Constrained Lagrangian submanifolds over singular constraining varieties and discriminant varieties

by

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Abstract. The notion of constrained lagrangian submanifold over regular constraining variety was introduced implicitely by Dirac [9] in his theory of generalized Hamiltonian dynamics. Following Dirac, many authors [4], [17], [26] consider constrained lagrangian submanifolds as the models for physical systems in classical mechanics and field theory. Quite elementary examples from: variational calculus with bypassing of obstacle [2]; geometrical approach to the thermodynamical phase transitions [16]; Kashiwara, Kawai, Pham theory of holonomic systems [22], show that the constrained lagrangian subsets over singular constraining varieties play an important role in various theories of mathematical physics. The aim of this paper is to give a precise approach to constrained lagrangian varieties and indicate their fundamental geometrical properties. We show that our notion of constrained lagrangian variety, restricted to the regular strata of constraint, reduces to the standard co-normal bundle notion. The homogeneous lagrangian varieties as the constrained lagrangian varieties over discriminant varieties were investigated and classified. Some immediate consequences of this classification for physical understanding of classical systems were established, especially for equilibrium of composite systems. The notion of Morse family on manifold with boundary

was introduced and classification theorem for normal forms of regular geometric interactions between holonomic components was proved. We propose also the geometrical framework for the recognition problem in the theory of constrained lagrangian varieties some advantages of which can be directly applied.

1. Introduction.

Let (M,ω) be a symplectic manifold. Let $K \subseteq M$ be a submanifold and let $H\colon K \longrightarrow \mathbb{R}$ be a differentiable function. The set

$$N = \{ w \in TM;\ \tau_M(w) \in K,\ \langle w \wedge u, \omega \rangle = - \langle u, dH \rangle \text{ for each } u \in TK \text{ such that } \tau_M(u) = \tau_M(w) \},$$

which is called a generalized Hamiltonian system in the symplectic manifold (M,ω) and was introduced by Dirac [9], is an example of a constrained lagrangian submanifold in the symplectic space $(TM,\tilde{\omega})$ - the tangent bundle with the canonical symplectic structure $\tilde{\omega} = \beta^*\omega_M$, where β is the morphism of fibre bundles; $\beta\colon TM \longrightarrow T^*M$, given by $\beta(u) = i_u\omega$ and ω_M is the standard symplectic form of the cotangent bundle T^*M. The constrained lagrangian submanifolds (c.l.s. for short) in some cotangent bundle, say (T^*Q,ω_Q), with a constraint K which is a submanifold of Q, were studied comprehensively in [25]. Many mechanical systems having c.l.s. as a constitutive set were given in [26].

Let us give now an introductory example, namely: wave front evolution as a partial motivation for investigations of c.l.s. with more general constraints, possibly exhibiting singularities.

Let Q be a configuration space (n-dimensional smooth manifold) for some optical system (cf. [14]). Let V_0 be a 1-codim.

normally oriented submanifold of Q. We shall consider a c.l.s. $L_{V_0} \subseteq T^*Q$ (see (2.1)) as an initial wave front (usually the submanifold V_0 together with a choice of a positively oriented co-normal element $\xi(x) \in \mathbb{P}T^*_xQ$ at every point of V_0 is taken as an initial wave front [14]). The evolution of the wave front is determined by a one-parameter family of symplectic relations (a symplectic relation is a certain lagrangian submanifold of the product of two symplectic manifolds, see also [4])

$$R_t \subseteq T^*M \times T^*M,$$

such that the wave front at time t is given as an image of the initial wave front

$$L_{V_t} = R_t(L_{V_0}).$$

Let us recall that the image of the subset $F \subset P_1$ with respect to the symplectic relation $R \subseteq (P_1 \times P_2, \pi_2^*\omega_2 - \pi_1^*\omega_1)$, where π_i: $P_1 \times P_2 \longrightarrow P_i$ are the respective canonical projections, is the set $R(F) = \{ p_2 \in P_2;$ there exists $p_1 \in F$ such that $(p_1, p_2) \in R \}$. Infinitesemally R_t can be given by a homogeneous Hamilton function H on $T^*Q - 0$ (since the positive reals operate on $T^*Q - 0$ by multiplication in the fibres we can write $H(\lambda\xi) = \lambda H(\xi)$ for all $\lambda > 0, \xi \in T^*Q - 0$), so R_t is defined by the flow

$$\mathbb{R}_+ \times (T^*Q - 0) \longrightarrow (T^*Q - 0)$$

obtained by integrating the corresponding Hamiltonian field X_H. We see that, for such flows, the mapping $\pi_Q \circ R_t : L_{V_0} \longrightarrow Q$ does not depend on $v \in L_{V_0} \cap T^*_xQ$, so we get a map $V_0 \ni x \longrightarrow \pi_Q \circ R_t|_{V_0}(x)$, the so-called ray map at time t (see [14]), which maps V_0 onto V_t. We know that usually at some times t_1 the ray map will have rank $< \dim Q - 1$ and in these points V_{t_1} has singularities (see Fig. below) and $R_{t_1}(L_{V_0})$ is a lagrangian submanifold defined over

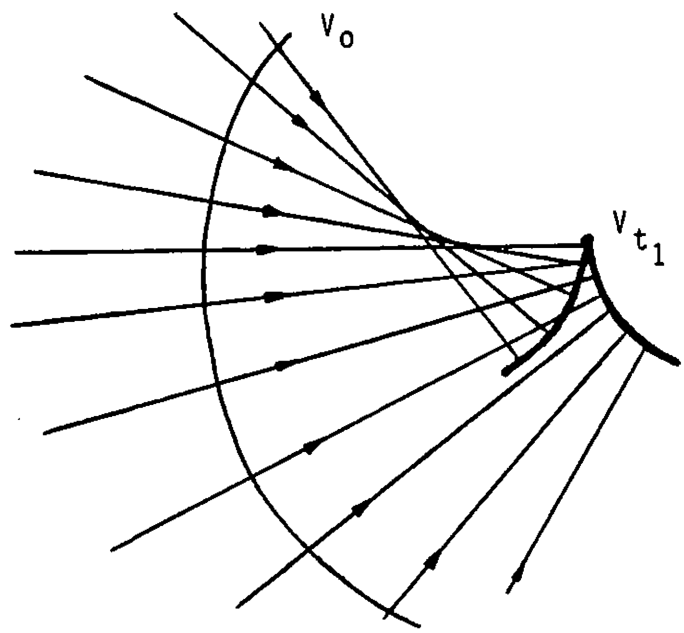

Fig. 1.

a singular constraint V_{t_1} (see §2). If we consider, by extensin the germ of L_{V_t} at the singular point in the zero section of T^*Q then this germ itself is singular. The purpose of this paper is to make precise the notion of c.l.s. over singular constraints and to study their geometrical properties in some applications.

One of the motivations for our investigations comes from the thermodynamics of phase transitions where the space of coexistence states (coexistence of phases) turns out to be a c.l.s. over a singular constraint which represents a possibly very complicated phase diagram (cf. [16], [17]).

The next important theory providing examples of singular lagrangian varieties (and c.l.s.) is the theory of linear differential systems (see [22], [18], [19]). A linear differential system is a left coherent D_X-module, say M, where D_X is the sheaf of differential operators of finite order with holomorphic coefficients on a smooth complex analytic manifold (X, ω_X) .

Remember that the characteristic variety of a differential operator $P = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha/\partial x^\alpha$ (a section of D_X in local coordinates) of order m is the hypersurface V(P) of the cotangent bundle T^*X defined by the principal symbol $\sigma(P) = \sum_{|\alpha|=m} a_\alpha(x)\zeta^\alpha$, which is a homogeneous function in coordinates $\zeta = (\zeta_1,\ldots,\zeta_n)$. For the module of type D_X/I (where I is a left ideal of finite type in D_X) the characteristic variety V of the system D_X/I is defined by the principal symbols $\sigma(P_1),\ldots,\sigma(P_p)$ of the generators $P_1,\ldots,P_p$ of I. The definition of the characteristic variety of a general differential system M can be found in [22]. It appears that the characteristic variety of a system M is an involutive subspace of T^*X (cf. [21]). For maximally overdetermined systems (called holonomic systems) dimV = dimX and V is a homogeneous lagrangian subset of T^*X. Singularities of characteristic varieties for holonomic systems have a special meaning as corresponding to the correct generalization of integrable connections (cf. [21], [22]). One kind of singular system, for which the characteristic variety V is a so-called regular analytic interaction, was considered in [20], [19]. As a main example of such systems one can take the following system

$$M : \begin{cases} (x_0 D_{x_0} - \alpha)u = 0 \\ D_{x_i} u = 0, \quad i = 1,\ldots,n. \end{cases}$$

In this paper we give the classification of normal forms of characteristic (lagrangian) varieties for such systems.

In Section 2 we introduce the notion of constrained lagrangian submanifold over singular constraint and describe the geometrical properties of such objects.

