# ON ANALYTIC CONTINUATION OF EULER PRODUCTS

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## § 1. Introduction:

As usual, $\mathbf{N}$, $\mathbf{Z}$; $\mathbf{Q}$, $\mathbf{R}_+$, $\mathbf{R}$, $\mathbf{C}$ denote the set of natural numbers, the ring of rational integers, the field of rational numbers, the multiplicative group of positive real numbers, the real number field and the complex number field, respectively. Let $k$ be a finite extension of $\mathbf{Q}$ and let $W(k)$ denote the (absolute) Weil group of $k$, [26]. For a finite extension $K \supseteq k$, let $G(K|k)$ and $W(K|k)$ denote the Galois group of $K$ over $k$ and the relative Weil group introduced in [29]. Let $C_k$ be the idèle-class group of $K$ and let $C_K^1$ denote the subgroup of idèle-classes having unit volume. Then $C_K \cong \mathbf{R}_+ \times C_K^1$, so that

$$W(K|k) \cong \mathbf{R}_+ \times W_1(K|k) \quad ,$$

where $W_1(K|k)$ is a compact group isomorphic to a certain extension of the Galois group $G(K|k)$ by $C_K^1$. The group $W(k)$ may be defined as a projective limit of the groups $W(K|k)$, where $K$ varies over finite extensions of $k$. Let

$$\rho : W(k) \longrightarrow GL(V) \tag{1}$$

be a continuous representation of $W(k)$ into the group of invertible linear operators of a finite dimensional complex vector space $V$ . There is a finite Galois extension $K$ of $k$ such that $\rho$ factors through $W(K|k)$ ; if $\mathbf{R}_+ \subseteq \operatorname{Ker}\rho$ , we say that $\rho$ is normalised. Let $X_1$ be the set of continuous normalised representations (1) and let $Y$ be the ring of virtual characters generated by the set of characters

$$\{\chi | \chi = \operatorname{tr}\rho \, , \, \rho \in X_1\} \quad .$$

Consider a polynomial

$$\Phi(t) = 1 + \sum_{j=1}^{\ell} t^j a_j \, , \; a_j \in Y \tag{2}$$

in $Y[t]$ and let

$$\Phi_g(t) = 1 + \sum_{j=1}^{\ell} t^j a_j(g) \tag{3}$$

for $g \in W(k)$ . The polynomial (2) is said to be unitary, if $\Phi_g(\alpha) \neq 0$ as soon as $|\alpha| \neq 1$ , $\alpha \in \mathbf{C}$, $g \in W(k)$ . Any $\rho$ in $X_1$ may be regarded as a representation of a compact group $W_1(K|k)$ , therefore it is semi-simple. Hence one can write

$$a_j = \sum_{\chi} m_j(\chi)\chi \quad , \quad m_j(\chi) \in \mathbf{Z} \quad ,$$

where $\chi$ varies over irreducible characters. Moreover, the set

$$X_o(\Phi) = \{\rho | m_j(\operatorname{tr}\rho) \neq 0 \quad \text{for some} \quad j\}$$

is finite. Given a prime divisor $p$ in $k$, let $\sigma_p$ and $I_p$ denote the Frobenius class and the inertia subgroup in $W(k)$ at the place $p$. Let $\rho \in X_1$ and let, as in (1), $V$ be the representation space of $\rho$. Consider a subspace

$$V^{I_p} = \{v \mid v \in V, \quad \rho(g)v = v \quad \text{for} \quad g \in I_p\}$$

of $I_p$-invariant vectors in $V$. Since the restriction $\rho(g)\big|_{V^{I_p}}$ of the operator $\rho(g)$ to $V^{I_p}$ does not depend on the choice of $g$ in $\sigma_p$, we may set

$$\rho(\sigma_p) = \rho(g)\big|_{V^{I_p}}, \quad g \in \sigma_p, \tag{4}$$

and extend (4) by linearity to $Y$. Furthermore, let

$$\Phi_p(t) = 1 + \sum_{j=1}^{\ell} t^j a_j(\sigma_p) \quad . \tag{5}$$

By (3) - (5), if $V^{I_p} = \{0\}$ for each $\rho$ in $X_0(\Phi)$, then

$$\Phi_p(t) = \Phi_g(t) \quad \text{for any} \quad g \quad \text{in} \quad \sigma_p . \tag{6}$$

In particular, relation (6) is satisfied for all but a finite number of primes $p$ in $k$. Let $F$ be a finite extension of $\mathbf{Q}$; we write

$$|A| =: N_{F/\mathbf{Q}} A \tag{7}$$

for any fractional ideal $\mathbf{A}$ in the ring of integers of $F$ .
In these notations, let

$$L(s,\Phi) = \prod_{p} \Phi_p(|p|^{-s})^{-1} \; , \quad \mathrm{Re}\, s > 1 \; , \; s \in \mathbf{C} \; , \tag{8}$$

where the product in (8) is extended over all the prime divisors $p$ in $k$ .

Theorem 1. The function $s \longmapsto L(s,\Phi)$ , defined for $\mathrm{Re}\, s > 1$ by an absolutely convergent product (8), can be meromorphically continued to the half-plane

$$\mathbf{C}_+ = \{s \,|\, \mathrm{Re}\, s > 0\} \; .$$

If $\Phi$ is unitary, this function can be meromorphically continued to the whole complex plane $\mathbf{C}$ ; if $\Phi$ is not unitary, then the function $L(s,\Phi)$ has a natural boundary

$$\mathbf{C}^\circ = \{s \,|\, \mathrm{Re}\, s = 0\}$$

and allows for no analytic continuation to the left half-plane

$$\mathbf{C}_- = \{s \,|\, \mathrm{Re}\, s < 0\} \; .$$

Take, in particular, $\Phi(t) = \det(1-t\rho)$ for some $\rho$ in $X_1$ , then equation (8) defines the Weil's L-function, [29],

$$L(s,\rho) = \prod_{p} \det(1-|p|^{-s}\rho(\sigma_p))^{-1} \quad , \ \mathrm{Re}\, s > 1 \ , \tag{9}$$

associated to $\rho$ . We develop the product (9) in an absolutely convergent for $\mathrm{Re}\, s > 1$ Dirichlet series

$$L(s,\rho) = \sum_{\mathfrak{n}} c(\mathfrak{n},\chi)|\mathfrak{n}|^{-s} \quad , \quad \chi =: \mathrm{tr}\,\rho \quad ,$$

where $\mathfrak{n}$ ranges over all the integral divisors of $k$ . Given $r$ representations $\rho_j$ , $1 \leq j \leq r$ , in $X_1$ with characters $\chi_j = \mathrm{tr}\,\rho_j$ , let

$$L(s,\vec{\chi}) = \sum_{\mathfrak{n}} \prod_{j=1}^{r} c(\mathfrak{n},\chi_j)|\mathfrak{n}|^{-s} \quad , \ \mathrm{Re}\, s > 1 \ , \tag{10}$$

be the convolution of the L-functions $L(s,\rho_j)$ , $1 \leq j \leq r$ , sometimes called the scalar product. Let $d_j = \chi_j(1)$ denote the dimension of the representation $\rho_j$ and assume, without a loss of generality, that

$$d_1 \geq \ldots \geq d_r \quad . \tag{11}$$

<u>Theorem 2</u>. The function $s \longmapsto L(s,\vec{\chi})$ defined for $\mathrm{Re}\, s > 1$ by an absolutely convergent Dirichlet series (10) can be analytically continued to $\mathbf{C}_+$ . If $r \geq 2$ and

$$(r \geq 3 \wedge d_3 \geq 2) \vee (d_1 \geq 3 \wedge d_2 \geq 2) \quad , \tag{12}$$

then this function has a natural boundary $\mathbf{C}°$ and can not be analytically continued to $\mathbf{C}_-$ . If (12) does not hold, the function (10) can be analytically continued to the whole plane $\mathbf{C}$ , namely,

$$L(s,\vec{\chi}) = L(s,\rho) \quad \text{when either} \quad r = 1 \quad \text{or} \quad d_2 = 1 \quad ;$$

$$L(s,\vec{\chi}) = L(s,\rho)L(2s,\det\rho)^{-1} \prod_{p \in S_\circ} \ell_p(|p|^{-s}) \quad \text{otherwise,}$$

where $\rho = \rho_1 \otimes \ldots \otimes \rho_r$ , $S_\circ$ is a finite set of primes in $k$ , and $\ell_p(t)$ is a rational function of $t$ satisfying the condition

$$\ell_p(\alpha) \neq 0,\infty \quad \text{when} \quad |\alpha| \neq 1 \quad . \tag{13}$$

Consider now $r$ finite extensions $k_j$ , $1 \leq j \leq r$ , of $k$ and let $d_j = [k_j : k]$ denote the degree of $k_j$ over $k$ . Given a Grossencharacter $\chi_j$ in $k_j$ , one defines an L-function Hecke

$$L(s,\chi_j) = \sum_{\mathfrak{A}} \chi_j(\mathfrak{A})|\mathfrak{A}|^{-s} = \sum_{\mathfrak{n}} c(\mathfrak{n},\chi_j)|\mathfrak{n}|^{-s} \text{ , Res} > 1 \text{ ,} \tag{14}$$

where $\mathfrak{A}$ and $\mathfrak{n}$ range over integral ideals of $k_j$ and $k$ , respectively. In particular,

$$c(\mathfrak{n},\chi_j) = \sum_{\mathfrak{a}} \chi_j(\mathfrak{a}) \text{ , } N_{k_j/k}\mathfrak{a} = \mathfrak{n} \text{ ,}$$

is a finite sum extended over integral ideals $A$ in $k_j$ whose relative norm to $k$ is equal to $\mathfrak{n}$ . We define the scalar product $L(s,\vec{\chi})$ of L-functions (14) by the equation (10). The grossencharacter $\chi_j$ can be regarded as an one-dimensional representation of $W(k_j)$ ; let $\rho_j$ be the representation of $W(k)$ induced by $\chi_j$ . Then

$$L(s,\chi_j) = L(s,\rho_j) ,$$

so that the scalar product $L(s,\vec{\chi})$ coincides with the scalar product (10) of L-functions $L(s,\rho_j)$ , $1 \leq j \leq r$ . By theorem 2, if $r \geq 2$ and the degrees $d_j$ satisfy (11) and (12), then $L(s,\vec{\chi})$ has $\mathbf{C}^\circ$ as its natural boundary and can not be continued analytically to $\mathbf{C}_-$ . This theorem has been proved by N. Kurokawa, [14], for Grossencharacters of finite order. The author has generalised the construction of Kurokawa's and has proved this result for arbitrary Grossencharacters assuming the validity of the Riemann Hypothesis for L-functions Hecke, [20]. The scalar product

$$\sum_{\mathfrak{n}} |\mathfrak{n}|^{-s} \prod_{j=1}^{r} c_{\mathfrak{n}}^{(j)}$$

of the Dirichlet series $\sum_{\mathfrak{n}} |\mathfrak{n}|^{-s} c_{\mathfrak{n}}^{(j)}$ , $1 \leq j \leq r$ , has been studied by many authors (see, for instance, [6], [25], [24], [23], [9], [19], [5]). The problem of analytic continuation of the scalar product (10) for L-functions (14) "mit Größencharakteren" has been posed by Yu.V. Linnik in the context of analytic arithmetic in algebraic number fields (cf. [18], [4]). P.K.J. Draxl, [2], has proved that $L(s,\vec{\chi})$ can be meromorphically continued to $\mathbf{C}_+$ for any set of Grossencharacters $\{\chi_j \mid 1 \leq j \leq r\}$. O.M. Fomenko, [4], has continued $L(s,\vec{\chi})$ meromorphically to the whole plane $\mathbf{C}$ in the case of two quadratic fields $r = d_1 = d_2 = 2$ (cf. also [9]), while the author, [19], has obtained an explicit expression for $L(s,\vec{\chi})$ in terms of L-functions Hecke in this case. Theorem 1 has been proved by N. Kurokawa, [12], [13], under an additional assumption that each of the characters in $X_o(\Phi)$ is of Galois type (so that the corresponding representation of $W(k)$ has a finite image). Here we remove this assumption. For this aid, the construction of [12], [13] is generalised to compact groups and a new equidistribution theorem for Frobenius classes in Weil groups is proved. This equidistribution theorem takes the place of the Chebotarev density theorem in [12]. In the case $k = \mathbf{Q}$ and for polynomials $\Phi$ with constant coefficients (that is, when $\Phi(t) \in \mathbf{Z}[t]$) theorem 1 has been known classically, [3] (see also [15], [1] for related results). A preliminary exposition of the results proved here has been given in the last paragraph of the book [21].

