (2.7) we find that $b = 1$ implies $\theta = \pm(\pi + \omega_-)$, and it follows from (2.8) that $k \equiv 1 \bmod 3$. The eigenvalue (5.14) and the Bethe Ansatz equations (5.16) are identical to in (5.20) and (5.22) by identifying $m = k + \frac{1}{2} \equiv \frac{3}{2} \bmod 3$. Similarly, the eigenvalue (5.15) and the Bethe Ansatz equations (5.17) are identical to (5.21) and (5.23) with L replaced by $L + 1$ and $m = k - 1 \equiv 0 \bmod 3$. Thus,

$$Z_{L=2r-1}^{b=1}(\omega_-, \delta_-, z) = Z_{L=2r-1}^{\mathrm{o},II}(\omega_-, \delta_-, z) + Z_{L=2r}^{\mathrm{e},I}(\omega_-, \delta_-, z). \tag{5.26}$$

  The second equality of (3.15) in conjecture 3 is a corollary.

- $b=0$, L even

  ¿From (2.7) we find that $b=0$ implies $\theta=\pm(\frac{2\pi}{3}+\omega_-)$, and it follows from (2.8) that $k\equiv 0 \bmod 3$. The eigenvalue (5.14) and the Bethe Ansatz equations (5.16) are identical to in (5.20) and (5.22) by identifying $m=k+1\equiv 1 \bmod 3$. Similarly, the eigenvalue (5.15) and the Bethe Ansatz equations (5.17) are identical to (5.21) and (5.23) with L replaced by $L+1$ and $\widetilde{m}=\frac{1}{2}-m=k\equiv 1 \bmod 3$. Hence,

$$Z^{b=0}_{L=2r}(\omega_-,\delta_-,z)=Z^{1/3_{\mathrm{e}}}_{L=2r}(\omega_-,\delta_-,z)+Z^{1/3_{\mathrm{o}}}_{L=2r+1}(\omega_-,\delta_-,z), \tag{5.27}$$

  Equation (3.17) in conjecture 4 follows as a corollary.

- $b=0$, L odd

  ¿From (2.7) we find that $b=0$ implies $\theta=\pm(-\frac{\pi}{3}+\omega_-)$, and it follows from (2.8) that $k\equiv 2 \bmod 3$. The eigenvalue (5.14) and the Bethe Ansatz equations (5.16) are identical to (5.20) and (5.22) by identifying $m=k+\frac{1}{2}$ and hence $\widetilde{m}=\frac{1}{2}-m=-k\equiv 1 \bmod 3$. Similarly, the eigenvalue (5.15) and the Bethe Ansatz equations (5.17) are identical to (5.21) and (5.23) with L replaced by $L+1$ and $m=k-1\equiv 1 \bmod 3$. Hence,

$$Z^{b=0}_{L=2r-1}(\omega_-,\delta_-,z)=Z^{1/3_{\mathrm{o}}}_{L=2r-1}(\omega_-,\delta_-,z)+Z^{1/3_{\mathrm{e}}}_{L=2r}(\omega_-,\delta_-,z). \tag{5.28}$$

  Equation (3.18) in conjecture 4 follows as a corollary.

5.2 $\gamma=\pi/2$

We take $\delta_+=\pi$ and $\omega_+=-2\gamma=-\pi$ and find

$$\frac{K_+(u+\pi)}{K_+(-u+\pi)}=\frac{K_+(u)}{K_+(-u)}=\left(\frac{\sin(\gamma/2-u)}{\sin(\gamma/2+u)}\right)^2=z(u)^{-2}, \tag{5.29}$$

As before, this allows us to identify (5.16) and (5.17) with (5.22) and (5.23) with L replaced by $L+1$. We will consider this correspondence in detail for $b=1$ and $b=0$.

- $b=1$, L odd

  ¿From (2.7) we find that $b=1$ implies $\theta=\pm(\omega_-+\pi)$, which means that k in (2.8) takes on only odd values. The eigenvalue (5.14) and the Bethe Ansatz equations (5.16) are identical to (5.20) and (5.22) by replacing L with $L+1$ and identifying $m=k+1$. Similarly, the eigenvalue (5.15) and the Bethe Ansatz equations (5.17) are identical to (5.21) and (5.23) with L replaced by $L+1$ but now $m=k-1$. We thus conclude

$$Z^{b=1}_{L=2r-1}(\omega_-,\delta_-,z)=\sum_{n\in\mathbb{Z}}Z^{1\mathrm{T}}_{L=2r,2n}(\omega_-,\delta_-,z). \tag{5.30}$$

  Here $Z^{b=1(\mathrm{or}\ 0)}_{L=2r-1}(\omega_-,\delta_-,z)$ (respectively, $Z^{1\mathrm{T}}_{L=2r,m}(\omega_-,\delta_-,z)$) denotes the total (resp., the charge m sector) partition functions of the Hamiltonian $H^{2\mathrm{T}}$ with $b=1$(or 0) (resp., $H^{1\mathrm{T}}$), where $\delta_+=\pi$ and $\omega_+=-2\gamma=-\pi$ are taken, but values of the left boundary parameters ω_- and δ_- are kept arbitrary.

  Conjecture 6 follows as a corollary.

- $b=0$, L odd

  This case is complementary to the previous case. Namely $b=0$ implies from (2.7) that $\theta=\pm\omega_-$, and now k in (2.8) takes on only even values. Using exactly the same argument as above, we conclude

$$Z^{b=0}_{L=2r-1}(\omega_-,\delta_-,z)=\sum_{n\in\mathbb{Z}}Z^{1\mathrm{T}}_{L=2r,2n+1}(\omega_-,\delta_-,z). \tag{5.31}$$

  Conjecture 7 follows as a corollary.

- $b = 0$, L even

  ¿From (2.7) we find that these values imply $\theta = \pm(\omega_- + \pi/2)$, which means that k in (2.8) takes on only even values. Using again the same reasoning as above, it follows that

$$Z^{b=0}_{L=2r}(\omega_-, \delta_-, z) = \frac{1}{2} \sum_{m \in \mathbb{Z}+1/2} Z^{1\mathrm{T}}_{L=2r+1,m}(\omega_-, \delta_-, z). \tag{5.32}$$

  Conjecture 8 follows as a corollary.

We would like to remark that in [13] for the case $\theta = \pi$, $\delta_+ = \delta_-$, $\omega_+ = \omega_-$, and $\gamma = \frac{\pi}{M+1}$ for any positive integer M (also an exceptional point of the 2BTL algebra (2.8)), the spectrum of H^{2T} on L sites (for L odd) is related to the spectra of diagonal chains, and therefore also of H^{1T}.

Let us finally comment on how the relations (5.25)–(5.28) can be interpreted from an algebraic point of view. In the 2BTL algebra when we have $e_i^2 = e_i$ and $e_L^2 = e_L$, i.e. $\gamma = \pi/3$ and $\omega_+ = -2\pi/3$, one can perform two types of quotient. In the first $e_L = 1$ and the L-site two-boundary Hamiltonian $H^{2\mathrm{T}}$ (2.14) becomes the L-site one-boundary Hamiltonian H^{1T} (2.13). In the second quotient we take $e_L e_{L-1} e_L = e_L$ and therefore the L-site two-boundary Hamiltonian $H^{2\mathrm{T}}$ (2.14) becomes the one-boundary Hamiltonian H^{1T} (2.13) now on $L+1$ sites. These two quotients, and relations between the Hamiltonians hold for generic values of the boundary parameters ω_- and δ_-. However we found that the magical connections between the spectra, (5.25)–(5.32), only occurred at the non-semisimple points (2.8). A proper understanding of this fact is still missing.

The conclusion of this section is that the spectrum of the 2BTL Hamiltonian (2.14) in the exceptional cases can be related to one-boundary Hamiltonian (2.13). Since the finite-size scaling limit of the spectra of the latter is known [7] this allows us to derive the finite size scaling limit of the 2BTL case. In the next section we shall do this.

6 Finite-size scaling limits of the spectra at $q = e^{\pi i/3}$ and $c = 0$ CFT

In the next two sections we discuss the finite-size scaling limit of the spectra discussed in Section 3 and 4. The case of finite size scaling of the Hermitian two-boundary chain, at the exceptional points (2.8), was considered in [23].

We start with some general observations. As is standard in finite size scaling the partition function (3.3), used in section 3 and 4, has to be substituted by a different one:

$$\widetilde{Z}_L(z) = \sum_i z^{\widetilde{E}_i}, \tag{6.1}$$

with

$$\widetilde{E}_i = \frac{(E_i - E_0)L}{\pi v}, \tag{6.2}$$

where E_0 is the ground-state energy, L is the size of the system and v is the sound velocity [22],

$$v = \frac{\pi}{\gamma} \sin\gamma. \tag{6.3}$$

Since all the spectra are contained in the H^{T} Hamiltonian, so are the ground-states. Therefore, for $q = e^{i\pi/(M+1)}$, in the finite size scaling limit one can use the result of [1] which states that the central charge c of the Virasoro algebra is:

$$c = 1 - \frac{6}{M(M+1)}. \tag{6.4}$$

We also expect the finite-size scaling limit of the spectrum of $H^{\rm d}$ in the sector m (eigenvalue of S^z called "electric" charge) to be given by a free boson field with Dirichlet-Dirichlet boundary conditions [21, 24]:

$$\chi_m = z^{\frac{M}{M+1}(m+\alpha)^2-\frac{1}{4M(M+1)}}P(z), \tag{6.5}$$

where

$$P(z) = \prod_{n\geq 1}(1-z^n)^{-1}, \tag{6.6}$$

and α depends on ω_- only and not on δ_- [26].