In Section 3 we show how to characterize the germs of homogeneous lagrangian varieties by the special blowing-up mappings and so-called prehomogeneous lagrangian submanifolds. The local structure of such varities is investigated. The special case of such varieties in generalized Hamiltonian systems is considered and the corresponding normal forms are indicated.

The work in Section 4 is the direct generalization of the notion of homogeneous lagrangian variety by means of the methods of composite systems, introduced in geometrical foundations of classical physics. While the facts obtained in this section may be of some interest in their own right it seems to us that the geometrical methods used to formulate them have independent physical interest. Here, in terms of constrained lagrangian varieties, we give the new formulation of the Gibbs phase rule and indicate the geometrical structure of the spaces of coexistence states. As an additional example of c.l.s. we give this one which appears in open swallowtail construction by symplectic triads.

In Section 5 we prove the classification theorem for generic pairs of the so-called regular geometric interactions and by the generalization of the standard notion of Morse family we write their polynomial normal forms. In Section 6 the recognition problem for germs of constrained lagrangian varieties is formulated and some basic results are established.

2. Lagrangian varieties over singular constraints.

Let Q be a smooth manifold and $L \subseteq T^*Q$ be a lagrangian submanifold of its cotangent bundle (for the basic definitions see [1]). If $\pi_Q(L) \subset K \subseteq Q$, where K is a submanifold of Q and π_Q is the cotangent bundle projection, then L is called a constrained lagrangian submanifold (c.l.s. for short) of T^*Q (cf. [17], [25]). In this paper we generalize the notion of c.l.s. by allowing K to have singularities. At first we generalize the notion of lagrangian submanifold itself by passing to the purely local objects.

Definition 2.1. Let N be (the germ of) a subset of T^*Q endowed with a stratification into smooth submanifolds, say $N = \bigcup_{i \in I} N_i$. N is called a lagrangian subset of T^*Q if every stratum N_i is an isotropic submanifold of (T^*Q, ω_Q) and $\dim N_i = \dim Q$ for the non-empty maximal strata of N.

Let N be a semialgebraic subset of T^*Q (see [11]). Then N is a lagrangian subset of (T^*Q, ω_Q) if and only if the maximal strata of some Whitney stratification of N are lagrangian (for the necessary basics of real algebraic geometry see e.g. [11], [6], [27]).

As we know ([25], Proposition 3.1), any c.l.s. L over a nonsingular constraint $K \subseteq Q$ can be described, using a smooth function F on K, in the following way

$$(2.1) \qquad L_{K,F} = \{ p \in T^*Q;\ \pi_Q(p) \in K \text{ and } \langle u,p \rangle = \langle u, dF \rangle \text{ for each } u \in TK \subset TQ \text{ such that } \tau_Q(u) = \pi_Q(p) \}.$$

Now we generalize this notion by taking more general constraints K.

Proposition 2.2. Let K be a semialgebraic subset of Q and $F\colon Q \longrightarrow \mathbb{R}$ a smooth function. Let $\{K_i^n\}_{i \in I}$ be maximal strata of some Whitney stratification of K. The set

$$\tilde{L}_{K,F} = \{ p \in T^*Q;\ \text{(i) } \pi_Q(p) \in K_i^n \text{ and } \langle u,p \rangle = \langle u, dF \rangle \text{ for each } u \in TK_i^n \subset TQ \text{ such that } \tau_Q(u) = \pi_Q(p) \text{ or (ii) } \pi_Q(p) = y \in Y \subset K - \bigcup_{i \in I} K_i^n,\ p \in \{dF(y)\} + V_y \},$$

where $V_y = \sum_i V_{y,i}$, $V_{y,i} = \left\{ p \in T_yQ;\ <u,p> = 0 \text{ for each } u \in \lim_{K_i^n \ni q \to y} T_qK_i^n \subset T_yQ, \text{ where } y \in \overline{K_i^n} \right\}$ is a lagrangian subset of (T^*Q,ω_Q).

Proof. The canonical strata of $\tilde{L}_{K,F}$, $L_{K_i^n,F}$, defined as in (2.1) are lagrangian (cf. [25] , Proposition 3.1). It is easy to check that the stratum $\tilde{L}'_{Y,F} = \{ p \in T^*Q;\ y = \tilde{\pi}_Q(p) \in Y,\ p \in \{dF(y)\} + V_y\} \subset \tilde{L}_{K,F}$ is contained in $L_{Y,F}$, which is lagrangian. Here the submanifold Y is a stratum of $K - \bigcup_{i \in I} K_i^n$. So the stratum $\tilde{L}'_{Y,F}$, as a submanifold of $L_{Y,F}$, is isotropic in (T^*Q,ω_Q).

<u>Remark 2.3.</u> (i) The function F appearing in (2.2) can be taken, in the more general situation, to be smooth only on the individual strata of K. This is the case for the singular homogeneous lagrangian sets introduced in the next sections.

(ii) We easily see that, in a neighbourhood of any point of K, $\tilde{L}_{K,F}$ can be described in the following form

$$p_i = \frac{\partial F}{\partial q_i}(q) + \sum_{j=1}^{k} \lambda_j e_{ij}(q) \ , \quad i = 1,\dots,\dim Q,\ \lambda_j \in \mathbb{R}, \qquad (2.3)$$
$$q \in K,$$

for some smooth functions e_{ij}.

Moreover, if $q \in K - \operatorname{sing} K$ we can take,

$$e_{ij}(q) = \frac{\partial}{\partial q_i} g_j(q)$$

in a neighbourhood of $\tilde{q}$, where $\{g_j\}$ are defining functions for the germ $(K,\tilde{q})$.

<u>Example 2.4.</u> Let $Q = \mathbb{R}^2$ and let K be defined by one of the equation

a) $g(x,y) = x^2 - y^3 = 0$, or

b) $g(x,y) = x^2 - y^2 = 0$.

In both cases 0 is an isolated singular point of K, but the dimensions of the respective singular fibres $L'_{\{0\},F} \subset T_0^*\mathbb{R}^2$ for these two cases, are different (no matter what F is), namely

a) $K_1^1 = \{x^2-y^3=0,\ x<0\}$, $K_2^1 = \{x^2-y^3=0,\ x>0\}$ and $V_{\{0\},1} = V_{\{0\},2}$, thus $\tilde{L}_{K,F}$ is described by the equations

$$p_x = \frac{\partial F}{\partial x}(x,y) + 2\lambda x,$$

$$p_y = \frac{\partial F}{\partial y}(x,y) - 3\lambda y^2,\ \lambda \in \mathbb{R},$$

if $(x,y) \in K_1^1 \cup K_2^1$, or

$$p_x = \frac{\partial F}{\partial x}(x,y) + \lambda\ ,$$

$$p_y = \frac{\partial F}{\partial y}(x,y)\ ,\quad \lambda \in \mathbb{R}$$

if $(x,y) = 0$ (cf. Fig. 1.).

b) In this case we have $V_{\{0\}} = \{(1,1),\ (-1,1)\}$, so we can write

$$\mathbb{R}^2 \cong \tilde{L}'_{\{0\},F} : \begin{cases} p_x = \frac{\partial F}{\partial x}(x,y) + \lambda - \mu \\ p_y = \frac{\partial F}{\partial y}(x,y) + \lambda + \mu;\ \lambda, \mu \in \mathbb{R},\ (x,y) = 0 \end{cases}$$

and for $(x,y) \in K$, $(x,y) \neq 0$ we have the standard representation formula (2.3) (see e.g. Fig. 2.)

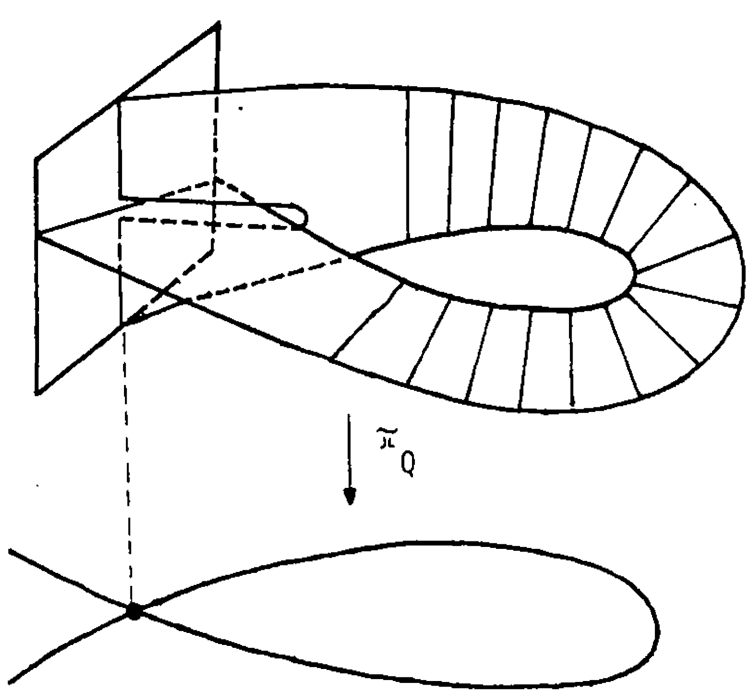

Fig. 2.