## § 2. On polynomials associated to representations of compact groups.

Consider a compact group $G$ and let $X$ be set of all the irreducible representations of $G$. Let

$$Y = \{ \sum_{\chi} m(\chi)\chi \mid m(\chi) \in \mathbb{Z} \ , \quad \chi = \mathrm{tr}\,\rho \ , \ \rho \in X\}$$

be the ring of virtual characters of $G$, so that $m$ ranges over all the functions $m : \check{X} \longrightarrow \mathbb{Z}$ on the set $\check{X} = \{\chi \mid \chi = \mathrm{tr}\,\rho \ , \ \rho \in X\}$ of irreducible characters of $G$, for which the set $\{\chi \mid m(\chi) \neq 0\}$ is finite. Given a polynomial $\Phi(t)$ of the form (2), we define $\Phi_g(t)$ by (3) and let

$$\Phi_g(t) = \prod_{j=1}^{\ell} (1-\alpha_j(g)t) \quad , \quad g \in G \quad . \tag{15}$$

Let, moreover,

$$\gamma = \sup\{ |\alpha_j(g)| \ \Big| \ 1 \leq j \leq \ell, \ g \in G\} \quad . \tag{16}$$

By lemma 14 in [20], we have

$$1 \leq \gamma < \infty \ . \tag{17}$$

A polynomial $\Phi(t)$ in $Y[t]$ is said to be unitary, if $\gamma = 1$. By (16) and (17), $\Phi(t)$ is unitary if and only if

$$\Phi_g(\alpha) \neq 0 \quad \text{whenever} \quad |\alpha| \neq 1 \quad \text{and} \quad g \in G \quad , \quad \alpha \in \mathbf{C} \; . \tag{18}$$

Write $a_j = \sum_{\chi} m_j(\chi)\chi$ with $\chi \in \check{X}$ and let, for $\Phi(t)$ having the form (2) ,

$$X_\circ(\Phi) = \{\varphi | \varphi \in X \; , \; m_j(\mathrm{tr}\varphi) \neq 0 \quad \text{for some} \quad j\}$$

be the set of all the irreducible representations of $G$ which are contained in one of the coefficients of $\Phi$ . By definition of $Y$ , the set $X_\circ(\Phi)$ is finite.

<u>Proposition 1</u>. Let $\Phi(t) \in Y[t]$ and suppose that $\Phi(0) = 1$ . There exists a sequence of integer valued functions

$$b_n : X \longrightarrow \mathbf{Z} \quad , \quad 1 \leq n < \infty \quad ,$$

such that

$$b_n(\varphi) = 0 \quad \text{for} \quad \varphi \notin X_\circ(\Phi) \; , \tag{19}$$

identity

$$\Phi(t) = \prod_{n=1}^{\infty} \prod_{\varphi \in X} \det(1-t^n\varphi)^{b_n(\varphi)} \tag{20}$$

holds formally in the ring of formal power series $Y[[t]]$ with coefficients in $Y$ , for each $g$ in $G$ the product

$$\Phi_g(t) = \prod_{n=1}^{\infty} \prod_{\varphi \in X} \det(1-t^n\varphi(g))^{b_n(\varphi)} \tag{21}$$

converges absolutely in the circle $|t| < \gamma^{-1}$, and the following estimates hold:

$$\left| \sum_{\varphi \in X} b_n(\varphi) \operatorname{tr}\varphi(g) \right| \leq \frac{\tau(n)}{n} \ell \gamma^n \quad , \quad n \in \mathbf{N} \ , \ g \in G \ , \tag{22}$$

and

$$\sum_{n \geq M} \sum_{\varphi \in X} \left| \log \det(1-t^n\varphi(g))^{b_n(\varphi)} \right| \leq \frac{\ell(|t|\gamma)^M}{(1-\gamma|t|)^2} \quad \text{when} \quad |t| < \gamma^{-1} \ , \tag{23}$$

where $\tau(n)$ denotes the number of positive divisors of $n$ and $\ell$ is the degree of $\Phi(t)$ .

<u>Proof</u>. To deduce (20) one constructs inductively two sequences

$$\{b_n \mid b_n : X \longrightarrow \mathbf{Z} \quad , \quad 1 \leq n \leq \infty\}$$

and

$$\{F_n \mid F_n(t) \in Y[t] \quad , \quad 1 \leq n < \infty\}$$

satisfying the following relations:

$$F_n(t) \equiv \Phi(t) \pmod{t^{n+1}} \tag{24}$$

and

$$F_n(t) = \prod_{\nu=1}^{n} \prod_{\varphi \in X} \det(1-t^{\nu}\varphi)^{b_{\nu}(\varphi)} \tag{25}$$

Let $F_\circ(t) = 1$ , suppose that (24), (25) hold and, moreover,

$$X_\circ(\Phi) \supseteq X_\circ(F_n) \quad . \tag{26}$$

Then, since $\Phi(0) = 1$ , we have, by (24),

$$F_n(t) \equiv (1 + bt^{n+1})\,\Phi(t) \pmod{t^{n+2}} \quad , \quad b \in Y \quad .$$

In view of (26), one can define $b_{n+1}$ by the relations:

$$b_{n+1}(\varphi) = 0 \quad \text{for} \quad \varphi \notin X_\circ(\Phi) \quad , \quad b = \sum_{\varphi \in X_\circ(\Phi)} b_{n+1}(\varphi)\,\mathrm{tr}\varphi \quad ;$$

let

$$F_{n+1}(t) = F_n(t) \prod_{\varphi \in X_\circ(\Phi)} \det(1-t^{n+1}\varphi)^{b_{n+1}(\varphi)} \quad .$$

Then (19) holds by construction, while (20) follows from (25). Write $\Phi(t)$ in the form (2) and define $\ell$ functions

$$\alpha_j : G \longrightarrow \mathbb{C} \quad , \quad 1 \leq j \leq \ell \quad ,$$

by (15); then (20) may be rewritten as

$$\prod_{j=1}^{\ell} (1-t\alpha_j) = \prod_{n=1}^{\infty} \prod_{\varphi\in X} \det(1-t^n\varphi)^{b_n(\varphi)} \tag{27}$$

We apply the operator

$$-t\frac{\partial}{\partial t} \log : Y[[t]] \to Y[[t]]$$

to both sides of (27)and obtain an identity

$$\sum_{j=1}^{\ell} \frac{t\alpha_j}{1-t\alpha_j} = \sum_{n=1}^{\infty} \sum_{\varphi\in X} nb_n(\varphi)\,\mathrm{tr}(t^n\varphi(1-t^n\varphi)^{-1}) \tag{28}$$

in $Y[[t]]$ . Let

$$\sigma(m,g) = \sum_{j=1}^{\infty} \alpha_j(g)^m \ , \ h_n(g) = n \sum_{\varphi\in X} b_n(\varphi)\,\mathrm{tr}\,\varphi(g)$$

for $g \in G$ .

It follows from (28) that, for any $g$ in $G$ ,

$$\sum_{m=1}^{\infty} t^m\sigma(m,g) = \sum_{m,n=1}^{\infty} t^{nm}h_n(g^m) \quad \text{in} \quad \mathbb{C}[[t]] \ ,$$

or equivalently,

$$\sigma(n,g) = \sum_{mm'=n} h_m(g^{m'}) \quad , \ m \in \mathbb{N} \ , \ m' \in \mathbb{N} \ . \tag{29}$$

Introducing the Möbius function $\mu : \mathbb{N} \to \{0,\pm 1\}$ one obtains from (29) an equation

$$\sum_{\nu | n} \mu(\nu)\, \sigma(\tfrac{n}{\nu}, g^{\nu}) = h_n(g) \ , \quad n \in \mathbb{N} \tag{30}$$

Since $|\sigma(m,g)| \leq \ell \gamma^m$, estimate (22) follows from (30). Estimate (23) is an easy consequence of (22) and the well known operator identity $\log \circ \det = \mathrm{tr} \circ \log$. The absolute convergence of (21) for $|t| < \gamma^{-1}$ follows from (23). This proves the proposition.

Proposition 2. If $\Phi$ is unitary, then there exists $n_o$ such that

$$b_n(\varphi) = 0 \quad \text{whenever either} \quad n > n_o \quad \text{or } \varphi \notin X_o(\Phi) \ , \tag{31}$$

and therefore

$$\Phi(t) = \prod_{n=1}^{n_o} \prod_{\varphi \in X_o(\Phi)} (1 - t^{n_\varphi})^{b_n(\varphi)} \tag{32}$$

Proof. By condition, $\gamma = 1$. Therefore it follows from (22) that one can find $n_o$ in $\mathbb{N}$ for which

$$\left| \sum_{\varphi \in X} b_n(\varphi) \text{ tr } \varphi(g) \right| < 1 \quad \text{whenever} \quad n > n_o, g \in G \,. \tag{33}$$

In view of orthogonality relations, (31) follows from (33) and (19). Identity (32) is a formal consequence of (20) and (31).

## § 3. Continuation of $L(s,\Phi)$ to $\mathbb{C}_+$ .

We return now to notations of § 1. In view of the remarks made in § 1 , any polynomial $\Phi$ in $Y[t]$ may be regarded as a polynomial with coefficients in the ring of virtual characters of a compact group $G = W_1(K|k)$ for some finite Galois extension $K \supseteq k$ . Given a representation (1) we denote by

$$S(\rho) = \{p \mid V^{I_p} \neq \{0\}\}$$

the set of all the primes $p$ in $k$ at which $\rho$ is ramified. It follows from the definitions, [29] , that $S(\rho)$ is a finite set. Indeed, let $\rho \in X$ and suppose that $\rho$ factors through $W(K|k)$ . Denote by $U_{\mathfrak{p}}$ the group of $\mathfrak{p}$- adic units in $K$ and regard $U_{\mathfrak{p}}$ as a subgroup of $C_K$ . By continuity of $\rho$ , we have $U_{\mathfrak{p}} \subseteq \operatorname{Ker} \rho$ for all but a finite number of prime divisors $\mathfrak{p}$ in $K$ . On the other hand, one can show (cf., for instance, [21], p.18) that if $K|k$ is unramified at $p$ and if $U_{\mathfrak{p}} \subseteq \operatorname{Ker} \rho$ for each $\mathfrak{p}$ dividing $p$ , then $S(\rho)$ does not contain the prime divisor $p$ of $k$ . Thus $S(\rho)$ is finite and, therefore, the set

$$S(\Phi) = \{p \mid p \in S(\rho) \text{ for some } \rho \text{ in } X_o(\Phi)\}$$

is also finite. Moreover, by (6) ,

$$\Phi_p(t) = \Phi_g(t) \quad \text{for} \quad p \notin S(\Phi) \ , \ g \in \sigma_p \ . \quad (34)$$

<u>Proposition 3</u>. If $\Phi$ is an unitary polynomial and $\Phi(0) = 1$, then $L(s,\Phi)$ can be meromorphically continued to the whole plane $\mathbb{C}$ .

