# Regular 2-forms on the moduli space of rank two stable bundles on an algebraic surface

by

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## 0. Introduction

We know already that the moduli space of stable bundles on an algebraic curve is always an unirational algebraic variety. In particular, there is no regular 2-form on it.

In the surface case the situation is more interesting. Ellingsrud showed that if the underlying surface is the projective plane, then the moduli space, like in the curve case, is a unirational variety.

But this is not always true. For an abelian or K3 surface, which has an everywhere non-degenerate 2-form, Mukai [Muk] proved that the moduli space has also such a 2-form. This shows that if the moduli space is compact, then it is an irrational variety.

In general we prove the following:

**Theorem 1**

*Let $X$ be an algebraic surface with a non-trivial regular 2-form and an ample divisor $H = K_X + H_0$ where $K_X$ is the canonical divisor and $H_0$ is an ample divisor on $X$. Given a line bundle $L$ on $X$ and an integer $k$, let $\tilde{M}(L,k)$ be a resolution of singularities of a compactification of the moduli space of rank 2 $H$-stable bundles with the determinant bundle $det = L$ and second Chern class $c_2 = k$, then there exists an integer $k_0$ such that for $k \geq k_0$ every irreducible component of $\tilde{M}(L,k)$ has a non-zero regular 2-form.*

We remark that

1) There always exists a stable bundle with large second Chern class. ([Mar], [Gi3], [Ta]).

2) It is not very difficult to show that at least one irreducible component has a regular 2-form, but in general the moduli space is not irreducible ([F]).

One has the following remarkable corollary:

*Every irreducible component of the moduli space of rank 2 stable bundles with large second Chern class on an algebraic surface with a non-trivial regular 2-form is an irrational variety.*

The idea of proof for theorem 1 is the following:

We construct a subvariety $\bar{V}$ in $Hilb^l(X)$, the Hilbert scheme of 0-dimensional subschemes of the length $l$, in $X$ with the properties:

1) The codimension of $\bar{V}$ in $Hilb^l(X)$ is smaller than $\frac{1}{2}$dim $Hilb^l(X)$.

2) There is a surjective rational map $\bar{e} : \bar{V} \longrightarrow \tilde{M}(L,k)$ with fibres birational to some projective space.

On the other hand, every regular 2-form on $X$ induces a quasi-symplectic structure on $Hilb^l(X)$. (see [Mum], [Be])

Because of 1), we get a non-trivial regular 2-form on $\bar{V}$ from the restriction of the above quasi-symplectic structure to $\bar{V}$, and this 2-form can be pushed down on $\tilde{M}(L,k)$ using $\bar{e}$.

The properties are a consequence of the following vanishing theorem of generic vector bundle form the moduli space $M(L,k)$.

Let $\mathcal{O}(D)$ be a line bundle on $X$, we define three subvarieties in the moduli space $M(L,k)$ respect to the twisting $\mathcal{O}(D)$ as:

$$\begin{aligned} M_1^s &:= \{ [E] \in M(L,k) \,|\, H^1(E(D)) \neq 0 \} \\ M_2^s &:= \{ [E] \in M(L,k) \,|\, H^1(E(D+K)) \neq 0 \} \\ M_3^s &:= \{ [E] \in M(L,k) \,|\, \exists p \in X \quad H^1(I_p \otimes E(D)) \neq 0 \} \quad , \end{aligned}$$

where $I_p$ is the ideal sheaf of all regular functions on $X$ vanishing at $p$.

Suppose that $[E] \in M(L,k)$, we denote $n_k$ is the smallest integer so that $\chi(E(n_k H)) \geq 1$, the Hirzebruch-Riemann-Roch-formula gives $n_k \approx \sqrt{k/H^2}$.

**Theorem 2**

*Suppose that $X$ has the non-negative Kodaira-dimension, then there exist two natural numbers $k_0$ and $m_0$ depending only on the Chern classes of $X$, $H$ and $L$ so that for any $k \geq k_0$ and $m \geq m_0$ the subvarieties $M_i^s$ $(1 \leq i \leq 3)$ respect to the twisting $\mathcal{O}(mn_k H)$ are proper in the each component of the moduli space $M(L,k)$.*

**Remark**

1) It is easy to see that for any $[E] \in M(L,k)$ $H^2(E(mn_k H))$ always vanishes.

2) D. Gieseker [Gi1] and M. Maruyama [Mar] proved that for each $M(L,k)$ there exists a sufficiently large integer $n(L,k)$ so that

$$H^1(E(nH)) = H^2(E(nH)) = 0 \quad ,$$

and $E(nH)$ is generated by its global sections, $\forall\,[E] \in M(L,k)$, $\forall\, n \geq n(L,k)$. Our theorem 2 gives an explicit smaller number $n(L,k)$ for generic $[E] \in M(L,k)$.

Another interesting consequence of theorem 2, which we will just mention here without proof, is the following:

*Taking a fix choosed integer $m\,(\geq m_0)$ in theorem 2, then there exists an integer $k_0$ so that for any integer $k \geq k_0$*

*1) A generic curve $C$ in the linear system $|L + 2mn_kH|$ is the degenerate curve of two linearly independent sections of a stable vector bundle $E(mn_kH)$, $[E] \in M(L,k)$.*

*2) Let $C^1_{c_2(E(mn_kH))}$ be the subvariety of the $c_2(E(mn_k))-$ fold symmetric product $C_{c_2(E(mn_kH))}$ parametrizing effective divisors of degree $c_2(E(mn_kH))$ on $C$ moving in a linear system of the dimension at least 1. Then there exist some components of $C^1_{c_2(E(mn_kH))}$ so that whose generic element is the zero locus of a section of $E(mn_kH)$.*

*3) $M(L,k)$ has the correct dimension is equivalent to say that the components in $C^1_{c_2(E(mn_kH))}$ have the correct dimension, namely the Brill-Noether number +1.*

Donaldson [D2] proved recently that $M(0,k)$ has the correct dimension, if $k$ is sufficiently large, hence the above components have the correct dimension.

Finally, I wish to thank Rebecca Barlow for introducing me to the work of Mumford. By this work I learned how to construct 2-form on a subvariety of $Hilb^l(X)$. I would like to express my gratitude to Professor F. Hirzebruch, and to the support of the Max-Planck-Institut für Mathematik in Bonn.

**Contents**

## 1. Varieties of 0-dimensional subschemes in the special position respect to a linear system

Let $X$ be a surface, $|L|$ be a non-empty linear system and $z$ be a 0-dimensional subscheme in $X$ with the length $l$.

The linear subsystem $|I_z \otimes L|$ is regarded as all curves from $|L|$ containing $z$.

Clearly, we have the inequality:

$$\dim |I_z \otimes L| \geq \dim |L| - l \quad .$$

We say that $z$ is in the general position respect to $|L|$, if the equality holds. Otherwise $z$ is in the special position. More precisely, we consider the restriction map

$$0 \longrightarrow H^0(I_z \otimes L) \longrightarrow H^0(L) \xrightarrow{r} H^0(\mathcal{O}_z \otimes L) \longrightarrow \dots .$$

Given a positive integer $\eta$ with $0 \leq l - \eta < h^0(L)$, we define the subvariety $V_\eta$ of 0-dimensional subschemes in the special position respect to the linear system $|L|$ of the special degree $\eta$ as

$$V_\eta := \{ z \in Hilb^l(X) \,|\, \dim (r : H^0(L) \to H^0(\mathcal{O}_z \otimes L)) = l - \eta \} \quad .$$

Of course, $V_\eta$ is a proper subvariety in $Hilb^l(X)$, but the interesting thing is to give an upper bound of the dimension of $V_\eta$. We give some answers in following two lemmas:

**Lemma 1.1**

*Let $X$ be a surface, $q$ be the irregularity of $X$, and $V_\eta$ be the variety defined in the above, then we have the inequality*

$$dim\, V_\eta \leq 2l - \eta + q \quad .$$

The proof of lemma 1.1 is more or less classical, but we need the following:

**Lemma** ( Iarrobino [I] )

*Let $Sym^l(X)$ be the $l$-th symmetric product of the surface $X$, ( which is the parameter space of all 0-cycles in $X$ of the length $l$ ) and*

$$Hilb^l(X) \xrightarrow{\pi} Sym^l(X)$$

*be the canonical resolution of the singularities of* $Sym^l(X)$.

*Suppose* $\pi(z) = n_1 z_1 + ... + n_s z_s$, *then the fibre* $\pi^{-1}(\pi(z))$ *has the dimension* $l - s$.

**Lemma** (Clifford theorem in the surface case)

*If* $D_1$ *and* $D_2$ *are effective divisors in* $X$, *then*

$$h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1 \quad .$$

The proof of Cliffords theorem in the surface case is exactly same as in the curve case ([Gi2]), but we can not find a reference, which gives a proof. So we would like to give following:

**Proof of the lemma** (Clifford theorem in the surface case)

Suppose that $h^0(D_1) = 1$, then it is clear that

$$h^0(D_1 + D_2) \geq h^0(D_2) = h^0(D_2) + h^0(D_1) - 1 \quad .$$

As for the case $h^0(D_1) \geq 2$.

Let $|M|$ be the moving part of $|D_1|$, then

$$h^0(M) = h^0(D_1) \quad ,$$

and

$$h^0(D_1 + D_2) \geq h^0(M + D_2) \quad .$$

Let $t_1, ..., t_{R_1}$ be a base of $H^0(M)$ and $s_1, ..., s_{R_2}$ be a base of $H^0(D_2)$.

Because $|M|$ is free from fixed components, we may choose $t_1$ so that the zero locus of $t_1$ and the zero locus of $s_1$ have non common components.

Suppose

$$a_1 s_1 t_1 + a_2 s_1 t_2 + ... + a_{R_1} s_1 t_{R_1} = b_2 s_2 t_1 + b_3 s_3 t_1 + ... + b_{R_2} s_{R_2} t_1 \quad ,$$

then

$$s_1 t = t_1 s \quad .$$

Because $s_1$ does not vanish along any component of the zero locus of $t_1$, therefore, $t$ vanishes along the above zero locus. We get $t = \lambda t_1, s = \lambda s_1$ and $b_2 s_2 + ... + b_{R_2} s_{R_2} = \lambda s_1$, hence $\lambda = b_2 = ... = b_{R_2} = 0$ and $a_1 = ... = a_{R_1} = 0$.

This means that the $R_1 + R_2 - 1$ sections $s_1 t_1, s_1 t_2, ..., s_1 t_{R_1}, s_2 t_1, s_3 t_1, ..., s_{R_2} t_1$ in $H^0(M + D_2)$ are linearly independent, in particular,

$$h^0(M + D_2) \geq R_1 + R_2 - 1 \quad .$$

The lemma is proved.

**Proof of lemma 1.1**

Consider the natural map

$$\begin{array}{ccc} Hilb^l(X) & \xrightarrow{\pi} & Sym^l(X) \\ \uplus & & \uplus \\ z & \longmapsto & \sum_{i=1}^m l_i z_i \,. \end{array}$$

We may assume that $m$ is constant for each $z$ from $V := V_\eta$. ( In fact $m$ is constant for $z$ from a Zariski open set of $V$.) The Iarrobino Lemma says:

$$\textbf{(1.1)} \qquad \dim V \leq \dim \pi(V) + l - m \quad .$$

Let $z_r := z_1 + ... + z_m$ be the reduced subscheme of $z$, then the natural inclusion $I_z \hookrightarrow I_{z_r}$ induces the following exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(I_z \otimes L) & \longrightarrow & H^0(L) & \xrightarrow{r} & H^0(\mathcal{O}_z \otimes L) & \longrightarrow & \\ & & \updownarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & H^0(I_{z_r} \otimes L) & \longrightarrow & H^0(L) & \xrightarrow{r} & H^0(\mathcal{O}_{z_r} \otimes L) & \longrightarrow & , \end{array}$$

and we get

$$\dim (r : H^0(L) \to H^0(\mathcal{O}_{z_r} \otimes L)) =: l - \eta_r \leq l - \eta, \quad \forall z \in V \quad .$$

We see that $\pi(V)$ is embedded in the subvariety

$$\textbf{(1.2)} \qquad V_r := \{z \in Hilb^m(X)_r | \dim (r : H^0(L) \to H^0(\mathcal{O}_z \otimes L)) = l - \eta_r\} \quad ,$$

where $Hilb(X)_r$ denotes the Zariski open set of $Hilb(X)$ contains all reduced subschemes. In particular, we have the estimate

$$\dim \pi(V) \leq \dim V_r \quad .$$

Consider the subvariety $V_r$, at first we want to separate the fixed components of the linear subsystem $|I_z \otimes L|$, $z \in V_r$.

Let $F_z$ be the fixed component of $|I_z \otimes L|$ ( $F_z$ can be empty). If we move $z$ in $V_r$, then $F_z$ is moved in a subvariety of the parameter space $\mathcal{C}$ of all curves in $X$ with the bounded degree $LH$, where $H$ is an ample divisor.

We know that the local dimension of $\mathcal{C}$ at the point $C$ has the following upper bound

$$\dim \mathcal{C}|_C \leq \dim |C| + q(X) \quad .$$

Suppose that there are $m-n$ points $\{z_{n+1}, ..., z_m\}$ of $z$, $z \in V_r$, which lie on $F_z$.
It is clear that the number $m-n$ is constant for all $z$ from a Zariski open set of $V_r$.

Sum up the above disscusion, we define (locally) a morphism as

$$\begin{array}{ccc} V_r & \xrightarrow{f} & Hilb^{m-n}(X)_r \times \mathcal{C} \\ \Psi & & \Psi \\ z_1 + ... + z_m & \longmapsto & (z_{n+1} + ... + z_m,\, F_z)\,. \end{array}$$

The image $f(V_r)$ lies in the subvariety

$$\{(z, C) \in Hilb^{m-n}(X)_r \times \mathcal{C} | z \subset C\} \quad , \tag{1.3}$$

which has the dimension $\leq (m-n) + (h^0(C) - 1 + q(X))$ at the point $(z, C)$.

We want to understand fibres of $f$. Let $z = z_1 + ... + z_n + z_{n+1} + ... + z_m \in V_r$ with $z_{n+1}, ..., z_m$ lieing on $F_z$, then we have the exact sequence

$$0 \longrightarrow H^0(I_{z_1 + ... + z_n} \otimes (L - F_z)) \longrightarrow H^0(I_z \otimes L) \xrightarrow{r} H^0(I_z \otimes L \otimes \mathcal{O}_{F_z}) \longrightarrow ...,$$

where $r$ is the restriction of sections from $H^0(I_z \otimes L)$ to the curve $F_z$, $i$ is the multiplication of sections from $H^0(I_{z_1 + ... z_n} \otimes (L - F_z))$ with the section from $H^0(I_{z_{n+1} + ... + z_m} \otimes F_z)$, which has the zero locus $F_z$ and the image $i(H^0(I_{z_1 + ... + z_n} \otimes (L - F_z)))$ can be regarded as the subspace of all sections from $H^0(I_z \otimes L)$, which vanish along $F_z$.