It is also very easy to see that the initial data for the wave front evolution (as in the Introduction) in a neighbourhood of a singular point of a wave front form a singular constrained lagrangian subset as introduced in Proposition 2.2.

3. Germs of homogeneous lagrangian varieties.

Let $\tilde{X}$, X be open subsets of $\mathbb{R}^{p+1}$ containing zero. We consider the following map

$$\chi : \tilde{X} \longrightarrow X,\ \chi_0(\tilde{x})=\tilde{x}_0,\ \chi_i(\tilde{x}) = \tilde{x}_0\tilde{x}_i,\quad i = 1,\dots,p$$

(the $\tilde{x}_i$ can be interpreted as densities, or one can look at χ as a chart in a blowing-up construction).

Definition 3.1. A germ of a lagrangian submanifold $(\tilde{L},(\tilde{x}_0',0)) \subseteq T^*\tilde{X}$ generated by a smooth function-germ $\tilde{F}(\tilde{x}) = \tilde{x}_0 f(\tilde{x}_1,\dots,\tilde{x}_p)$ is called a regular, prehomogeneous lagrangian submanifold.

Proposition 3.2. Let $\tilde{x}_0' \neq 0$. Then $(T^*\chi(\tilde{L}),(0;\tilde{x}_0',0))$ is the germ of the smooth homogeneous lagrangian submanifold given by the following equations

$$(3.1)\begin{cases} y_0 = f(\tilde{x}_1,\dots,\tilde{x}_p) - \sum_1^p \tilde{x}_i y_i,\ y_j = \dfrac{\partial f}{\partial x_j}(\tilde{x}_1,\dots,\tilde{x}_p),\ j=1,\dots,p \\ x_i = x_0\tilde{x}_i,\ i=1,\dots,p. \end{cases}$$

If $\tilde{x}_0' = 0$ then $(T^*\chi(\tilde{L}),0)$ is the germ of the singular lagrangian subset described by the following equations

$$(3.2)\begin{cases} y_0 = f(\tilde{x}_1,\dots,\tilde{x}_p) - \sum_1^p \tilde{x}_i y_i,\ \tilde{x}_0\left(y_j - \dfrac{\partial f}{\partial \tilde{x}_j}(\tilde{x}_1,\dots,\tilde{x}_p)\right)=0,\ j=1,\dots,p \\ x_0 = \tilde{x}_0,\ x_i = \tilde{x}_0\tilde{x}_i,\ i = 1,\dots,p. \end{cases}$$

Proof. We see that $T^*\chi$ can be written in the following form

$$T^*\chi: \quad \begin{aligned} \tilde{y}_0 &= y_0 + \sum_1^p \tilde{x}_i y_i \\ \tilde{y}_j &= \tilde{x}_0 y_j,\ j = 1,\dots,p, \\ x_0 &= \tilde{x}_0,\ x_i = \tilde{x}_0\tilde{x}_i,\ i = 1,\dots,p. \end{aligned}$$

Now taking $\tilde{L} = \left\{ (\tilde{y},\tilde{x}) \in T^*\tilde{X};\ \tilde{y}_0 = f(\tilde{x}_1,\ldots,\tilde{x}_p),\ \tilde{y}_i = \tilde{x}_0 \frac{\partial f}{\partial \tilde{x}_i}(\tilde{x}_1,\ldots,\tilde{x}_p),\ i=1,\ldots,p \right\}$ and substituting into the equations for $T^*\mathcal{X}$ we obtain immediately the equations (3.1) and (3.2).

Remark 3.3. A germ of a homogeneous lagrangian submanifold is generated with respect to the canonical special symplectic structure of T^*X, by the germ at $(\tilde{x}_0',0) \neq 0$ of a generating function of the form

$$G(x_0,x_1,\ldots,x_p) = x_0 f(\frac{x_1}{x_0},\ldots,\frac{x_p}{x_0}).$$

The singular germ $(T^*\mathcal{X}(\tilde{L}),\ 0)$ has no generating function with respect to T^*X. However with respect to the special symplectic structure α: $T^*X \longrightarrow T^*Y$; $\alpha:(y,x) \longrightarrow (x,y)$ its generating family can be written in the following form

$$(3.3) \qquad \mathcal{F}(y_0,\ldots,y_p;\lambda_0,\lambda_1,\ldots,\lambda_p) = -\lambda_0(f(\lambda_1,\ldots,\lambda_p) - y_0 - \sum_1^p \lambda_i y_i).$$

Everywere, except zero, the germ of $\mathcal{F}$ is the germ of a Morse family (see [28]).

Corollary 3.4. Let (L,p) be the germ of a homogeneous lagrangian submanifold (h.l.s. for short) in T^*X. There exists a special symplectic structure α' on T^*X, which is equivalent to α (see Remark 3.3) and such that the generating family for the h.l.s. germ $(\alpha'(L),\alpha'(p))$ has the form (3.3) (equivalence of special symplectic structures means composition with symplectomorphisms preserving the fibre structure $T^*Y \longrightarrow Y$). Thus with respect to some special symplectic structure $\tilde{\alpha}$: $T^*X \longrightarrow T^*X$, which respects the canonical action of the positive reals on the fibres of T^*X, the h.l.s. (L,p) can be written in the form:

$$(\tilde{\alpha}(L),\tilde{\alpha}(p)) = (T^*\mathcal{X}(\tilde{L}),\tilde{\alpha}(p)),$$

for some pre-h.l.s. $\tilde{L} \subseteq (T^*\tilde{X},\omega_{\tilde{X}})$.

We see that to (the germ of) any h.l.s. in T^*X we can associate the following map germ (cf. [22])

$$F: (\mathbb{R}^p \times \mathbb{R}^p, 0) \longrightarrow Y \tag{3.4}$$

$$F(\tilde{x}, y) = (f(\tilde{x}_1, \dots, \tilde{x}_p) - \sum_1^p \tilde{x}_i y_i,\ y_1, \dots, y_p).$$

Let us denote by C_F the set of critical points of F and by $\Delta_F \subset \mathbb{R}^{p+1}$ the set of critical values of F (the discriminant of F) Then we have immediately

<u>Proposition 3.5.</u> Any germ of a h.l.s. in T^*X, has the structure of a germ of a co-normal bundle (def. see e.g. [22])

$$(T^*_{\Delta_F}Y,\ p')$$

with respect to an appropriate special symplectic structure $\alpha: T^*X \longrightarrow T^*Y$ on T^*X.

<u>Remark 3.6.</u> In the classical thermodynamics of phase transitions the singular germ $(T^*X(\tilde{L}), 0)$ has an important meaning (the point when the new coexisting phase appears [16]). It is easily seen that this germ has two components

$$A = (T^*_{\Delta_F}Y,\ 0) = (\{(x,y);\ x_i = x_0 \tilde{x}_i,\ y_0 = f(\tilde{x}_1, \dots, \tilde{x}_p) - \sum_1^p \tilde{x}_i y_i,$$
$$y_i = \frac{\partial f}{\partial \tilde{x}_i}(\tilde{x}_1, \dots, \tilde{x}_p) \text{ for } (\tilde{x}_1, \dots, \tilde{x}_p) \in \mathbb{R}^p\},\ 0)$$

$$B = (\{(x,y);\ x_0 = 0,\ x_i = 0,\ i = 1, \dots, p,\ y_0 = f(\tilde{x}_1, \dots, \tilde{x}_p) - \sum_1^p \tilde{x}_i y_i$$
$$\text{for } (\tilde{x}_1, \dots, \tilde{x}_p) \in \mathbb{R}^p\}, 0) = (\{(x,y); y \in \operatorname{Image} F,\ x = 0\}, 0),$$

which are lagrangian and intersect along Δ_F.

It turns out that the homogeneous generalized Hamiltonian systems (introduced in §1) provide the examples of simple Darboux normal forms. In what follows we will engage in the local analysis of such systems.

Let (S, ω) be a symplectic manifold. Let $D \subset (T^*S, \omega_S)$ be a h.l.s. (also singular in the previous sense). The corresponding generalized Hamiltonian system $D' \subset (TS,\omega)$ is defined (cf. [9], [25]) by the morphism of fibre spaces β: $TS \rightarrow T^*S$, i.e. $\omega = \beta^*\omega_S$ and $D' = \beta^{-1}(D)$. We also D will call the generalized Hamiltonian system.