Proof. It follows from the relations (8), (9)', (32) and (34) that

$$L(s,\Phi) = \prod_{\rho \in X_0(\Phi)} (L^{\Phi}(ns,\rho))^{b_n(\rho)} \prod_{p \in S(\Phi)} \Phi_p(|p|^{-s})^{-1} \ , \quad (35)$$

where

$$L^{\Phi}(s,\rho) =: L(s,\rho) \prod_{p \in S(\Phi)} \det(1-\rho(\sigma_p)|p|^{-s}) \ .$$

Since $L(s,\rho)$ is a meromorphic function, [29], and the set $X_0(\Phi)$ is finite, the assertion follows from (35).

<u>Remark 1</u>. The product $\prod_{p \in S(\Phi)}$ appears in (35) because $\Phi_p(t)$ can not be evaluated by (32) when $p \in S(\Phi)$ .

Choose two rational integers $M$ and $N$ subject to the condition:

$$M > 0 \ , \ \gamma^M < N \ , \ N > |p| \quad \text{for each} \quad p \ \text{in} \ S(\Phi) \quad (36)$$

with $\gamma$ defined by (16) and let, in notations of (20) and (4),

$$f_{n,p}(t) = \prod_{\varphi\in X} \det(1-t^n\varphi(\sigma_p))^{b_n(\varphi)} . \tag{37}$$

We define, generalising the construction of [12] , two finite products

$$Z_N(s) = \prod_{|p|<N} \Phi_p(|p|^{-s})^{-1} \quad \text{and} \tag{38.1}$$

$$R_{N,M}(s) = \prod_{\substack{p\notin S(\Phi)\\ |p|<N}} \prod_{n<M} f_{n,p}(|p|^{-s}) , \tag{38.2}$$

and two infinite products

$$U_M(s) = \prod_{n<M} \prod_{p\notin S(\Phi)} f_{n,p}(|p|^{-s})^{-1} , \tag{38.3}$$

$$T_{N,M}(s) = \prod_{n\geq M} \prod_{|p|\geq N} f_{n,p}(|p|^{-s})^{-1} \tag{38.4}$$

It follows from (38) and (20) that

$$L(s,\Phi) = Z_N(s)\ R_{N,M}(s)\ U_M(s)\ T_{N,M}(s) \tag{39}$$

as a formal Euler product. Moreover, it follows from (9) that

$$U_M(s) = \prod_{n<M} \prod_{\rho\in X_o(\Phi)} L(ns,\rho)^{b_n(\rho)} \prod_{p\in S(\Phi)} f_{n,p}(|p|^{-s}) , \tag{40}$$

since, by (19), $b_n(\rho) = 0$ when $\rho \notin X_o(\Phi)$ .

Lemma 1. The functions

$$s \mapsto R_{N,M}(s) \ , \quad s \mapsto U_M(s) \ , \quad s \mapsto Z_N(s)$$

are meromorphic in $\mathbb{C}$ .

Proof. Since $L(s,\rho)$ is meromorphic in $\mathbb{C}$ , [29], the assertion follows from (38.1), (38.2), and (40).

Lemma 2. Suppose that M , N satisfy (36). Then the product $T_{N,M}(s)$ converges absolutely for $\operatorname{Re} s > \frac{1}{M}$ .

Proof. By (36), we have

$$\gamma|p|^{-n \operatorname{Re} s} < 1 \quad \text{for} \quad \operatorname{Re} s > \frac{1}{M} \ , |p| \geq N \ . \tag{41}$$

In view of (41), we deduce from (23) and (37) that

$$\sum_{n \geq M} |\log f_{n,p}(|p|^{-s})| \leq \frac{\ell(\gamma|p|^{-n \operatorname{Re} s})^M}{(1-\gamma|p|^{-\operatorname{Re} s})^2} \quad \text{for} \quad \operatorname{Re} s > \frac{1}{M} \ .$$

Therefore, if $\operatorname{Re} s > \frac{1}{M}$ , then

$$\sum_{n\geq M} \sum_{|p|\geq N} |\log f_{n,p}(|p|^{-s})| \leq \frac{\ell\gamma^{M}[k:\mathbb{Q}]}{(1-\gamma N^{-1/M})^2} \sum_{n=1}^{\infty} n^{-M\,\mathrm{Re}\,s} , \quad (42)$$

since there are no more than $[k : \mathbb{Q}]$ prime divisors $p$ in $k$ such that $|p| = n$ , $n \in \mathbb{N}$ . The assertion of lemma 2 follows from (42) and (38.4).

Proposition 4. Let $\Phi(t) \in Y[t]$ , $\Phi(o) = 1$ . The function defined by (8) for $\mathrm{Re}\, s > 1$ can be meromorphically continued to the right half-plane $\mathbb{C}_+$ .

Proof. Choose $M,N$ satisfying (36). By lemma 1 and lemma 2 , equation (39) defines a meromorphic continuation of $L(s,\Phi)$ to the half-plane

$$\mathbb{C}_{1/M} = \{s \mid \mathrm{Re}\ s > \frac{1}{M}\} .$$

Therefore the assertion follows from an obvious relation:

$$\mathbb{C}_+ = \bigcup_{M=1}^{\infty} \mathbb{C}_{1/M} .$$

## § 4. A general prime number theorem

Let $K \supseteq k$ be a fixed throughout this paragraph finite Galois extension of $k$ of degree $n+1 = [K : \mathbb{Q}]$ over $\mathbb{Q}$ and let $\mathfrak{M}$ be a finite subset of normalised irreducible representations each of which factors through $W(K|k)$. Thus $\mathfrak{M}$ may be regarded as a subset of $X$. Let $Gr(K)$ denote the group of all the grossencharacters in $K$ trivial on $\mathbb{R}_+$, so that $Gr(K)$ is a discrete group isomorphic to the group of characters of $C_K^1$. For $\psi \in Gr(K)$, let $\mathfrak{f}(\psi)$ denote the conductor of $\psi$; given $\rho$ in $X_1$ which factors through $W(K|k)$, we write

$$\rho|_{C_K} = \psi_1 \oplus \dots \oplus \psi_\ell \ , \ \psi_j \in Gr(K),\ 1 \leq j \leq \ell \ ,$$

and denote by $\mathfrak{f}(\rho)$ the least common multiple of $\mathfrak{f}(\psi_1), \dots, \mathfrak{f}(\psi_r)$. We fix an integral divisor $\mathfrak{f}_0$ in $k$ satisfying the condition

$$\mathfrak{f}_0 \equiv 0(\mathfrak{f}(\rho)) \quad \text{for each } \rho \text{ in } \mathfrak{M} \ , \tag{43}$$

and let

$$S(\mathfrak{M}) = \bigcup_{\rho \in \mathfrak{M}} S(\rho)$$

be the finite set of primes outside of which any representation in $\mathfrak{M}$ is unramified. Let $\check{\mathfrak{M}} = \{\chi \mid \chi = \operatorname{tr}\rho\ ,\ \rho \in \mathfrak{M}\}$ be the set of characters of representations in $\mathfrak{M}$ . For $g \in W(K|k)$ , $\varepsilon > 0$ let

$$\mathfrak{P}(g,\varepsilon) = \{ p \mid p \notin S(\mathfrak{M}),\ |\chi(\sigma_p) - \chi(g)| < \varepsilon \ \text{ for each } \chi \text{ in } \check{\mathfrak{M}}\}$$

and let, for $x_2 > x_1 > 0$ ,

$$\mathfrak{P}(g,\varepsilon;x_1,x_2) = \{ p \mid p \in \mathfrak{P}(g,\varepsilon)\ ,\ x_1 \leq |p| < x_2 \}\ ,$$

where, as usual, $p$ varies over prime divisors of $k$ . We denote by $P(g,\varepsilon;x_1,x_2)$ the cardinality of the set $\mathfrak{P}(g,\varepsilon;x_1,x_2)$ . The main purpose of this paragraph is the proof of the following statement.

Proposition 5. There are two positive numbers $c_1$ and $c_2$ such that

$$P(g,\varepsilon;x_1,x_2) \geq c_1\varepsilon^n \int_{x_1}^{x_2} \frac{du}{\log u} + O(x_2 \exp(-c_2\sqrt{\log x_2})) \qquad (44)$$

for every $\varepsilon, g, x_1, x_2$ subject to the conditions

$$1 > \varepsilon > 0\ ,\ x_2 > x_1 > 0\ ,\ \ g \in W(K|k)\ , \qquad (45)$$

where the O - constant does not depend on $\varepsilon, g, x_1, x_2$ .

Remark 2. The constants in (44) may depend on the set $\mathfrak{M}$ . We believe to be true, but couldn't prove, the following statement.

Conjecture. There is a function $c(\mathfrak{M},\varepsilon)$ such that

$$P(g,\varepsilon; x_1, x_2) = c(\mathfrak{M},\varepsilon) \int_{x_1}^{x_2} \frac{du}{\log u} + O(x_2 \exp(-c_2 \sqrt{\log x_2})), \quad c_2 > 0 , \tag{46C}$$

where $c_2$ and the O - constant do not depend on the data (45). Obviously, (46C) and (44) imply the inequality

$$c(\mathfrak{M},\varepsilon) \geq c_1 \varepsilon^n \quad \text{for any } \mathfrak{M} . \tag{47}$$

We deduce Proposition 5 from Proposition 6 to be stated below. Let $\mathfrak{a}$ be an integral divisor in K and let

$$\mathcal{G}(\mathfrak{a}) = \{\psi \mid \psi \in Gr(K) , \mathfrak{f}(\psi) \mid \mathfrak{a}\}$$

be the group of those grossencharacters whose conductor divides $\mathfrak{a}$ . By a theorem of Hecke, [8] (cf. also [7], § 9), $\mathcal{G}(\mathfrak{a})$ is an abelian group of rank n , so that

$$\mathcal{G}(\mathfrak{a}) = \mathcal{G}_0(\mathfrak{a}) \times \mathcal{G}_1(\mathfrak{a}) , \quad \mathcal{G}_1(\mathfrak{a}) \cong \mathbb{Z}^n$$

and $\mathfrak{g}_o(\mathfrak{a})$ is a finite group. We choose a system

$$\{\lambda_j \mid 1 \leq j \leq n\}$$

of free generators of $\mathfrak{g}_1(\mathfrak{a})$ and write

$$\lambda_j(\alpha) = \exp(2\pi i \varphi_j(\alpha)), \alpha \in C_K, 1 \leq j \leq n, -\frac{1}{2} \leq \varphi_j(\alpha) < \frac{1}{2} . \quad (48)$$