Because $F_z$ is the fixed component of $|I_z \otimes L|$, $r$ is zero map, hence we have the isomorphism

(1.4)
$$H^0(I_{z_1+...+z_n} \otimes (L - F_z)) \xrightarrow{\ i\ } H^0(I_z \otimes L)$$

and the linear subsystem $|I_{z_1+...+z_n} \otimes (L - F_z)|$ has non more fixed components.

Let $R = h^0(L)$ and $R' = h^0(L - F_z)$, from (1.4) and (1.2) we get

(1.5)
$$h^0(I_{z_1+...+z_n} \otimes (L - F_z)) = h^0(I_z \otimes L) = R - (l - \eta_r),$$

and

$$\begin{aligned} \dim (r: \ H^0(L - F_z) \to & H^0(\mathcal{O}_{z_1+...+z_n} \otimes (L - F_z))) \\ &= R' + l - \eta_r - R \quad (< R') \\ &= n - (n + R + \eta_r - R' - l) \\ &= : n - \eta' \quad (\eta' \geq 0) \end{aligned}$$

with the following restriction map

$$0 \longrightarrow H^0(I_{z_1+...+z_n} \otimes (L - F_z)) \longrightarrow H^0(L - F_z) \longrightarrow H^0(\mathcal{O}_{z_1+...+z_n} \otimes (L - F_z)) \longrightarrow ... \quad .$$

After the above discussion we see easily that the fibre of $f$ is embedded in the following subvariety

(1.6)
$$f^{-1}(f(z)) \hookrightarrow$$
$$\{z_1 + ... + z_n \in Hilb^n(X)_r | \dim (r: \ H^0(L - F_z) \to H^0(\mathcal{O}_{z_1+...+z_n} \otimes (L - F_z)) = n - \eta'\}$$

with $|L - F_z|$ is free from the fixed component.

The number $n$ can be zero. ( for examples if $|L - F_z|$ is composed with pencil or $L - F_z = \mathcal{O}_X$. )
In this case we have

$$\dim V_r|_z = \dim f(V_r)|_{f(z)} \leq m + h^0(F_z) - 1 + q(X).$$

Since $R' - (R - l + \eta_r) = 0$ from (1.4) and Clifford lemma, we obtain

$$\begin{aligned}\dim V_r|_z \leq& m + h^0(F_z) - 1 + q(X) + R' - R + l - \eta_r \\ =& l + m - \eta_r + q(X) + (h^0(F_z) + h^0(L - F_z) - 1 - h^0(L) \\ \leq& l + m - \eta_r + q(X).\end{aligned}$$

From (1.1) and (1.2) we have

$$\dim V \leq 2l - \eta_r + q(X) \leq 2l - \eta + q(X).$$

Now suppose $n > 0$. We want to bound the dimension of the fibre $f^{-1}$.

We analyse carefully the fibre $f^{-1}f(z)$. The linear system $|L - F_z|$ is free from fixed components, because its linear subsystem $|I_{z_1 + \ldots + z_n} \otimes (L - F_z)|$ in (1.4) is already free from fixed components. Let $B$ be the set of base points of $|L - F_z|$.

Suppose there are $n - s$ points $\{z_{s+1}, \ldots, z_n\}$ from $z_1 + \ldots + z_n \in f^{-1}f(z)$, which lie in $B$. The number $n - s$ is constant for $z_1 + \ldots z_n$ from a Zariski open set of $f^{-1}f(z)$.

Because $z_{s+1}, \ldots, z_n$ are base points of $|L - F_z|$, the exact sequence:

$$0 \to H^0(I_{z_1 + \ldots z_n} \otimes (L - F_z)) \to H^0(I_{z_1 + \ldots + z_s} \otimes (L - F_z)) \to H^0(\mathcal{O}_{z_{s+1} + \ldots + z_n} \otimes (L - F_z)) \to \ldots$$

induces the isomorphism

$$H^0(I_{z_1 + \ldots + z_n} \otimes (L - F_z)) \xrightarrow{\simeq} H^0(I_{z_1 + \ldots + z_s} \otimes (L - F_z)) \quad . \tag{1.7}$$

Compare (1.5), we obtain

$$\dim (r : H^0(L - F_z) \to H^0(\mathcal{O}_{z_1 + \ldots + z_s} \otimes (L - F_z))) = n - \eta',$$

in the restriction map

$$0 \to H^0(I_{z_1 + \ldots + z_s} \otimes (L - F_z)) \to H^0(L - F_z) \xrightarrow{r} H^0(\mathcal{O}_{z_1 + \ldots + z_s} \otimes (L - F_z)) \to \ldots.$$

Therefore, we may define the morphism as

$$b: f^{-1}f(z) \to \{z \in Hilb^s(X \setminus B)_r | \dim (r: H^0(L - F_z) \to H^0(\mathcal{O}_{z_1+\ldots+z_s} \otimes (L - F_z))) = n - \eta'\}$$
$$z_1 + \ldots + z_n \mapsto \qquad z_1 + \ldots + z_s \, .$$

Because the fibre of $b$ lies in the set of the base points $B$ which has dimension 0, hence $b^{-1}$ also has dimension 0, and we have

(1.8)
$$\dim f^{-1}f(z) = \dim bf^{-1}f(z) \quad .$$

Now consider the regular map induced by the linear system $|L - F_z|$

$$X \setminus B \xrightarrow{\ \phi\ } P^{R'-1} \quad .$$

$\phi$ maps the points $\{z_1, ..., z_s\}$ to $t$ different points $\{w_1, ..., w_t\}$, and $t$ is constant for each $z_1+...+z_s$ element of a Zariski open set of $bf^{-1}f(z)$.

Let $H$ denote the hyperplane section of $P^{R'-1}$, then $\phi$ induces the isomorphism by pull back

(1.9)
$$|I_{w_1+\ldots+w_t} \otimes H| \xrightarrow{\ \phi^*\ } |I_{z_1+\ldots+z_s} \otimes (L - F_z)| \quad .$$

This shows, the points $\{w_1, ..., w_t\}$ span a proper subspace $P^{n-\eta'-1}$ in $P^{R'-1}$.

$\phi$ induces also a morphism of Hilbert schemes in the following way

$$bf^{-1}f(z) \xrightarrow{\ \phi_*\ } \{w \subset Hilb^t(P^{R'-1})_r | w \in \phi(X \setminus B),\ \bar{w} = P_w^{n-\eta'-1}\}$$
$$z_1 + \ldots + z_s \longmapsto \qquad w_1 + \ldots + w_t \quad .$$

We see easily that the fibre of $\phi_*$ is contained in the following subvariety

$$\phi_*^{-1}(w) \subset \{z \in Hilb^s(X \setminus B)_r | z \subset \phi^{-1}(w_1) \cup \ldots \cup \phi^{-1}(w_t),\ z \cap \phi^{-1}(w_i) \neq \emptyset \, (1 \le i \le t)\} \quad .$$

The dimension of $\phi^{-1}(w_i)$, $(1 \le i \le t)$ must be zero. Otherwise the linear subsystems $|I_{z_1+\ldots+z_s} \otimes (L - F_z)| \hookleftarrow |I_{z_1+\ldots+z_n} \otimes (L - F_z)|$ would have some fixed components $\phi^{-1}(w_i)$ from (1.9) and (1.7), but the above second linear subsystem is free from fixed components from (1.4).

Therefore, we obtain

$$\dim \phi_*^{-1} = 0 \quad ,$$

hence

(1.10) $$\dim bf^{-1}f(z) = \dim \phi_* bf^{-1}f(z) \quad .$$

The image $\phi_* bf^{-1}f(z)$ can be described better more in the following way:

Suppose the first $n-\eta'$ points $\{w_1, ..., w_{n-\eta'}\}$ of $w$, $w \in \phi_* bf^{-1}f(z)$ span already the linear subspace $P_w^{n-\eta'-1}$.

If we move $w$ in a Zariski open set of $\phi_* bf^{-1}f(z)$, then the sum of its first $n-\eta'$ points is naturally moved in a subvariety of the following variety:

$$\{w \in Hilb^{n-\eta'}(P^{R'-1})_r | w \subset \phi(X \setminus B)\} \quad ,$$

and they span always a subspace of dimension $n-\eta'-1$.

It means that we can (locally) define a morphism " the projection to the first $n-\eta'$ points"

$$\begin{array}{ccc} \phi_* bf^{-1}f(z) & \xrightarrow{p} & \{w \in Hilb^{n-\eta'}(P^{R'-1})_r | w \subset \phi(X \setminus B)\} \\ \Psi & & \Psi \\ w & \longmapsto & w_1 + ... + w_{n-\eta'} \quad . \end{array}$$

The fibre of $p$ is embedded in the following subvariety

$$p^{-1}(p(w)) =$$

$$\{p(w) + w_{n-\eta'+1} + ... + w_t \in Hilb^t(P^{R'-1})_r | \{w_{n-\eta'+1}, ..., w_t\} \subset \phi(X \setminus B) \cap P_w^{n-\eta'-1}\} .$$

The dimension of the intersection $\phi(X \setminus B) \cap P_w^{n-\eta'-1}$ must be zero. Otherwise the linear subsystem $|I_w \otimes H|$ in (1.9) would have a fixed component, whose pull back under $\phi$ is a fixed component in the linear subsystems $|I_{z_1+...+z_s} \otimes (L - F_z)| \overset{\sim}{\leftarrow} |I_{z_1+...+z_n} \otimes (L - F_z)|$, this is impossible.

Therefore, we obtain

$$\dim p^{-1} = 0 \quad ,$$

this implies

$$\dim \phi_* bf^{-1}f(z) = \dim p\phi_* bf^{-1}f(z) \leq 2(n-\eta') \quad .$$

From (1.10), (1.8) and (1.5) we get an upper bound of the fibre dimension

$$\begin{aligned}\dim f^{-1}(f(z)) \leq & 2(n-\eta') \\ \leq & 2n-\eta' \quad (\eta' \geq 0) \\ = & 2n-(n+R+\eta_r-R'-l) \\ = & n+l-\eta_r+h^0(L-F_z)-h^0(L) \quad ,\end{aligned}$$

hence from (1.3) and Clifford lemma we obtain

$$\begin{aligned}\dim V_r|_z = & \dim f(V_r)|_{f(z)} + \dim f^{-1}(f(z)) \\ \leq & (m-n)+h^0(F_z)-1+q(X)+n+l-\eta_r+h^0(L-F_z)-h^0(L) \\ \leq & m+l-\eta_r+q(X) \quad .\end{aligned}$$

Finally (1.2) and (1.1) follow the inequality

$$\dim V \leq 2l-\eta_r+q(X) \leq 2l-\eta+q(X) \quad .$$

Lemma 1.1 is done.

Under some stronger conditions on $L$ we have the following observation :

Suppose that $L$ is ample line bundle and $z$ is 0-dimensional subscheme of the length $l$. We look at the exact sequence

$$\begin{aligned}0 \to & H^0(I_z \otimes (L+K)) \to H^0(\mathcal{O}(L+K)) \to H^0(\mathcal{O}_z \otimes (L+K)) \to \\ \to & H^1(I_z \otimes (L+K)) \to H^1(\mathcal{O}(L+K)) \to 0 \quad ,\end{aligned}$$

clearly, we have $H^1(\mathcal{O}(L+K))=0$ from the Kodaira-vanishing theorem.

If we have some furthermore suitable assumptions for $L$ so that $h^0(L+K)>l$, then $z$ is in the special position respect to $|L+K|$ if and only if $H^1(I_z \otimes (L+K)) \neq 0$.

Motivated by the above observation we have the following:

**Lemma 1.2**

*Let $X$ be a surface of the non-negative Kodaira dimension with canonical divisor $K$, and the irregularity $q$. Suppose that $L$ is an ample divisor in $X$ with*

$$L^2 > 4l$$

*and*

$$\delta(L) := \; min.\{\, LC \,|\, all\ curves\ C \subset X\,\} \geq 4 \quad ,$$

*then the subvariety*

$$V := \{\, z \in Hilb^l(X) \,|\, H^1(I_z \otimes (L+K) \neq 0\,\}$$

*has*

$$dim\, V \leq 2l - \delta(L)/4 + q \quad .$$

The proof ot lemma 1.2 is a consequence of the Bogomolov T-stability theorem [Bo], [Reid] and the technique due to I. Reider, M. Beltrametti, P. Francia and A. J. Sommese in the proof of vanishing theorem of rank 1 torsion free sheaves [Reider], [BFS]. We have to use again Iarrobino lemma and the following:

**Lemma** ( Gieseker [Gi2] )
*Let $X$ be a surface of the non-negative Kodaira dimension, and $D$ be a divisor in $X$ with $D^2 > 0$, whose linear system has non fixed components and non base points, then*

$$h^0(D) \leq \frac{D^2}{2} + 2 \quad .$$

**Proof of lemma 1.2**

It is clear that $Ext^1_{\mathcal{O}}(I_z \otimes L, \mathcal{O}) \simeq H^1(I_z \otimes (L+K))^\vee$ has a constant positive dimension for all $z$ form a Zariski open set in $V$ by using the upper semi-continous theorem.
Noting ampleness of $L$, we have always $Ext^0_{\mathcal{O}}(I_z \otimes L, \mathcal{O}) \simeq H^0(-L) = 0$. This implies, there exists a family of rank 2 torsion free sheaves $\mathcal{E}$ on $X \times V_0'$ so that $\mathcal{E}|_{X,z} =: E_z$ comes from the following extension:

$$0 \longrightarrow \mathcal{O} \longrightarrow E_z \longrightarrow I_z \otimes L \longrightarrow 0 \quad ,$$

with a non-trivial extension class in $Ext^1_{\mathcal{O}}(I_z \otimes L, \mathcal{O})$, where $V_0'$ is a little smaller open set in $V$. ( In claim 2.1 in 2.1 we give an exact proof of existence of such a family.)