In the case of $H^{2\rm T}$, the situation is more complicated. One expects, depending on the boundary parameters, the finite-size scaling spectrum to be given by the free boson field theory either with Neumann-Neumann boundary conditions or with Neumann-Dirichlet boundary conditions. The connection between the conformal field theory and the boundary parameters of the XXZ Hamiltonian are known only at the decoupling point $q = e^{i\pi/2}$ [31]. For $q = e^{i\pi/3}$ such a connection is not known and one of the results of the magic shown in Section 3 is that in four cases (the four exceptional points of the 2BTL algebra) we will get it. For what we need, it is sufficient to mention what we expect in the case of Neumann-Neumann boundary conditions in the sector μ [21, 24]

$$\chi_\mu = z^{\frac{M+1}{M}(\mu+\beta)^2-\frac{1}{4M(M+1)}}P(z). \tag{6.7}$$

Here the sectors are specified by the values μ (the "magnetic" charge) which are the eigenvalues of an operator related to a $U(1)$ symmetry not seen in the XXZ chain and β is an unknown parameter depending on the boundary parameters.

We consider the case $M = 2$. We have, using (6.4), $c = 0$. In the continuum, the spectrum of $H^{\rm T}$ in the sector specified by spin S gives the partition function (we use (6.1)) [29]:

$$\chi_S^{\rm T}(z) = \lim_{L\to\infty}\widetilde{Z}_{L,S}^{\rm T}(z) = \left(z^{S(2S-1)/3} - z^{(S+1)(2S+3)/3}\right)P(z). \tag{6.8}$$

Using the results of [26] one can fix the value of α in (6.5) and for the sector m of $H^{\rm d}$ one obtains:

$$\chi_m^{1\rm T}(z) = \lim_{L\to\infty}\widetilde{Z}_{L,m}^{1\rm T}(z) = z^{2(m-1/4)^2/3-1/24}P(z) = z^{m(2m-1)/3}P(z). \tag{6.9}$$

Using (6.8) and (6.9) one can check the following identity:

$$\chi_m^{1\rm T} = \chi_{1/2-m}^{1\rm T} = \sum_{n\geq 0}\left(\chi_{m+3n}^{\rm T} + \chi_{3/2+m+3n}^{\rm T}\right), \tag{6.10}$$

which corresponds to the finite size scaling limit of (3.5) and (3.6) in section 3.

We define:

$$\chi_{\rm e}^{\rm T}(z) = \sum_{S\geq 0}(2S+1)\chi_S^{\rm T}, \qquad \chi_{\rm o}^{\rm T}(z) = \sum_{S\geq 1/2}(2S+1)\chi_S^{\rm T}, \tag{6.11}$$

$$\chi_{\rm e}^{1\rm T}(z) = \sum_{m\in\mathbb{Z}}\chi_m^{1\rm T}, \qquad \chi_{\rm o}^{1\rm T}(z) = \sum_{m\in\mathbb{Z}+1/2}\chi_m^{1\rm T}. \tag{6.12}$$

where e and o correspond to the $L\to\infty$ limits for even and odd length chains respectively.

Using (6.8), (6.9) and (6.11), (6.12) one gets:

$$\chi_{\rm e}^{1\rm T}(z) = \chi_{\rm o}^{1\rm T}(z) = \frac{1}{3}\left(\chi_{\rm e}^{\rm T}(z) + \chi_{\rm o}^{\rm T}(z)\right), \tag{6.13}$$

which corresponds to the finite size scaling limit of (3.7).

In order to understand the spectra of $H^{2\mathrm{T}}$ at the exceptional points it is important to make contact with characters of $N=2$ superconformal field theory. It is useful to first define the following functions (with $u = y^{2/3}$):

$$\begin{aligned}
\chi^{\mathrm{e},I}(y,z) &= \sum_{\mu\in\mathbb{Z}} \chi^{1\mathrm{T}}_{3\mu}(z) y^{3\mu} = u^{1/6}\sum_{\mu\in\mathbb{Z}} u^{2\mu-1/6} z^{3(2\mu-1/6)^2/2-1/24} P(z),\\
\chi^{\mathrm{e},II}(y,z) &= \sum_{\mu\in\mathbb{Z}} \chi^{1\mathrm{T}}_{3\mu+2}(z) y^{(3\mu+2)} = u^{1/6}\sum_{\mu\in\mathbb{Z}} u^{2\mu+7/6} z^{3(2\mu+7/6)^2/2-1/24} P(z),\\
\chi^{\mathrm{o},I}(y,z) &= \sum_{\mu\in\mathbb{Z}} \chi^{1\mathrm{T}}_{3\mu+1/2}(z) y^{(3\mu+1/2)} = u^{1/6}\sum_{\mu\in\mathbb{Z}} u^{2\mu+1/6} z^{3(2\mu+1/6)^2/2-1/24} P(z),\\
\chi^{\mathrm{o},II}(y,z) &= \sum_{\mu\in\mathbb{Z}} \chi^{1\mathrm{T}}_{3\mu+3/2}(z) y^{(3\mu+3/2)} = u^{1/6}\sum_{\mu\in\mathbb{Z}} u^{2\mu+5/6} z^{3(2\mu+5/6)^2/2-1/24} P(z),\\
\chi^{1/3,\mathrm{e}}(y,z) &= \sum_{\mu\in\mathbb{Z}} \chi^{1\mathrm{T}}_{3\mu+1}(z) y^{(3\mu+1)} = u^{1/6}\sum_{\mu\in\mathbb{Z}} u^{2\mu+1/2} z^{3(2\mu+1/2)^2/2-1/24} P(z),\\
\chi^{1/3,\mathrm{o}}(y,z) &= \sum_{\mu\in\mathbb{Z}} \chi^{1\mathrm{T}}_{3\mu+5/2}(z) y^{(3\mu+5/2)} = u^{1/6}\sum_{\mu\in\mathbb{Z}} u^{2\mu+3/2} z^{3(2\mu+3/2)^2/2-1/24} P(z),
\end{aligned} \tag{6.14}$$

It was pointed out by Eguchi and Yang [34] (see also [35]) that the Ramond sector of an $N=2$ superconformal field theory for any value of the central charge is related to a $c=0$ conformal field theory. Here we consider the $N=2$ superconformal for the case $c=1$ only.

In the Ramond sector, the $N=2$ superconformal algebra has a super sub-algebra with four generators; L_0 (belonging to the Virasoro algebra), two supercharges G_0^i ($i=1,2$) and a $U(1)$ charge T_0:

$$\begin{aligned}
&\frac{1}{2}\left\{G_0^i, G_0^j\right\} = \delta_{ij}\left(L_0 - \frac{1}{24}\right), \qquad \left[T_0, G_0^i\right] = \mathrm{i}\epsilon_{ij} G^j,\\
&\left[L_0, G_0^i\right] = \left[L_0, T_0\right] = 0.
\end{aligned} \tag{6.15}$$

We are going to use this observation later in section 8.

The Ramond sector has three representations, $(1/24, +1/6)$, $(1/24, -1/6)$ and $(3/8, 1/2)$ (the first figure indicates the scaling dimension and the second the charge of the primary fields) [36].

If we define $\tilde{L}_0 = L_0 - 1/24$ then the central charge is shifted from the value $c=1$ to the value zero and the three representations become:

$$(1/24, \pm 1/6) \rightarrow (0, \pm 1/6), \qquad (3/8, 1/2) \rightarrow (1/3, 1/2) \tag{6.16}$$

With this shift in ground state energy the superalgebra (6.15) becomes the superalgebra of quantum mechanics with two supercharges. This superalgebra has two-dimensional representations if the eigenvalues of $\tilde{L}_0$ are different of zero. If the eigenvalue of $\tilde{L}_0$ is zero, one has a one dimensional representation if the supersymmetry is unbroken or a two-dimensional representation if the supersymmetry is broken (keep in mind that we have two representations of scaling dimension $1/24$). Moreover the spectrum of $\tilde{L}_0$ is non-negative.

The characters in any $N=2$ representation R are defined as:

$$\chi(u,z) = \underset{R}{Tr}\left(u^{T_0} z^{L_0}\right) \tag{6.17}$$

The characters corresponding to the three representations in the Ramond sector are given by [36]:

$$\chi_{0,\pm 1/6}(u,z) = \sum_{\mu\in\mathbb{Z}} u^{\mu\pm 1/6} z^{3(\mu\pm 1/6)^2/2-1/24} P(z), \tag{6.18}$$

$$\chi_{1/3,1/2}(u,z) = \sum_{\mu\in\mathbb{Z}} u^{\mu+1/2} z^{3(\mu+1/2)^2/2-1/24} P(z). \tag{6.19}$$

Notice that the expressions of the characters are similar to those shown in (6.14) and therefore we expect to see them in the XXZ model with non-diagonal boundary conditions.