Proposition 3.7. Normal forms for the generic germs of generalized Hamiltonian systems defined over a smooth hypersurface, the cusp variety and the swallowtail variety are generated by the following generating families

Hypersurface

$(\tilde{A}_1)$ $$G(\lambda_0,q,p) = \lambda_0 p_1,$$

cusp

$(\tilde{A}_2)$ $$G(\lambda_0,\lambda,q,p) = \lambda_0(\lambda^3+\lambda p_1+q_1)$$

swallowtail

$(\tilde{A}_3)$ $$G(\lambda_0,\lambda,q,p) = \lambda_0(\lambda^4+\lambda^2 p_1+\lambda q_2+p_2),$$

where (S,ω) is endowed with the Darboux form $\sum_{i=1}^{n} dp_i \wedge dq_i = \omega$

Proof. Let D be a germ of homogeneous generalized Hamiltonian system. For the generic D by diffeomorphic change of variables in S we can reduce the corresponding mapgerm (3.4) to one of the standard normal forms (see [11] , [29]). For the first three stable Whitney maps, by the standard method of reduction of parameters (so-called stable equivalence [29]), we obtain the three normal forms for the corresponding generating families as in Proposition 3.7. However by such procedure the symplectic form ω is not longer in Darboux form. So we have to ask for the normal form of ω with respect to the group of diffeomorphisms of S preserving the respective constraining varieties (hypersurface, generalized cusp and generalized swallowtail [3]). Here we can apply the following known result of Arnold ([3], Theorem 1 and Theorem 2, in the smooth case proved by Melrose): if Δ_i, $\Delta_1 = \{(q,p) \in S;\ p_1 = 0\}$, $\Delta_2 = \{(q,p) \in S;\ p_1^3 - q_1^2 = 0\}$, $\Delta_3 = \{(q,p) \in S;\ x^4 + p_1 x^2 + q_2 x + p_2$ has a root of order $\geq 2\}$, is one of the constraining hypersurfaces mentioned in the proposition , then generically the symplectic form ω can be reduced to the Darboux normal form by a diffeomorphism preserving the respective constraining variety Δ_i. So we obtain mutually the symplectic structure ω and generalized Hamiltonian system D in the desired normal form.

<u>Remark 3.8.</u> $D' \subset (TS, \sum_{i=1}^{n} (\dot{p}_i dq_i - \dot{q}_i dp_i))$ is a Hamiltonian dynamical system. Let us assume that n=2 then the one-parameter families of integral curves of D' form the lagrangian varieties appearing in the variational problem of bypassing of obstacle. In the $(\tilde{A}_3)$ - case these varieties are called open swallowtails [3],[2] (see Fig. 3.)

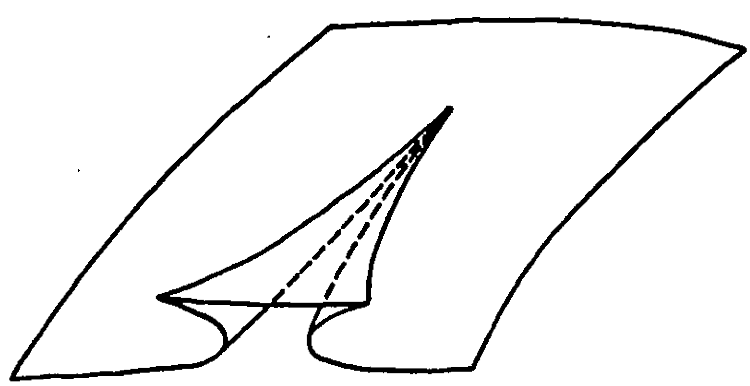

Fig. 3.

<u>4. Composite homogeneous systems.</u>

A simple generalization of the preceding notions, useful in describing the space of coexistence states in classical phase transitions (cf. [16], [17]), can be carried out by considering the following control map,

$$(4.1)\quad \bar{\chi} : \prod_{i=1}^{k}\tilde{X}_i \longrightarrow X,\quad \bar{\chi}(\tilde{x}^1,\dots,\tilde{x}^k) = \Big(\sum_1^k \tilde{x}_o^i, \sum_1^k \tilde{x}_1^i\tilde{x}_o^i,\dots,\sum_1^k \tilde{x}_p^i\tilde{x}_o^i\Big).$$

Now the prehomogeneous composite lagrangian submanifold $\tilde{L}_k \subset \prod_{i=1}^{k} T^*\tilde{X}_i$ is generated by the function

$$G(\tilde{x}^1,\dots,\tilde{x}^k) = \sum_{i=1}^{k}\tilde{x}_o^i f(\tilde{x}^i),$$

for some smooth function $f: \tilde{X} \longrightarrow \mathbb{R}$. The corresponding homogeneous lagrangian subset is obtained as an image $T^*\bar{\chi}(\tilde{L}_k)$.

A closed, composite homogeneous system is defined to be a pair $(\tilde{L},D)$, where $\tilde{L}_k$ is a prehomogeneous composite lagrangian submanifold and D is a coisotropic submanifold of $\prod_{i=1}^{k} T^*\tilde{X}_i$ defined by an equation $\sum_{i=1}^{k}\tilde{x}_o^i$ = const. = $c \neq 0$. The motivation for this terminology comes from the geometrical formalism of classical thermodynamics (cf. [16], [25]), namely we have,

<u>Corollary 4.1.</u> The space of equilibrium states Ξ for a thermodynamical closed system has the form

$$(4.2)\qquad \pi(\tilde{l}) \subset T^*\bar{\chi}(D)/\sim \;\cong T^*Y',$$

where $\tilde{l} = T^*\bar{\chi}(\tilde{L}_k \cap D)$, $T^*\bar{\chi}(D)/\sim$ is a canonical symplectic manifold associated to the coisotropic submanifold $T^*\bar{\chi}(D) \subset T^*X$ (cf. [4]) and π is its characteristic projection.

Suppose given a germ of a homogeneous lagrangian subset in T^*X (§3) such that the corresponding map-germ F is stable (cf. [11]). We can parametrize the set of critical points of F by $(\tilde{x}_1,\dots,\tilde{x}_p) = \tilde{x}$. Thus we have a mapping

$$(4.3)\quad g(\tilde{x}) = F|_{C_F}(\tilde{x}) = \Big(f(\tilde{x}) - \sum_{i=1}^{p}\tilde{x}_i\frac{\partial f}{\partial\tilde{x}_i}(\tilde{x}), \frac{\partial f}{\partial\tilde{x}_1}(\tilde{x}),\dots,\frac{\partial f}{\partial\tilde{x}_p}(\tilde{x})\Big).$$

We define:

$$(4.4)\qquad M_r(g) = \left\{\tilde{x}\in C_F;\ \#(g^{-1}(g(\tilde{x}))) = r\right\},$$

(the r-fold points in the source of g),

(4.5) $$M = \bigcup_{r \geqslant 2} M_r(g).$$

<u>Proposition 4.2.</u> For any stable homogeneous lagrangian subset in T^*X, there exists $\nu \in \mathbb{N}$ such that for every $k \geqslant \nu$ we have

(4.6) $$T^*Y \supset L_{\Delta_F, 0} = T^*\bar{\chi}(\tilde{L}_k)$$

Proof. Let us fix $k \in \mathbb{N}$. Then for $T^*\bar{\chi}(\tilde{L}_k)$ we can write the equations

$$(\alpha) \quad \begin{cases} f(\tilde{x}^1) = y_0 + \sum_{i=1}^{p} \tilde{x}_i^1 y_i \\ \quad \cdots \\ f(\tilde{x}^k) = y_0 + \sum_{i=1}^{p} \tilde{x}_i^k y_i , \end{cases}$$

$$(\beta) \quad \begin{cases} \tilde{x}_o^1 (y_1 - \frac{\partial f}{\partial \tilde{x}_1^1}(\tilde{x}^1)) = 0, \ldots, \tilde{x}_o^1 (y_p - \frac{\partial f}{\partial \tilde{x}_p^1}(\tilde{x}^1)) = 0 \\ \quad \cdots \\ \tilde{x}_o^k (y_1 - \frac{\partial f}{\partial \tilde{x}_1^k}(\tilde{x}^k)) = 0, \ldots, \tilde{x}_o^k (y_p - \frac{\partial f}{\partial \tilde{x}_p^k}(\tilde{x}^k)) = 0 \end{cases}$$

$$(\gamma) \quad \begin{cases} x_o = \sum_{j=1}^{k} \tilde{x}_o^j , \\ x_1 = \sum_{j=1}^{k} \tilde{x}_1^j \tilde{x}_o^j \\ \quad \cdots \\ x_p = \sum_{j=1}^{k} \tilde{x}_p^j \tilde{x}_o^j . \end{cases}$$

In the case $x_o \neq 0$, the equations (α), (β) can be rewritten in the form

$$g(\tilde{x}^1) = \ldots = g(\tilde{x}^k).$$

For stable F and a sufficiently small neighbourhood of zero U

$$\max_{y \in U} \left\{ \# g^{-1}(y) \right\} = \dim_{\mathbb{R}} \mathcal{E}_p / J(f) = s < \infty \ , \quad \text{(cf. [6], [11])} ,$$

where $\mathcal{E}_p$ is the ring of germs at zero of smooth functions $\mathbb{R}^p \longrightarrow \mathbb{R}$, and J(f) is its Jacobi ideal (generated by the germs of the first-order partial derivatives of f). Thus obviously we can take $\nu = s \leqslant p+1$.