Consider, for $\varepsilon > 0$, an $\varepsilon$ - neighbourhood $V(\varepsilon;\mathfrak{a})$ of the neutral element in $C_K$ consisting of the idele-classes satisfying the following condition:

$$\lambda(\gamma\alpha) = 1 , |\varphi_j(\gamma\alpha)| < \frac{\varepsilon}{2} \text{ whenever } 1 \leq j \leq n , \gamma \in G(K|k) , \lambda \in \mathfrak{g}_o(\mathfrak{a}) ,$$

where $C_K$ is regarded as a left $G(K|k)$-module. For each prime divisor $p$ in $k$ we choose an element $\tau_p$ in $\sigma_p$ fixed throughout this paragraph and, for each $t$ in $W(K|k)$, let

$$\mathfrak{A}(g,t,\varepsilon;x) = \{p \mid t^{-1}\tau_p t \in V(\varepsilon;\mathfrak{f}_o)g , p \notin S(\mathfrak{m}), |p| < x\} ,$$

where the divisor $\mathfrak{f}_o$ defined by (43) is regarded as a $G(K|k)$ invariant integral divisor in $k$ and $p$ ranges over primes in $k$ . Let $A(g,t,\varepsilon;x)$ denote the cardinality of the finite set $\mathfrak{A}(g,t,\varepsilon;x)$ and let

$$A_o(g,\varepsilon;x) = \int_{W_1(K|k)} A(g,t,\varepsilon;x)\ d\mu(t) \ , \tag{49}$$

where $\mu$ denotes the normalised by the condition $\mu(W_1(K|k))=1$ Haar measure on $W_1(K|k)$ .

Proposition 6. The function $t \mapsto A(g,t,\varepsilon;x)$ is $\mu$ - measurable, so $A_o(g,\varepsilon;x)$ is well defined. Moreover, there are two positive constants $c_3, c_4$ such that for any $\varepsilon$ in the interval $0 < \varepsilon < 1$ we have

$$A_o(g,\varepsilon;x) = c_3\varepsilon^n \int_2^x \frac{du}{\log u} + O(x \exp(-c_4\sqrt{\log x})) \tag{50}$$

with an O - constant independent on $g,\varepsilon,x$ .

The proof of Proposition 6 depends on a prime number theorem generalising both the Chebotarev density theorem and the classical estimates, [8], for grossencharacters. Let us recall that any $\psi$ in $Gr(K)$ may be regarded (cf., for instance, [7], § 9) as a character of the group of fractional ideals generated by the set of all those prime divisors in K which do not divide the conductor $\mathfrak{f}(\psi)$ of $\psi$ . Write, in particular,

$$\psi((\alpha)) = \prod_{\gamma} |\gamma\alpha|^{it(\gamma)} \left(\frac{\gamma\alpha}{|\gamma\alpha|}\right)^{\nu(\gamma)} \quad \text{for} \quad ((\alpha),\mathfrak{f}(\psi))=1, \tag{51}$$

where $(\alpha)$ denotes the principal ideal generated by $\alpha \neq 0$, $\alpha \in K$, and $\gamma$ varies over all the $n+1$ distinct isomorphisms

$$\gamma : K \to \mathbb{C}$$

of $K$ into $\mathbb{C}$. Here $t(\gamma) \in \mathbb{R}$, $\nu(\gamma) \in \mathbb{Z}$, and $\nu(\gamma) \in \{0,1\}$ when $\gamma$ corresponds to a real place of $K$, so that $\gamma(K) \subseteq \mathbb{R}$. The exponents $t(\gamma)$, $\nu(\gamma)$ are known, [8] (or [7], §9), to satisfy certain normalisation conditions. For $\psi \in Gr(K)$ we let

$$v(\psi) = \prod_{\gamma} (|t(\gamma)| + 1)(|\nu(\gamma)| + 1) \tag{51.1}$$

in notations of (51). Suppose that $\rho \in X_1$, $\rho$ factors through $W(K|k)$ and

$$\rho|_{C_K} = \psi_1 \oplus \ldots \oplus \psi_\ell \, , \; \psi_j \in Gr(K) \quad \text{when} \quad 1 \leq j \leq \ell \, .$$

We define then the weight of $\rho$ by

$$v(\rho) = \max_{1 \leq j \leq \ell} v(\psi_j) \tag{51.2}$$

with $v(\psi_j)$ given by (51.1). For brevity, we write

$$v(\chi) = v(\rho) \quad \text{when} \quad \chi = \operatorname{tr} \rho \, , \; \rho \in X_1 \, .$$

<u>Theorem 3</u>. Let $\mathfrak{N} \subseteq X_1$ and suppose that each $\rho$ in $\mathfrak{N}$ factors through $W(K|k)$ for a finite extension $K \supseteq k$ and that there exists an integral ideal $\mathfrak{a}$ in $K$ satisfying the condition

$$\mathfrak{a} \equiv 0(\mathfrak{f}(\rho)) \quad \text{whenever} \quad \rho \in \mathfrak{N} .$$

Then there is $c_5 > 0$ such that

$$\sum_{|p|<x} \chi(\sigma_p) = g(\chi)\int_2^x \frac{du}{\log u} + O(x \exp(-c_5 \frac{\log x}{\sqrt{\log x} + \log v(\chi)})), \quad (52)$$

whenever $\chi = \operatorname{tr} \rho$ , $\rho \in \mathfrak{N}$. Here $p$ ranges over prime divisors in $k$ ; the $O$ -constant and $c_5$ may depend on $\mathfrak{N}$ but not on a particular representation $\rho$ in $\mathfrak{N}$ ; $g(\chi)$ denotes the multiplicity of the identical representation in $\rho$ .

Proof. Since $v(\rho_1 \oplus \rho_2) \geq v(\rho_j)$ , $j=1,2$ , and both $\rho_1$ and $\rho_2$ factor through $W(K|k)$ as soon as $\rho_1 \oplus \rho_2$ does, it is enough to prove (52) for irreducible representations. Suppose now that $\rho \in \mathfrak{N}$ and $\rho$ is induced by another representation, say,

$$\rho = \operatorname{Ind}_{W(K|k')}^{W(K|k)} \rho' \quad , \quad k \subseteq k' \subseteq K \ , \quad \rho' \in X_1 \ ,$$

and $\rho'$ factors through $W(K|k')$ . Then $L(s,\rho) = L(s,\rho')$ , so taking the logarithmic derivative in the Euler product decomposition (9) one obtains an estimate

$$\sum_{|p|<x} \chi(\sigma_p) = \sum_{|p'|<x} \chi'(\sigma_{p'}) + O_\alpha(x^{1/2+\alpha})$$

for any $\alpha > 0$, where $\chi = \mathrm{tr}\,\rho$, $\chi' = \mathrm{tr}\,\rho'$, the $O_\alpha$ - constant depends only on $\alpha$ and the degree of $K$ over $\mathbb{Q}$; $p$ and $p'$ vary over prime divisors of $k$ and $k'$, respectively. Moreover, since

$$\chi(\alpha) = \sum_{\gamma} \chi'(\gamma\alpha), \quad \alpha \in C_K,$$

where $\gamma$ ranges over a system of representatives of the, say, right classes of $G(K|k')$ in $G(K|k)$, we conclude that $v(\chi) = v(\chi')$. Thus passing, if necessary, to an intermediate field $k'$ one may assume that $\rho$ is a primitive irreducible representation of $W(K|k)$. A classical argument (cf.[29], p. 32-34) shows then that $\rho$ may be written in the form

$$\rho = \sum_{j=1}^{\ell} a_j \psi_j, \quad a_j \in \mathbb{Z}, \qquad (53)$$

where $\psi_j$, $1 \leq j \leq \ell$, is a monomial representation of $W(K|k)$ induced by a grossencharacter $\lambda_j$ in $Gr(k_j)$, $k \subseteq k_j \subseteq K$, and

$$\psi_j(\alpha) = n_j \omega(\alpha) \quad \text{for } \alpha \in C_K, \qquad (54)$$

where $\omega$ is a grossencharacter in $Gr(K)$ . It follows from (53) and (54) that $v(\rho) = v(\psi_j) = v(\omega)$ . On the other hand, by (53),

$$L(s,\rho) = \prod_{j=1}^{\ell} L(s,\lambda_j)^{a_j} \quad , \tag{55}$$

where $L(s,\lambda_j)$ is a Hecke L-function in $k_j$ , $1 \leq j \leq r$ . Taking the logarithmic derivatives in (55) one obtains an estimate

$$\sum_{|p|<x} \chi(\sigma_p) = \sum_{j=1}^{\ell} a_j \sum_{|\mathfrak{p}_j|<x} \lambda_j(\mathfrak{p}_j) + O_\alpha(x^{1/2+\alpha}), \ \alpha > 0, \tag{56}$$

where $p$ and $\mathfrak{p}_j$ range over prime divisors of $k$ and $k_j$ , respectively, and the $O_\alpha$ - constant depends only on $\alpha > 0$ and on the degree of $K$ over $\mathbb{Q}$ . By a classical theorem, [8] (cf. also [11], [17]) ,

$$\sum_{|\mathfrak{p}_j|<x} \lambda_j(\mathfrak{p}_j) = g(\lambda_j)\int_2^x \frac{du}{\log u} + O(x\exp(-c_6\frac{\log x}{\sqrt{\log x} + \log v(\lambda_j)})) \tag{57}$$

with $g(\lambda_j) = \begin{cases} 0 \ , \ \lambda_j \neq 1 \\ 1 \ , \ \lambda_j = 1 \end{cases}$ , $c_6 > 0$ , where $c_6$ and the $O$ - constant depend only on the conductor of $\lambda_j$ (and the field $k_j$). Since $\psi_j = \mathrm{Ind}_{W(K|k_j)}^{W(K|k)}(\lambda_j)$ , it follows from (54) that

$$\omega((\alpha)) = \lambda_j((N_{K/k_j}\alpha)) \quad \text{for} \quad \alpha \in K^*, \ (\alpha,\mathfrak{a}) = 1 \tag{58}$$

when one regards $\omega$ and $\lambda_j$ as characters of fractional ideals in $K$ and $k_j$. By (58) and (51),

$$v(\chi) = v(\rho) = v(\omega) \geq v(\lambda_j) \quad . \tag{59}$$

Estimate (52) follows from (56), (57) and (59).

Remark 3. Theorem 3 emphasizes the dependence of the error term in the prime number theorem on the weight $v(\chi)$ of the character but not on its conductor. We could not apply the Brauer's theorem (53) directly to an arbitrary representation in $\mathfrak{M}$ because it is not a priori clear how to relate $v(\rho)$ and $v(\psi_j)$ in this case.