The $N = 2$ superconformal algebra has a sub-superalgebra which is the $N = 1$ superconformal algebra. The last one has, in the Ramond sector, a sub-superalgebra with only one supercharge:

$$G_0^2 = L_0 - 1/24 = \widetilde{L}_0. \tag{6.20}$$

Being the square of an Hermitian operator $\tilde{L}_0$ in (6.20) has a non-negative spectrum.

The $N = 1$ superconformal algebra has only one representation with scaling dimension $1/24$ and one representation with scaling dimension $3/8$. The two representations $(1/24, \pm 1/6)$ remain irreducible but coincide in the case of $N = 1$ superconformal (the charge which distinguished them in the case $N = 2$ is not present in the case $N = 1$). The representation $(3/8, 1/2)$ splits into two identical representations in the case $N = 1$.

The characters in any $N = 1$ representation R are now defined as:

$$\chi(z) = \underset{R}{Tr}\, z^{L_0} \tag{6.21}$$

The new characters are:

$$\chi_{1/24}^{(N=1)} \to \chi_0^{(N=1)}(z) \quad = \quad \sum_{\mu \in \mathbb{Z}} z^{3(\mu \pm 1/6)^2/2 - 1/24} P(z) \tag{6.22}$$

$$\chi_{3/8}^{(N=1)} \to \chi_{1/3}^{(N=1)}(z) \quad = \quad \sum_{\mu \in \mathbb{Z}} z^{3(2\mu + 1/2)^2/2 - 1/24} P(z) \tag{6.23}$$

We make now the connection between $N = 2$ and $N = 1$ superconformal theories and the XXZ chain with boundaries.

We start with the finite-size spectra of $H^{1\mathrm{T}}$. Comparing the partition functions (6.14) and the character expressions (6.18) and (6.19) one sees that there is mis-match of the charges and it is not possible to write the partition functions of L even or L odd separately in terms of $N = 2$ characters. One is able to do it however in terms of $N = 1$ superconformal characters:

$$\sum_m \chi_m^{1\mathrm{T}}(z) = \chi_0^{(1)}(z) + \chi_{1/3}^{(1)}(z). \tag{6.24}$$

for both m even and m odd.

If, however we combine the spectra for L even and odd, the sum of the partition functions can be expressed in terms of $N = 2$ characters:

$$\sum_{m \in Z} \chi_m^{1\mathrm{T}} y^m + \sum_{m \in Z + 1/2} \chi_m^{1\mathrm{T}} y^m = \chi_{0,+1/6}(u, z) + \chi_{0,-1/6}(u, z) + \chi_{1/3,1/2}(u, z). \tag{6.25}$$

where $u = y^{3/2}$.

We have to keep in mind that we are in the finite-size scaling limit. However, since the Hamiltonians describe stochastic processes, the ground-state energy is zero and the spectrum is positive both for $H^{1\mathrm{T}}$ and $H^{2\mathrm{T}}$ (if $b = 1$) and one can ask which role the superalgebras (6.20) and (6.15) can play for the finite chains. For $H^{1\mathrm{T}}$ for both L even and odd, the superalgebra (6.20) does not give anything new. If we combine the spectra of L even and L odd however, if magic exists, the superalgebra (6.15) might become relevant even for finite chains and we are going to show in section 8 that this is indeed the case. A closer inspection of equation (6.24) suggests that if one excludes from the spectra of the finite chains for L even and odd separately those states which contribute to the character with scaling dimensions $1/3$, one has a chance to be able to use the superalgebra (6.15). This is due to the fact that the two representations $(0, \pm 1/6)$ remain irreducible for $N = 1$. In the 'cleaned' spectra we will see, in the finite chain for L even and L odd separately, the ground state as a singlet and the rest of the spectrum has degeneracies which are multiples of 2.

We consider now $H^{2\mathrm{T}}$. Conjectures 3 and 4 of section 3 give us only the spectra but not any assignment of the "magnetic" charge $\tilde{m}$ since no $U(1)$ is known (except at the decoupling point). We can however "export" the $U(1)$ known for H^{d} and define magnetic charges in this way.

For $b=1$ one finds using (3.14), (3.15), (6.14) and (6.18):

$$\begin{aligned}\widetilde{Z}^{b=1}_{\mathrm{even}}(y,z) &\approx \widetilde{Z}^{b=1}_{\mathrm{odd}}(y,z) \approx \chi^{\mathrm{e},I}(y,z)+\chi^{\mathrm{o},II}(y,z) \approx \chi^{\mathrm{e},II}(y,z)+\chi^{\mathrm{o},I}(y,z) \\ &\approx u^{1/6}\chi_{0,\pm 1/6}(y,z). \end{aligned} \tag{6.26}$$

where the sign $\approx$ implies equality modulo a redefinition of charges.

An amusing observation: in spite of the fact that for $b=1$ one can expect the superalgebra (6.15) to play a role for finite chains (the ground state is indeed a singlet), it does not. There are singlets for energy non-zero.

For $b=0$ one finds using (3.17), (3.18), (6.14) and (6.19):

$$\widetilde{Z}^{b=0}_{\mathrm{even}}(y,z)=\widetilde{Z}^{b=0}_{\mathrm{odd}}(y,z)=\chi^{1/3,\mathrm{e}}(y,z)+\chi^{1/2,\mathrm{o}}(y,z)=u^{1/6}\chi_{1/3,1/2}(y,z). \tag{6.27}$$

The important conclusion of our discussion is that the four exceptional points of the 2BTL algebra can be related to representations of $N=2$ superconformal algebra.

7 Finite-size scaling limits of the spectra at $q=e^{i\pi/2}$ and $c=-2$ CFT

¿From (6.4) one obtains $c=-2$, a case much studied in the framework of logarithmic conformal field theory (see [37] and [38] for reviews). We are going to touch this topic at the end of the section. Firstly we have to derive the finite-size scaling limit of the identities discussed in section 4.

¿From [29] and [31] we have:

$$\chi^{\mathrm{T}}_S(z)=\left(z^{S(S-1)/2}-z^{(S+2)(S+1)/2}\right)P(z), \tag{7.1}$$

and:

$$\chi^{1\mathrm{T}}_m(z)=z^{(m^2-1/4)/2}P(z)=\chi^{1\mathrm{T}}_{-m}(z) \tag{7.2}$$

To derive (7.2) we have used (6.5) and the results of [26]. Equation (7.2) corresponds to the finite size scaling limit of (4.2). From (7.1) and (7.2) we get:

$$\chi^{1\mathrm{T}}_m(z)=\sum_{n\geq 0}\chi^{\mathrm{T}}_{1/2+m+2n}, \tag{7.3}$$

which corresponds to the finite size scaling limit of (4.3). We define the total characters for H^{T}

$$\chi^{\mathrm{T}}_{\mathrm{e}}(z)=\sum_{S=0}(2S+1)\chi^{\mathrm{T}}_S, \qquad \chi^{\mathrm{T}}_{\mathrm{o}}(z)=\sum_{S=1/2}(2S+1)\chi^{\mathrm{T}}_S, \tag{7.4}$$

and $H^{1\mathrm{T}}$

$$\chi^{1\mathrm{T}}_{\mathrm{e}}(z)=\sum_{m\in\mathbb{Z}}\chi^{1\mathrm{T}}_m, \qquad \chi^{1\mathrm{T}}_{\mathrm{o}}(z)=\sum_{m\in\mathbb{Z}+1/2}\chi^{1\mathrm{T}}_m, \tag{7.5}$$

Using (7.3)–(7.5) we derive the relations:

$$\chi^{1\mathrm{T}}_{\mathrm{e}}(z)=\frac{1}{2}\chi^{\mathrm{T}}_{\mathrm{o}}(z), \qquad \chi^{1\mathrm{T}}_{\mathrm{o}}(z)=\frac{1}{2}\chi^{\mathrm{T}}_{\mathrm{e}}(z) \tag{7.6}$$

These relations correspond to (4.5).

We now check if (4.9)–(4.11), conjectured for the finite chains, are valid in the finite size scaling limit. We use here the results of [31] which give the values of β in (6.7) as a function of the angle θ

appearing in the expression of $H^{2\mathrm{T}}$. We also show that for each value of θ the partition functions in the finite-size scaling limit are given by one character only of the $c=-2$ theory.