Let $E_z^{\vee\vee}$ be the bidual of $E_z$, then $E_z^{\vee\vee}$ is rank 2 vector bundle, and the canonical map $E_z \to E_z^{\vee\vee}$ induces the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & E_z & \longrightarrow & I_z \otimes L & \longrightarrow & 0 \\ & & \| & & \downarrow & & \uparrow\downarrow \varphi & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & E_z^{\vee\vee} & \longrightarrow & I_{z'} \otimes L & \longrightarrow & 0 \end{array} ,$$

where $z'$ is a subscheme of $z$. The inclusion $\varphi$ defines the homomorphisms

$$\tilde{\varphi} : Ext^1_{\mathcal{O}}(I_{z'} \otimes L, \mathcal{O}) \longrightarrow Ext^1_{\mathcal{O}}(I_z \otimes L, \mathcal{O}) \quad ,$$

which maps the extension class of the second exact sequence $e'$ to the non trivial extension class of the first one. (see [Ty], prop.1.2 and lemma 1.2)

In particular, $e' \neq 0$. This implies $I_{z'} \neq \mathcal{O}$, otherwise $Ext^1_{\mathcal{O}}(I_{z'} \otimes L, \mathcal{O}) \simeq H^1(L+K)^{\vee} = 0$.

The difference of the two lengthes $l - |z'|$ is just the length of the singularities locus of $E_z$, and it is a constant number $l - m$ for all $z$ from a little smaller Zariski open set $V_0$ of $V_0'$.

Therefore the rk-2 bundles $E_z^{\vee\vee}$, $z \in V_0$ have the same determinant bundle $L$ and the second Chern number $|z'| = m$, and it forms a family of bundles. ( more exacly we should say, in some smaller open set of $V_0$.)

Formally we also define a "bidual " map

(1.11)
$$\begin{array}{ccc} V_0 & \xrightarrow{\vee\vee} & Hilb^m(X) \\ \Psi & & \Psi \\ z & \longmapsto & z' \end{array} .$$

We want to understand better more the subvariety $\vee\vee(V_0)$.

Look at the exact sequence:

$$0 \longrightarrow \mathcal{O} \longrightarrow E_z^{\vee\vee} \longrightarrow I_{z'} \otimes L \longrightarrow 0 \quad .$$

Because $E_z^{\vee\vee}$ is a rank 2 vector bundle and $c_1^2(E_z^{\vee\vee}) = L^2 > 4l \geq 4m = 4c_2(E_z^{\vee\vee}) > 0$, from the prop. (1.4) in [BFS] we get the following:

There exists a curve $D_{z'}$ in $X$, which contains the subscheme $z'$ and satisfies the inequalities

(1.12)
$$LD_{z'} - m \leq D_{z'}^2 < LD_{z'}/2 < m \quad .$$

By using the Hodge index-theorem and the Gieseker lemma we have the following:

**Claim**

The linear system $|D_{z'}|$ has the dimension $\leq m/2$.

**Proof**

Suppose $|M|$ be the moving part of $|D_{z'}|$, then

$$LM/2 \leq LD_{z'}/2 < m$$

from (1.12).

The Hodge index-theorm gives:

$$L^2M^2/4 \leq (LM)^2/4 < m^2 \quad .$$

Since $L^2/4 > l \geq m$, we obtain $M^2 < m$.

**Case 1**

$|M|$ is not composed with pencil.

Blow-up the base points of $|M|$

$$\sigma : \widehat{X} \longrightarrow X \quad ,$$

then the linear system of proper transformation $|\widehat{M}|$ has following properties:

1) $|\widehat{M}|$ is free from base points.

2) $0 < \widehat{M}^2 \leq M^2$.

3) $|\widehat{M}|$ and $|M|$ have the same dimension

Therefore, we can apply the Gieseker lemma and get:

$$h^0(M) = h^0(\widehat{M}) \leq \widehat{M}^2/2 + 2 \leq M^2/2 + 2 < m/2 + 2 \quad .$$

**Case 2**

$|M|$ is composed with pencil. (see [BPV] page 113-114)

There exists 1-dimensional algebraic system in $X$, whose generic element is a smooth irreducible curve $F$, so that $M$ is algebraic equivalent to $lF$ with $\dim|M| \leq l$.

From the following inequalities

$$lFL = ML \leq D_{z'}L < 2m, \quad FL \geq 4 \quad ,$$

we obtain

$$\dim |M| \leq l \leq m/2 \quad .$$

The claim is done.

Now we want to bound $\dim V$ by using the similar method in the proof of lemma 1.1.

Consider the canonical maps:

(1.13)
$$\begin{array}{ccc} Hilb^l(X) & \xrightarrow{\pi_l} & Sym^l(X) \\ \cup & & \cup \\ V_0 & \longrightarrow & \pi_l(V_0) \\ \Psi & & \Psi \\ z & \longmapsto & \sum_{i=1}^{s} l_i z_i \end{array} \quad ,$$

then we have

$$\dim V_0 \leq \dim \pi_l(V_0) + l - s$$

from the Iarrobino lemma. So we have to estimate $\dim \pi_l(V_0)$. It is easy to see that $\pi_l(V_0)$ is canonically embedded in $Hilb^s(X)$.

On the other hand we look at the bidual map (1.1), $z' = \vee\vee(z)$ is a subscheme of $z$. This follows that under the map

$$\begin{array}{ccc} Hilb^m(X) & \xrightarrow{\pi_m} & Sym^m(X) \\ \cup & & \cup \\ \vee\vee(V_0) & \longrightarrow & \pi_m(\vee\vee(V_0)) \\ \Psi & & \Psi \\ z' & \longmapsto & \sum_{i=1}^{n} l'_i z'_i \, . \end{array}$$

we have

(1.14)
$$\{z'_1, ..., z'_n\} \subset \{z_1, ..., z_s\}$$

$$l_i' \leq l_i, \quad (1 \leq i \leq n) \quad .$$

The prop. (1.4) and our claim just say that the points $\{z_1', ..., z_n'\}$ lie on the curve $D_{z'}$ with $\dim |D_{z'}| \leq m/2$. Therefore, $\pi_l(V_0)$ lies in the subvariety:

$$\{z_1 + ... + z_n + ... + z_s \in Hilb^s(X)_r | \{z_1, ..., z_n\} \subset C, \, C \in \mathcal{C}_{m/2}\} \quad ,$$

where $\mathcal{C}_{m/2}$ is the parameter space of all curves $C$ in $X$ with $\dim |C| \leq m/2$.
We see easily that the above subvariety has dimension $\leq 2(s-n) + n + m/2 + q(X)$.

From (1.13) we get:

$$\begin{aligned}
\dim V_0 \leq & \sum_{i=1}^{s} l_i - s + (2(s-n) + n + m/2 + q(X)) \\
= & m/2 + q(X) + \sum_{i=1}^{n} l_i + \sum_{i=n+1}^{s} l_i + (s-n) \\
= & m/2 + q(X) - \sum_{i=1}^{n} l_i' + (\sum_{i=1}^{n} l_i' - \sum_{i=1}^{n} l_i) + 2\sum_{i=1}^{n} l_i + \sum_{i=n+1}^{s} l_i + (s-n) \quad .
\end{aligned}$$

Because $m \geq LD_{z'}/2 \geq \frac{1}{2}\text{min.}\{LC \,|\, \text{all curves } C \subset X\} =: \delta(L)$ in (1.12), $\sum_{i=1}^{n} l_i' = m$, $l_i' \leq l_i$, $(1 \leq i \leq n)$ in (1.14) and $l_i \geq 1$, $(1 \leq i \leq s)$, the above last inequality implies:

$$\begin{aligned}
\dim V_0 \leq & m/2 + q(X) - m + 2\sum_{i=1}^{n} l_i + 2\sum_{i=n+1}^{s} l_i \\
= & 2l + q(X) - m/2 \\
\leq & 2l + q(X) - \frac{1}{4}\delta(L) \quad .
\end{aligned}$$

Lemma 1.2 is proved.

## 2. Basic definitions, constructions and a vanishing theorem for generic rank 2 stable bundles

The goal of this section is to show theorem 2. We give the outline of our proof as the following:

Let $M(L,k)$ be the moduli space of rank 2 $H$-stable bundles with $det = L$ and $c_2 = k$. By twisting $E$ with $\mathcal{O}(n_0 H)$ we may assume $L$ is ample. This is nothing but because of some technique reasons in the proof of lemma 2.1. We will often denote $E(nH)$ by $E(n)$, $det(E(n))$ by $det(n)$, and $c_2(E(n))$ by $c_2(n)$. Suppose that $n_k$ is the smallest integer so that $\chi(E(n_k H)) \geq 1$, the Hirzebruch-Riemann-Roch-formula gives $n_k \approx \sqrt{k/H^2}$. If $k$ is sufficiently large, then for any $[E]$ from $M(L,k)$ the twisted bundle $E(n_k H)$ has at least one non-trivial section, this is because of the Serre-duality and the stability of $E$.

A section of $E(n_k)$ with 1-dimensional zero locus $C$ is naturally regarded as a section of $E(n_k)(-C)$ with the isolated zero locus $z$, and induces the exact sequence

$$(2.1, n_k) \qquad 0 \to \mathcal{O} \to E(n_k)(-C) \to I_z \otimes (det(n_k) - 2C) \to 0 \quad .$$

With another words we say, all elements $[E]$ from $M(L,k)$ come from the extensions of the rank 1 torsion-free sheaves $I_z \otimes (det(n_k) - 2C)$ by the structure sheaf $\mathcal{O}$.

So it is natural to study the moduli space $F^0$ of all extensions (2.1, $n_k$). By standard arguments, there exists a stratification of $F^0$.

$$F^0 = \bigcup_{\delta,\eta,i} F^0_{\delta,\eta,i} \quad .$$

Roughly say, the moduli space $F^0_{\delta,\eta}$ comes from three contributions. The first part is the global extension group $Ext^1_{\mathcal{O}}(I_z \otimes (det(n_k) - 2C), \mathcal{O})$ $(\simeq H^1(I_z \otimes (det(n_k) + K - 2C))^\vee \simeq C^\eta)$ with the fixed torsion-free sheaf $I_z \otimes (det(n_k) - 2C)$.

The second part is the moduli space of all torsion-free sheaves $I_z \otimes (det(n_k) - 2C)$ with the fixed line bundle $\mathcal{O}(C)$, $HC = \delta$, and $\dim H^1(I_z \otimes (det(n_k) + K - 2C) = \eta$. This moduli space is the following subvariety of $Hilb^{|z|}(X)$

$$(2.2) \qquad V_{\delta,\eta,i}(C) := \{ z \in Hilb^{|z|}(X) \,|\, \dim H^1(I_z \otimes (det(n_k) + K - 2C)) = \eta \} \quad .$$

And the third part is the irreducible component $\mathcal{L}_{\delta,i}$ of the moduli space $\mathcal{L}_\delta$ of line bundles $\mathcal{O}(C)$ with the degree $HC = \delta$, which is relatively smaller and has the bounded dimension $\leq q(X)$.

In 2.1 we show that there exist two canonical morphisms in a obvious way

$$F^0_{\delta,\eta,i} \xrightarrow{(e,\tilde{p})} M(L,k) \times \mathcal{L}_{\delta,i} \longrightarrow M(L,k) \quad ,$$

the fibre of the morphism $(e,\tilde{p})$ over $([E],\mathcal{O}(C))$ is identified with a Zariski open set of the global sections space $H^0(E(n_k)(-C))$ via the block-map.

More deeply, we want to know that for which $F^0_{\delta,\eta,i}$, the image $e(F^0_{\delta,\eta,i})$ is a Zariski open set in $M(L,k)$. We analyse carefully the extension group $Ext^1_{\mathcal{O}}(I_z \otimes (det(n_k) - 2C), \mathcal{O})$ $\simeq H^1(I_z \otimes (det(n_k) + K - 2C))^\vee \simeq C^\eta$, and the variety $V_{\delta,\eta,i}(C)$.
Looking at the exact sequence

$$0 \to H^0(\mathcal{O}_z \otimes (det(n_k) + K - 2C))/r(H^0(det(n_k) + K - 2C)) \to H^1(I_z \otimes (det(n_k) + K - 2C)) \to$$
$$\to H^1(det(n_k) + K - 2C) \to 0 \quad ,$$

and by using the Riemann-Roch-theorem, the Hodge-index-theorem, and the Gieseker Lemma we have an upper bound of $h^1(det(n_k) + K - 2C)$.
The quotient space

$$H^0(\mathcal{O}_z \otimes (det(n_k) + K - 2C))/r(H^0(det(n_k) + K - 2C)) \simeq C^{\eta'}$$

just measures the special position of the subscheme $z$ respect to the linear system $|det(n_k)+K-2C|$. Our lemma 1.1 gives the upper bound

$$\dim V_{\delta,\eta,i}(C) \leq 2|z| - \eta' + q(X).$$

We put all inequalities together and get in lemma 2.1 the following estimate

$$\dim F_{\delta,\eta,i}(C) \leq \text{the virtual dimension of}\, M(L,k) - c\sqrt{k}HC + d\sqrt{k} \quad ,$$

where $c$ and $d$ are some positive constants only depending on the Chern classes of $X$, $H$, and $L$. This shows that
*There exist two constants $k_0$ and $\delta_0$ only depending on Chern-classes of $X$, $H$ and $L$ so that for any $k \geq k_0$ the variety $\bigcup_{\delta \leq \delta_0} e(F^0_{\delta,\eta,i}) =: M_0(L,k)$ is a Zariski open and dense set in $M(L,k)$.*

The result is expected. To get the extension (2.1, $n_k$) for a generic element $[E]$ from $M(L,k)$ we have to twist $E$ with line bundle $\mathcal{O}(n_kH-C)$, which is not very different to the line bundle $\mathcal{O}(n_kH)$

arising from the inequality $\sum_{i=0}^{2} h^i(E(n_k)) = \chi(E(n_k)) \geq 1$ from the Hirzebruch-Riemann-Roch-formula.

Now we limit our attention to the subvarieties $V_{\delta,\eta,i}$ (2.2). By Standard arguments we show in lemma 2.2:

*The subvarieties $V_{\delta,\eta,i}(C)$, $\delta \leq \delta_0$ defined in (2.2) has the following upper bound of the codimension in $Hilb^{|z|}(X)$*

$$\textbf{(2.3)} \qquad codim_{Hilb^{|z|}(X)} V_{\delta,\eta,i}(C) \leq d\sqrt{k} \quad ,$$

*where $d$ is a constant only depending on the Chern-classes of $X$, $H$ and $L$.*

Now we twist again $E(n_k)$, $[E] \in M_0(L,k)$ with $\mathcal{O}((m-1)n_kH)$ and get the twisted exact sequence from $(2.1,n_k)$

$$\textbf{(2.4)} \quad 0 \to \mathcal{O}((m-1)n_kH+C-K+K) \to E(mn_k) \to I_z \otimes ((m-1)n_kH+det(n_k)-C-K+K) \to 0 \quad ,$$

where $z \in V_{\delta,\eta,i}(C)$ and $HC \leq \delta_0$.