After these preparations we are ready to prove Proposition 6. Let us define two infinitely differentiable functions $f_\pm : \mathbb{R} \to \mathbb{R}$ subject to the following conditions:

1) $0 \leq f_\pm(t) \leq 1$, $f_\pm(t) = f_\pm(t+1)$ for each $t$ in $\mathbb{R}$,

and

2) $f_+(t) = 1$ for $|t| < \frac{\varepsilon}{2}$, $f_+(t) = 0$ for $\frac{\varepsilon}{2} + \Delta \leq |t| \leq \frac{1}{2}$,
$f_-(t) = 1$ for $|t| < \frac{\varepsilon}{2} - \Delta$, $f_-(t) = 0$ for $\frac{\varepsilon}{2} \leq |t| \leq \frac{1}{2}$,

where $\varepsilon$ and $\Delta$ are two real numbers satisfying the inequalities

$$0 < \Delta < \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \Delta < \frac{1}{2} .$$

We denote by $n_o(\mathfrak{a})$ the order of the group $\mathfrak{g}_o(\mathfrak{a})$ and, in notations of (48), define another two functions

$$\tilde{h}_{\pm} : C_K \to \mathbb{R}$$

by letting

$$\tilde{h}_{\pm}(\alpha) = \frac{1}{n_o(\mathfrak{a})} \sum_{\lambda \in \mathfrak{g}_o(\mathfrak{a})} \lambda(\alpha) \prod_{j=1}^{n} \prod_{\gamma \in G(K|k)} f_{\pm}(\varphi_j(\gamma\alpha)) . \qquad (60)$$

It follows from (60) and the definition of $f_{\pm}$ that if $\mathfrak{a}$ is $G(K|k)$ - invariant, then

$$\tilde{h}_{+}(\alpha) = 1 \quad \text{for} \quad \alpha \in V(\varepsilon;\mathfrak{a}) ,$$
$$\tilde{h}_{-}(\alpha) = 0 \quad \text{for} \quad \alpha \notin V(\varepsilon;\mathfrak{a}) , \qquad (61.1)$$

and

$$0 \leq \tilde{h}_{\pm}(\alpha) \leq 1 \quad \text{for} \quad \alpha \in C_K \qquad (61.2)$$

We substitute the Fourier expansion of $f_{\pm}$ , say

$$f_{\pm}(t) = \sum_{\ell=-\infty}^{\infty} b_{\pm}(\ell) \exp(2\pi i \ell t) , \tag{62}$$

in (60) to obtain a Fourier series for $\tilde{h}_{\pm}$ :

$$\tilde{h}_{\pm}(\alpha) = \frac{1}{n_o(\alpha)} \sum_{\lambda \in \mathcal{G}_o(\alpha)} \sum_{m} \tilde{b}_{\pm}(m)\lambda(\alpha) \prod_{j=1}^{n} \prod_{\gamma \in G(K|k)} \lambda_j(\gamma\alpha)^{m(j,\gamma)} , \tag{63}$$

where $m$ ranges over all the functions of the form

$$m : \{j|1 \leq j \leq n\} \times G(K|k) \rightarrow \mathbb{Z} ,$$

and

$$\tilde{b}_{\pm}(m) =: \prod_{j=1}^{n} \prod_{\gamma \in G(K|k)} b_{\pm}(m(j,\gamma)) . \tag{64}$$

Consider the character $\psi_\lambda^{(m)}$ in $Gr(K)$ defined by

$$\psi_\lambda^{(m)} : \alpha \mapsto \lambda(\alpha) \prod_{j=1}^{n} \prod_{\gamma \in G(K|k)} \lambda_j(\gamma\alpha)^{m(j,\gamma)} \quad \text{for } \alpha \in C_K, \lambda \in \mathcal{G}_o(\alpha) ,$$

and let $\rho_\lambda^{(m)}$ denote the representation of $W(K|k)$ induced by $\psi_\lambda^{(m)}$ .

In these notations, we define a function

$$h_{\pm} : W(K|k) \rightarrow \mathbb{C}$$

by an absolutely convergent series

$$h_{\pm}(\alpha) = \frac{1}{n_o(\mathfrak{a})\cdot d} \sum_{\lambda\in \mathfrak{g}_o(\mathfrak{a})} \sum_m \tilde{b}_{\pm}(m)\chi_\lambda^{(m)}(\alpha) \quad \text{for}$$

$$\alpha \in W(K|k) \quad , \tag{65}$$

where $d = [K : k]$ denotes the order of $G(K|k)$ and $\chi_\lambda^{(m)} = \operatorname{tr} \rho_\lambda^{(m)}$. It follows from (63)-(65) that if $\mathfrak{a}$ is $G(K|k)$ invariant, then

$$h_{\pm}(\alpha) = \tilde{h}_{\pm}(\alpha) \quad \text{for} \quad \alpha \in C_K . \tag{66}$$

From now on we let $\mathfrak{a} = \mathfrak{f}_o$, in notations of (43), and define, for $x > 0$ and $t \in W(K|k)$, a sum

$$A_{\pm}(t,x) = \sum_{|p|<x}^{*} \sum_{\chi} \chi(t^{-1}\tau_p t g^{-1}) h_{\pm}(t^{-1}\tau_p t g^{-1}) \frac{d(\chi)}{d} , \tag{67}$$

where $\sum_{|p|<x}^{*}$ is extended over prime divisors of $k$ such that $p \notin S(\mathfrak{m})$ and $\chi$ ranges over irreducible characters of $W(K|k)$ trivial on $C_K$ of dimension $d(\chi)$ (so that $\sum_\chi$ is a finite sum). It follows from the orthogonality relations

$$\frac{1}{d}\sum_\chi d(\chi)\chi(\alpha) = \begin{cases} 1 & \text{for} \quad \alpha \in C_K \\ 0 & \text{for} \quad \alpha \notin C_K \end{cases} ,$$

relations (61) and definition (67) that

$$A_-(t,x) \leq A(g,t,\varepsilon;x) \leq A_+(t,x) \ .$$

Therefore

$$A_-^o(x) \leq A_o(g,\varepsilon;x) \leq A_+^o(x) \ , \tag{68}$$

where, in notations of (49) , we let

$$A_\pm^o(x) = \int_{W_1(K|k)} d\mu(t) \ A_\pm(t,x) \tag{69}$$

For brevity, we have suppressed the variables $g,\varepsilon$ in the notations: $A_\pm(t,x)$ and $A_\pm^o(x)$ . We need the following simple lemma.

Lemma 3. Let $\chi$ be an irreducible character of a compact group $G$ and let $\mu$ be the normalised by the condition $\mu(G) = 1$ Haar measure on $G$ . We have

$$\int_G (t^{-1}h_1 \, t h_2^{-1}) d\mu(t) = \chi(h_1)\overline{\chi(h_2)} \ d(\chi)^{-1} \ , \tag{70}$$

where $h_1,h_2 \in G$ , and $d(\chi)$ denotes the dimension of $\chi$ .

Proof of lemma 3. Write $\chi = \mathrm{tr}\,\rho$, $\rho(h) = (a_{ik}(h))$ for $h \in G$, $1 \leq i$, $k \leq d(\chi)$. Without loss of generality, we can assume that $\rho$ is an unitary representation, so that $(a_{ik}(h))$ is an unitary matrix for each $h$. By orthogonality relations,

$$\int_G \overline{a_{ik}(h)}a_{\ell k}(h)\ d\mu(h) = \frac{1}{d(\chi)}\delta_{i\ell}\ \delta_{kj} \quad . \tag{71}$$

On the other hand, since $a_{ik}(h)$ is unitary,

$$\chi(t^{-1}\ h_1 t\, h_2^{-1}) = \mathrm{tr}(\rho(t)^{-1}\rho(h_1)\rho(t)\rho(h_2)) =$$

$$\sum_{i,k,\ell,j} \overline{a_{ik}(t)}\ a_{ij}(h_1)a_{j\ell}(t)\overline{a_{k\ell}(h_2)} .$$

Therefore (70) follows from (71).

Write, decomposing the product $\chi\,\chi_\lambda^{(m)}$ into irreducible components,

$$\chi(\alpha)\chi_\lambda^{(m)}(\alpha) = \sum_i \psi_{\lambda,\chi,i}^{(m)}(\alpha) \quad \text{for} \quad \alpha \in W(K|k) \tag{72}$$

with $\psi_{\lambda,\chi,i}^{(m)} \in \check{X}$, and substitute (65) into (67). This gives

$$A_\pm(t,x) = \frac{1}{n_o(\alpha)d^2} \sum_{|p|<x}\ \sum_{\lambda,\chi,m,i} \tilde{b}_\pm(m)d(\chi)\psi_{\lambda,\chi,i}^{(m)}(t^{-1}\tau_p t\, g^{-1}) . \tag{73}$$

Let

$$c_{\pm}(m;\lambda,\chi,i) = \frac{d(\chi)}{n_o(\mathfrak{a})\, d^2 d(\psi^{(m)}_{\lambda,\chi,i})}\ \overline{\psi^{(m)}_{\lambda,\chi,i}(g)}\ \tilde{b}_{\pm}(m) \ . \tag{74}$$

It follows from (69), (73) and (70) that, in notations (74),

$$A^o_{\pm}(x) = \sum_{\lambda,\chi,m,i} c_{\pm}(m;\lambda,\chi,i) \sum_{|p|<x}{}^{*} \psi^{(m)}_{\lambda,\chi,i}(\tau_p) \ . \tag{75}$$

By construction, $\psi^{(m)}_{\lambda,\chi,i}$ is unramified at $p$ whenever $p \nmid D\mathfrak{f}_o$, where $D$ denotes the discriminant of the extension $K|k$. Thus

$$\psi^{(m)}_{\lambda,\chi,i}(\tau_p) = \psi^{(m)}_{\lambda,\chi,i}(\sigma_p) \quad \text{for } p \nmid \mathfrak{f}_o D, \text{ and}$$

$$\mathfrak{f}_o \equiv O(\mathfrak{f}(\psi^{(m)}_{\lambda,\chi,i})) \ . \tag{76}$$

Since $\chi$ is trivial on $C_K$ and the character $\chi^{(m)}_{\lambda}$ is induced by the grossencharacter

$$\psi^{(m)}_{\lambda} : \alpha \mapsto \lambda(\alpha) \prod_{j,\lambda} \lambda_j(\gamma\alpha)^{m(j,\gamma)} \quad \text{with } \lambda \in \mathcal{Y}_o(\mathfrak{f}_o) \ , \tag{77}$$

we have

$$v(\psi^{(m)}_{\lambda,\chi,i}) \leq v(\psi^{(m)}_{\lambda}) \ . \tag{78}$$

In view of (76) and (78), one deduces from (52) in theorem 3 an estimate

$$\sum_{|p|<x} \psi^{(m)}_{\lambda,\chi,i}(\tau_p) = g(\psi^{(m)}_{\lambda,\chi,i})\int_2^x \frac{du}{\log u} + O(x\exp(-c_5\frac{\log x}{\sqrt{\log x + \log v(\psi_\lambda^{(m)})}})) \tag{79}$$

where $c_5 > 0$ ; the O-constant and $c_5$ depend on $\mathfrak{F}_o$ and $\mathfrak{m}$ only. It follows from (72) that

$$g(\psi^{(m)}_{\lambda,\chi,i}) = 0 \quad \text{when} \quad \psi_\lambda^{(m)} \neq 1$$

and that $g(\psi^{(m)}_{\lambda,\chi,i}) = 1$ for exactly one $i$ in (72) when $\psi_\lambda^{(m)} = 1$ . Therefore one obtains from (75), (74) and (79) an estimate

$$A^o_\pm(x) = \frac{1}{n_o(\mathfrak{a})d}\,(\sum_m{}^* \tilde{b}_\pm(m))\int_2^x \frac{du}{\log x} + O(xq_\pm(x)), \tag{80}$$

where $\sum_m^*$ is extended over those functions $m$ for which the character

$$\lambda^{(m)} : \alpha \mapsto \prod_{j=1}^{n} \prod_{\gamma\in G(K|k)} \lambda_j(\gamma\alpha)^{m(j,\gamma)} \tag{81}$$

is trivial, and

$$q_{\pm}(x) = \sum_{m,\lambda} |\tilde{b}_{\pm}(m)| \exp(-c_5 \frac{\log x}{\sqrt{\log x} + \log v(\psi_\lambda^{(m)})}) . \qquad (82)$$