One has:

$$\begin{aligned}
\lim_{k\to\infty} \widetilde{Z}^{b=1}_{L=2r-1}(z) &= \sum_{n\in\mathbb{Z}} \chi^{1\mathrm{T}}_{2n}(z) \qquad (\theta=0)\\
&= \sum_{n\in\mathbb{Z}} z^{2n^2-1/8}P(z) = z^{-1/8}\Theta_{0,2}(z)P(z) \qquad (7.7)\\
\lim_{k\to\infty} \widetilde{Z}^{b=0}_{L=2r-1}(z) &= \sum_{n\in\mathbb{Z}} \chi^{1\mathrm{T}}_{2n+1}(z) \qquad (\theta=\pi)\\
&= \sum_{n\in\mathbb{Z}} z^{2(n+1/2)^2-1/8}P(z) = z^{-1/8}\Theta_{2,2}(z)P(z). \qquad (7.8)\\
\lim_{k\to\infty} \widetilde{Z}^{b=0}_{L=2r}(z) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \chi^{1\mathrm{T}}_{n+1/2}(z) \qquad (\theta=\frac{\pi}{2})\\
&= \sum_{n\in\mathbb{Z}} z^{2(n+1/4)^2-1/8}P(z) = z^{-1/8}\Theta_{1,2}(z)P(z). \qquad (7.9)
\end{aligned}$$

where

$$\Theta_{\lambda,\kappa}(z) = \sum_{n\in\mathbb{Z}} z^{(2\kappa n+\lambda)^2/4\kappa}. \qquad (7.10)$$

In the continuum there is a well-studied example of a $c=-2$ logarithmic CFT. This theory has an extended W-symmetry with only a finite number of irreducible and indecomposable representations [37, 38]. There are four irreducible fields conventionally written as $V_0, V_1, V_{-1/8}, V_{3/8}$ where the subscript is the conformal dimension. There are also two reducible but indecomposable modules R_0 and R_1. The characters ($\chi = Trz^{L_0}$) are given by:

$$\begin{aligned}
\chi_{V_0} &= \frac{1}{2}z^{-1/8}\left(\Theta_{1,2}(z)+\partial\Theta_{1,2}(z)\right)P(z) \qquad (7.11)\\
\chi_{V_1} &= \frac{1}{2}z^{-1/8}\left(\Theta_{1,2}(z)-\partial\Theta_{1,2}(z)\right)P(z) \qquad (7.12)\\
\chi_{V_{-1/8}} &= z^{-1/8}\Theta_{0,2}(z)P(z) \qquad (7.13)\\
\chi_{V_{3/8}} &= z^{-1/8}\Theta_{2,2}(z)P(z) \qquad (7.14)\\
\chi_R &= 2z^{-1/8}\Theta_{1,2}(z)P(z) \qquad (7.15)
\end{aligned}$$

where $\chi_R \equiv \chi_{R_0} = \chi_{R_1} = 2\left(\chi_{V_0}+\chi_{V_1}\right)$ and:

$$\partial\Theta_{\lambda,\kappa} = \sum_{n\in\mathbb{Z}} (2\kappa n+\lambda) z^{(2\kappa n+\lambda)^2/4\kappa}. \qquad (7.16)$$

The finite size scaling of the lattice partition functions can now be identified with the continuum results:

$$\begin{aligned}
\lim_{k\to\infty} \widetilde{Z}^{b=1}_{L=2r-1}(z) &= \chi_{V_{-1/8}} \qquad (7.17)\\
\lim_{k\to\infty} \widetilde{Z}^{b=0}_{L=2r-1}(z) &= \chi_{V_{3/8}} \qquad (7.18)\\
\lim_{k\to\infty} \widetilde{Z}^{b=0}_{L=2r}(z) &= \chi_{V_0}+\chi_{V_1} = \frac{1}{2}\chi_R \qquad (7.19)
\end{aligned}$$

A very relevant observation is that to each of the three exceptional points of the 2BTL algebra corresponds to a representation of the $c=-2$ theory in the finite size scaling limit.

The finite-size scaling limit of the spectra of $H^{1\mathrm{T}}$ taken separately for L even and odd can also be expressed in terms of $c=-2$ characters:

$$\begin{aligned}
\chi^{1\mathrm{T}}_{\mathrm{e}}(z) &= \chi_{-1/8}(z)+\chi_{3/8}(z), \qquad (7.20)\\
\chi^{1\mathrm{T}}_{\mathrm{o}}(z) &= \chi_{\mathrm{R}}(z). \qquad (7.21)
\end{aligned}$$

We know that H^{1T} contains indecomposable representations and one can explicitly compute wave-functions using the methods of [31]. Further work is required to understand properly the continuum limit of this well defined lattice model.

8 Symmetries in the H^{1T} and H^d quantum chains for $q = e^{i\pi/3}$

In this section we are going to discuss the degeneracies occurring in the spectra of the Hamiltonian H^{1T} given by 2.13. First we are going to take $a_- = 1$ ($\delta_- = \pi$). Before we present our conjectures, it is instructive to consider the cases $L = 5$ and $L = 6$ and compare the characteristic polynomials appearing for different values of m.

In Appendix A, for each value of L, the characteristic polynomials for different values of m are separated into two groups. For L even, the group called 0_e in Section 3.2, is given first and contains the characteristic polynomials for $m = 3k$ and $m = 3k + 2$ ($k \in \mathbb{Z}$); the second group called $1/3_e$ contains the characteristic polynomials corresponding to $m = 3k + 1$. For L odd, we have the group 0_o with $m = 3k + 1/2$ and $m = 3k + 3/2$ and the group $1/3_o$ with $m = 3k + 5/2$.

For $L = 5$ the group 0_o contains the levels with $m = -5/2, -3/2, 1/2$ and $3/2$. We notice that the ground state (energy zero) appears as a singlet, all the other states appear as doublets except the level of energy 5 which appears as a quadruplet. The group $1/3_o$ ($m = -1/2$ and $5/2$) contains the energy level 4 twice, the others as singlets. Notice that the energy level 4 appears in both groups.

What happens with the degeneracies if we change the value of a_- ($\delta_- \neq \pi$) in H^{1T}? The degeneracies within each group stay unchanged but the energy level 4 which was common to the two groups splits into two values, one for the doublet in 0_o and a second one for the doublet in $1/3_o$. This observation is true for all values of L: the degeneracies within each of the two groups are independent on a_- and therefore have to be related to properties of the 1BTL algebra. This is indeed the case. Degeneracies appearing between both groups are accidental.

The 1BTL algebra has a center containing several elements. These elements, which can be written as a linear combination of words of the 1BTL algebra, have the property that they commute with *all* elements in the algebra. If one specifies the XXZ representation of the 1BTL algebra (2.9) and (2.10) , one obtains operators (centralizers) which commute with the Hamiltonian H^{1T} (2.13) (2.16) written in the same representation. At the exceptional points of the 1BTL algebra (2.3) where the algebra is not semi-simple, the centralizers which commute with H^{1T} are of Jordan form. It is easy to check that if an $N \times N$ matrix is a Jordan cell, any matrix commuting with it (in particular H^{1T}) has to have an N-fold degeneracy. One centralizer found by Doikou [25] was discussed in detail in [6], the construction of the centralizers and their properties will be published elsewhere. The main message is that the use of the centralizers allows to get all the degeneracies observed in the H^{1T} quantum chain.

Part of the symmetries observed in the groups 0_e and 0_o can be understood in the following way. We consider again the $L = 5$ example shown in Appendix A. It is convenient to order the characteristic polynomials according to the value of $\tilde{m}$. We notice that apart the ground-state which appears at $\tilde{m} = 0$, one sees doublets at $\Delta\tilde{m} = \pm 1$. For other values of L one can define a new "charge" such that doublets occur in the same way. This implies that one obtains representations of the superalgebra (6.20). The fact that the spectrum in the sectors 0_e and 0_o is composed of doublets and one singlet is contained in the following conjecture:

Conjecture 9 *The partition functions of H^{1T} satisfy the following identity for all values of L and δ:*

- *L even*

$$\sum_{n\in\mathbb{Z}} Z^{1T}_{L,3n}(\delta_-) = 1 + \sum_{n\in\mathbb{Z}} Z^{1T}_{L,3n+2}(\delta_-) \tag{8.22}$$

- *L odd*

$$\sum_{n\in\mathbb{Z}} Z^{1T}_{L,3n+1/2}(\delta_-) = 1 + \sum_{n\in\mathbb{Z}} Z^{1T}_{L,3n+3/2}(\delta_-) \tag{8.23}$$

Notice that the sectors $1/3_e$ and $1/3_0$ contain many singlets and therefore have nothing to do with the superalgebra (6.15). H^{1T} is compatible with the superalgebra (6.20) for any value of δ in a trivial way since it describes a stochastic process and therefore the spectrum is non-negative. We have not tried to find the odd generators of the superalgebras (6.15) and (6.20).

We are going to compare now the spectra for system sizes $L = 2p - 1$ and $L = 2p$ taking again $a_- = 1$ and using the tables of Appendix A. We start with the example $p = 3$. We notice that not only the sectors 0_e and 0_o have common energy levels but so do the sectors $1/3_e$ and $1/3_o$. Combining the spectra of the $L = 5$ and $L = 6$ one obtains only doublets (including the ground-state!). Some doublets may have the same energy. These observations are valid for any value of p and for $a_- = 1$ only. The super-algebra (6.15) gives doublets in the ground-state only if the supersymmetry is spontaneously broken. The fact that for $a_- = 1$, combining the spectra of the sectors $1/3_e$ for $2p$ sites and $1/3_o$ for $2p - 1$ sites, one obtains doublets is the basis of the following conjecture:

Conjecture 10 *The partition functions of H^{1T} for $a_- = 1$ only, satisfy the following identity:*

$$\sum_{n\geq 0}\left(Z^{1T}_{2p,3n+1} + Z^{1T}_{2p-1,3n+5/2}\right) = \sum_{n<0}\left(Z^{1T}_{2p,3n+1} + Z^{1T}_{2p-1,3n+5/2}\right) \tag{8.24}$$

Let us observe that relations between spectra of different sizes which are only valid for $a_- = 1$ are accidental if one thinks of symmetries coming from the 1BTL algebra. In a different approach [19] relations to supersymmetry were found for the Hamiltonian H^d (see (2.17)) for the value $a_- = 1$.