We find two big but constant integers $k_0$ and $m_0$ so that for any $k \geq k_0$, $m \geq m_0$, and any curve $C$ in $X$ with $HC \leq \delta_0$ the line bundles $\mathcal{O}((m-1)n_kH+C-K)$ and $\mathcal{O}((m-1)n_kH+det(n_k)-C-K)$ in (2.4) are both ample. Furthermore, the second line bundle also satisfies the coditions in lemma 1.2 namely, for $z \in V_{\delta,\eta,i}(C)$

$$[(m-1)n_kH + det(n_k) - C - K]^2 > 4|z|$$

and for any curve $D$ in $X$.

$$[(m-1)n_kH + det(n_k) - C - K]D > 4h\sqrt{k} \quad ,$$

where $h$ is a constant bigger than the coefficient $d$ in (2.3).

We look at the cohomology exact sequence induced by the exact sequence (2.4)

$$\to H^1((m-1)n_kH+C-K+K) \to H^1(E(mn_k)) \to H^1(I_z \otimes ((m-1)n_kH+det(n_k)-C-K+K)) \to .$$

The ampleness of the line bundle $\mathcal{O}((m-1)n_kH + C - K)$ implies $H^1(E(mn_k))$ is embedded in

$H^1(I_z \otimes ((m-1)n_k H + det(n_k) - C - K + K))$.

Therefore if $H^1(E(mn_k)) \neq 0$, then the subscheme $z$ in the above exact sequence must lie on the following subvariety of $Hilb^{|z|}(X)$

$$V := \{ z \in Hilb^{|z|}(X) \mid H^1(I_z \otimes ((m-1)n_k H + det(n_k) - C - K + K)) \neq 0 \} \quad .$$

On the other hand, lemma 1.2 gives the lower bound of the codimension of $V$ in $Hilb^{|z|}(X)$

$$\text{codim}_{Hilb^{|z|}(X)} V > h\sqrt{k} - q(X) \quad .$$

Hence from (2.3) we obtain

$$\text{codim}_{V_{\delta,\eta,i}(C)} V_{\delta,\eta,i}(C) \cap V > (h-d)\sqrt{k} - q(X) > 0 \quad .$$

This shows that all bundles $[E]$ from $M_0(L,k)$ with $H^1(E(mn_k) \neq 0$ form a proper subvariety of $M_0(L,k)$. With similar arguments we prove also the rest statements in theorem 2.

**2.1** We begin to exactly construct the moduli space $F^0$ of all extensions $(2.1, n_k)$. More generally, suppose that $n$ is a fixed integer so that for any element $[E] \in M(L,k)$ there exists the exact sequence

**(2.1,n)** $$0 \to \mathcal{O} \to E(n)(-C) \to I_z \otimes (det(n) - 2C) \to 0 \quad ,$$

where the subscheme $z$ in (2.1) has the length

**(2.5)** $$|z| = c_2(n) + C^2 - det(n)C \quad .$$

The stability of $E$ and effectivity of $C$ give the following inequalities

**(2.6)** $$0 \leq \delta := HC < Hdet(n)/2 \quad .$$

Let $\mathcal{L}_\delta$ $(0 \leq \delta < Hdet(n)/2)$ be the moduli space of all line bundles $\mathcal{O}(C)$ on $X$ with at least one non-trivial section and the fixed degree $HC = \delta$. It is clear that $\mathcal{L}_\delta$ is a quasi-projective variety of dimension $\leq q(X)$. We have the decomposition of the irreducible components

$$\mathcal{L}_\delta = \bigcup_i \mathcal{L}_{\delta,i} \quad .$$

We consider the product space $Hilb^{l_i}(X) \times \mathcal{L}_{\delta,i}$ with $l_i := c_2(n) + C^2 - det(n)C, \quad \mathcal{O}(C) \in \mathcal{L}_{\delta,i}$.

There is a universal subscheme

$$Z^{l_i}(X) \subset X \times Hilb^{l_i}(X)$$

flat of degree $l_i$ over $Hilb^{l_i}(X)$ so that for each locally noether scheme $Z \subset X \times T$, whose direct image $p_{2*}(\mathcal{O}_Z)$ is a locally free $\mathcal{O}_T$-modul of rank $l_i$, there exists exactly one morphism

$$f : T \longrightarrow Hilb^{l_i}(X)$$

satisfying

$$Z = (1_X \times f)^*(Z^{l_i}(X))$$

The ideal sheaf $\mathcal{I}$ of $Z^{l_i}(X)$ is just the universal ideal sheaf of all ideal sheaves, which defines 0-dimensional subschemes of the length $l_i$ in $X$ (see [Gö], page 19).

Let $\mathcal{C}$ be the universal line bundle on $X \times \mathcal{L}_{\delta,i}$. From the projections $q_1$, $q_2$ and $p_X$

$$\begin{array}{ccccc} X \times (Hilb^{l_i}(X) \times \mathcal{L}_{\delta,i}) & \xrightarrow{q_2} & X \times \mathcal{L}_{\delta,i} & \xrightarrow{p_X} & X \\ {\scriptstyle q_1}\downarrow & & & & \\ X \times Hilb^{l_i}(X) & & & & \end{array}$$

we get a family of sheaves $I_z \otimes (det(n) - 2C)$

$$\begin{array}{c} q_1^*(\mathcal{I}) \otimes q_2^*(p_X^*(det(n)) - 2\mathcal{C}) \\ \downarrow \\ X \times (Hilb^{l_i}(X) \times \mathcal{L}_{\delta,i}). \end{array}$$

For a fixed integer $\eta$ the subset

$$V_{\delta,\eta,i} := \{ (z, \mathcal{O}(C)) \in Hilb^{l_i}(X) \times \mathcal{L}_{\delta,i} \mid \dim Ext^1_{\mathcal{O}}(I_z \otimes (det(n) - 2C), \mathcal{O}) = \eta \}$$

is a quasi-projective subvariety of $Hilb^{l_i}(X) \times \mathcal{L}_{\delta,i}$ from the upper semi-continuous theorem.

By taking an elemment $(z, \mathcal{O}(C)) \in V_{\delta,\eta,i}$ and an extension class $e \in Ext^1_{\mathcal{O}}(I_z \otimes (det(n) - 2C), \mathcal{O})$, we get a rank 2 torsion-free sheaf $E$ with $det(E) = L$ and $c_2(E) = k$ from the extension (2.1,$n$). To get the moduli space of all such extensions is just glueing all elements $(z,\, \mathcal{O}(C),\, e)$ together. More exactly, let us look at the subfamily

$$q_1^*(\mathcal{I}) \otimes q_2^*(p_X^*(det(n)) - 2\mathcal{C})$$
$$\downarrow$$
$$X \times V_{\delta,\eta,i}.$$

Because for any $(z, \mathcal{O}(C)) \in V_{\delta,\eta,i}$ the extension group $Ext^1_{\mathcal{O}}(I_z \otimes (det(n) - 2C), \mathcal{O})$ has the constant dimension $\eta$, the relative extension group $Ext^1_{\pi_V}(q_1^*(\mathcal{I}) \otimes q_2^*(p_X^*(det(n)) - 2\mathcal{C}), \mathcal{O})$ respect to the projection $\pi_V : X \times V_{\delta,\eta,i} \to V_{\delta,\eta,i}$ is a rank $\eta$ vector bundle $F_{\delta,\eta,i}$ on $V_{\delta,\eta,i}$. We see that $F_{\delta,\eta,}$ is the moduli space of all extensions (2.1, $n$) with the fixed dates $(\delta, \eta, i)$. Furtheremore, we have the following:

**Claim 2.1**

*There exists a family of torsion-free sheaves $\mathcal{E}$ on $X \times F_{\delta,\eta,i}$, so that the restriction $\mathcal{E}|_{(X,z,\mathcal{O}(C),e)}$ is isomorphic to $E(n)(-C)$ coming from the extension (2.1, $n$) with the extension class $e$.*

The proof is a combination of the argument for the case $\eta = 1$ ( see [OV], page 366) and the argument for the case $I_z \simeq \mathcal{O}$ ( see [NR], page 19, prop. 3.1).

**Proof**

By [BPS], page 137 there exists a spectral sequence

$$H^p(Ext^q_{\pi_V}(q_1^*(\mathcal{I}) \otimes q_2^*(p_X^*(det(n)) - 2\mathcal{C}), \pi_V^*(F^\vee_{\delta,\eta,i}))$$
$$\Rightarrow Ext^{p+q}_{\mathcal{O}_{X \times V_{\delta,\eta,i}}}(q_1^*(\mathcal{I}) \otimes q_2^*(p_X^*(det(n)) - 2\mathcal{C}), \pi_V^*(F^\vee_{\delta,\eta,i})) \quad .$$

Because for any $(z, \mathcal{O}(C)) \in V_{\delta,\eta,i}$ noting $(2C - det(n))H < 0$ we have

$$Ext^0_{\mathcal{O}}(I_z \otimes (det(n) - 2C), \mathcal{O}) \simeq H^0(\mathcal{O}(2C - det(n)) = 0 \quad ,$$

therefore $Ext^0_{\pi_V}(q_1^*(\mathcal{I}) \otimes q_2^*(p_X^*(det(n)) - 2\mathcal{C}), \pi_V(F^\vee_{\delta,\eta,i})) = 0$, hence from the above spectral sequence we obtain the isomorphism

$$\begin{aligned} & Ext^1_{\mathcal{O}_{X \times V_{\delta,\eta,i}}}(q_1^*(\mathcal{I}) \otimes q_2^*(p_X^*(det(n)) - 2\mathcal{C}), \pi_V^*(F^\vee_{\delta,\eta,i})) \\ \textbf{(2.7)} \qquad & \simeq H^0(Ext^1_{\pi_V}(q_1^*(\mathcal{I}) \otimes q_2^*(p_X^*(det(n)) - 2\mathcal{C}), \pi_V^*(F^\vee_{\delta,\eta,i})) \\ & \simeq H^0(F_{\delta,\eta,i} \otimes F^\vee_{\delta,\eta,i})) \quad . \end{aligned}$$

On the other hand, let

$$\pi : X \times F_{\delta,\eta,i} \longrightarrow X \times V_{\delta,\eta,i}$$

be the natural projection, then the composition of the base change morphism (see [BPS], page 137)

$$Ext^1_{\mathcal{O}_{X\times V_{\delta,\eta,i}}}(q_1^*(\mathcal{I})\otimes q_2^*(p_X^*(det(n))-2\mathcal{C}),\pi_*(\mathcal{O}_{X\times F_{\delta,\eta,i}}))$$

$$\to Ext^1_{\mathcal{O}_{X\times F_{\delta,\eta,i}}}(\pi^*(q_1^*(\mathcal{I})\otimes q_2^*(p_X^*(det(n))-2\mathcal{C})),\pi^*(\pi_*(\mathcal{O}_{X\times F_{\delta,\eta,i}})))$$

and the canonical morphism

$$\pi^*(\pi_*(\mathcal{O}_{X\times F_{\delta,\eta,i}}))\to\mathcal{O}_{X\times F_{\delta,\eta,i}}\simeq\pi^*(\mathcal{O}_{X\times V_{\delta,\eta,i}})$$

gives the morphism

$$Ext^1_{\mathcal{O}_{X\times V_{\delta,\eta,i}}}(q_1^*(\mathcal{I})\otimes q_2^*(p_X^*(det(n))-2\mathcal{C}),\pi_*(\mathcal{O}_{X\times F_{\delta,\eta,i}}))$$
$$\to Ext^1_{\mathcal{O}_{X\times F_{\delta,\eta,i}}}(\pi^*(q_1^*(\mathcal{I})\otimes q_2^*(p_X^*(det(n))-2\mathcal{C})),\pi^*(\mathcal{O}_{X\times V_{\delta,\eta,i}})) \quad .$$

Noting the inclusion $\pi_V^*(F^\vee_{\delta,\eta,i})\hookrightarrow\pi_*(\mathcal{O}_{X\times F_{\delta,\eta,i}})$ we get the morphism

**(2.8)**
$$Ext^1_{\mathcal{O}_{X\times V_{\delta,\eta,i}}}(q_1^*(\mathcal{I})\otimes q_2^*(p_X^*(det(n))-2\mathcal{C}),\pi_V^*(F^\vee_{\delta,\eta,i}))$$
$$\to Ext^1_{\mathcal{O}_{X\times F_{\delta,\eta,i}}}(\pi^*(q_1^*(\mathcal{I})\otimes q_2^*(p_X^*(det(n))-2\mathcal{C})),\pi^*(\mathcal{O}_{X\times V_{\delta,\eta,i}})) \quad .$$

The canonical element in $H^0(F_{\delta,\eta,i}\otimes F^\vee_{\delta,\eta,i})$ gives rise to an element in $Ext^1_{\mathcal{O}_{X\times F_{\delta,\eta,i}}}(\pi^*(q_1^*(\mathcal{I})\otimes q_2^*(p_X^*(det(n))-2\mathcal{C})),\pi^*(\mathcal{O}_{X\times V_{\delta,\eta,i}}))$ via the ismorphism (2.7) and the morphism (2.8). Finally, the extension $\mathcal{E}$

$$0\to\pi^*(\mathcal{O}_{X\times V_{\delta,\eta,i}})\to\mathcal{E}\to\pi^*(q_1^*(\mathcal{I})\otimes q_2^*(p_X^*(det(n))-2\mathcal{C}))\to 0$$

corresponding the above extension class is exactly as required in our claim. Claim 2.1 is done.

Because "locally free" and "$H$-stable" are both open conditions in the parameter space of a family of torsion free sheaves, we get a Zariski open set $F^0_{\delta,\eta,i}\subset F_{\delta,\eta,i}$, so that all extensions with $(z,\,\mathcal{O}(C),\,e)\in F^0_{\delta,\eta,i}$ are $H$-stable vector bundles.

The universal extension $\mathcal{E}$ induces a morphism

$$e:\,F^0_{\delta,\eta,i}\longrightarrow M(L,k) \quad .$$

In fact, $e$ is just the correspondence:

$$(z,\,\mathcal{O}(C),\,e)\mapsto[E] \quad ,$$

where $E$ comes from the extension (2.1, $n$) with the extension class $e$.