In writing out the first term of (80) a well known identity

$$\sum_{\chi} d(\chi) = d$$

has been used to carry out summation over $\chi$ . To estimate $q_{\pm}(x)$ we notice that, since $f_{\pm}$ is assumed to be smooth, it follows from (62) and (64) that

$$\tilde{b}_{\pm}(m) = O(\|m\|^{-3}\Delta^{-3}) \text{ , where } \|m\| =: \prod_{j=1}^{n} \prod_{\gamma \in G(K|k)} |m(j,\gamma)| . \qquad (83)$$

By definition of the weight (51.1), one obtains from (77) and (81):

$$v(\psi_\lambda^{(m)}) = O(v(\lambda^{(m)})) = O(\|m\|^{nd}) . \qquad (84)$$

Relations (82)-(84) give

$$q_{\pm}(x) = O(\Delta^{-3} \sum_{m} \|m\|^{-3} \exp(-c_7 \frac{\log x}{\sqrt{\log x} + \log m})) \text{ , } c_7 > 0 \text{ , } \qquad (85)$$

where the O - constants in (83)-(85) and $c_7$ depend on $\mathcal{F}_0$

and $\mathfrak{M}$ . Since the number of functions

$$m : \{j \mid 1 \leq j \leq n\} \times G(K|k) \to \mathbb{Z}$$

with $\|m\| = \ell$ can be estimated like $O_\varepsilon(\ell^\varepsilon)$ for every positive $\varepsilon$ , we obtain from (85)

$$q_\pm(x) = O(\sum_{\ell=1}^{\infty} \Delta^{-3}\ell^{-2}\exp(-c_7\frac{\log x}{\sqrt{\log x} + \log \ell})) ,$$

so that

$$q_\pm(x) = O(\exp(-c_8\sqrt{\log x})\Delta^{-3}) , \; c_8 > 0 , \qquad (86)$$

with the $O$ - constant and $c_8$ depending on $\mathcal{F}_o$ and $\mathfrak{M}$ only. Write $\psi^\gamma(\alpha) = \psi(\gamma\alpha)$ for $\gamma \in G(K|k)$ , $\alpha \in C_K$ and $\psi \in Gr(K)$ and define a set of integers

$$\{a(\ell;\, j,\gamma) \mid \gamma \in G(K|k), 1 \leq j \leq n, 1 \leq \ell \leq n\}$$

by the equations

$$\lambda_j^\gamma = \prod_{\ell=1}^{n} \lambda_\ell^{a(\ell;j,\gamma)} . \qquad (87)$$

Since, by (81), condition $\lambda^{(m)}=1$ is equivalent to the equations

$$\sum_{j=1}^{n} \sum_{\gamma \in G(K\ k)} a(\ell;j,\gamma)m(j,\gamma) = 0 \ , \quad 1 \le \ell \le n \ , \qquad (88)$$

we have

$$\sum_{m} \tilde{b}_{\pm}(m) = \sum_{m} \int du\ \tilde{b}_{\pm}(m) \ \exp(2\pi i \sum_{\ell,j,\gamma} a(\ell;j,\gamma)m(j,\gamma)u_{\ell}) \ , \qquad (89)$$

where the integration in $\int du$ is taken over the cube

$$\mathfrak{b} =: \{u \mid -\tfrac{1}{2} \le u_{\ell} \le \tfrac{1}{2} \ , \ 1 \le \ell \le n\} \ .$$

By (62) and (64), it follows from (89) that

$$\sum_{m}^{*} \tilde{b}_{\pm}(m) = \int_{\mathfrak{b}} du \prod_{j=1}^{n} \prod_{\gamma \in G(K|k)} f_{\pm}(\sum_{=1}^{n} u_{\ell} a(\ell;j,\gamma)) . \qquad (90)$$

Combining relations (80), (86) and (90) one obtains an estimate

$$A_{\pm}^{o}(x) = (c_3 \varepsilon^n + O(\Delta)) \int_2^x \frac{du}{\log u} + O(\Delta^{-3} \exp(-c_8 \sqrt{\log x})) \ , \quad c_8 > 0 \ , \qquad (91)$$

where $c_3 = c_9 (n_o(\alpha) d)^{-1}$ and $c_9$ denotes the volume of the set

$$\mathfrak{b}_1 =: \{u \mid -\tfrac{1}{2} \le \sum_{\ell=1}^{n} u_{\ell} a(\ell;j,\gamma) \le \tfrac{1}{2} \ , \gamma \in G(K|k) \ , \ 1 \le j \le n\} \ .$$

In particular, since $\mathfrak{b}_1$ contains the origin in $\mathbb{R}^n$, we have

$$c_3 > 0 . \qquad (92)$$

Proposition 6 follows from (68), (92) and (91) with a properly adjusted $\Delta$ .

Proof of Proposition 5. Choose $\varepsilon_1 > 0$ and suppose that for some $t$ in $W(K|k)$ we have

$$p \in \mathfrak{A}(g,t,\varepsilon_1;x_2) \setminus \mathfrak{A}(g,t,\varepsilon_1;x_1) . \qquad (93)$$

Then $\chi(\sigma_p) = \chi(t^{-1}\tau_p t) = \chi(\alpha_p g)$ for some $\alpha_p$ in $V(\varepsilon_1;\mathfrak{f}_o)$ and for each $\chi$ in $\check{\mathfrak{M}}$ . Write $\chi = \operatorname{tr}\rho$ , $\rho(\alpha) = (r_{ik}(\alpha))$ for $\alpha \in W(K|k)$ and let

$$\chi(\alpha) = \sum_{j=1}^{d(\chi)} \psi_j^\rho(\alpha) , \psi_j^\rho \in \mathfrak{g}(\mathfrak{f}_o) , \text{ for } \alpha \in C_K . \qquad (94)$$

By construction,

$$\psi_j^\rho = \lambda^{j,\rho} \prod_{i=1}^{n} \lambda_i^{\overline{m}(j,\rho;i)} \qquad (95)$$

for some $\lambda^{j,\rho}$ in $\mathfrak{g}_o(\mathfrak{f}_o)$ and some integers $\overline{m}(j,\rho;i)$ . It follows from (94) and (95) that

$$\chi(\alpha_p g) = \sum_{j=1}^{d(\chi)} r_{jj}(g)\psi_j^{\rho}(\alpha_p) = \sum_{j=1}^{d(\chi)} r_{jj}(g) \prod_{i=1}^{n} \lambda_i^{\overline{m}(j,\rho;i)}(\alpha_p) , \tag{96}$$

where $d(\chi)$ denotes the dimension of $\rho$ . Moreover, since $\alpha_p \in V(\varepsilon_1;\mathcal{F}_o)$ , we have

$$\left| \prod_{i=1}^{n} \lambda_i^{\overline{m}(j,\rho,i)}(\alpha_p) - 1 \right| < \pi\varepsilon_1 \sum_{i=1}^{n} |\overline{m}(j,\rho;i)| . \tag{97}$$

Let $M$ denote the largest of the finite set

$$\{\pi|r_{jj}(g)| \sum_{i=1}^{n} |\overline{m}(j,\rho;i)|\}$$

of positive real numbers, where $\rho$ and $j$ vary over $\mathfrak{M}$ and the interval $1 \leq j \leq d(\chi)$ , respectively. Without loss of generality, we may assume that each $\rho$ is unitary and, therefore, $|r_{ij}(g)| \leq 1$ . Thus $M$ may be chosen independent of $g$ in $W(K|k)$ . Relations (96) and (97) give:

$$|\chi(\alpha_p g) - \chi(g)| < M\varepsilon_1 \quad \text{for} \quad \chi \in \check{\mathfrak{M}} . \tag{98}$$

Inequality (98) shows that $p \in \mathcal{P}(g,M\varepsilon_1;x_1,x_2)$ whenever (93) holds for some $t$ in $W(K|k)$ . Therefore

$$P(g,\varepsilon;\, x_1,x_2) \geq A(g,t,\varepsilon M^{-1};x_2) - A(g,t,\varepsilon M^{-1};x_1) \tag{99}$$

for each $t$ in $W(K|k)$ . Integrating in (99) over $W_1(K|k)$ and recalling (49) we obtain

$$P(g,\varepsilon;x_1,x_2) \geq A_o(g,\varepsilon M^{-1};x_2) - A_o(g,\varepsilon M^{-1};x_1). \qquad (100)$$

Relation (44) follows from (50) and (1oo) with $c_1 = c_3 M^{-n}$ . This proves Proposition 5.

## § 5. Proof of theorem 1 .

For $s \in \mathbb{C}$ we denote by $\operatorname{Re} s$ and $\operatorname{Im} s$ the real and imaginary parts of $s$ , respectively. Given $t_o \in \mathbb{R}$ , $\delta \in \mathbb{R}_+$ , $\nu \in \mathbb{R}_+$ , let

$$D_\nu(\delta,t_o) = \{s \mid s \in \mathbb{C}, \frac{1}{\nu+1} < \operatorname{Re} s \leq \frac{1}{\nu}, t_o < \operatorname{Im} s \leq t_o+\delta\}.$$

Consider a polynomial $\Phi(t)$ in $Y[t]$ and suppose that $\Phi(0) = 1$ and that $X_o(\Phi) \subseteq X$ for some finite Galois extension $K \supseteq k$ , in notations of § 3 , so that $\Phi$ may be regarded as a polynomial with coefficients in the ring of virtual characters of $W_1(K|k)$ .

<u>Proposition 7</u>. If $\Phi$ is not unitary, then there is $\nu_o$ in $\mathbb{R}$ such that the function $s \mapsto L(s,\Phi)$ has at least one pole in $D_\nu(\delta,t_o)$ as soon as $\nu > \nu_o$ .

We retain the notations of § 3 . In particular, let $N,M \in \mathbb{N}$ and suppose that (36) is satisfied, so that equation (39) defines a meromorphic continuation of $L(s,\Phi)$ to $\mathbb{C}_{1/M}$ . Let, moreover, $M = \nu+1$ , so that $D_\nu(\delta,t_o) \subseteq \mathbb{C}_{1/M}$ . For a meromorphic function $f$ we denote by $n(f,T)$ the number of zeros of $f$ in the rectangle

$$\{s \mid 0 \leq \operatorname{Re} s \leq 1 , 0 \leq |\operatorname{Im} s| \leq |T|, T \cdot \operatorname{Im} s \geq 0\} .$$

Let $a_1(\nu;\delta,t_o)$ and $a_2(\nu;\delta,t_o)$ denote the number of distinct zeros of $U_M$ in $D_\nu(\delta,t_o)$ and the number of distinct poles of $Z_N$ in $D_\nu(\delta,t_o)$ , respectively. To simplify our notations let us assume that $t_o(t_o+\delta) \geq 0$ .