Let us define

$$\Gamma = g(M), \quad \Gamma_i = g(M_i(g)), \quad (i \geqslant 2).$$

Then it is obvious, on the basis of equations (γ) that if $y \in \Gamma_i$ then $T^*_y Y \cap T^* \bar{\chi}(\tilde{L}_k)$ is an i-dimensional vector subspace of $T^*_y Y$. It is easily seen that for the smooth stratum Γ_i we have

$$\bigcup_{y \in \Gamma_i} T^*_y Y \cap T^* \bar{\chi}(\tilde{L}_k) = L_{\Gamma_i, 0}, \quad (k \geqslant \nu)$$

and this completes the proof.

The co-normal bundle over a semialgebraic set considered in §3 (see [21], [22]) is not a constrained lagrangian subset in our sense. These two notions coincide only on the nonsingular strata of the constraint. The aim of this paper is to provide a direct motivation for the use of the symplectic geometrical notions even though they exhibit singularities.

Let us consider a closed system (e.g. in physics, a system with a fixed number of particles or moles [16]). The corresponding equilibrium state space is defined in Corollary 4.1. On the basis of this corollary and Proposition 4.2 we have

<u>Corollary 4.3.</u> (i) $\pi(\tilde{l}) = L_1 \cup L_2$, where L_1, L_2 are lagrangian submanifolds in T^*Y' defined as follows

$$L_1 = \left\{ (x,y) \in T^*Y'; \; y_i = \frac{\partial f}{\partial \tilde{x}_i}(\tilde{x}), \; x_i = c\tilde{x}_i, \; i = 1,\dots,p, \; \tilde{x}_i \in \mathbb{R} \right\},$$

$$L_2 = L_{\varkappa(\Gamma), G}.$$

Here $\varkappa$ is the projection $\varkappa : (y_0, y_1, \dots, y_p) \longrightarrow (y_1, \dots, y_p)$ and the function G does not necessary extend to a smooth function on Y'.

(ii) If Γ is a smooth submanifold of Y then L_1 and L_2 intersect regularly (cf. [20]) (along the so-called binodal curve in the two dimensional case [17], see Fig. 4, in the simpler case).

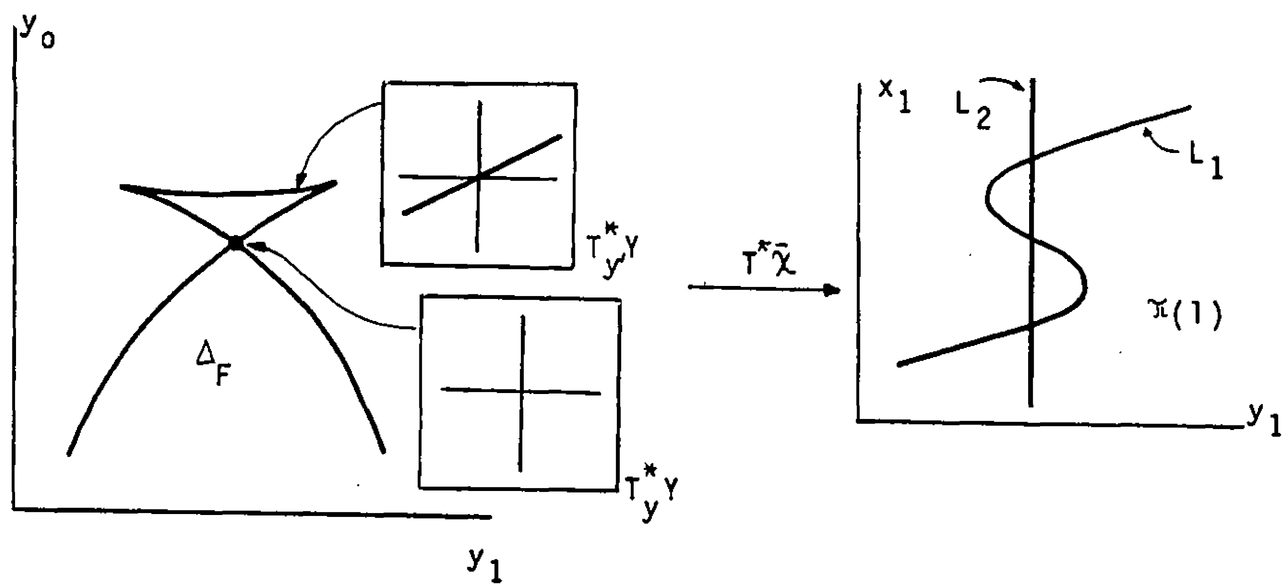

Fig. 4.

(iii) $B = \pi_Y(C_{\pi_Y}|L_1 \cup C_{\pi_Y}|L_2)$ is a full bifurcation diagram for F (see e.g. [5]).

(iv) Gibbs phase rule [24] :

The number of coexisting phases $\nu \leqslant \dim(T^*_y Y \cap L_{\Delta_F,0})$.

This is a symplectic version of the catastrophe-theoretic formulation of the Gibbs phase rule

$$\nu \leqslant \operatorname{codim}_{\mathbb{R}^{\mu-1}} T_0 + 1$$

(introduced in [24] p.663). In our terms the analog of this inequality is the following

$$\nu \leqslant \operatorname{codim} C_{\pi_Y}|L_2 + 1 .$$

To be more precise we can formulate the following

Proposition 4.4. L_2 is a constrained lagrangian subset over $K = \mathcal{H}(\Gamma)$ (the phase diagram [17]) with a generating function

$$F(y) = f(\tilde{x}) - \sum_{i=1}^{p} \tilde{x}_i y_i,$$

where $K \ni y \longrightarrow \tilde{x}(y)$ is given by an isomorphism g on the smooth connected components of M.

Proof. Immediate on the basis of the elementary properties of discriminant varieties (see e.g. [5], [24]) and Legendre transformation of generating functions (cf. [25]).

Remark 4.5. At every point of the phase diagram K for L_2 we can write

$$y_i = \frac{\partial f}{\partial \tilde{x}_i}(\tilde{x}), \quad i = 1,\dots,p$$

$$x_i = (\tilde{x}_i - \tilde{x}_i^k)\lambda_1 + \dots + (\tilde{x}_i^{k-1} - \tilde{x}_i^k)\lambda_k + c\tilde{x}_i^k,$$

where $g(\tilde{x}) = g(\tilde{x}^1) = \dots = g(\tilde{x}^k)$, $\tilde{x}^i \neq \tilde{x}^j$ when $i \neq j$, and λ_i are Morse parameters (cf. [28]). Hence taking the basis vectors

$$v = \tilde{x} - \tilde{x}^k, \quad v_i = \tilde{x}^i - \tilde{x}^k, \quad i = 1,\dots,k-1,$$

we obtain the Clapeyron-Clausius formula in completely general form:

$$\text{if } w \in T_y K \text{ then } w \perp \{v, v_1, \dots, v_{k-1}\},$$

which can be written also in the form appearing in handbooks

$$\sum_{j=1}^{p} (\tilde{x}_j - \tilde{x}_j^k)\delta y_j = 0$$

$$\dots$$

$$\sum_{j=1}^{p} (\tilde{x}_j^{k-1} - \tilde{x}_j^k)\delta y_j = 0, \qquad \delta y_j \in \mathbb{R}.$$

We give now the example of c.l.s. appearing in the variational problem of bypassing of obstacle in Euclidean space. In [2] (p. 45) the following notion was introduced

Definition 4.6. A symplectic triad in the symplectic manifold (P,ω) is the triplet (H,L,l) which consists of a smooth hypersurface H in P and of a lagrangian manifold $L \subseteq P$ tangent to H with first order tangency along a lagrangian manifold hypersurface l.

Let U be a domain on the hypersurface (obstacle) in $\mathbb{R}^n$. Let us consider a geodesic flow γ on U with the given initial front. We consider the distance along the geodesics of U, to the initial front as a function $s:U \longrightarrow \mathbb{R}$, such that $(\nabla s)^2 = 1$ on U. Let us consider a smooth extension of s, say $\bar{s}: \mathbb{R}^n \longrightarrow \mathbb{R}$. Thus we can define:

$$L_{U,\bar{s}} = \left\{ (x,p) \in T^*\mathbb{R}^n; \quad x \in U, \langle v,p \rangle = \langle v, d\bar{s} \rangle \text{ for each } v \in T_x U \subset T_x\mathbb{R}^n \right\}$$

and hypersurface $\tilde{H}$:

$$\tilde{H} = \left\{ (x,p) \in T^*\mathbb{R}^n; \quad |p|^2 = 1 \right\} .$$

Proposition 4.7. The triplet $(\tilde{H}, L_{U,\bar{s}}, \tilde{H} \cap L_{U,\bar{s}})$ is the symplectic triad. It generates the variety of rays tangent to the geodesics of our geodesic flow γ on U.