The projection $p: Hilb^{l_i}(X) \times \mathcal{L}_{\delta,i} \to \mathcal{L}_{\delta,i}$ induces the following morphisms:

$$\begin{array}{ccc} F^0_{\delta,\eta,i} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\tilde{p}} & \\ V^0_{\delta,\eta,i} & \xrightarrow{p} & \mathcal{L}_{\delta,i} \end{array}$$

**(2.9)**

$$\begin{array}{ccc} F^0_{\delta,\eta,i} & \xrightarrow{(e,\tilde{p})} & M(L,k) \times \mathcal{L}_{\delta,i} \\ & \searrow^{e} & \downarrow \\ & & M(L,k)\,. \end{array}$$

We describe fibres of the morphism $(e, \tilde{p})$ in the following way:

Given an element $([E], \mathcal{O}(C)) \in (e, \tilde{p})(F^0_{\delta,\eta,i})$, then $E$ has the representation (2.1,$n$).

Let $H^0(E(n)(-C))^0$ be the set of all sections in $H^0(E(n)(-C))$ with isolated zero locus, which is a non-empty Zariski open set, because the above extension just gives such a section.

Suppose that $s$ is a section from the above Zariski open set, then $s$ induces the exact sequence (2.1, $n$) with the zero locus $z$ of $s$ and the extension class $e$.

We compare the block-map defined in [T], and define the generalized block-map simply as:

$$\begin{array}{ccc} H^0(E(n)(-C))^0 & \xrightarrow{\varphi} & F^0_{\delta,\eta,i} \\ \cup & & \cup \\ s & \longmapsto & (z, \mathcal{O}(C), e) \end{array} \quad .$$

It is easy to see that the image $\varphi(H^0(E(n)(-C))^0)$ is eaxctly the fibre $(e, \tilde{p})^{-1}([E], \mathcal{O}(C))$.

Becuase $E(n)(-C)$ is stable, hence is simple, it implies that $\varphi$ is injective (see [T], lemma 2.3). In particular, the fibre $(e, \tilde{p})^{-1}([E], \mathcal{O}(C))$ has the dimension $h^0(E(n)(-C))$.

**2.2** Now let $n_k$ be the smallest integer so that $\chi(E(n_k)) \geq 1$, the Hirzebruch-Riemann-Roch-formula and some calculations give $n_k = \sqrt{k/H^2} + ak^{-1} + b$, where $a$, $b$ are numbers depending on $k$ but are bounded.

If $k$ is sufficiently large, then $H^2(E(n_k)) = 0$ from the Serre duality and the stability of $E$, hence $H^0(E(n_k)) \neq 0$, and we have the moduli space $F^0 = \bigcup_{\delta,\eta,i} F^0_{\delta,\eta,i}$ of all extensions (2.1, $n_k$), which are $H$-stable and locally free. We want to know that for which $F^0_{\delta,\eta,i}$, the image $e(F^0_{\delta,\eta,i})$ of the morphism $e$ in (2.9) is a Zariski open set in the moduli space $M(L,k)$. This is the following:

**Lemma 2.1**

*There exist positive integers $k_0$, $c$ and $d$ which depend only on the Chern classes of $X$, $H$ and $L$ so that for any $k \geq k_0$ and any $F^0_{\delta,\eta,i}$ we have*

$$dim\, F^0_{\delta,\eta,i} \leq 4k - c\sqrt{k}\delta + d\sqrt{k}.$$

We know that the moduli space $M(L,k)$ has the dimension $\geq 4k$ + constant . Lemma 2.1 just means that generic $[E] \in M(L,k)$ come from the extension (2.1, $n_k$) with the curve $C$ of smaller degree $HC = \delta \leq d/c + 1 =: \delta_0$.

**Proof**

We look at the diagram (2.9). Since $\mathcal{L}_{\delta,i}$ has bounded dimension $\leq q(X)$, it is sufficient to show that the dimension of the fibre $\tilde{p}^{-1}(C) =: F^0_{\delta,\eta,i}(C)$ has the following upper bound

$$\dim F^0_{\delta,\eta,i}(C) \leq 4k - c\sqrt{k}\delta + d\sqrt{k} \quad .$$

We analyes carefully the fibration

$$\pi : F^0_{\delta,\eta,i}(C) \longrightarrow p^{-1}(C) =: V^0_{\delta,\eta,i}(C) \quad .$$

The bace space $V^0_{\delta,\eta,i}(C)$ is a Zariski open set in the subvariety

$$V_{\delta,\eta,i}(C) =: \{ z \in Hilb^{l_i}(X) \mid h^1(I_z \otimes (det(n_k) + K - 2C)) = \eta \} \quad ,$$

with

$$\text{(2.10)} \qquad \begin{aligned} l_i &= c_2(E(n_k)) + C^2 - det(E(n_k))C \\ &= n_k^2 H^2 + n_k HL + k + C^2 - C(2n_k H + L) \end{aligned}$$

and the fibre $\pi^{-1}(z)$ is a Zariski open set in the global extension group $Ext^1_{\mathcal{O}}(I_z \otimes (det(n_k) - 2C), \mathcal{O}) \simeq H^1(I_z \otimes (det(n_k) + K - 2C))^{\vee}$ of the dimension $\eta$.
So it is sufficient to estimate

$$\dim V_{\delta,\eta,i}(C) + \eta \leq 4k - c\sqrt{k}\delta + d\sqrt{k} \quad . \tag{2.11}$$

Applying the Riemann-Roch-theorem to the line bundle

$$\mathcal{O}(det(n_k) + K - 2C) \simeq \mathcal{O}(2n_kH + L + K - 2C) \quad ,$$

noting $h^2(det(n_k) + K - 2C) = h^0(2C - det(n_k)) = 0$, since $(2C - det(n_k))H < 0$, we get

$$\begin{aligned}
&h^0(det(n_k) + K - 2C) - h^1(det(n_k) + K - 2C) \\
=&\chi(\mathcal{O}) + \frac{1}{2}[(2n_kH + L + K - 2C)^2 - (2n_kH + L + K - 2C)K] \\
=&\chi(\mathcal{O}) + \frac{1}{2}L(L + K) + 2n_k^2H^2 + n_kH(2L + K) - CK + 2[C^2 - C(2n_kH + L)] \quad .
\end{aligned}$$

We replace the last term $2[C^2 - C(2n_kH + L)]$ in the last equality by (2.10) and get

$$\begin{aligned}
&h^0(det(n_k) + K - 2C) - h^1(det(n_k) + K - 2C) \\
=&2l_i - 2k - CK + n_kHK + \chi(\mathcal{O}) + \frac{1}{2}L(L + K) \quad .
\end{aligned} \tag{2.12}$$

We relate the dimension $\eta = h^1(I_z \otimes (det(n_k) + K - 2C))$ with the degree $\eta''$ of the special position of the scheme $z$ respect to the linear system $|det(n_k) + K - 2C|$ by the following exact sequence

$$\begin{aligned}
&0 \to H^0(I_z \otimes (det(n_k) + K - 2C)) \to H^0(det(n_k) + K - 2C) \to \\
&\to H^0(\mathcal{O}_z \otimes (det(n_k) + K - 2C)) \to H^1(I_z \otimes (det(n_k) + K - 2C)) \to \\
&\to H^1(det(n_k) + K - 2C) \to 0, \qquad z \in V_{\delta,\eta,i} \quad .
\end{aligned} \tag{2.13}$$

We have two cases:

1) $h^0(det(n_k) + K - 2C) \leq l_i$.

2) $h^0(det(n_k) + K - 2C) > l_i$.

As for the first case. Suppose in the exact sequence (2.13) we have

$$\begin{aligned}
&\dim\{r : H^0(det(n_k) + K - 2C)) \to H^0(\mathcal{O}_z \otimes (det(n_k) + K - 2C))\} \\
&=h^0(det(n_k) + K - 2C) - \eta' \\
&=l_i - (l_i - h^0(det(n_k) + K - 2C) + \eta') \\
&=l_i - \eta'' \quad ,
\end{aligned} \tag{2.14}$$

here $\eta\prime$ and $\eta''$ are non-negative integers, then from (2.13) and (2.12) we get the relations bewteen $\eta$, $\eta'$ and $\eta''$

$$
\begin{aligned}
\eta =& h^1(I_z \otimes (det(n_k) + K - 2C)) \\
=& l_i - (h^0(det(n_k) + K - 2C)) - \eta') + h^1(det(n_k) + K - 2C) \\
=& l_i - (h^0(det(n_k) + K - 2C) - \eta') \\
& + h^0(det(n_k) + K - 2C) + 2k - 2l_i + CK - n_k HK - \chi(\mathcal{O}) - L(L+K)/2 \\
=& 2k - l_i + \eta' + CK - n_k HK - \chi(\mathcal{O}) - L(L+K)/2 \quad .
\end{aligned} \tag{2.15}
$$

On the other hand, from (2.14) we see that $V_{\delta,\eta,i}(C)$ lies on the following subvariety

$$\{ z \in Hilb^{l_i}(X) \,|\, \dim \{ r : H^0(det(n_k) + K - 2C) \to H^0(\mathcal{O}_z \otimes (det(n_k) + K - 2C)\} = l_i - \eta'' \} \quad .$$

If $\eta' > 0$, then $\eta'' > 0$ and $l_i - \eta'' < h^0(det(n_k) + K - 2C)$. The above subvariety is just defined in lemma 1.1, hence we obtain

$$
\begin{aligned}
\dim V_{\delta,\eta,i}(C) \leq & 2l_i - \eta'' + q(X) \\
=& 2l_i - (l_i - h^0(det(n_k) + K - 2C) + \eta') + q(X) \\
\leq & 2l_i - \eta' + q(X) \quad (l_i - h^0(det(n_k) + K - 2C) \geq 0) \quad .
\end{aligned} \tag{2.16}
$$

If $\eta' = 0$, we have automatically the last estimate.

Hence from (2.15) and (2.16) we get

$$\eta + \dim V_{\delta,\eta,i}(C) \leq 2k + (l_i + CK) - n_k HK - \chi(\mathcal{O}) - L(L+K)/2 + q(X) \quad . \tag{2.17}$$

To bound the term $l_i + CK$ in (2.17).

Since $CH < det(n_k)H/2 = n_k H^2 + LH/2$,

the Hodge- index-theorem gives

$$C^2 \leq \frac{(CH)(CH)}{H^2} \leq (n_k + \frac{LH}{2H^2})CH \quad .$$

Noting $H = K + H_0$, we have simply

$$CK = C(H - H_0) \leq CH \quad .$$

Combining the above two inequalities we obtain

$$\begin{aligned} l_i + CK =& n_k^2 H^2 + k + n_k HL - C(2n_k H + L) + C^2 + CK \\ \le& n_k^2 H^2 + k + n_k HL - C(2n_k H + L) + (n_k + \frac{LH}{2H^2})CH + CH \\ =& n_k^2 H^2 + k + n_k HL - (n_k - \frac{LH}{2H^2} - 1)CH - CL \\ \le& n_k^2 H^2 + k + n_k HL - (n_k - \frac{LH}{2H^2} - 1)CH \quad (CL \ge 0) \quad , \end{aligned}$$

and from (2.17) noting $n_k = \sqrt{k/H^2} + ak^{-1} + b$, we find positive integers $k_0$, $c$ and $d$ depending only on the Chern classes of $X$, $H$ and $L$ so that if $k \ge k_0$, then

$$\eta + \dim V_{\delta,\eta,i}(C) \le 4k - c\sqrt{k}CH + d\sqrt{k} \quad .$$

We have proved (2.11) for case 1.

To case 2. Suppose in the exact sequence (2.13) we have

$$\begin{aligned} &\dim \{r : H^0(det(n_k) + K - 2C)) \to H^0(\mathcal{O}_z \otimes (det(n_k) + K - 2C))\} \\ =& l_i - \eta' \quad , \end{aligned}$$

where $\eta'$ is a non-negative integer. Then the exact sequence (2.13) gives

$$\begin{aligned} \eta =& h^1(I_z \otimes (det(n_k) + K - 2C)) \\ =& h^1(det(n_k) + K - 2C) + \eta' \quad . \end{aligned} \tag{2.18}$$

Similar as in the first case $V_{\delta,\eta,i}(C)$ is embedded in the subvariety

$$\{\, z \in Hilb^{l_i}(X) \,|\dim \{r : (H^0(det(n_k) + K - 2C)) \to H^0(\mathcal{O}_z \otimes (det(n_k) + K - 2C))\} = l_i - \eta' \,\} \quad .$$

In this case we have always $l_i - \eta' \le l_i < h^0(det(n_k) + K - 2C)$. If $\eta' > 0$, then applying lemma 1.1 again we get

$$\dim V_{\delta,\eta,i}(C) \le 2l_i - \eta' + q(X) \quad .$$

If $\eta' = 0$, we have trivially the above inequality.

Using (2.18) and (2.12) we obtain

$$
\begin{aligned}
&\eta + \dim V_{\delta,\eta,i}(C) \\
(2.19) \qquad &\le 2l_i + h^1(det(n_k) + K - 2C) + q(X) \\
&= 2k + (h^0(det(n_k) + K - 2C) + KC) - n_k HK - \chi(\mathcal{O}) - L(L+K)/2 + q(X) \quad .
\end{aligned}
$$

We have to bound the term $h^0(det(n_k) + K - 2C) + KC$ in (2.19)

Let $|M|$ be the moving part of the linear system $|det(n_k) + K - 2C|$.

If $|M|$ is not composed with pencil, we can apply the Gieseker lemma same as in the proof of lemma 1.2 and obtain:

$$h^0(det(n_k) + K - 2C) = h^0(M) \le M^2/2 + 2 \quad .$$

Using the Hodge-index-theorem again we have a upper bound of $M^2/2$

$$
\begin{aligned}
M^2/2 \le& \frac{(MH)^2}{2H^2} \\
\le& \frac{[(det(n_k) + K - 2C)H]^2}{2H^2} \\
=& \frac{[2n_k H^2 - 2CH + (L+K)H]^2}{2H^2} \\
=& 2n_k^2 H^2 - 4n_k CH + \frac{2CH}{H^2} CH + d_1 CH + d_2 n_k + d_3 \\
\le& 2n_k^2 H^2 - 4n_k CH + (2n_k + \frac{LH}{H^2}) CH + d_1 CH + d_2 n_k + d_3 \quad (CH < n_k H^2 + LH/2) \\
=& 2n_k^2 H^2 + (\frac{LH}{H^2} - 2n_k) CH + d_1 HC + d_2 n_k + d_3 \quad ,
\end{aligned}
$$

where $d_i$ are some constants depending only on the Chern classes of $H$, $L$ and $K$.