Lemma 4. The following estimate holds

$$a_1(\nu;\delta,t_o) = O(\nu^2\sqrt{\log\nu}) \quad , \tag{101}$$

where the O - constant does not depend on $\nu$ (but may depend on $\Phi$ , $t_o$ and $\delta$ ) .

Proof. Since $f_{n,p}(|p|^{-s}) \neq 0$ when $\operatorname{Re} s \neq 0$ and $L(\rho,s) \neq 0$ when $\operatorname{Re} s > 1$ for any $\rho$ in $X_1$ , it follows from (40) that

$$a_1(\nu;\delta,t_o) \leq \sum_{1\leq m<\nu+1} \sum_{\rho\in X_o(\Phi)} |n(L(\rho,\cdot),m(t_o+\delta))-n(L(\varphi,\cdot),mt_o)| \ .$$

Since an L - function Hecke $L(\chi,s)$ , $\chi \in Gr(K)$ , grows in the critical strip $0 \leq \operatorname{Re} s \leq 1$ not faster than a power of $\operatorname{Im} s$, a classical argument (see, e.g.,[27], § 9.2, or [21], lemma 1, p. 146-147) shows that

$$n(L(\rho,\cdot),T+1)-n(L(\rho,\cdot),T) = O(\log|T|) \tag{102}$$

for an one dimensional $\rho$ in $X_1$ . By a theorem of Weil's , [29], one can write

$$L(s,\rho) = \prod_{i=1}^{\ell} L(s,\chi_i)^{b_i} \;, b_i \in \mathbb{Z} \;,\; \chi_i \in Gr(k_i) \;,\; \rho \in X_o(\Phi)$$

for some intermediate fields $k_i$ , $k \subseteq k_i \subseteq K$ ; therefore (102) holds for any $\rho$ in $X_o(\Phi)$ . Thus

$$a_1(\nu;\delta,t_o) = O(\sum_{1\leq m<\nu+1} (m\delta)\log|m(t_o+\delta)|) = O(\nu^2 \log \nu) \;,$$

as claimed.

In notations of § 4 , let $\mathfrak{M} = X_o(\Phi)$ . Write

$$\Phi_g(t) = (1-\alpha(g)t)^b \tilde{\Phi}_g(t), \; |\alpha(g)| = \gamma \;,\; b \geq 1 \;;\; \tilde{\Phi}_g(\alpha(g)^{-1}) \neq 0 \tag{103}$$

for some $g$ in $W_1(K|k)$ , so that $\alpha(g)^{-1}$ is a root of $\Phi_g$ whose multiplicity is equal to $b$ . By definition, $S(\mathfrak{M}) = S(\Phi)$ .

<u>Lemma 5</u>. There exists $\varepsilon_o$ in $\mathbb{R}_+$ such that for every $\varepsilon$ in the interval $0 < \varepsilon < \varepsilon_o$ and for each $p$ in $\mathfrak{P}(g,\varepsilon^{b+2})$ the polynomial $\Phi_p$ has a root $\kappa(p)^{-1}$ satisfying the condition

$$|\log|\kappa(p)| - \log \gamma| < \varepsilon \quad . \tag{104}$$

Proof. Choose $\varepsilon_1$ in the interval $0 < \varepsilon_1 < 1$ in such a way that $\tilde{\Phi}_g(t) \neq 0$ in the circle: $|t - \alpha(g)^{-1}| \leq \varepsilon_1$ and let

$w$ be the minimum of $|\tilde{\Phi}_g(t)|$ in this circle. Obviously, $w > 0$. Choose $w_1 > 0$ so that

$$|a_j(p) - a_j(g)| < w_1 \varepsilon \quad \text{for} \quad p \in \mathcal{P}(g,\varepsilon)\ ,\ 1 \leq j \leq \ell\ ,\ \varepsilon > 0\ , \tag{105}$$

where

$$\Phi_g(t) = 1 + \sum_{j=1}^{\ell} t^j a_j(g)\ ,\ \Phi_p(t) = 1 + \sum_{j=1}^{\ell} t^j a_j(p)\ .$$

For each $\varepsilon$ in the interval $0 < \varepsilon < \varepsilon_1$ we get an estimate

$$|\Phi_g(t)| \geq w\,\gamma^b \varepsilon^b \quad \text{on the circle: } |t - \alpha(g)^{-1}| = \varepsilon\ .$$

Write $\Phi_p(t) = \Phi_g(t) + h_p(t)$. By (105), for $p \in \mathcal{P}(g,\varepsilon^b)$ we have

$$|h_p(t)| < w_1(1+\gamma)^{\ell} \ell\, \varepsilon^b \quad \text{on the circle: } |t - \alpha(g)^{-1}| = \varepsilon,$$

as soon as $0 < \varepsilon < 1$. Therefore there exists a positive $\varepsilon_2$ such that

$$|h_p(t)| < |\Phi_g(t)| \quad \text{when} \quad p \in \mathcal{P}(g,\varepsilon^{b+1}) \quad \text{and} \quad |t - \alpha(g)^{-1}| = \varepsilon\ , \tag{106}$$

as soon as $0 < \varepsilon < \varepsilon_2$. By a well known lemma (cf.,e.g.,[28], § 3.42), (106) implies that $\Phi_p$ has a root $\kappa(p)^{-1}$ satisfying the inequality $|\kappa(p)^{-1} - \alpha(g)^{-1}| \leq \varepsilon$. This implies the assertion of lemma 5 .

<u>Lemma 6</u>. If $\gamma > 1$ , then there are two positive numbers $c_o$ and $\bar{\nu}_o$ such that

$$a_2(\nu;\delta,t_o) > c_o\nu^3 \quad \text{when} \quad \nu > \bar{\nu}_o . \tag{107}$$

Proof. Let $\varepsilon = \nu^{-4}$, $\lambda = 4\varepsilon(\nu+1)$ , and define a finite set

$$\mathcal{L} = \{ j \mid j \in \mathbb{N} , \exp((j+1)\lambda) \leq \gamma \exp(-\varepsilon(2\nu+1))\} .$$

Obviously, there are $c_9 > 0$ and $\nu_1 > 0$ such that

$$|\mathcal{L}| > c_9\nu^3 \quad \text{when} \quad \nu > \nu_1 , \tag{108}$$

where $|\mathcal{L}|$ denotes the cardinality of $\mathcal{L}$ . In notations of (103), let

$$Q_j(\nu) =: \mathcal{P}(g,\varepsilon^{b+2};\gamma^\nu \exp(j\lambda + \nu\varepsilon),\gamma^\nu \exp((j+1)\lambda + \nu\varepsilon)) ,$$

and let

$$Q(\nu) =: \mathcal{P}(g, \varepsilon^{b+2}; (\gamma \exp \varepsilon)^{\nu}, (\gamma \exp(-\varepsilon))^{\nu+1}) .$$

It follows from the definition of $\mathcal{L}$ that

$$Q_j(\nu) \subseteq Q(\nu) \quad \text{for} \quad j \in \mathcal{L} . \tag{109}$$

By (44) in Proposition 5, one can find $\nu_2$ such that

$$|Q_j(\nu)| \geq 1 \quad \text{when} \quad \nu > \nu_2 , \ j \in \mathcal{L} . \tag{110}$$

Since $Q(\nu) \subseteq \mathcal{P}(g, \varepsilon^{b+2})$, it follows from lemma 5 that for each $p$ in $Q(\nu)$ there exists $\kappa(p)$ satisfying (104) and such that $\Phi_p(\kappa(p)^{-1}) = 0$, as soon as $\varepsilon < \varepsilon_o$. Let $\kappa(p) = |p|^{s(p)}$. It follows from (104) that

$$\frac{1}{\nu+1} < \operatorname{Re} s(p) \leq \frac{1}{\nu} \quad \text{when} \quad p \in Q(\nu) . \tag{111}$$

If

$$\frac{2\pi}{\nu \log \gamma} < \delta \quad , \tag{112}$$

then we can choose $s(p)$ in such a way that

$$t_o < \operatorname{Im} s(p) \leq t_o + \delta . \tag{113}$$

In view of (111), (113) and (36) with $M = \nu+1$, we conclude that for each $p$ in $Q(\nu)$ the function $Z_N(s)$ has a pole $s(p)$ in $D_\nu(\delta, t_o)$ as soon as $\nu$ satisfies (112) and the inequality

$$\varepsilon = \nu^{-4} < \varepsilon_o. \tag{114}$$

Moreover, since $\mathrm{Re}\, s(p) = (\log|\kappa(p)|)(\log|p|)^{-1}$, it follows from (104) that if (112), (114) hold, then condition

$$s(p) = s(q) \ ,\ p \in Q(\nu) \ ,\ q \in Q(\nu)$$

implies an inequaltiy

$$\big|\log|p| - \log|q|\big| \leq 2\varepsilon(\nu+1) \ .$$

Therefore, since $\lambda > 2\varepsilon(\nu+1)$ by definition,

$$s(p) \neq s(q) \quad \text{when} \quad p \in Q_j(\nu),\ q \in Q_{j'}(\nu), |j-j'| \geq 2 \ , \tag{115}$$

as soon as (112), (114) are satisfied. In view of (115), (110) and (108), we can choose

$$\bar{\nu}_o = \max\,\{\nu_1, \nu_2, \delta^{-1}(\log\gamma)^{-1}(2\pi),\ \varepsilon_o^{-1/4}\}$$

and obtain (107) with $c_o = \frac{1}{2} c_9$ . This proves the lemma.

Proof of Proposition 7. It follows from (38.2) and lemma 2 that, in notations of (38),

$$R_{N,M}(s)T_{N,M}(s) \neq 0 \quad \text{for} \quad s \in D_\nu(\delta,t_o) \ . \qquad (116)$$

By (101) of lemma 4 and (107) of lemma 6, there exists $\nu_o$ for which

$$a_2(\nu;\delta,t_o) > a_1(\nu;\delta,t_o) \quad \text{when} \quad \nu > \nu_o \ . \qquad (117)$$

The assertion of Proposition 7 follows from (116), (117) and (39).

Corollary 1. If $\Phi$ is not unitary, then $\mathbb{C}^o = \{s \mid \mathrm{Re}\ s = 0\}$ is the natural boundary of the function $s \mapsto L(s,\Phi)$ defined in $\mathbb{C}_+$ by a sequence of equations (39) when $M$ varies over $\mathbb{N}$ .

Proof. Let $s \in \mathbb{C}^o$ . Each neighbourhood of $s$ contains $D_\nu(\delta,t_o)$ for some $\delta$ in $\mathbb{R}_+$ , some $t_o$ in $\mathbb{R}$ , and some $\nu > \nu_o$ ; therefore, by Proposition 7, it contains a pole of $L(s,\Phi)$ . Thus $\mathbb{C}^o$ is contained in the closure of the set of poles of $L(s,\Phi)$ , and the assertion follows.

Theorem 1 follows from Proposition 3, Proposition 4 and Corollary 1 .

## § 6. On scalar product of L - functions; proof of theorem 2.

We start with a few simple remarks concerning convolutions of L - functions (cf. [20]; [21], Ch.II § 1,2). Given $r$ power series

$$f_j(t) = \sum_{n=0}^{\infty} a(n,j)t^n , \quad 1 \leq j \leq r ,$$

one defines their Hadamard convolution (cf. [6]) by letting

$$(f_1 * \ldots * f_r)(t) =: \sum_{n=0}^{\infty} \left(\prod_{j=1}^{r} a(n,j)\right)t^n . \tag{118}$$

The following assertion can be deduced by simple calculations in the ring $\mathbb{C}[[t]]$ of formal power series with constant coefficients (cf. [20], § 3).