Proof. We see that $L_{U,\bar{s}}$ (cf. (2.1)) forms the set of all extensions of the 1-forms ds from U to the whole ambient space. Thus the hypersurface $l = \tilde{H} \cap L_{U,\bar{s}} \subset L_{U,\bar{s}}$ consists all extensions of ds which are vanishing on the fibres of normal bundle to U (because of $(\nabla s)^2 = 1$) (see also [2], p.45). The first order tangency of $L_{U,\bar{s}}$ to $\tilde{H}$ is easily seen.

The new class of singular lagrangian sets, so-called open swallowtails (cf. [3]) is provided by this kind of symplectic triads, namely the lagrangian variety generated by the triad is the image of l, say $\pi(l)$, in the canonical symplectic manifold of characteristics of $\tilde{H}$.

Hierarchy of the generic singularities provided by the symplectic triads is determined by the mutual positions of the flow γ and the line of asymptotic points on the obstacle surface U. Let us fix n=3. If the source point, say $x_0 \in U$, of the germ of geodesic flow (γ, x_0) is outside of the line of asymptotic points on U then $\pi(1)$ has no singularities. If (γ, x_0) is transversal to the line of asymptotic points at x_0 then the corresponding germ of $\pi(1)$ has the cusp structure (described conveniently in the appropriate space of polynomials [2]) i.e. is the product of the usual cusp singularity and Euclidean space. If γ is not transversal in x_0 to the mentioned line of asymptotic points (which happens generically in isolated points of this line) then $\pi(1)$ has a structure of open swallowtail (introduced in [3]), as in Fig.3.

5. On regular geometric interactions between holonomic components.

In this section we consider all of the previously introduced objects in the complex analytic category. We will study another example of a singular lagrangian subset which appeared in the microlocal analysis of differential systems (cf. [18], [22]).

On the basis of §3 we see that every homogeneous lagrangian subset of T^*Y can be described locally by the following generating family (see Remark 3.3)

(5.1) $$\mathcal{F}(y_0,\dots,y_p,\lambda_0,\lambda_1,\dots,\lambda_m) = \lambda_0 G(\lambda, y), \quad \lambda = (\lambda_1,\dots,\lambda_m).$$

Thus we can use the formalism of generating families (see [28], [29]) to classify normal forms for the regularly intersecting homogeneous lagrangian (holonomic [21], [22], [19]) components.

We say the the pair (L_1,L_2) of h.l.s. is a regular geometric interaction (intersection [20]) if $L_1 \cap L_2$ is a submanifold of L_2 of codimension 1 and for every point $x \in L_1 \cap L_2$ we have

$$T_x(L_1 \cap L_2) = T_x L_1 \cap T_x L_2.$$

Let $G\colon (\Lambda \times Y, 0) \to \mathbb{C}$ be an anylytic function-germ. Let $\Lambda_0 \subset \Lambda$ be a hypersurface of Λ, $0 \in \Lambda_0$. We can choose an appropriate coordinate system on Λ such that

$$\Lambda_0 = \left\{ (\lambda_1, \ldots, \lambda_m) \in \Lambda;\ \lambda_1 = 0 \right\}.$$

As we know from the standard theory of generating families for lagrangian submanifolds (see e.g. [28]), the minimal number m of parameters for which all singularities of the described lagrangian submanifold can be generated in this way is greater than or equal to p (see [28]). However if we also allow an arbitrary hypersurface $\Lambda_0 \subset \Lambda$ as an eventual parameter space, we have to increase the minimal dimension of Λ to p+1.

Definition 5.1. A function-germ $G\colon(\Lambda \times Y, 0) \to \mathbb{C}$ is called a Morse family on the manifold Λ with boundary Λ_0 if in appropriate coordinates on Λ and Y we have

$$\frac{\partial G}{\partial y_0}(0) \neq 0, \quad \frac{\partial G}{\partial \lambda_i}(0) = 0, \quad \frac{\partial G}{\partial y_j}(0) = 0, \quad 1 \leqslant j \leqslant p, \quad 1 \leqslant i \leqslant p+1,$$

and

$$\operatorname{rank}\left(\frac{\partial^2 G}{\partial \lambda \partial \bar{\lambda}}, \frac{\partial^2 G}{\partial \lambda \partial \bar{y}}\right)(0) = p+1,$$

where $\Lambda_0 = \left\{ \lambda_1 = 0 \right\}$, $\bar{\lambda} = (\lambda_2, \ldots, \lambda_{p+1})$, $\bar{y} = (y_1, \ldots, y_p)$.

Proposition 5.2. Let $G\colon (\Lambda \times Y, 0) \to \mathbb{C}$ be a Morse family on a manifold with boundary. Then the pair (L_1,L_2) generated by $\mathcal{F}_1(y,\lambda_0,\lambda) = \lambda_0 G(\lambda,y)$ and $\mathcal{F}_2(y,\lambda_0,\bar{\lambda}) = \lambda_0 G(0,\bar{\lambda},y)$ respectively is a regular geometric interaction.

Proof. Taking into account the condition $\frac{\partial G}{\partial y_0}(0) \neq 0$, we can directly build up the corresponding analytic, nondegenerate (see [21]) map-germ on manifold $\mathbb{C}^{p+1}\times\mathbb{C}^p$ with the boundary $\{\lambda_1=0\}$.

$$F: (\mathbb{C}^{p+1}\times\mathbb{C}^p,0) \longrightarrow (\mathbb{C}^{p+1},0)$$

$$F(\lambda_1,\bar{\lambda},\bar{y}) = (f(\lambda_1,\bar{\lambda},\bar{y}),\bar{y}).$$

The corresponding characteristic variety (see [20] , [21]) of this mapping forms a regular geometric interaction. The corresponding components of this variety are generated in the standard way by the families $\mathcal{F}_1$, $\mathcal{F}_2$ mentioned in the proposition.

By [20] , [21] we have a direct correspondence between generating families on manifolds with boundary and the corresponding mappings associated to the holonomic components of an interaction. Hence after straightforward calculations we obtain immediately

Proposition 5.3. For a germ of a regular geometric interaction (L_1,L_2) in T^*Y there exists a Morse family $\mathcal{F}$ on a manifold with boundary generating the pair (L_1,L_2).

All properties of regularly interacting pairs can be formulated in the language of Morse families on manifolds with boundary, which is especially convenient in the classification of their local normal forms. Following [20] , [29] we introduce the notion of equivalence of Morse families,

Definition 5.4. Let $\mathcal{F}_1(y,\lambda_0,\lambda)=\lambda_0 G_1(\lambda,y)$, $\mathcal{F}_2(y,\lambda_0,\lambda)=\lambda_0 G_2(\lambda,y)$ be Morse families for the two geometric interactions. We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent iff there exists an isomorphism $(\lambda,y) \longrightarrow \Phi(\lambda,y)=(\varphi(\lambda,y),\psi(y))\in\mathbb{C}^{p+1}\times\mathbb{C}^p$, preserving $\Lambda_0\times\mathbb{C}^{p+1}$ and such that

$$G_1(\lambda,y) = G_2(\Phi(\lambda,y)) + \alpha(y),$$

where $\alpha\in\langle y_0^2,y_1,\dots,y_p\rangle\mathcal{O}_{p+1}$, ($\mathcal{O}_{p+1}$ is the ring of holomorphic function-germs on $\mathbb{C}^{p+1}$ and $\langle y_0^2,y_1,\dots,y_p\rangle\mathcal{O}_{p+1}$ is the ideal generated by $y_0^2, y_1, \ldots, y_p$).

Proposition 5.5. Let $n \leqslant 5$. For a generic set of regularly interacting pairs (V_1, V_2) of lagrangian submanifolds of $T^*\mathbb{C}^n$ we can reduce the corresponding generating family $\mathcal{F}$ in a neighbourhood of any point of $V_1 \cap V_2$, using the equivalence and a defined above standard reduction of parameters, to one of the following normal forms:

n=2

$$\mathcal{F}(y_0,y_1,\lambda_0,\lambda_1)=\lambda_0(\lambda_1^2+ y_1\lambda_1+ y_0)$$

n=3, additionally

$$\mathcal{F}(y_0,y_1,y_2,\lambda_0,\lambda_1)=\lambda_0(\lambda_1^3+\lambda_1^2y_1+\lambda_1y_2+y_0)$$

$$\mathcal{F}(y_0,y_1,y_2,\lambda_0,\lambda_1,\lambda_2)=\lambda_0(\lambda_2\lambda_1+\lambda_2^3+\lambda_2y_1+\lambda_2^2y_2+y_0)$$

n=4, additionally

$$\mathcal{F}(y_0,y_1,y_2,y_3,\lambda_0,\lambda_1)=\lambda_0(\lambda_1^4+y_1\lambda_1^3+y_2\lambda_1^2+y_3\lambda_1+y_0)$$