We conclude this section with an amusing observation. If one looks at the spectra of H^{2T} shown in Appendix A although in the finite-size scaling limit they give the same characters as those of H^{1T}, the degeneracies are very different. New degeneracies appear and some of the energy levels coincide between systems of different sizes. This suggests, that as in the case of the H^{1T} Hamiltonian for which the symmetries were given by the center of the 1BTL algebra, the center of the 2BTL algebra will play an important role for understanding the new degeneracies. The way to make a connection between the spectra of systems of different sizes has also to be understood.

9 Conclusion

We have observed that for the finite open XXZ spin chain with boundaries at $\Delta = 0$ and $\Delta = -1/2$ magic occurs. This magic was discovered accidentally when several computer outputs were lying on one desk.

We have first noticed that the same energy levels appearing in the $U_q(sl(2))$ symmetric chain with an even and odd number of sites can be seen in an non-$U_q(sl(2))$ invariant XXZ chain with certain diagonal boundary conditions. The degeneracies are not the same and energy levels seen in chains with an even and odd number of sites get mixed up. The main point is that the rules how to obtain the spectra of the diagonal chain (including the degeneracies) from the spectra of the $U_q(sl(2))$ invariant chain are simple (see conjecture 1 and eqs 4.2-4.5). The same phenomenon appears if we look at the XXZ chain with some non-diagonal boundary conditions (see conjectures 3, 4, 6, 7, and 8).

We can, at least partially, understand the origin of this magic in the following way. The $U_q(sl(2))$ symmetric XXZ chain is written in terms of a representation of the Temperley-Lieb algebra. The exceptional points of this algebra where the algebra is not semisimple (it contains indecomposable representations) occur if q is a root of unity. This is the case for $q = e^{i\pi/3}$ and $q = e^{i\pi/2}$. The XXZ chain with diagonal boundary conditions can be expressed in terms of generators of the one-boundary Temperley-Lieb algebra. This algebra has also exceptional points

2.3 and it is at these exceptional points that the magic was observed. In the case of the chain with non-diagonal boundary conditions, by accident, we have chosen the parameters such that we have exhausted the exceptional points of the two-boundary Temperley-Lieb algebra compatible with our choice of the boundary terms of the diagonal chain. There are four exceptional points in the case $q = e^{i\pi/3}$ and three for $q = e^{i\pi/3}$. For the exceptional points one expects special properties of the representations of the algebras. It is the interplay of the representations which produces the magic. This insight in the problem was possible because we were able to find an expression for the exceptional points of the two-boundary Temperley-Lieb algebra (see eq.2.8).

Once we understood the connection between the quantum chains and the Temperley-Lieb algebra and its extensions, we were able to extend the "domain of magic" beyond our original observations (see conjectures 2 and 5). Next, we wanted to go beyond numerics, and use the Bethe Ansatz in order to check the conjectures. It turned out that precisely at the exceptional points of the two-boundary Temperley-Lieb algebra (again magic!) one can write the Bethe Ansatz equations. This is possible because at the exceptional points, the Hamiltonian has a block triangular form, and that in order to compute the spectrum, one can disregard the off-diagonal blocks and keep only the diagonal ones. At least in the cases discussed here, the Bethe Ansatz equations for each block coincided with the equations obtained for a block of the Hamiltonian with diagonal boundary conditions. In the latter case, the diagonal blocks are labeled by the eigenvalues m of S^z (eq. 2.18). In this way, part of the conjectures were confirmed by the Bethe Ansatz equations (see section 5). A look at the Bethe Ansatz equations for more general boundary conditions suggested a further extension of the 'magic' (see eqs.(5.25)–(5.32)). The origin of this new extension could also be located using the one and two-boundary Temperley-Lieb algebra (see comment in the end of sec.5.2).

The "magic" was observed on finite chains. We have extended it to the finite-size scaling limit. Part of the "magic" was confirmed by using known characters of conformal field theory for the $U_q(sl(2))$ symmetric case and for the Gauss model (see sections 6 and 7).

In our opinion, two remarkable observations have to be mentioned. In the finite-size scaling limit to each of the exceptional points of the two-boundary Temperley-Lieb algebra corresponds a representation of a chiral conformal field theory: $c = 0$ in the Ramond sector of a $N = 2$ superconformal for $q = e^{i\pi/3}$. In the case $c = -2$ observed for $q = e^{i\pi/2}$ the situation is more subtle since the quantum field theory is less under control. One obtains some combinations of a particular $c = -2$ conformal field theory. The origin of this mismatch is not clear but can in principle be fixed since for $q = e^{i\pi/2}$ one is at the decoupling point and one can easily derive the continuum theory from its lattice realization. The second observation has to do with Jordan cell structure. In non-minimal models, character functions don't characterize alone the conformal field theory. This phenomenon can be seen explicitly in our study. For example, the chains with diagonal boundary conditions we considered are Hermitian. They have however the same spectra as some chains with non-diagonal boundary conditions on one side of the chain, corresponding to another representation of the one-boundary Temperley-Lieb algebra [6]. In this different representation on has Jordan cells and therefore a different model.

Finally, let us observe, that understanding the connection between the quantum chains and the Temperley-Lieb algebra and its extensions has another bonus. The quantum chains H^{1T} and H^{2T} (see eqs. (2.16) and (2.19) have in general spectra with no degeneracies. This is not the case at the exceptional points of the one and two-boundary Temperley Lieb algebras. An example of this kind is discussed in detail in section 8 where part of the degeneracies are encoded in Conjectures 9 and 10.

Acknowledgment

JdG and VR are grateful for financial support from the Deutsche Forschungsgemeinschaft and the Australian Research Council. AN and VR gratefully acknowledge support from the EU network *Integrable models and applications: from strings to condensed matter* HPRN-CT-2002-00325. PP was partially supported by the Heisenberg-Landau grant and by the RFBR grant No. 05-01-01086.

Appendix A Characteristic polynomials in the case $\gamma = \pi/3$

Here we present characteristic polynomials, factorised over the integers, for the Hamiltonians of the open (i.e. $U_q(sl(2))$ invariant) chains (see (2.9), (2.12)), the 1-boundary chains (see (2.13), (2.16)) with $\omega_- = -2\pi/3$, $a_- = 1$ ($\delta_- = \pi$), and the 2-boundary chains (see (2.14), (2.19)) for $\omega_\pm = -2\pi/3$, $a_\pm = 1$ ($\delta_\pm = \pi$) in two cases $b = 1$ and $b = 0$. In all cases we fix anisotropy $\Delta = -1/2$ ($q = e^{\pi/3}$). We collect data for the chains of sizes $1 \le L \le 8$ and organize them according to values of spin S for the open chains (each sector S comes with multiplicity $2S + 1$) and according to values of the charge m for the 1-boundary chains.

We give characteristic polynomials factorized over the integers. For each factor we use its total degree adding subscripts to distinguish between different factors of the same total degree. For example 4_a and 4_b are two different factors of degree 4. Groups of factors which appear always together are taken in parentheses. Explicit expressions for factors are presented at the end of the appendix.

$\mathbf{L=1}$

S		open chain
1/2		$(\mathbf{1_0})$
m	$\tilde{m}$	1-boundary chain
1/2	0	$(\mathbf{1_0})$
−1/2	1	$(\mathbf{1_1})$
2-boundary chain, $b = 1$ case		
$(\mathbf{1_2})\ (\mathbf{1_0})$		
2-boundary chain, $b = 0$ case		
$(\mathbf{1_1})^2$		

$\mathbf{L=2}$

S	open chain
0	$(\mathbf{1_0})$
1	$(\mathbf{1_1})$
m	1-boundary chain
0	$(\mathbf{1_2})\ (\mathbf{1_0})$
−1	$(\mathbf{1_2})$
1	$(\mathbf{1_1})$
2-boundary chain, $b = 1$ case	
$(\mathbf{1_3})\ (\mathbf{1_2})^2\ (\mathbf{1_0})$	
2-boundary chain, $b = 0$ case	
$(\mathbf{2_a})\ (\mathbf{1_2})(\mathbf{1_1})$	