<u>Proposition 8</u>. Suppose that $f_j$ , $1 \leq j \leq r$ , has the form

$$f_j(t) = \prod_{i=1}^{d_j} (1 - \alpha(i,j)t)^{-1} , \quad \alpha(i,j) \in \mathbb{C} , \tag{119}$$

and let

$$d = \prod_{j=1}^{r} d_j , \quad d_1 \geq \ldots \geq d_r , \quad n = \sum_{j=1}^{r} d_j - r+1 . \tag{120}$$

The following identity holds formally in $\mathbb{C}[[t]]$ :

$$(f_1 * \ldots * f_r)(t) = (f_1 \bullet \ldots \bullet f_r)(t)h(t) ,$$

$$h(t) \equiv 1 \pmod{t^2} , \qquad (121)$$

where $h(t)$ is a polynomial of degree not higher than $d-1$ and

$$(f_1 \bullet \ldots \bullet f_r)(t) =: \prod_{\nu} \left(1 - t \prod_{j=1}^{r} \alpha(\nu(j),j)\right)^{-1} \qquad (122)$$

with $\nu$ ranging over the set of sequences

$$\{(\nu(1),\ldots,\nu(r)) \mid 1 \leq \nu(j) \leq d_j , \nu(j) \in \mathbb{N}\} .$$

In particular, if $f_j(t) = (1-t)^{-d_j}$ , $1 \leq j \leq r$ , so that $\alpha(i,j) = 1$ for each pair $(i,j)$ , then

$$(f_1 * \ldots * f_r)(t) = (1-t)^{-n} h_r(t) ,$$

$$h_r(t) \equiv 1 + (d-n)t \pmod{t^2} , \qquad (123)$$

where $h_r(t)$ is a polynomial of degree not higher than $n-d_1$ .

<u>Corollary 2</u>. If $r \geq 2$ and condition (12) is satisfied, then the polynomial $h_r(t)$ in (123) has a root $\beta$ with $|\beta| < 1$ .

Proof. By (123), we can write

$$h_r(t) = \prod_{j=1}^{n-d_1} (1+\beta_j t) \quad , \quad \sum_{j=1}^{n-d_1} \beta_j = d - n \ ,$$

so that

$$\max_j |\beta_j| \geq \frac{d-n}{n-d_1} \quad .$$

On the other hand, conditions $r \geq 2$ and (12) imply the inequality

$$\frac{d-n}{n-d_1} > 1 \ ,$$

and the assertion follows.

To prove theorem 2 let, for $\rho \in X_1$ ,

$$f_p(\rho, t) = \det(1-t\,\rho(\sigma_p))^{-1}$$

and let, in notations of (10) ,

$$f_p(\vec{\chi}, t) = f_p(\rho_1, t) * \ldots * f_p(\rho_r, t)$$

and

$$f_p^o(\vec{\chi},t) = f_p(\rho_1,t)\cdot \ \dots \ \cdot f_p(\rho_r,t) \ ,$$

where $p$ ranges over prime divisors of $k$ . Let, furthermore,

$$\rho = \rho_1 \otimes \dots \otimes \rho_r$$

and let

$$S(\vec{\chi}) =: \bigcup_{j=1}^{r} S(\rho_j) \ .$$

By (122),

$$f_p^o(\vec{\chi},t) = \det(1-t\rho_1(\sigma_p) \otimes \dots \otimes \rho_r(\sigma_p))^{-1}$$

therefore, recalling (4) and the definition of $S(\rho)$ , we get

$$f_p^o(\vec{\chi},t) = f_p(\rho,t) \quad \text{for} \quad p \notin S(\vec{\chi}) \ . \tag{124}$$

By (121), there is $h_p(t)$ in $\mathbb{C}[t]$ for which

$$f_p(\vec{\chi},t) = f_p^o(\vec{\chi},t)h_p(t) \ . \tag{125}$$

Lemma 7. There exists a polynomial $\Phi \in Y[t]$ such that $S(\Phi) \subseteq S(\vec{\chi})$ and

$$h_p(t) = \Phi_p(t) \quad \text{for} \quad p \notin S(\vec{\chi}) .$$

Moreover, if $r \geq 2$ and (12) holds, then $\Phi$ is not unitary.

Proof. Let $T^m A$ and $\Lambda^m A$ denote the m-th symmetric and exterior powers of a linear operator $A$ in a finite dimensional complex vector space. By well known identities of linear algebra,

$$\det(1+At) = \sum_{m=0}^{\infty} t^m \operatorname{tr}(\Lambda^m A) \ , \ \det(1-At)^{-1} = \sum_{m=0}^{\infty} t^m \operatorname{tr}(T^m A)$$

in $\mathbb{C}[[t]]$. Since, by Proposition 8, the degree of $h_p(t)$ does not exceed $d-1$, it follows from (124) and (125) that $h_p(t) = \Phi_p(t)$ for $p \notin S(\vec{\chi})$, where $\Phi(t) = 1 + \sum_{\ell=1}^{d-1} b_\ell t^\ell$ with

$$b_\ell = \sum_{\ell_1=0}^{\ell} (-1)^{\ell_1} \operatorname{tr}(\Lambda^{\ell_1} \rho) \prod_{j=1}^{r} \operatorname{tr}(T^{\ell-\ell_1} \rho_j) .$$

In particular, taking $g$ to be the unit element in $W_1(K|k)$ one obtains $\Phi_g(t) = ((1-t)^{-d_1} * \ldots * (1-t)^{-d_r})(1-t)^d$. Therefore, by Corollary 2, $\Phi$ is not unitary when $r \geq 2$ and (12) holds. This proves the lemma.

We notice now that, by definition,

$$L(s,\vec{\chi}) = \prod_p f_p(\vec{\chi},|p|^{-s}) ,$$

where $p$ varies over prime divisors of $k$ . Therefore (124) and (125) give

$$L(s,\vec{\chi}) = L(s,\rho) \prod_{p\in S(\vec{\chi})} \ell_p(|p|^{-s}) \prod_p h_p(|p|^{-s}) , \qquad (126)$$

where

$$\ell_p(t) = f_p^o(\vec{\chi},t) \det (1-t\rho(\sigma_p)) . \qquad (127)$$

It follows from (126), (127), lemma 7 and Theorem 1 that the function $s\mapsto L(s,\vec{\chi})$ can be continued meromorphically to $\mathbb{C}_+$ and has a natural boundary $\mathbb{C}^o$ when $r \geq 2$ and (12) holds. If $r = 1$ , then $L(s,\vec{\chi}) = L(s,\rho_1)$ by definition. Suppose that $r \geq 2$ and (12) doesn't hold. It follows from (10) that $L(s,\vec{\chi}) = L(s,\rho)$ if $d_2 = 1$ . In the remaining fall $d_1=d_2=2$ and either $r = 2$ or $d_3=1$ , so taking $\rho_2' = \rho_2 \prod_{j\geq 3} \rho_j$ when $r > 2$ we reduce the problem to the case $d_1=d_2=r=2$ . In this case, however, a direct calculation shows that $f_p(\chi,t) = f_p(\rho,t)(1-t^2\det\rho(\sigma_p))$ for $p \notin S(\vec{\chi})$ . This completes the proof of theorem 2.

We should like to conclude this article with a few remarks concerning scalar products of L - functions "mit Grössencharakteren". Let $k_j$ , $1 \leq j \leq r$ , be an extension of $k$ and let $d_j = [k_j : k]$ denote its degree; let $\chi_j \in Gr(k_j)$ . We define the scalar product $L(s,\vec{\chi})$ of L - functions Hecke (14) by equation (10).

<u>Corollary 3</u>. Suppose that $d_1 \geq \ldots \geq d_r$ . The function $s \mapsto L(s,\vec{\chi})$ can be meromorphically continued to $\mathbb{C}_+$ . If $r \geq 2$ and (12) holds, then $\mathbb{C}^o$ is the natural boundary of this function. If either $r = 1$ or $r \geq 2$ but (12) does not hold, then $L(s,\vec{\chi})$ is meromorphic in $\mathbb{C}$ .

Proof. Regard $\chi_j$ as an one-dimensional representation of $W(k_j)$ , $1 \leq j \leq r$ , denote by $\rho_j$ the representation of $W(k)$ induced by $\chi_j$ and apply theorem 2 taking into account that

$$L(s, \chi_j) = L(s, \rho_j) \ , \ 1 \leq j \leq r \ ,$$

as formal Dirichlet series.

Remark 4. One can prove (cf. [20] or [21], Ch.II § 3) that, in fact,

$$L(s,\vec{\chi}) = \prod_{i=1}^{t} L(s, \psi_i) L(s,\Phi)^{-1} \ell(s,\vec{\chi}) \ ,$$

where $\Phi(t) \in Y[t]$ , $\ell(s,\vec{\chi}) = \prod_{p \in S_o} \ell_p(|p|^{-s})$ for a finite set $S_o$ of prime divisors in $k$ , $\ell_p(t)$ is a rational function of $t$ and $\psi_i \in Gr(K_i)$ , $k \subseteq K_i \leq K$ , $K$ denote the smallest Galois extension of $k$ containing $\hat{k} = k_1 \cdot \ldots \cdot k_r$ , the composite field of $k_1,\ldots,k_r$ . Moreover, if $k_1,\ldots,k_r$ are linearly disjoint over $k$ , so that $[\hat{k} : k] = \prod_{j=1}^{r} d_j$ , then $t = 1$ , $K_1 = \hat{k}$ and

$$\psi_1 = \prod_{j=1}^{r} \chi_j \circ N_{K_1/k_j} \quad .$$

A more careful calculation (cf. [21], p.90, Corollary 2) shows that $\ell(s,\vec{\chi}) = 1$ when the fields $k_1,\ldots,k_r$ are arithmetically independent over $k$ . We say that $k_1,\ldots,k_r$ are arithmetically independent over $k$ (cf.[16], [22]) when $[\hat{k} : k] = \prod_{j=1}^{r} d_j$ and

$$(e_i(\mathfrak{p}_i), e_j(\mathfrak{p}_j)) = 1 \quad \text{whenever} \quad 1 \leq i < j \leq r , \quad \mathfrak{p}_i \mid p , \ \mathfrak{p}_j \mid p$$

for each prime divisor $p$ in $k$ , where $\mathfrak{p}_j$ ranges over prime divisors in $k_j$ and $e_j(\mathfrak{p}_j)$ denotes the ramification index of $\mathfrak{p}_j$ in the extension $k_j \supseteq k$ . In particular, if $r=2$ , $d_1=d_2=2$ , and the discriminants of the quadratic extensions $k_1 \supseteq k$ and $k_2 \supseteq k$ are coprime, then (cf. [19])

$$L(s,\vec{\chi}) = L(s,\psi)L(2s,\psi_o)^{-1},$$

where $\psi = (\chi_1 \circ N_{\hat{k}/k_1})(\chi_2 \circ N_{\hat{k}/k_2})$ and $\psi_o$ is a grossencharacter in $k$ (depending on $\chi_1$ and $\chi_2$ ).

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