Noting $CK \le CH$, the above two inequalities give

$$(2.20) \qquad h^0(det(n_k) + K - 2C) + CK \le 2n_k^2 H^2 + (\frac{LH}{H^2} + d_1 + 1 - 2n_k) CH + +d_2 n_k + d_3 \quad .$$

If $|M|$ is composed with pencil, then there exists a 1-dimensional algebraic system, whose generic element $F$ is a smooth irreducible curve so that $M$ is algebraic equivalent to $lF$, and $\dim |M| \le l$ (see [BPV], page 113-114). $l$ is easy to bound as the following

$$
\begin{aligned}
l =& \frac{MH}{FH} \\
\leq& MH \\
\leq& (det(n_k) + K - 2C)H \\
=& (2n_k H + L + K - 2C)H \\
\leq& 2n_k H^2 + d_4 \\
=& 2n_k^2 H^2 - 2n_k^2 H^2 + 2n_k CH - 2n_k CH + 2n_k H^2 + d_4 \\
\leq& 2n_k^2 H^2 - 2n_k CH + n_k(2H^2 + LH) + d_4 \\
& (2n_k CH < 2n_k^2 H^2 + n_k LH) \quad .
\end{aligned}
\tag{2.21}
$$

Combine (2.20) and (2.21) we have always

$$h^0(det(n_k) + K - 2C) + CK \leq 2n_k^2 H^2 - d'_1 n_k HC + d'_2 n_k + d'_3 \quad ,$$

and from (2.19) and $n_k = \sqrt{k/H^2} + ak^{-1} + b$ we find two constant $c$ and $d$ so that

$$\eta + \dim V_{\delta,\eta,i}(C) \leq 4k - c\sqrt{k}HC + d\sqrt{k}$$

for case 2.

Lemma 2.1 is completed.

We look at the diagram

$$
\begin{array}{ccccc}
F^0_{\delta,\eta,i} & \xrightarrow{(e,\tilde{p})} & M(L,k) \times \mathcal{L}_{\delta,i} & \longrightarrow & M(L,k) \\
\pi \downarrow & & & & \\
V^0_{\delta,\eta,i} & \longrightarrow & Hilb^{l_i}(X) \times \mathcal{L}_{\delta,i} & \longrightarrow & \mathcal{L}_{\delta,i} \\
& & \underbrace{\qquad\qquad\qquad}_{p} & &
\end{array}
$$

Lemma 2.1 just says that the variety $\bigcup_{\substack{\delta,\eta,i \\ \delta \leq \delta_0}} e(F^0_{\delta,\eta,i})$ is a Zariski open dense set in the moduli space $M(L,k)$, where $\delta_0$ is a constant depending only on Chern classes of $X$, $H$ and $L$.

Now we restrict our attention to the subvariety $p^{-1}(\mathcal{O}(C)) =: V^0_{\delta,\eta,i}(C) \subset Hilb^{l_i}(X)$. More precisely, we have the following:

**Lemma 2.2**

*There exist two constants $k_0$ and $d$ so that for any $k \geq k_0$ and any $\delta \leq \delta_0$ if $e(F^0_{\delta,\eta,i})$ is a Zariski open set in $M(L,k)$, then*

$$codim_{Hilb^{l_i}(X)} V^0_{\delta,\eta,i}(C) \leq d\sqrt{k} \quad .$$

**Proof**

Under the assumption $HC \leq \delta_0$, and by standard arguments we show little later that

$$\dim Hilb^{l_i}(X) \leq 4k + d_1\sqrt{k} = \dim M(L,k) + d_1\sqrt{k} + \text{constant} \quad , \tag{2.22}$$

and

$$\dim H^0(E(n_k)(-C)) \geq \eta - d_2\sqrt{k}, \quad \forall\, [E] \in e(F^0_{\delta,\eta,i}) \quad . \tag{2.23}$$

On the other hand the fibre $(e,\tilde{p})^{-1}([E],\mathcal{O}(C))$ is identified with a Zariski open set of $H^0(E(n_k)(-C))$ via the block-map (see the end of 2.1), hence from (2.23) it holds

$$\begin{aligned} \dim e^{-1} &\geq \dim\,(e,\tilde{p})^{-1} \\ &\geq \eta - d_2\sqrt{k} \\ &= \dim \pi^{-1} - d_2\sqrt{k} \quad . \end{aligned}$$

If $\dim e(F^0_{\delta,\eta,i}) = \dim M(L,k)$, then

$$\begin{aligned} \dim F^0_{\delta,\eta,i} &= \dim M(L,k) + \dim e^{-1} \\ &\geq \dim M(L,k) + \delta - d_2\sqrt{k} \\ &= \dim M(L,k) + \dim \pi^{-1} - d_2\sqrt{k} \quad , \end{aligned}$$

hence

$$\begin{aligned} \dim V^0_{\delta,\eta,i} &= \dim F^0_{\delta,\eta,i} - \dim \pi^{-1} \\ &\geq \dim M(L,k) - d_2\sqrt{k} \quad . \end{aligned}$$

Applying the sime-continuous theorem to the morphism $p : V^0_{\delta,\eta,i} \to p(V^0_{\delta,\eta,i})$ we get for any $\mathcal{O}(C) \in p(V_{\delta,\eta,i})$

$$\dim V^0_{\delta,\eta,i}(C) \geq \dim V^0_{\delta,\eta,i} - \dim p(V^0_{\delta,\eta,i})$$
$$\geq \dim M(L,k) - d_2\sqrt{k} - q(X) \quad .$$

Finally from (2.22) we obtain

$$\mathrm{codim}_{Hilb^{l_i}(X)} V^0_{\delta,\eta,i}(C) \leq d\sqrt{k} \quad .$$

Now we begin to prove (2.22) and (2.23).

It is well known that all curves $C$ in $X$ with the bounded degree $HC \leq \delta_0$ form an algebraic variety. This shows that for any such curve the numbers $C^2$, $KC$ and $LC$ are bounded.

From (2.10) and noting $n_k = \sqrt{k/H^2} + ak^{-1} + b$ we get easily

$$\begin{aligned}\dim Hilb^{l_i}(X) &= 2l_i \\ &= 2(n_k^2H^2 + k + n_kHL + [C^2 - C(2n_kH + L)]) \\ &\leq 4k + d_1\sqrt{k} \quad .\end{aligned}$$

Similar as the above, we find a positive integer $k_0$ so that for any $k \geq k_0$ and any curve $C$ in $X$ with $HC \leq \delta_0$ the line bundle $\mathcal{O}(det(n_k) - 2C)$ is ample. Kodaira vanishing theorem gives $H^1(det(n_k) + K - 2C) = 0$. Therefore from the exact sequence

$$\begin{aligned}0 \to H^0(I_z \otimes (det(n_k) + K - 2C)) \to H^0(det(n_k) + K - 2C) \to \\ \to H^0(\mathcal{O}_z \otimes (det(n_k) + K - 2C)) \to H^1(I_z \otimes (det(n_k) + K - 2C)) \to 0\end{aligned}$$

we obtain

$$\begin{aligned}&h^0(I_z \otimes (det(n_K) + K - 2C)) \\ =&h^1(I_z \otimes (det(n_k) + K - 2C)) + (h^0(det(n_k) + K - 2C) - l_i) \\ =&\eta + (h^0(det(n_k) + K - 2C) - l_i) \quad .\end{aligned}$$

Noting (2.12), (2,10) and $n_k = \sqrt{k/H^2} + ak^{-1} + b$ we get an upper bound

$$\begin{aligned}&|h^0(det(n_k) + K - 2C) - l_i| \\ =&| - 2k + l_i - CK + n_kHK + \chi(\mathcal{O}) + L(L+K)/2| \\ =&| - 2k + k + n_k^2H^2 + n_kHL + C^2 - C(2n_kH + L) - CK + n_kHK + \chi(\mathcal{O}) + L(L+K)/2| \\ \leq& a_1\sqrt{k},\end{aligned}$$

hence

$$h^0(I_z \otimes (det(n_k) + K - 2C)) \geq \eta - a_1\sqrt{k} \quad . \tag{2.24}$$

We want to compare $H^0(I_z \otimes (det(n_k) + K - 2C))$ and $H^0(I_z \otimes (det(n_k) - 2C))$. We take a fixed choosed effective divisor $D$ in $X$ so that linear systems $|D|$ and $|D - K|$ are free from fixed components and base points.
For each $z \in V^0_{\delta,\eta,i}(C)$ we find $D \in |D|$ and $D' \in |D - K|$ so that $z \cap D = \emptyset = z \cap D'$.
Look at the following exact sequences

$$\begin{aligned} 0 \to H^0(I_z \otimes (det(n_k) + K - D - 2C)) \to H^0(I_z \otimes (det(n_k) + K - 2C)) \to \\ \to H^0(\mathcal{O}(D) \otimes (det(n_k) + K - 2C)) \to \dots \quad , \end{aligned}$$

and

$$\begin{aligned} 0 \to H^0(I_z \otimes (det(n_k) + K - D - 2C)) \to H^0(I_z \otimes (det(n_k) - 2C)) \to \\ \to H^0(\mathcal{O}(D') \otimes (det(n_k) - 2C)) \to \dots \quad . \end{aligned}$$

From the first exact sequence and (2.24), noting the Clifford theorem in the curve case

$$\begin{aligned} & h^0(\mathcal{O}(D) \otimes (det(n_k) + K - 2C)) \\ \leq & \frac{1}{2}(det(n_k) + K - 2C)D \\ \leq & a_2\sqrt{k} \quad , \end{aligned}$$

we obtain

$$\begin{aligned} & h^0(I_z \otimes (det(n_k) + K - D - 2C)) \\ \geq & h^0(I_z \otimes (det(n_k) + K - 2C)) - (a_2\sqrt{k}) \\ \geq & \eta - (a_1 + a_2)\sqrt{k} \quad , \end{aligned}$$

hence from the second sequence we get immediately

$$h^0(I_z \otimes (det(n_k) - 2C)) \geq \eta - (a_1 + a_2)\sqrt{k} \quad .$$

Finally from the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow E(n_k)(-C) \longrightarrow I_z \otimes (det(n_k) - 2C)) \longrightarrow 0$$

we have

$$\begin{aligned} h^0(E(n_k)(-C)) &\geq h^0(I_z \otimes (det(n_k) - 2C)) - h^1(\mathcal{O}) \\ &\geq \eta - (a_1 + a_2)\sqrt{k} - q(X) \\ &\geq \eta - d_2\sqrt{k} \quad . \end{aligned}$$

lemma 2.2 is proved.

By taking a line bundle $\mathcal{O}(D)$ on $X$ we define three subvarieties in the moduli space $M(L,k)$ respect to the twisting $\mathcal{O}(D)$ as the following:

$$\begin{aligned} M_1^s &:= \{ [E] \in M(L,k) \mid H^1(E(D)) \neq 0 \} \\ M_2^s &:= \{ [E] \in M(L,k) \mid H^1(E(D+K)) \neq 0 \} \\ M_3^s &:= \{ [E] \in M(L,k) \mid \exists p \in X \quad H^1(I_p \otimes E(D)) \neq 0 \} \quad . \end{aligned}$$

We have the main theorem in this section:

**Theorem 2**

*Let $n_k(\approx \sqrt{k/H^2})$ be the smallest number so that $\chi(E(n_k)) \geq 1$, then there exist two positive integers $k_0$ and $m_0$ so that for any $k \geq k_0$ and any $m \geq m_0$ the subvarieties $M_1^s$, $M_2^s$, and $M_3^s$ respect to the twisting $\mathcal{O}(mn_kH)$ are proper in the each component of the moduli space $M(L,k)$.*

**Proof**

We prove at first the statement for $M_1^s$, and will see later that the rest cases are easily reduced to this case. We consider little more general situation:

*Suppose $c$ is fix choosed positive integer, then there exist two positive integers $k_1$ and $m_1$ so that for any integers $k \geq k_1$, $m \geq m_1$ and $-c \leq n \leq c$ the subvariety $M_1^s$ respect to the twisting $\mathcal{O}(m(n_k + n)H)$ is proper in $M(L,k)$.*

Because all curves $C$ in $X$ with the bounded degree $HC \leq \delta_0$ form an algebraic variety and $n_k = \sqrt{k/H^2} + ak^{-1} + b$, we may find two positive integers $k_0$ and $m_0$ so that for any integers $k \geq k_0$, $m \geq m_0$, $-c \leq n \leq c$ and any curve $C$ in $X$ with $HC \leq \delta_0$ the line bundles $\mathcal{O}(((m-1)n_k + mn)H + C - K)$ and $\mathcal{O}(det(n_k) + ((m-1)n_k + mn)H - C - K)$ are ample.
Furtheremore, noting $l_i = n_k^2H^2 + n_kHL + k + C^2 - C(2n_kH + L)$ we may assume $k_0$ and $m_0$ are big enough so that the second bundle satisfies the conditions in lemma 1.2 namely,

$$[det(n_k) + ((m-1)n_k + mn)H - C - K)]^2 > 4l_i,$$

and for any curve $D$ in $X$ it holds

$$[det(n_k) + ((m-1)n_k + mn)H - C - K]D > 4h\sqrt{k},$$

here $h$ is an integer biger then the coefficient $d$ in lemma 2.2.

We look at the Zariski-open dence set $\bigcup_{\substack{\delta,\eta,i \\ \delta\leq\delta_0}} e(F^0_{\delta,\eta,i})$ in $M(L,k)$ from lemma 2.1. So it is sufficient to show that for any Zariski-open set $e(F^0_{\delta,\eta,i})$ the intersection $M^s_1 \cap e(F^0_{\delta,\eta,i})$ is a proper subvariety. Let $F^0_{\delta,\eta,i}$ be such a variety, we consider the following diagram

$$\begin{array}{ccccc} F^0_{\delta,\eta,i} & \xrightarrow{(e,\tilde{p})} & M(L,k)\times\mathcal{L}_{\delta,i} & \longrightarrow & M(L,k) \\ \pi\downarrow & & & & \\ V^0_{\delta,\eta,i} & \xrightarrow{p} & \mathcal{L}_{\delta,i} & & \end{array} .$$

It is also enough to prove for each fibre $\tilde{p}^{-1}(\mathcal{O}(C)) =: F^0_{\delta,\eta,i}(C)$ the intersection $M^s_1 \cap e(F^0_{\delta,\eta,i}(C))$ is a proper subvariety in $e(F^0_{\delta,\eta,i}(C))$.