$$\mathcal{F}(y_0,y_1,y_2,y_3,\lambda_0,\lambda_1,\lambda_2)=\lambda_0(\lambda_2\lambda_1+\lambda_2^4+y_1\lambda_2^3+y_2\lambda_2^2+y_3\lambda_2+y_0)$$

$$\mathcal{F}(y_0,y_1,y_2,y_3,\lambda_0,\lambda_1,\lambda_2)=\lambda_0(\lambda_2^3+\lambda_1^2+y_1\lambda_2+y_2\lambda_1+y_3\lambda_1\lambda_2+y_0)$$

n=5, additionally

$$\mathcal{F}(y_0,\ldots,y_4,\lambda_0,\lambda_1)=\lambda_0(\lambda_1^5+y_1\lambda_1^4+y_2\lambda_1^3+y_3\lambda_1^2+y_4\lambda_1+y_0)$$

$$\mathcal{F}(y_0,\ldots,y_4,\lambda_0,\lambda_1,\lambda_2)=\lambda_0(\lambda_2\lambda_1+\lambda_2^5+y_1\lambda_2^4+y_2\lambda_2^3+y_3\lambda_2^2+y_4\lambda_2+y_0)$$

$$\mathcal{F}(y_0,\ldots,y_4,\lambda_0,\lambda_1,\lambda_2)=\lambda_0(\lambda_2^3+\varepsilon\lambda_2\lambda_1^2+\varphi_1(y)\lambda_1^3+y_1\lambda_2+\lambda_1y_2+y_3\lambda_1^2+ +y_4\lambda_2\lambda_1+y_0),\quad \varepsilon^3=\varepsilon,\quad 4\varepsilon+27\varphi_1^2(0)\neq 0,$$

$$\mathcal{F}(y_0,\ldots,y_4,\lambda_0,\lambda_1,\lambda_2)=\lambda_0(\lambda_2^4+\varphi_2(y)\lambda_2^2\lambda_1+\lambda_1^2+\lambda_1y_1+\lambda_2y_2+\lambda_2^2y_3+ +\lambda_1\lambda_2y_4+y_0),\quad \varphi_2^2(0)\neq 4,$$

$$\mathcal{F}(y_0,\ldots,y_4,\lambda_0,\lambda_1,\lambda_2,\lambda_3)=\lambda_0(\lambda_2^3+\varphi_3(y)\lambda_2\lambda_3^2+\varepsilon\lambda_3^3+\lambda_3\lambda_1+y_1\lambda_2+y_2\lambda_3+ +y_3\lambda_3^2+y_4\lambda_2\lambda_3+y_0),\quad \varepsilon^3=\varepsilon,\quad 27\varepsilon^2+4\varphi_3^3(0)\neq 0.$$

Proof. (an outline). Following [15], [23] we can carry out the classification of analytic function-germs on manifolds with boundary and can immediately obtain their universal unfoldings. Hence we obtain the classification of the corresponding normal forms for bundle codimension $\leqslant 4$ (if c is a codimension of the germ and m its modality then bundle codimension $b=c-m$ [23]), namely

germ	c	b	unfolding parameters	conditions
λ_1^2	1	1	λ_1	
λ_1^3	2	2	λ_1, λ_1^2	
λ_1^4	3	3	$\lambda_1, \lambda_1^2, \lambda_1^3$	
λ_1^5	4	4	$\lambda_1, \lambda_1^2, \lambda_1^3, \lambda_1^4$	
$\lambda_1\lambda_2 + \lambda_2^3$	2	2	λ_2, λ_2^2	
$\lambda_1\lambda_2 + \lambda_2^4$	3	3	$\lambda_2, \lambda_2^2, \lambda_2^3$	
$\lambda_1\lambda_2 + \lambda_2^5$	4	4	$\lambda_2, \lambda_2^2, \lambda_2^3, \lambda_2^4$	
$\lambda_2^3 + \lambda_1^2$	3	3	$\lambda_2, \lambda_1, \lambda_1\lambda_2$	
$\lambda_2^3 + \varepsilon\lambda_2\lambda_1^2 + a_1\lambda_1^3$	5	4	$\lambda_2, \lambda_1, \lambda_1^2, \lambda_1\lambda_2$	$\varepsilon^3 = \varepsilon,\ 4\varepsilon + 27a_1^2 \neq 0$
$\lambda_2^4 + a_2\lambda_2^2\lambda_1 + \lambda_1^2$	5	4	$\lambda_2, \lambda_1, \lambda_2^2, \lambda_1\lambda_2$	$a_2^2 \neq 4$
$\lambda_2^3 + a_3\lambda_2\lambda_3^2 + \varepsilon\lambda_3^3 + \lambda_3\lambda_1$	5	4	$\lambda_2, \lambda_3, \lambda_3^2, \lambda_2\lambda_3$	$\varepsilon^3 = \varepsilon, 27\varepsilon^2 + 4a_3^3 \neq 0$

Applying analogous arguments as for the classification of ordinary lagrangian mappings (see [29] Theorems 5, 6, 7.), after straightforward calculations we obtain the classification list of Proposition 5.5.

Remark 5.6. The next important notion in the investigation of Gauss-Manin systems corresponding to regular geometric interactions (cf. [20], [21]) is the notion of an universal unfolding of such a system or equivalently the notion of unfoldings of regular geometric interactions. First we have to introduce unfoldings of lagrangian submanifolds. Let $L \subset (P,\omega)$ be a lagrangian submanifold; an unfolding of L is a triplet $((\tilde{P},\tilde{\omega}),\tilde{P}_Z,\tilde{L})$, where $(\tilde{P},\tilde{\omega})$ is a symplectic manifold, $\tilde{P}_Z$ is a fibre bundle with base space Z whose fibres are coisotropic submanifolds of $(\tilde{P},\tilde{\omega})$ and $\tilde{L} \subset (\tilde{P},\tilde{\omega})$ is a lagrangian submanifold such that

$$\pi_{z_0} : \tilde{P}_{z_0} \longrightarrow \tilde{P}_{z_0}/\sim \;\cong P, \quad \pi_{z_0}(\tilde{L}\cap\tilde{P}_{z_0}) = L$$

for an initial point $z_0 \in Z$ of the unfolding. Extending this notion to regularly intersecting pairs (L_1,L_2), $L_i \subset T^*Y$, and using our approach by Morse families on manifolds with boundary we can give the classification of more degenerate intersecting pairs. By specializing the above notion we obtain exactly the notion of unfoldings of interacting holonomic components introduced in [20]. The classification of normal forms for the corresponding universal unfoldings can be carried out by adapting Wassermann's results [27] . We shall leave this for a forthcoming paper.

6. The local classification of constrained lagrangian varieties, the recognition problem.

In this section we consider only the c.l.s. in (T^*X,ω_X) which are determined by pairs (G,f), where $G:(X,0) \longrightarrow (\mathbb{R}^m,0)$ is a map-germ representing the constraint $K = G^{-1}(0)$ and $f:(X,0) \longrightarrow \mathbb{R}$ is a function-germ generating the corresponding c.l.s.

Let H be the group of germs of diffeomorphisms $h:(X,0) \longrightarrow (X,0)$, S the group of invertible $(m+1)\times(m+1)$ matrices $M = (M_{ij})$

over $\mathcal{E}(X)$ (i.e. with entries which are smooth function germs on X) such that $M_{i,m+1} = 0$ $(i=1,\dots,m)$, $M_{m+1,m+1} = 1$.
We define the group

(6.1) $$\mathcal{G} = H \times S$$

and the action

$$\mu : \mathcal{G} \times B \longrightarrow B, \quad (h,M)\cdot F = M\cdot(F\circ h),$$

where $B = m_n\mathcal{E}_n^m \times m_n^2$ (m_n is the maximal ideal of the local ring $\mathcal{E}_n$) and we think here of $F \in B$ as a column vector $\binom{G}{f}$.

<u>Definition 6.1.</u> Let $F_1=(G_1,f_1)$, $F_2=(G_2,f_2)$ represent the two c.l.s. say L_{F_1}, L_{F_2}. We say that L_{F_1}, L_{F_2} are equivalent iff F_1 and F_2 are in the same $\mathcal{G}$-orbit of the action μ, i.e. for some $(h,M)\in\mathcal{G}$, $(h,M)\cdot F_1 = F_2$.

We know that the above equivalence implies the symplectic equivalence of L_{F_1} and L_{F_2} by symplectic lifting of the diffeomorphism $h:X \to X$. In analogy to the complex case [8] and using the standard constructions in [11] we can introduce the tangent space at the map $F = (G,f)$ to its orbit

$$TF = m_n J(F) + \langle G \rangle^{m+1},$$

where $J(F)$ is the $\mathcal{E}_n$-submodule of $m_n\mathcal{E}_n^{m+1}$ generated by $\partial F/\partial x_i$ $(i=1,\dots,n)$. So we can define

$$\operatorname{codim} F = \operatorname{cod} L_F = \dim_{\mathbb{R}} B/TF$$

if this is finite.