$\mathbf{L=3}$

S		open chain
1/2		$(\mathbf{1_2})\ (\mathbf{1_0})$
3/2		$(\mathbf{1_2})$
m	$\tilde{m}$	1-boundary chain
−3/2	2	$(\mathbf{1_3})$
1/2	0	$(\mathbf{1_3})\ (\mathbf{1_2})\ (\mathbf{1_0})$
3/2	−1	$(\mathbf{1_2})$
−1/2	1	$(\mathbf{2_a} \cdot \mathbf{1_2})$
2-boundary chain, $b = 1$ case		
$(\mathbf{2_b} \cdot \mathbf{2_c})\ (\mathbf{1_3})^2\ (\mathbf{1_2})\ (\mathbf{1_0})$		
2-boundary chain, $b = 0$ case		
$(\mathbf{2_a})^2\ (\mathbf{1_4})^2(\mathbf{1_2})^2$		

$\mathbf{L=4}$

S	open chain
0	$(\mathbf{1_3})\ (\mathbf{1_0})$
1	$(\mathbf{2_a} \cdot \mathbf{1_2})$
2	$(\mathbf{1_3})$
m	1-boundary chain
2	$(\mathbf{1_3})$
0	$(\mathbf{2_b} \cdot \mathbf{2_c})\ (\mathbf{1_3})\ (\mathbf{1_0})$
−1	$(\mathbf{2_b} \cdot \mathbf{2_c})$
1	$(\mathbf{2_a} \cdot \mathbf{1_2})(\mathbf{1_4})$
−2	$(\mathbf{1_4})$
2-boundary chain, $b = 1$ case	
$(\mathbf{2_b} \cdot \mathbf{2_c})^2\ (\mathbf{2_d} \cdot \mathbf{1_3} \cdot \mathbf{1_4})\ (\mathbf{1_3})\ (\mathbf{1_5})^2\ (\mathbf{1_0})$	
2-boundary chain, $b = 0$ case	
$(\mathbf{6} \cdot \mathbf{3_a})\ (\mathbf{2_a} \cdot \mathbf{1_2})\ (\mathbf{1_4})^4$	

L = 5

S		open chain
$1/2$		$(\mathbf{2_b \cdot 2_c})\ (\mathbf{1_0})$
$3/2$		$(\mathbf{2_b \cdot 2_c})$
$5/2$		$(\mathbf{1_4})$
m	$\tilde{m}$	1-boundary chain
$-5/2$	3	$(\mathbf{1_5})$
$-3/2$	2	$(\mathbf{2_d \cdot 1_3 \cdot 1_4})\ (\mathbf{1_5})$
$1/2$	0	$(\mathbf{2_b \cdot 2_c})\ (\mathbf{2_d \cdot 1_3 \cdot 1_4})(\mathbf{1_5})\ (\mathbf{1_0})$
$3/2$	-1	$(\mathbf{2_b \cdot 2_c})\ (\mathbf{1_5})$
$-1/2$	1	$(\mathbf{6 \cdot 3_a})\ (\mathbf{1_4})$
$5/2$	-2	$(\mathbf{1_4})$
2-boundary chain, $b = 1$ case		
$(\mathbf{7 \cdot 5 \cdot 1_5})\ (\mathbf{2_b \cdot 2_c})\ (\mathbf{2_d \cdot 1_3 \cdot 1_4})^2$ $(\mathbf{1_5})^4\ (\mathbf{1_6})^2\ (\mathbf{1_0})$		
2-boundary chain, $b = 0$ case		
$(\mathbf{6 \cdot 3_a})^2\ (\mathbf{3_b \cdot 3_c})^2\ (\mathbf{1_4})^2$		

L = 6

S	open chain
0	$(\mathbf{2_d \cdot 1_3 \cdot 1_4})\ (\mathbf{1_0})$
1	$(\mathbf{6 \cdot 3_a})$
2	$(\mathbf{2_d \cdot 1_3 \cdot 1_4})\ (\mathbf{1_5})$
3	$(\mathbf{1_5})$
m	1-boundary chain
3	$(\mathbf{1_5})$
2	$(\mathbf{2_d \cdot 1_3 \cdot 1_4})\ (\mathbf{1_6})\ (\mathbf{1_5})$
0	$(\mathbf{7 \cdot 5 \cdot 1_5})\ (\mathbf{2_d \cdot 1_3 \cdot 1_4})\ (\mathbf{1_6})$ $(\mathbf{1_5})\ (\mathbf{1_0})$
-1	$(\mathbf{7 \cdot 5 \cdot 1_5})\ (\mathbf{1_6})\ (\mathbf{1_5})$
-3	$(\mathbf{1_6})$
1	$(\mathbf{6 \cdot 3_a})\ (\mathbf{3_b \cdot 3_c})$
-2	$(\mathbf{3_b \cdot 3_c})$
2-boundary chain, $b = 1$ case	
$(\mathbf{9 \cdot 4_a})\ (\mathbf{7 \cdot 5 \cdot 1_5})^2\ (\mathbf{4_b \cdot 2_e \cdot 1_6})^2$ $(\mathbf{2_d \cdot 1_3 \cdot 1_4})\ (\mathbf{1_5})^2\ (\mathbf{1_6})^4(\mathbf{1_0})$	
2-boundary chain, $b = 0$ case	
$(\mathbf{16 \cdot 10 \cdot 1_4})\ (\mathbf{6 \cdot 3_a})\ (\mathbf{3_b \cdot 3_c})^4\ (\mathbf{1_7})^4$	

L = 7

S		open chain
$1/2$		$(\mathbf{7 \cdot 5 \cdot 1_5})\ (\mathbf{1_0})$
$3/2$		$(\mathbf{7 \cdot 5 \cdot 1_5})\ (\mathbf{1_6})$
$5/2$		$(\mathbf{3_b \cdot 3_c})$
$7/2$		$(\mathbf{1_6})$
m	$\tilde{m}$	1-boundary chain
$-5/2$	3	$(\mathbf{4_b \cdot 2_e \cdot 1_6})$
$-3/2$	2	$(\mathbf{9 \cdot 4_a})\ (\mathbf{4_b \cdot 2_e \cdot 1_6})\ (\mathbf{1_6})$
$1/2$	0	$(\mathbf{9 \cdot 4_a})\ (\mathbf{7 \cdot 5 \cdot 1_5})\ (\mathbf{4_b \cdot 2_e \cdot 1_6})$ $(\mathbf{1_6})\ (\mathbf{1_0})$
$3/2$	-1	$(\mathbf{7 \cdot 5 \cdot 1_5})\ (\mathbf{4_b \cdot 2_e \cdot 1_6})\ (\mathbf{1_6})$
$7/2$	-3	$(\mathbf{1_6})$
$-7/2$	4	$(\mathbf{1_7})$
$-1/2$	1	$(\mathbf{16 \cdot 10 \cdot 1_4})\ (\mathbf{3_b \cdot 3_c})\ (\mathbf{1_7})^2$
$5/2$	-2	$(\mathbf{3_b \cdot 3_c})\ (\mathbf{1_7})$
2-boundary chain, $b = 1$ case		
$(\mathbf{20_a \cdot 20_b \cdot 1_8})\ (\mathbf{9 \cdot 4_a})^2\ (\mathbf{7 \cdot 5 \cdot 1_5})$ $(\mathbf{4_b \cdot 2_e \cdot 1_6})^4\ (\mathbf{3_d \cdot 3_e \cdot 1_6})^2\ (\mathbf{1_6})^2\ (\mathbf{1_8})^3\ (\mathbf{1_0})$		
2-boundary chain, $b = 0$ case		
$(\mathbf{16 \cdot 10 \cdot 1_4})^2\ (\mathbf{15 \cdot 12})^2\ (\mathbf{3_b \cdot 3_c})^2\ (\mathbf{1_7})^8$		

L = 8

S	open chain
0	$(\mathbf{9 \cdot 4_a})\ (\mathbf{1_0})$
1	$(\mathbf{16 \cdot 10 \cdot 1_4})\ (\mathbf{1_7})$
2	$(\mathbf{9 \cdot 4_a})\ (\mathbf{4_b \cdot 2_e \cdot 1_4})$
3	$(\mathbf{4_b \cdot 2_e \cdot 1_6})$
4	$(\mathbf{1_7})$
m	1-boundary chain
3	$(\mathbf{4_b \cdot 2_e \cdot 1_6})\ (\mathbf{1_8})$
2	$(\mathbf{9 \cdot 4_a})\ (\mathbf{4_b \cdot 2_e \cdot 1_6})\ (\mathbf{3_d \cdot 3_e \cdot 1_6})\ (\mathbf{1_8})$
0	$(\mathbf{20_a \cdot 20_b \cdot 1_8})\ (\mathbf{9 \cdot 4_a})\ (\mathbf{4_b \cdot 2_e \cdot 1_6})$ $(\mathbf{3_d \cdot 3_e \cdot 1_6})\ (\mathbf{1_8})\ (\mathbf{1_0})$
-1	$(\mathbf{20_a \cdot 20_b \cdot 1_8})\ (\mathbf{4_b \cdot 2_e \cdot 1_6})$ $(\mathbf{3_d \cdot 3_e \cdot 1_6})\ (\mathbf{1_8})$
-3	$(\mathbf{3_d \cdot 3_e \cdot 1_6})\ (\mathbf{1_8})$
-4	$(\mathbf{1_8})$
4	$(\mathbf{1_7})$
1	$(\mathbf{16 \cdot 10 \cdot 1_4})\ (\mathbf{15 \cdot 12})\ (\mathbf{1_7})^2$
-2	$(\mathbf{15 \cdot 12})\ (\mathbf{1_7})$
2-boundary chain, $b = 1$ case	
———	
2-boundary chain, $b = 0$ case	
———	