Let $[E] \in e(F^0_{\delta,\eta,i}(C))$, we look at the following twisted cohomology exact sequence induced by (2.1)

$$\to H^1(((m-1)n_k+mn)H+C-K+K) \to H^1(E(m)(n_k+n)) \to H^1(I_z\otimes(det(n_k)+((m-1)n_k+mn)H-C-K+K) \to$$

The Kodaira vanishing theorem gives $H^1((mn_k + mn)H + C) = 0$. If the $[E] \in M^s_1 \cap e(F^0_{\delta,\eta,i}(C))$, then from the above exact sequence we see that the subvariety $\pi e^{-1}(M^s_1 \cap e(F^0_{\delta,\eta,i}(C)) \subset V^0_{\delta,\eta,i}(C)$ $\subset Hilb^{l_i}(X)$ lies also on the subvariety

$$\{\, z \in Hilb^{l_i}(X) \,|\, H^1(I_z \otimes (det(n_k) + ((m-1)n_k + mn)H - C) \neq 0 \,\} \quad ,$$

which is exactly defined in lemma 1.2.

Therefore we obtain

$$\mathrm{codim}_{Hilb^{l_i}(X)} \pi e^{-1}(M^s_1 \cap e(F^0_{\delta,\eta,i}(C)) \geq h\sqrt{k} - q(X) \quad .$$

By using lemma 2.2 we get

$$\mathrm{codim}_{V^0_{\delta,\eta,i}(C)} \pi e^{-1}(M^s_1 \cap e(F^0_{\delta,\eta,i}(C)) \geq (h-d)\sqrt{k} - g(X) > 0 \quad .$$

Look at the pull back of $\pi$, we have

$$\operatorname{codim}_{F^0_{\delta,\eta,i}(C)} e^{-1}(M_1^s \cap e(F^0_{\delta,\eta,i}(C)) > 0 \quad ,$$

hence

$$\operatorname{codim}_{e(F_{\delta,\eta,i}(C))} M_1^s \cap e(F^0_{\delta,\eta,i}(C)) > 0 \quad .$$

The statement is proved for $M_1^s$.

By the exactly same argument we also show that $M_2^s$ is proper in $M(L,k)$.

As for $M_3^s$. We replace rank 2 $H$-stable bundle $E$, $[E] \in M(L,k)$ by rank 2 $H$-stable torsion-free sheaf $I_p \otimes E$, $p \in X$, $[E] \in M(L,k)$.

The bidual of $I_p \otimes E$ is just $E$. This shows that $I_{p_1} \otimes E_1 \simeq I_{p_2} \otimes E_2$ iff $I_{p_1} \simeq I_{p_2}$, and $E_1 \simeq E_2$, hence the moduli space $M'(L,k)$ of all such sheaves is isomorphic to $M(L,k) \times X$. In particular $\dim M'(L,k) = \dim M(L,k) + 2$.

For $M'(L,k)$ we have the exactly same construction as (2.9)

$$\begin{array}{ccc} F'^0_{\delta,\eta,i} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\tilde{p}} & \\ V'^0_{\delta,\eta,i} & \xrightarrow{\ p\ } & \mathcal{L}_{\delta,i} \end{array} \quad ,$$

$$\begin{array}{ccc} F'^0_{\delta,\eta,i} & \xrightarrow{(e,\tilde{p})} & M'(L,k) \times \mathcal{L}_{\delta,i} \\ & \searrow^{e} & \downarrow \\ & & M'(L,k) \end{array} \quad .$$

Let $n'_k$ be the smallest integer so that $\chi(I_p \otimes E(n'_k H)) \geq 1$, it is easy to see that $|n'_k - n_k|$ is constant.

By the same argument in lemma 2.1 we have the following estimate of the dimension for $F'^0_{\delta,\eta,i}$ respect the twisting $\mathcal{O}(n'_k H)$

$$\dim F'^0_{\delta,\eta,i} \leq 4k - d'\sqrt{k} + c'\sqrt{k} \quad .$$

Hence we may take the Zariski open dense set $M'_0(L,k) := \bigcup_{\substack{\delta,\eta,i \\ \delta \geq \delta_0}} e(F'^0_{\delta,\eta,i})$ in $M'(L,k)$ so that

$$\textbf{(2.25)} \qquad \dim_{M'(L,k)}(M'(L,k) \setminus M'_0(L,k)) \geq 3 \quad .$$

If $e(F'^0_{\delta,\eta,i})$ is a Zariski open set in $M'(L,k)$, we have also the same inequality as in lemma 2.2

$$\mathrm{codim}_{Hilb^{l_i}(X)} V'^{0}_{\delta,\eta,i}(C) \geq d'\sqrt{k} \quad .$$

Taking a line bundle $\mathcal{O}(D)$ on $X$ we define also a subvariety in $M'(L,k)$ respect to the twisting $\mathcal{O}(D)$

$$M_1'^s := \{ [I_p \otimes E] \in M'(L,k) \,|\, H^1(I_p \otimes E(D)) \neq 0 \} \quad .$$

Let $c$ be a fix choosed positive integer. By the same argument in the proof of lemma 2.3 for $M_1^s$ we find two constants $k_3$ and $m_3$ so that for any $k \geq k_3$, $m \geq m_3$ and $-c \leq n \leq c$ the subvariety $M_1'^s$ respect to the twisting $\mathcal{O}(m(n_k' + n)H)$ satisfies

$$\mathrm{codim}_{M_0'(L,k)} M_1'^s \cap M_0'(L,k) \geq 3 \quad .$$

From (2.25) we get also

$$\mathrm{codim}_{M'(L,k)} M_1'^s \cap M'(L,k) \geq 3 \quad .$$

On the other hand the bidual $I_p \otimes E \rightarrow (I_p \otimes E)^{\vee\vee} = E$ induces the projection

$$p_M : M'(L,k) \rightarrow M(L,k) \quad .$$

We see that the image $p_M(M_1'^s)$ is exactly $M_3^s$ respect to the tiwsting $\mathcal{O}(m(n_k' + n)H)$, and is a proper subvariety, since $\dim M_1'^s \leq \dim M'(L,k) - 3 < \dim M(L,k)$.

Finally if we take $k_0 := \max(k_1, k_2, k_3)$ and $m_0 := \max(m_1, m_2, m_3)$ and $c = |n_k - n_k'|$, then for any two integers $k \geq k_0$ and $m \geq m_0$ the subvarieties $M_i^s$, $1 \leq i \leq 3$ respect to the twisting $\mathcal{O}(mn_kH)$ are proper in $M(L,k)$. Theorem 2 is proved.

## 3. Regular 2-form on the moduli space of rank 2 stable bundles on an algebraic surface

In this section we are going to prove theorem 1.

By taking $m_k := m_0 n_k$ in theorem 2 we have the Zariski open dense set $M(L,k) \setminus \cup_{i=1}^{3} M_i^s$ in the moduli space $M(L,k)$. Let $F_{0,1}^0$ be the moduli space defined in 2.1 of all extensions (2.1,$m_k$) with $C = 0$ and $\eta = \dim Ext_{\mathcal{O}}^1(I_z \otimes det(n_k), \mathcal{O}) = 1$. Furthermore, suppose that $e : F_{0,i}^0 \to M(L,k)$ is the morphism in (2.9), then we have simply the following:

**Proposition 3.1**

$$e(F_{0,1}^0) = M(L,k) \setminus \cup_{i=1}^{3} M_i^s \quad .$$

**Proof**

Let $[E] \in M(L,k) \setminus \cup_{i=1}^{3} M_i^s$, then $H^1(I_p \otimes E(m_k)) = 0, \quad \forall p \in X$, and $H^1(E(m_k)(K)) = 0$. The vanishing of the first cohomology group implies that $E(m_k)$ is generated by its global sections. Applying the Bertinis theorem to the map (see [GH2])

$$X \longrightarrow G(2, H^0(E(m_k))^\vee)$$

follows that the set of sections from $H^0(E(m_k))$ with the isolated zero locus is a non-empty Zariski open set. Thus each section from the above set induces the exact sequence (2.1,$m_k$) with $C = 0$

$$0 \to \mathcal{O} \to E(m_k) \to I_z \otimes det(m_k) \to 0 \quad .$$

We twist the above exact sequence with canonical divisor $K$, hence get the cohomology exact sequence

$$\to H^1(E(m_k)(K)) \to H^1(I_z \otimes (det(m_k) + K) \to H^2(K) \to \quad .$$

Because $H^1(E(m_k)(K)) = 0$, $H^1(I_z \otimes (det(m_k) + K)) \simeq Ext_{\mathcal{O}}^1(I_z \otimes det(m_k), \mathcal{O})^\vee \neq 0$ and $H^2(K) \simeq H^0(\mathcal{O}) \simeq C$, we have $\eta = \dim Ext_{\mathcal{O}}^1(I_z \otimes det(m_k), \mathcal{O}) = 1$.

This means that all vector bundles $[E] \in M(L,k) \setminus \cup_{i=1}^{3} M_i^s$ come from the extensions (2.1,$m_k$) with $C = 0$ and $\eta = 1$, i.e. $e(F_{0,1}^0) = M(L,k) \setminus \cup_{i=1}^{3} M_i^s$. Proposition 3.1 is proved.

We denote simply that $F := F^0_{0,1}$ and $V := V^0_{0,1}$. In 2.1 we have constructed an universal extension $\mathcal{E}$ on $X \times F$. Because any non-zero extension classes from $Ext^1_{\mathcal{O}}(I_z \otimes det(m_k), \mathcal{O}) \simeq C, \quad z \in V$ give the isomorphic bundles $E$ form the extensions (2.1,$m_k$), hence it is easy to see that the universal extension $\mathcal{E}$ on $X \times (F \setminus$ the zero section of $F)$ can be pushed down on $X \times V$. This is in fact the construction of the universal extension in [OV], page 366. Therefore we get a morphism

$$e : V \longrightarrow M(L,k) \quad ,$$

the image $e(V)$ is the Zariski open dense set $M(L,k) \setminus \cup_{i=1}^3 M_i^s$, the fibre $e^{-1}([E])$ is identified with the Zariski open set of the projective space $P(H^0(E(m_k)))$ via the block-map (see end of 2.1). By taking the Zariski-close of $V$ respect $e(V)$ in $Hilb^l(X)$ respect in $\bar{M}(L,k)$, a compactification of $M(L,k)$ (for example, the Gieseker-compactification), we get a surjective rational map

$$\bar{e} : \bar{V} \longrightarrow \bar{M}(L,k) \quad .$$

Let $\bar{V}_i$ be an irreducible component of $\bar{V}$ and $\bar{M}_i := \bar{e}(\bar{V}_i)$ be the irreducible component of $\bar{M}(L,k)$, then we have the following lemma, perhaps shoud be called as the global block-map:

**Lemma 3.2**

*There exists a finite rational map $g : \widehat{M_i} \longrightarrow \bar{M}_i$ induced by a Galois-extension of the function field $K(\bar{M}_i)$ so that for the fibre-product $\bar{V}_i \times_{\bar{M}_i} \widehat{M_i} =: \widehat{V_i}$ we have the following diagram of rational maps*

**(3.1)**

$$\begin{array}{ccccccc} P \times \widehat{M_i} & \xrightarrow{\bar{\Psi}} & \widehat{V_i} & \xrightarrow{g} & \bar{V}_i & \longrightarrow & Hilb^l(X) \\ & \searrow & \downarrow \hat{e} & & \downarrow \bar{e} & & \\ & & \widehat{M_i} & \xrightarrow{g} & \bar{M}_i & & , \end{array}$$

*where $P$ is the projective space identified by $P(H^0(E(m_k)))$, and $\bar{\Psi}$ is a birational map.*

**Proof**

We consider the fibration $\bar{V}_i \longrightarrow \bar{M}_i$ we may find a subvariety $S \subset \bar{V}_i$ of the dimmension $= \dim \bar{M}_i$ and meeting generic fibre of $\bar{V}_i \longrightarrow \bar{M}_i$ in $n$ points.

By the exactly same argument " Branched covering trick" in [BPV], page 43, theorem (18.3) for the $P^1$-bundle case we have a finite rational map $g : \widehat{M_i} \longrightarrow \bar{M}_i$ coming from a Galois-extension of the function field $K(\bar{M}_i)$

$$\begin{array}{ccc}\widehat{V}_i & \xrightarrow{g} & \bar{V}_i \\ \hat{e}\downarrow & & \downarrow\bar{e} \\ \widehat{M}_i & \xrightarrow{g} & \bar{M}_i\end{array}$$

so that the pull back of $S$ in the fibre-product space $\bar{V}_i \times_{\widehat{M}_i} \bar{M}_i =: \widehat{V}_i$ splits into $n$ subvarieties $g^*(S) = S_1 + ... + S_n$, each $S_i$ meets generic fibre of $\widehat{V}_i \to \widehat{M}_i$ in one point.

The fibre of $\widehat{V}_i \to \widehat{M}_i$ over $\widehat{[E]}$ is the fibre of $\bar{V}_i \to \bar{M}_i$ over $[E]$ via the map $g : \widehat{M}_i \to \bar{M}_i$, which is the image $\varphi(PH^0(E(m_k))^0)$, $E \in [E]$ of the block-map $\varphi : PH^0(E(m_k))^0 \to \bar{V}_i$.

On the other hand, let $V_i := e^{-1}(\bar{M}_i \cap (M(L,k) \setminus \cup_{i=1}^3 M_i^s))$, then any bundle $E_z(m_k)$ from the extensions $(2.1,m_k)$

$$0 \to \mathcal{O} \to E_z(m_k) \to I_z \otimes det(m_k) \to 0, \quad z \in V_i$$

has the global sections space of the constant dimension $\chi(E(m_k))$. Therefore the direct image $p_*(\mathcal{E})$ of the universal bundle $\mathcal{E} \to X \times V_i$ under the projection $p : X \times V_i \to V_i$ is a rank $h^0(E_z(m_k))$ vector bundle $W_i$ on $V_i$.

Because $S_1$ and $\widehat{M}_i$ are birational, the pull back vector bundle $g^*(W_i)$ on on $S_1$ induces a vector bundle $\widehat{W}_i$ on a Zariski open set $\widehat{M}_i^0 \subset \widehat{M}_i$, the fibre of $\widehat{W}_i$ over $\widehat{[E]}$ is identified with the fibre of $W_i$ over $E$ via the map $g\hat{e}^{-1} : \widehat{M}_i^0 \to \bar{V}_i$, which is just $H^0(E(m_k))$.