Let F have finite codimension, and let $\tilde{F}: X \times \mathbb{R}^k \longrightarrow \mathbb{R}^{m+1}$ be a k-parameter unfolding of F (for the definitions see [11], [27]), then for the universal and stable unfoldings in this case we have the standard result.

<u>Proposition 6.2.</u> Let $\tilde{F}$ be a k-parameter unfolding of F. $\tilde{F}$ is a universal unfolding of F if and only if TG, $\frac{\partial\tilde{F}}{\partial u_1}\Big|_{u=0}, \dots, \frac{\partial\tilde{F}}{\partial u_k}\Big|_{u=0}$

together span $\mathcal{E}_n^{m+1}$. A universal unfolding is locally stable and the minimal number of its parameters is the codimension of F.

Proof. The universality of such unfoldings is immediate on the basis of standard results in [11], [12]. The local stability of F follows from [10] (Proposition 2.2). We must only consider the G(n,k) - stability problem with G the group of matrices defined in (6.1).

Example 6.3. Let us take $F = (G(x,y), f(x,y)) = (xy, x^2 + y^2)$. The tangent space to the orbit of F is following

$$TF = \langle (xy, 2x^2), (y^2, 2xy), (x^2, 2xy), (xy, 2y^2), (xy, 0), (0, xy) \rangle \mathcal{E}_{xy} .$$

We can easily see that TF is exactely equal to $\mathfrak{m}_{xy}^2 \times \mathfrak{m}_{xy}^2 = B$, in fact it is enough to show that the corresponding map defined by the following equality

$$(ax^2+bxy+cy^2, dx^2+exy+fy^2) = (\alpha xy+\beta y^2+\gamma x^2+\delta xy+\varrho xy, \alpha 2x^2+\beta 2xy+\gamma 2yx+ +\delta 2y^2+\sigma xy),$$

namely

$$(\alpha, \beta, \gamma, \delta, \varrho, \sigma) \longrightarrow (a,b,c,d,e,f) = (\gamma, \alpha+\delta+\varrho, \beta, 2\alpha, 2\beta+2\gamma+\sigma, 2\delta)$$

has a maximal rank. But it is easy to check that it is so. Hence for this germ of c.l.s. we have

$$\operatorname{codL}_F = 0.$$

We can also show that the considered germ is simple according to our equivalence relation (cf. [8]).

A more complete approach to the recognition problem for c.l.s. can be done by using the notion of generating families defined over constraints. Thus, in the general case, a germ of c.l.s. is defined by a pair $E = (G,f)$, where $G: (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^m, 0)$ is as above and $f: X \times \Lambda \longrightarrow \mathbb{R}$ is the corresponding generating family over a constraint $K = G^{-1}(0) \subset X$. This generating-family approach is not so direct and close to the standard one (cf. [28]) as it was before for generating functions only.

In analogy to Definition 6.1, we define the corresponding equivalence notion for generating families. We define the group

$$\tilde{\mathcal{G}} = \tilde{H} \times \tilde{S}$$

and the action

$$\tilde{\mu}: \tilde{\mathcal{G}} \times \tilde{B} \longrightarrow \tilde{B}, \ (\tilde{h},\tilde{M})\cdot E = \tilde{M}\cdot(E\circ\tilde{h}),$$

where $\tilde{B} = m_n \mathcal{E}_n^m \times m_{n+k}^2$, $\tilde{H}$ is the group of germs of diffeomorphisms $\tilde{h}:(\mathbb{R}^{n+k},0) \to (\mathbb{R}^{n+k},0)$, $\pi_{\mathbb{R}^n}\circ\tilde{h} = \varphi\circ\pi_{\mathbb{R}^n}$ $(\pi_{\mathbb{R}^n}:\mathbb{R}^{n+k} \to \mathbb{R}^n)$, for some diffeomorphism $\varphi:(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ and $\tilde{S}$ is the group of invertible $(m+1)\times(m+1)$ matrices

$$\begin{pmatrix} A(x), & 0 \\ w(x,\lambda), & 1 \end{pmatrix}, \ A(x) \text{ a smooth m m matrix,}$$

with smooth entries. We think here of $E \in \tilde{B}$ as an element of $\pi_n^*(m_n\mathcal{E}_n^m) \times m_{n+k}^2$.

<u>Definition 6.4.</u> Let L_{E_1}, L_{E_2} be c.l.s. determined by E_1 and E_2 respectively. We say that L_{E_1}, L_{E_2} are equivalent iff E_1, E_2 are in the same $\tilde{\mathcal{G}}$-orbit of the $\tilde{\mu}$-action, i.e. for some $(\tilde{h},\tilde{M}) \in \tilde{\mathcal{G}}$,

$$(\tilde{h},\tilde{M})\cdot E_1 = E_2$$

<u>Remark 6.5.</u> On a smooth stratum of K, L_E is defined by the following standard generating family

$$\mathcal{F}(x,\lambda,\mu) = f(x,\lambda) + \sum_{i=1}^{m} \mu_i G_i(x).$$

This provides some justification for the notion introduced above of a generating family over a constraint.

In analogy to the standard tangent space of a map-germ (cf. [11]) we define the following tangent space for a map-germ $E = (G,f) \in \tilde{B}$.

$$TE = T_1E + T_2E, \quad \text{where}$$

$$T_1E = \sum_{i=1}^{m} \langle G_1,\dots,G_m\rangle \mathcal{E}_n e_i + \mathfrak{m}_n\left\{\frac{\partial E}{\partial x_1},\dots,\frac{\partial E}{\partial x_n}\right\},$$

$$T_2E = \left(\langle G_1,\dots,G_m\rangle \mathcal{E}_{n+k} + \mathfrak{m}_{n+k}\left\langle \frac{\partial f}{\partial \lambda_1},\dots,\frac{\partial f}{\partial \lambda_k}\right\rangle\right)e_{m+1},$$

$$e_i = (0\dots\underset{\hat{i}}{1}\dots0).$$

This space has a standard interpretation in terms of orbits for the equivalence relation defined above.

We say that E (as well as L_E) have finite codimension if $\dim_{\mathbb{R}} \tilde{B}/TE$ is finite. If this is so we define $\mathrm{cod}E = \dim_{\mathbb{R}} \tilde{B}/TE$. One can check (cf. [7]) that E has finite codimension if and only if

$$\dim_{\mathbb{R}} \mathcal{E}_{n+k}/\left(\langle G_1,\dots,G_m\rangle \mathcal{E}_{n+k} + \mathfrak{m}_{n+k}\left\langle\frac{\partial f}{\partial\lambda_1},\dots,\frac{\partial f}{\partial\lambda_k}\right\rangle\right) < \infty$$

and

$$\dim_{\mathbb{R}} \mathcal{E}_n^m/\left(\langle G_1,\dots,G_m\rangle^m \mathcal{E}_n + \mathfrak{m}_n\left\{\frac{\partial G}{\partial x_1},\dots,\frac{\partial G}{\partial x_n}\right\}\right) < \infty \quad .$$

Using standard procedures one can write down the classification of normal forms for generating families over constraints (cf. [13]). The complete classification in the case of small codimensions will be left to a forthcoming paper.

Our group of equivalences is a subgroup of the group of contact equivalences (cf. [11], [7]). It is easy to check the following

<u>Proposition 6.6.</u> The group $\tilde{\mathcal{G}}$ is a geometric subgroup (according to Damon [7]) of the contact group.

Hence we can apply the methods of [7] and construct the corresponding unfolding theory. Let $u \in \mathbb{R}^r$ and set $\tilde{B}_r = \mathfrak{m}_{n+r}\mathcal{E}_{n+r}^m \times \mathfrak{m}_{n+k+r}^2$. We say that $\tilde{E} \in \tilde{B}_r$ is an unfolding of $E \in \tilde{B}$ if $\tilde{E}|_{u=0} = E$. We write $E_u(\lambda,x) = \tilde{E}(\lambda,x,u) = (G(x,u), f(\lambda,x,u))$.

<u>Proposition 6.7.</u> Let $\tilde{E} \in \tilde{B}_r$ be an r-parameter unfolding of $E \in B$ and let E have finite codimension. Then $\tilde{E}$ is versal if and only if

$$\sum_{i=1}^{m} \langle G_1,\ldots,G_m\rangle \xi_n e_i + \left\{\frac{\partial E}{\partial x_1},\ldots,\frac{\partial E}{\partial x_n}\right\}\xi_n + (\langle G_1,\ldots,G_m\rangle \xi_{n+k} + \xi_{n+k}\langle \frac{\partial f}{\partial \lambda_1},\ldots,\frac{\partial f}{\partial \lambda_k}\rangle)e_{m+1} + \mathbb{R}\left\{\frac{\partial \tilde{E}}{\partial u_i}\Big|_{u=0}\right\} = B.$$

The proof of this basic unfolding theorem follows in the traditional way from the standard theory of unfoldings (see e.g.[7], [12],[11]).

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