Here

$$
\begin{aligned}
&\mathbf{1_n} := x - n,\\
&\mathbf{2_a} := x^2 - 4x + 2, \qquad \mathbf{2_b} := x^2 - 5x + 5, \qquad \mathbf{2_c} := x^2 - 7x + 11,\\
&\mathbf{2_d} := x^2 - 8x + 13, \qquad \mathbf{2_e} := x^2 - 12x + 34,\\
&\mathbf{3_a} := x^3 - 10x^2 + 30x - 26, \qquad \mathbf{3_b} := x^3 - 14x^2 + 63x - 91,\\
&\mathbf{3_c} := x^3 - 16x^2 + 83x - 139, \qquad \mathbf{3_d} := x^3 - 21x^2 + 144x - 321,\\
&\mathbf{3_e} := x^3 - 21x^2 + 144x - 323,\\
&\mathbf{4_a} := x^4 - 18x^3 + 117x^2 - 324x + 321,\\
&\mathbf{4_b} := x^4 - 24x^3 + 212x^2 - 816x + 1154,\\
&\mathbf{5} := x^5 - 21x^4 + 170x^3 - 661x^2 + 1229x - 867,\\
&\mathbf{6} := x^6 - 20x^5 + 157x^4 - 610x^3 + 1204x^2 - 1078x + 278,\\
&\mathbf{7} := x^7 - 28x^6 + 323x^5 - 1983x^4 + 6962x^3 - 13868x^2 + 14323x - 5770,\\
&\mathbf{9} := x^9 - 45x^8 + 882x^7 - 9872x^6 + 69450x^5 - 317988x^4 + 945869x^3 - 1758591x^2 +\\
&\qquad 1849158x - 834632,\\
&\mathbf{10} := x^{10} - 46x^9 + 933x^8 - 10972x^7 + 82698x^6 - 416480x^5 + 1415230x^4 -\\
&\qquad 3192220x^3 + 4551945x^2 - 3680424x + 1268488,\\
&\mathbf{12} := x^{12} - 75x^{11} + 2562x^{10} - 52701x^9 + 726928x^8 - 7081826x^7 + 49954540x^6 -\\
&\qquad 257012522x^5 + 956940353x^4 - 2513874287x^3 + 4421248479x^2 -\\
&\qquad 4672270934x + 2242636033,\\
&\mathbf{15} := x^{15} - 93x^{14} + 4008x^{13} - 106163x^{12} + 1932458x^{11} - 25600562x^{10} +\\
&\qquad 254927932x^9 - 1942567842x^8 + 11417435665x^7 - 51744755105x^6 +\\
&\qquad 179295171389x^5 - 466283136174x^4 + 880666793685x^3 - 1139877184096x^2 +\\
&\qquad 903639748800x - 330565630976,\\
&\mathbf{16} := x^{16} - 76x^{15} + 2670x^{14} - 57512x^{13} + 849351x^{12} - 9109048x^{11} +\\
&\qquad 73289680x^{10} - 450525464x^9 + 2134046231x^8 - 7794633798x^7 +\\
&\qquad 21803583759x^6 - 45993980288x^5 + 71220198638x^4 - 77391639144x^3 +\\
&\qquad 54589655356x^2 - 21469924224x + 3193100216,\\
&\mathbf{20_a} := x^{20} - 107x^{19} + 5390x^{18} - 169909x^{17} + 3757766x^{16} - 61956744x^{15} +\\
&\qquad 789839374x^{14} - 7968451646x^{13} + 64579301106x^{12} - 424317702016x^{11} +\\
&\qquad 2271049739581x^{10} - 9910587278544x^9 + 35165896339844x^8 -\\
&\qquad 100788840091272x^7 + 230719077491798x^6 - 414591247028377x^5 +\\
&\qquad 569799308661865x^4 - 575489965955241x^3 + 400153245868113x^2 -\\
&\qquad 169764861535134x + 32741611046721,\\
&\mathbf{20_b} := x^{20} - 109x^{19} + 5600x^{18} - 180287x^{17} + 4078540x^{16} - 68907116x^{15} +\\
&\qquad 901976466x^{14} - 9365114226x^{13} + 78317050846x^{12} - 532575594652x^{11} +\\
&\qquad 2960318111641x^{10} - 13469643127548x^9 + 50063934003660x^8 -\\
&\qquad 151115303430784x^7 + 366651617570206x^6 - 703734083636583x^5 +\\
&\qquad 1042848523879615x^4 - 1149127999221811x^3 + 885073633040283x^2 -\\
&\qquad 424451387835254x + 95207779114473.
\end{aligned}
\tag{A.1}
$$

Appendix B Characteristic polynomials in the case $\gamma = \pi/2$

Here we present characteristic polynomials for the Hamiltonians of the open (i.e. $U_q(sl(2))$ invariant) chains (see (2.9), (2.12)), the 1-boundary chains (see (2.13), (2.16)) with $\omega_- = 0$, $a_- = 1$ ($\delta_- = \pi$), and the 2-boundary chains (see (2.14), (2.19)) for $\omega_\pm = 0$, $a_\pm = 1$ ($\delta_\pm = \pi$) and in cases $b = 1$ and $b = 0$ (for L even we only consider $b = 0$). In all cases we fix anisotropy $\Delta = 0$ ($q = e^{\pi/2}$).

As for the previous appendix we collect data for the chains of sizes $1 \le L \le 8$ and organize them according to values of spin S for the open chains (each sector S comes with multiplicity $2S + 1$) and according to values of the charge m for the 1-boundary chains. Explicit expressions for factors are presented at the end.

$\mathbf{L = 1}$

S	open chain
1/2	$(\mathbf{1_0})$
m	1-boundary chain
1/2	$(\mathbf{1_1})$
$-1/2$	$(\mathbf{1_1})$
2-boundary chain, $b = 1$ case	
$(\mathbf{1_1})\ (\mathbf{1_3})$	
2-boundary chain, $b = 0$ case	
$(\mathbf{1_2})^2$	

$\mathbf{L = 2}$

S	open chain
0	$(\mathbf{1_1})$
1	$(\mathbf{1_1})$
m	1-boundary chain
0	$(\mathbf{1_1} \cdot \mathbf{1_3})$
1	$(\mathbf{1_2})$
-1	$(\mathbf{1_2})$
2-boundary chain, $b = 1$ case	
———	
2-boundary chain, $b = 0$ case	
$(\mathbf{2_a})\ (\mathbf{1_3})^2$	

$\mathbf{L = 3}$

S	open chain
1/2	$(\mathbf{1_1} \cdot \mathbf{1_3})$
3/2	$(\mathbf{1_2})$
m	1-boundary chain
1/2	$(\mathbf{2_a})\ (\mathbf{1_3})$
$-1/2$	$(\mathbf{2_a})\ (\mathbf{1_3})$
3/2	$(\mathbf{1_3})$
$-3/2$	$(\mathbf{1_3})$
2-boundary chain, $b = 1$ case	
$(\mathbf{2_b} \cdot \mathbf{1_3} \cdot \mathbf{1_4} \cdot \mathbf{1_5})\ (\mathbf{1_4})^3$	
2-boundary chain, $b = 0$ case	
$(\mathbf{2_c} \cdot \mathbf{2_d})^2$	

$\mathbf{L = 4}$

S	open chain
0	$(\mathbf{2_a})$
1	$(\mathbf{2_a})\ (\mathbf{1_3})$
2	$(\mathbf{1_3})$
m	1-boundary chain
0	$(\mathbf{2_b} \cdot \mathbf{1_3} \cdot \mathbf{1_4} \cdot \mathbf{1_5})(\mathbf{1_4})$
1	$(\mathbf{2_c} \cdot \mathbf{2_d})$
-1	$(\mathbf{2_c} \cdot \mathbf{2_d})$
2	$(\mathbf{1_4})$
-2	$(\mathbf{1_4})$
2-boundary chain, $b = 1$ case	
———	
2-boundary chain, $b = 0$ case	
$(\mathbf{2_e} \cdot \mathbf{2_f} \cdot \mathbf{1_5})\ (\mathbf{2_g} \cdot \mathbf{1_4} \cdot \mathbf{1_6})^2\ (\mathbf{1_5})^3$	