Let $\widehat{W}_i^0$ denote the Zariski open set containing all sections with isolated zero locus. We define the global block-map

$$\begin{array}{ccc}P(\widehat{W}_i^0) & \xrightarrow{\Psi} & \widehat{V}_i \\ & \searrow \quad \swarrow & \\ & \widehat{M}_i^0 & \end{array}$$

as following

$$(\widehat{[E]}, s) \mapsto (\widehat{[E]}, z),$$

where $z$ is the zero locus of the section $s$ of the bundle $E(m_k)$. The map $\Psi$ is well defined, and similar as the local block map $\Psi$ is an isomorphism.

Because the projective space bundle $P(\widehat{W}_i)$ is birational to the trivial projective space bundle $P \times \widehat{M}_i^0$ and $\widehat{M}_i \setminus \widehat{M}_i^0$ is a proper subvariety of $\widehat{M}_i$, we get the diagram (3.1). Lemma 3.2 is done.

Let $\omega$ be a non-zero regular 2-form on $X$, then $\omega$ induces a regular 2-forme $\varphi$ on $Hilb^l(X)$ in a canonical way. If $\omega$ is everywhere non zero, this is exactly the cases that $X$ is either a $K$ 3 surface or an abelian surface, then Beauville [Be] proved $\varphi$ is everywhere non-degenerate on $Hilb^l(X)$, i.e. the skew-symmetric form $\varphi_p$ on the tangent space $T_p(Hilb^l(X))$ defined by $\varphi$ is everywhere non-degenerate. In general we have the following statement:

**Lemma 3.3**

*Let $X$ be an algebraic surface, $\omega$ be a regular 2-form on $X$ with the zero locus $(\omega)_0$, and $X_0 := X \setminus (\omega)_0$. Then the regular 2-form $\varphi$ induced by $\omega$ is everywhere non-degenerate on $Hilb^l(X_0)$.*

The following proof is same as in the compact case, namely, if $X$ is a $K$ 3 surface or an abelian surface ( see [B] ).

**Proof**

Let $X^l_*$ denote the set of $l$-tuples $(x_1, ..., x_l)$ with at most two $x_i$s equal, the $l$- th symmetric group $\Sigma_l$ operates naturally on $X^l_*$.

We have the canonical resolution of the singularities of $Sym^l(X)_* := X^l_*/\Sigma_l$,

$$\begin{array}{ccc} & & X^l_* \\ & & \downarrow{\scriptstyle \sigma} \\ Hilb^l(X)_* & \xrightarrow{\pi} & Sym^l(X)_*. \end{array}$$

The map $\pi$ is easy to understand, the fibre over each $2z_1 + \sum_{i=3}^l z_i$ is just identified with the exceptional divisor $E_{z_1}$ of the blowing-up $\widehat{X} \to X$ at the point $z_1$,

and $\pi$ is the blowing-up of $D \cap Sym^l(X)_*$ in $Sym^l(X)_*$, where $\Delta = \sigma^{-1}(D)$ is the diagonal of $X^l$.

Noting $\Delta \cap X^l_*$ is smooth of codimension 2 in $X^l_*$, if

$$B_\Delta(X^l_*) \xrightarrow{\eta} X^l_*$$

denotes the blowing-up of $X^l_*$ along $\Delta$, then we have the following diagram

$$\begin{array}{ccccc} B_\Delta(X^l_*) & & \xrightarrow{\eta} & & X^l_* \\ & B_\Delta(X^l_{0*}) & \longrightarrow & X^l_{0*} & \\ \rho\downarrow & \downarrow & & \downarrow & \downarrow\sigma \\ & Hlib^l(X_0)_* & \longrightarrow & Sym^l(X_0)_* & \\ Hilb^l(X)_* & & \xrightarrow{\pi} & & Sym^l(X)_*, \end{array}$$

where $\rho$ is a Galois-covering with $\Sigma_l$ ramified simply along the exceptional divisor $E'$ of $\eta$.

From $\omega$ we deduce a 2-form $\phi$ on $X^l$ by $\phi := \sum_{i=1}^{l} p_i^*(\omega)$, which is everywhere non degenerate on $X_0^l$, where $p_i : X^l \to X$ is the $i$-th projection.

The pull back $\eta^*(\phi)$ is invariant under the $\Sigma_l$-action, thus descents a holomorphic 2-form $\varphi_*$ on $Hilb^l(X)_*$, with $\rho^*(\varphi_*) = \eta^*(\phi)$.

Let $\phi^l$ and $\varphi_*^l$ be the $l$-times wedge product of $\phi$ and $\varphi$.

From the generalized Riemann-Hurwitz formular of canonical divisor for branch covering, and the formular of canonical divisor for blowing-up along a sub-manifold of codimension 2 (see [GH2], page 608), we have:

$$\begin{aligned}\rho^*(div(\varphi_*^l)) =& div(\rho^*(\varphi^l)) - E' \\ =& div(\eta^*(\phi^l)) - E' \\ =& \eta^*(div(\phi^l)) + E' - E' \\ =& \eta^*(div(\phi^l)).\end{aligned}$$

Because $\phi$ is a symplectic structure on $X_0^l$, thus

$$\rho^*(div(\varphi_*^l|_{Hilb^l(X_0)_*}) = \eta^*(div(\phi^l|_{X_{0*}}) = 0,$$

this shows that $\varphi_*|_{Hilb^l(X_0)_*}$ is also a symplectic structure on $Hilb^l(X_0)_*$.

Because the subvariety $Hilbl(X) \setminus Hilb^l(X)_*$ has codimension $\geq 2$, by the Hartoges theorem $\varphi_*$ extends to a holomorphic 2-form $\varphi$ on $Hilb^l(X)$.

We say $div(\varphi^l|_{Hilb^l(X_0)}) = 0$. Otherewise, it would be true

$$div(\varphi_*^l|_{Hilb^l(X_0)_*}) = div(\varphi^l|_{Hilb^l(X_0)_*}) > 0,$$

because the subvariety $Hilb^l(X_0) \setminus Hilb^l(X_0)_*$ has codimension $\geq 2$, it can not contain wholly the divsor $div(\varphi^l|_{Hilb^l(X_0)})$. But we know already, $div(\varphi_*^l|_{Hilb^l(X_0)_*}) = 0$, this is a contradiction. Lemme 3.3 is proved.

**Remark**

Mumford considered the restriction of $\varphi$ to the little smaller Zariski open set $Hilb^l(X_0)_r$ of all reduced 0-dimensional subschemes. ([Mu])

Now theorem 1 is a direct consequence of theorem 2, lemma 3.2 and 3.3.

**Proof of theorem 1**

We consider the morphisms

$$\begin{array}{ccc} \bar{V}_i & \longrightarrow & Hilb^l(X) \\ {\scriptstyle \bar{e}}\downarrow & & \\ \bar{M}_i & & \end{array} ,$$

where $\bar{M}_i$ is an irreducible component of $\bar{M}(L,k)$ and $\bar{e}$ is a surjective rational map, the generic fibre $\bar{e}^{-1}([E])$ is identified with $PH^0(E(m_k))$ of the dimension $\chi(E(m_k)) - 1$ via the block-map. Therefor we get

$$\begin{aligned} \dim \bar{V}_i =& \dim \bar{M}_i + \dim \bar{e}^{-1} \\ & \geq 4k + \text{constant} + \chi(E(m_k)) - 1 \\ & \geq (4 + m_0^2 H^2)k + a\sqrt{k} + b \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\dim Hilb^l(X) &= l \\ &= c_2(E(m_k)) \\ &\leq (1 + m_0^2 H^2)k + a'\sqrt{k} + b' \quad , \end{aligned}$$

where $a$, $b$, $a'$ and $b'$ are some constants depending only on the Chern-classes of $X$, $H$, $L$ and the number $m_0$ in theorem 2. We see easily that if $k$ is sufficently large, then

$$\dim \bar{V}_i > \frac{1}{2}\dim Hilb^l(X) \quad .$$

On the other hand, let $\omega$ be a non-zero regular 2-form on $X$, form lemma 3.3 $\omega$ induces a regular 2-form $\varphi$ on $Hilb^l(X)$, which defines a non-degenerate skew-symmetric form $\varphi_p$ on the tangent space $T_p(Hilb^l(X))$, $\forall p \in Hilb^l(X \setminus (\omega)_0)$. In particular, the maximal isotropic subspaces of $\varphi_p$ have the dimension $l$.

We show that the intersection $\bar{V}_i \cap Hilb^l(X \setminus (\omega)_0) =: \bar{V}_{i0}$ is a non empty open set by the following simple argument:

Let $[E]$ be a generic element $\bar{M}_i \cap (M(L,k) \setminus \cup_{i=1}^3 M_i^s)$, then $E(m_k)$ is generated by its global sections. By using the Bertinis theorem again to the map

$$X \longrightarrow G(2, H^0(E(m_k))^\vee)$$

we may find a section $s \in H^0(E(m_k))$ with isolated zero locus $z$ such that $z \cap (\omega)_0 = \emptyset$. It just means that $z \in \bar{V}_i \cap Hilb^l(X \setminus (\omega)_0)$

Let $i$ be the the inclusion map of the smooth part $\bar{V}_{0ir}$ of $\bar{V}_{0i}$ in $Hilb^l(X_0)$, then the restriction $i^*(\varphi)$ is not zero, since $dim\, \bar{V}_{0ir} > \frac{1}{2}Hilb^l(X_0)$.

By taking resolutions of singularities of the varieties $\bar{M}_i$, $\bar{V}_i$, $\widehat{M}_i$, and $\widehat{V}_i$ in the diagram (3.1) we get

$$\begin{array}{ccccccc} P \times \widehat{M}_i' & \xrightarrow{\bar{\Psi}} & \widehat{V}_i' & \xrightarrow{g} & \tilde{V}_i & \xrightarrow{i} & Hilb^l(X) \\ & \searrow & \downarrow \hat{e} & & \downarrow \bar{e} & & \\ & & \widehat{M}_i' & \xrightarrow{g} & \tilde{M}_i. & & \end{array}$$

Thus we obtain a non-zero 2-form $i^*(\varphi)$ on $\tilde{V}_i$, hence a non-trivial 2-form $\bar{\Psi}^* g^* i^*(\varphi)$ on $P \times \widehat{M}_i'$. The isomorphisms $H^0(\widehat{M}_i', \Omega^2) \longrightarrow H^0(P \times \widehat{M}_i', \Omega^2)$ and $\Psi^* : H^0(P \times \widehat{M}_i', \Omega^2) \to H^0(\widehat{V}_i', \Omega^2)$ follow that $\bar{\Psi}^* g^* i^*(\varphi)$ hence $g^* i^*(\varphi)$ is pull back of a 2-form on $\widehat{M}_i'$.

Because $g^* i^*(\varphi)$ is $Gal(\widehat{V}_i'/\tilde{V}_i)$−invariant, therefore the above 2-form on $\widehat{M}_i'$ is $Gal(\widehat{M}_i'/\tilde{M}_i)$−invariant, hence descends a non-trivial 2-form on $\tilde{M}_i$. Theorem 1 is proved.

## Reference

[Be] A. Beauville: Some remarks on Kaehler manifolds with $c_1 = 0$. Classification of algebraic and analytic manifolds. Progress in Mathematics Vol. 39.

[BFS] M. Beltrametti, P. Francia and A. J. Sommese: On Reiders method and higher order embeddings. MPI preprint.

[Bo] F. A. Bogomolov: Holomorphic tensor and vector bundles on projective varieties. Math. USSR Izvestija 13 (1979), No. 3 499-555.

[BPV] W. Barth, C. Peters and A. Van de Ven: Compact complex surfaces. Springer-Verlag Berlin Heidelberg New York Tokyo.

[D1] S. K. Donaldson: Irrationality and h-cobordism conjecture. J. Diff. Geo. 26 (1987), 141-168.

[D2] S. K. Donalsdon: Polynomial invariants for smooth four-manifolds. preprint.

[E] G. Ellingsrud: Sur l'irréducibilité du module des fibrés stables sur $P^2$. Math. Z. 182 (1983), 189-192.

[F] R. Friedman: Rank two vector bundles over regular elliptic surfaces. preprint.

[Gi1] D. Gieseker: On the moduli of vector bundles on an algebraic surface. Ann. of Math. 106 (1977), 45-60.

[Gi2] D. Gieseker: Global moduli for surfaces of general type. Inv. Math. 43 (1977), 233-283.

[Gi3] D.Gieseker: A construction of stable vector bundles on an algebraic surface. J. Diff. Geo. 27 (1988), 137-154.

[Gö] L. Göttsche: Die Betti-Zahlen des Hilbert-Schemas fuer Unterschemata der Laenge $n$ einer glatten Flaeche. Diplomarbeit. Bonn 1988.

[GH1] P. Griffiths and J. Harris: Residues and zero-cycles on algebraic varieties. Ann. of Math. 108 (1978), 461-505.

[GH2] P. Griffithis and J. Harris: Principles of algebraic geometry. A Wiley-Interscience publication John Wiley Sons New York. Chichester. Brisbane. Toronto.

[H] R. Hartshorne: Algebraic geometry. Graduate texts in Mathematics Spring-Verlag.

[I] A. Iarrobino: Punctual Hilbert schemes. Bull. Amer. Math. Soc. 78 (1972), 819-823.

[K] D. Kotschick: On manifolds homeomorphic to $CP^2 \sharp 8C\bar{P}^2$. To appear in Inv. Math..

[Mar] M. Maruyama: Stable vector bundles on an algebraic surface. Nagoya Math. J. 58 (1975), 25-68.

[Muk] S. Mukai: Symplectic structure of the moduli space of sheaves on an abelian or K-3 surface. Inv. Math. 77 (1984), 101-116.

[Mum] D. Mumford: Rational equivalence of 0-cycles on surfaces. J. Math. Kyoto Univ. 9-2 (1969), 195-204.

[NR] M. S. Narasimhan and S. Ramanan: Moduli of vector bundles on a compact Riemann surface. Ann. of Math. 89 (1969), 14-51.

[OV] C. Okonek and A. Van de Ven: Stable bundles and differentiable structures on certain ellptic surfaces. Inv. Math. 86 (1986), 357-370.

[Reid] M. Reid: Bogomolovs theorem $c_1^2 \leq 4c_2$. Intl. Symp. Algebraic geometry. Kyoto, 1977, 623-642.

[Reider] I. Reider: Vector bundles of rank 2 and linear systems on algebraic surfaces. Ann. of Math. 127 (1988), 309-316.

[Ta] C. Taubes: Self-dual connections on 4-manifolds with indefinite intersection matrix. J. Diff Geo. 19(1984), 517-560.

[Ty] A. N. Tyarin: Cycles, curves and vector bundles on an algebraic surface. Duke Math. J. 54 (1987), 1-26.