$\mathbf{L=5}$

S	open chain
1/2	$(\mathbf{2_b}\cdot\mathbf{1_3}\cdot\mathbf{1_4}\cdot\mathbf{1_5})$
3/2	$(\mathbf{2_c}\cdot\mathbf{2_d})$
5/2	$(\mathbf{1_4})$
m	1-boundary chain
1/2	$(\mathbf{2_e}\cdot\mathbf{2_f}\cdot\mathbf{1_5})\ (\mathbf{2_g}\cdot\mathbf{1_4}\cdot\mathbf{1_6})\ (\mathbf{1_5})$
−1/2	$(\mathbf{2_e}\cdot\mathbf{2_f}\cdot\mathbf{1_5})\ (\mathbf{2_g}\cdot\mathbf{1_4}\cdot\mathbf{1_6})\ (\mathbf{1_5})$
3/2	$(\mathbf{2_g}\cdot\mathbf{1_4}\cdot\mathbf{1_6})\ (\mathbf{1_5})$
−3/2	$(\mathbf{2_g}\cdot\mathbf{1_4}\cdot\mathbf{1_6})\ (\mathbf{1_5})$
5/2	$(\mathbf{1_5})$
−5/2	$(\mathbf{1_5})$
2-boundary chain, $b=1$ case	
$(\mathbf{3_a}\cdot\mathbf{3_b})^4\ (\mathbf{3_c}\cdot\mathbf{3_d}\cdot\mathbf{1_5}\cdot\mathbf{1_7})$	
2-boundary chain, $b=0$ case	
$(\mathbf{3_e}\cdot\mathbf{3_f}\cdot\mathbf{3_g}\cdot\mathbf{3_h}\cdot\mathbf{1_6^2})^2\ (\mathbf{1_6})^4$	

$\mathbf{L=6}$

S	open chain
0	$(\mathbf{2_e}\cdot\mathbf{2_f}\cdot\mathbf{1_5})$
1	$(\mathbf{2_e}\cdot\mathbf{2_f}\cdot\mathbf{1_5})\ (\mathbf{2_g}\cdot\mathbf{1_4}\cdot\mathbf{1_6})$
2	$(\mathbf{2_g}\cdot\mathbf{1_4}\cdot\mathbf{1_6})\ (\mathbf{1_5})$
3	$(\mathbf{1_5})$
m	1-boundary chain
0	$(\mathbf{3_a}\cdot\mathbf{3_b})^2\ (\mathbf{3_c}\cdot\mathbf{3_d}\cdot\mathbf{1_5}\cdot\mathbf{1_7})$
1	$(\mathbf{3_e}\cdot\mathbf{3_f}\cdot\mathbf{3_g}\cdot\mathbf{3_h}\cdot\mathbf{1_6^2})(\mathbf{1_6})$
−1	$(\mathbf{3_e}\cdot\mathbf{3_f}\cdot\mathbf{3_g}\cdot\mathbf{3_h}\cdot\mathbf{1_6^2})(\mathbf{1_6})$
2	$(\mathbf{3_a}\cdot\mathbf{3_b})$
−2	$(\mathbf{3_a}\cdot\mathbf{3_b})$
3	$(\mathbf{1_6})$
−3	$(\mathbf{1_6})$
2-boundary chain, $b=1$ case	
——	
2-boundary chain, $b=0$ case	
$(\mathbf{4_a}\cdot\mathbf{2_h})^4\ (\mathbf{4_d}\cdot\mathbf{4_e}\cdot\mathbf{4_f}\cdot\mathbf{1_7^2})^2$ $(\mathbf{4_b}\cdot\mathbf{4_c})\ (\mathbf{1_7})^4$	

Here

$$\mathbf{1_n} := x-n,$$

$$\begin{aligned}
&\mathbf{2_a} := x^2-6x+7, && \mathbf{2_b} := x^2-8x+11, && \mathbf{2_c} := x^2-7x+11,\\
&\mathbf{2_d} := x^2-9x+19, && \mathbf{2_e} := x^2-8x+13, && \mathbf{2_f} := x^2-12x+33,\\
&\mathbf{2_g} := x^2-10x+22, && \mathbf{2_h} := x^2-14x+47,
\end{aligned}$$

$$\begin{aligned}
&\mathbf{3_a} := x^3-17x^2+94x-169, && \mathbf{3_b} := x^3-19x^2+118x-239,\\
&\mathbf{3_c} := x^3-17x^2+87x-127, && \mathbf{3_d} := x^3-19x^2+111x-197,\\
&\mathbf{3_e} := x^3-16x^2+83x-139, && \mathbf{3_f} := x^3-18x^2+101x-167,\\
&\mathbf{3_g} := x^3-18x^2+101x-181, && \mathbf{3_h} := x^3-20x^2+131x-281,
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&\mathbf{4_a} := x^4-28x^3+290x^2-1316x+2207,\\
&\mathbf{4_b} := x^4-28x^3+282x^2-1188x+1697,\\
&\mathbf{4_c} := x^4-28x^3+282x^2-1220x+1921,\\
&\mathbf{4_d} := x^4-28x^3+286x^2-1252x+1951,\\
&\mathbf{4_e} := x^4-28x^3+286x^2-1260x+2017,\\
&\mathbf{4_f} := x^4-28x^3+286x^2-1268x+2063,
\end{aligned}$$

References

[1] V. Pasquier and H. Saleur, Nucl. Phys. B **330** (1990), 523.

[2] D. Levy, Phys. Rev. Lett., **67** (1991), 1971-1974.

[3] D. Levy, Int. J. Mod. Phys. A **6** (1991), 5127-5154.

[4] P.P. Martin and H. Saleur, Lett. Math. Phys. **30** (1994), 189, hep-th/9302094.

[5] P. Martin and H. Saleur, J. Algebra **225** (2000), 957-988.

[6] A. Nichols, V. Rittenberg and J. de Gier, J. Stat. Mech. (2005) P03003, cond-mat/0411512.

[7] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W Quispel, J. Phys. A **20** (1987), 6397.

[8] J. de Gier, to appear in Discr. Math., math.CO/0211285.

[9] A. Ram, Proc. London Math. Soc. (3) **75** (1997), no. 1, 99-133.

[10] A. Ram, J. Algebra **260** (2003), no. 1, 367-415.

[11] P.P. Martin and D. Woodcock, J. Algebra **225** (2000), 957.

[12] J. Cao, H.-Q. Lin, K.-J. Shi, and Y. Wang, Nucl. Phys. **B663**, (2003), 487, cond-mat/0212163.

[13] R. I. Nepomechie, J. Statist. Phys. **111**, 1363 (2003), hep-th/0211001.

[14] R. I. Nepomechie, J. Phys. **A37**, 433 (2004), hep-th/0304092.

[15] R. I. Nepomechie and F. Ravanini, J. Phys. **A36**, 11391 (2003), hep-th/0307095.

[16] J. de Gier and P. Pyatov, J. Stat. Mech.: Theor. Exp. , P03002 (2004), hep-th/0312235.

[17] Yu. A. Drozd and V. V. Kirichenko *Finite dimensional algebras*, Springer-Verlag Berlin, 1994.

[18] S. Mitra, B. Nienhuis, J. de Gier and M.T. Batchelor, J. Stat. Mech. (2004), P09010, cond-mat/0401245.

[19] P. Fendley, K. Schoutens and B. Nienhuis, J. Phys. A **36** (2002), 12399; X. Yang and P. Fendley, 2004, cond-mat/0404682.

[20] P. Pyatov, J. Stat. Mech. (2004), P09003, math-ph/0406025.

[21] Saleur H. *Lectures on Non Perturbative Field Theory and Quantum Impurity Problems*, cond-mat/9812110.

[22] C. J. Hamar, J. Phys. A **18** (1985) L1133.

[23] C. Ahn and R. I. Nepomechie, Nucl.Phys. B676 (2004) 637-658, hep-th/0309261.

[24] Oshikawa M., Affleck I. 1997 Nucl. Phys. B 495 533, cond-mat/9612187.

[25] Doikou A., math-ph/0402067.

[26] F.C. Alcaraz, M. Baake, U. Grimm and V. Rittenberg, J. Phys. A **22** (1989), L5; U. Grimm and V. Rittenberg, Int. J. Mod. Phys. B **4** (1990), 669.

[27] A.A. Belavin, J.Phys. A **37** (2004), 317, hep-th/0305209.

[28] P.A. Pearce, V. Rittenberg, J. de Gier and B. Nienhuis, J. Phys. A **35** (2002), math-ph/0209017; J. de Gier, B. Nienhuis, P.A. Pearce and V. Rittenberg, Phys. Rev. E **67** (2003), 016101, cond-mat/0205467; J. Stat. Phys. **114** (2004), 1, cond-mat/0301430.

[29] M. Bauer and H. Saleur, Nucl. Phys. B **320** (1989), 591.

[30] U. Bilstein and B. Wehefritz, J. Phys. A **32** (1999), 191, cond-mat/9807166.

[31] U. Bilstein, J. Phys. A **33** (2000), 4437, cond-mat/0002162.

[32] A.P. Isaev, private communication.

[33] U. Bilstein and V. Rittenberg, J. Phys. A **33** (2000), L381, cond-mat/0004241.

[34] T. Eguchi and S-K. Yang, Mod. Phys. Lett. **5** (1990), 1693–1701;

[35] H. Saleur and N.P. Warner, 1993, hep-th/9311138.

[36] F. Ravanini and S.-K. Yang, Nucl. Phys. B **295** (1988), 262;

[37] M. Gaberdiel, Int. J. Mod. Phys. A **18** (2003), 4593, hep-th/0111260.

[38] M. Flohr, Int. J. Mod. Phys. A **18** (2003), 4497.