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ALGEBRAIC K-THEORY, K-REGULARITY, AND $\mathbb{T}$-DUALITY OF $\mathcal{O}_\infty$-STABLE C^*-ALGEBRAS

SNIGDHAYAN MAHANTA

Dedicated to Professor Marc A. Rieffel on the occasion of his 75th birthday.

Abstract. We study the algebraic K-theory of certain $\mathcal{O}_\infty$-stable C^*-algebras. The key ingredient is the canonical comparison map from algebraic to topological K-theory of C^*-algebras, which is shown to be an isomorphism for $\mathcal{O}_\infty$-stable C^*-algebras. As a consequence we obtain an explicit description of the algebraic K-theory of $ax+b$-semigroup C^*-algebras coming from number theory and that of $\mathcal{O}_\infty$-stabilized noncommutative tori. Along the way we obtain a new functorial model for the topological K-theory spectrum of a C^*-algebra as well as K-regularity for $\mathcal{O}_\infty$-stable C^*-algebras. The article finishes by extending some earlier results of the author on topological $\mathbb{T}$-duality and noncommutative motives.

Introduction

It is well-known that the algebraic K-theory groups of number fields (and rings) contain a lot of number theoretic information [31]. Now there is also a functorial construction of a purely infinite separable $ax+b$-semigroup C^*-algebra starting from any number ring [12, 23]. While complete computation of the algebraic K-theory of number rings is a formidable task (see, for instance, [43] for a survey), we are able to carry out an explicit computation of the algebraic K-theory of $ax+b$-semigroup C^*-algebras associated to number rings (cf. Theorem 2.1). Noncommutative tori constitute arguably the most widely studied class of noncommutative spaces. Geometric invariants of them were studied extensively by Connes and Rieffel (see, for instance, [6, 7, 34]). They are also relevant from the viewpoint of number theory via the real multiplication program [27, 28]. We show in the sequel that the algebraic K-theory of noncommutative tori are explicitly computable after $\mathcal{O}_\infty$-stabilization (cf. Theorem 2.2). Using some powerful results of Rieffel [34], one also obtains a clear understanding of the elements in the algebraic K-theory groups (in low degrees).

Many of our results rely on the canonical comparison map from algebraic to topological K-theory, which was conjectured to be an isomorphism for stable C^*-algebras by Karoubi [19]. The conjectures were eventually proved by Suslin–Wodzicki [39, 40], building upon some earlier work [11, 17]. In an unpublished manuscript Cortiñas–Phillips [8] recently showed that the comparison map is also an isomorphism for purely infinite C^*-algebras. In the sequel a similar result is proven using rather elementary methods. More precisely, it is shown that the comparison map is an isomorphism for $\mathcal{O}_\infty$-stable C^*-algebras (cf. Theorem 1.2), which can easily be generalized to a much wider class of C^*-algebras (see the Remark below). The class of $\mathcal{O}_\infty$-stable C^*-algebras turns out to be enough to carry out the computations

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mentioned above. We obtain a new functorial model for the topological K-theory spectrum of a C^*-algebra (cf. Theorem 3.3). Using the same circle of ideas we show that $A\hat{\otimes}\mathcal{O}_\infty$ is K-regular for any C^*-algebra A (cf. Theorem 4.1), supporting a conjecture of Rosenberg [37]. Finally, in Section 5 some results (Theorem 5.4 and Theorem 5.6) are proven extending the authors earlier work on topological $\mathbb{T}$-duality [26]. The generic nature of the results are as follows: if certain stable or $\mathcal{O}_\infty$-stable C^*-algebras are KK-equivalent, then their noncommutative motives (constructed earlier by the author) are isomorphic. It so happens that $\mathbb{T}$-duality is a rich source of KK-equivalence between C^*-algebas.

Remark. Most of the arguments below are based on one simple trick (see Lemma 1.1). The range of applicability of this trick is much wider than the case explored here (see, for instance, Proposition 1.1.2. of [35]). In fact, Theorem 1.2 effortlessly generalizes to all properly infinite C^*-algebras using similar arguments. The author is grateful to D. Enders for pointing it out.

Notations and Conventions: In the sequel we denote the category of all C^*-algebras by $\mathtt{C}^*$ and $\hat{\otimes}$ stands for the minimal C^*-tensor product. We denote by $\mathbf{K}(-)$ [resp. $\mathrm{K}_n(-)$] the nonconnective algebraic K-theory spectrum [resp. algebraic K-theory group] functor on $\mathtt{C}^*$. Finally, we are going to denote by $\mathtt{hSp}$ the triangulated stable homotopy category. All spaces are assumed to be Hausdorff.

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1. The comparison map

Thanks to the Karoubi conjecture we know that the nonconnective algebraic K-theory of a stable C^*-algebra is isomorphic to its topological K-theory. In fact, there is a canonical comparison map $c_n(A) : \mathrm{K}_n(A) \to \mathrm{K}_n^{\mathrm{top}}(A)$ that induces the isomorphism when A is stable [19] (see also [37]). The comparison map $c_0(A) : \mathrm{K}_0(A) \to \mathrm{K}_0^{\mathrm{top}}(A)$ is always an isomorphism. Recall that the Cuntz algebra $\mathcal{O}_\infty$ is the universal unital C^*-algebra generated by a set of isometries $\{s_i \,|\, i \in \mathbb{N}\}$ with mutually orthogonal range projections $s_i s_i^*$ [10]. Observe that $\mathcal{O}_\infty$ is a unital C^*-algebra, so that $\mathcal{O}_\infty$-stabilization preserves unitality (unlike $\mathbb{K}$-stabilization). The following lemma is crucial and it demonstrates that $\mathcal{O}_\infty$ possesses enough *built-in stability.*

Lemma 1.1. There is a commutative diagram in $\mathtt{C}^*$

(1)
$$\begin{array}{ccc} \mathcal{O}_\infty & \xrightarrow{\quad\iota\quad} & \mathcal{O}_\infty \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\kappa} & \\ & \mathcal{O}_\infty\hat{\otimes}\mathbb{K}, & \end{array}$$

where the top horizontal arrow $\iota : \mathcal{O}_\infty \to \mathcal{O}_\infty$ is an inner endomorphism.

Proof. Observe that the subset $\{s_i s_j^* \,|\, i,j \in \mathbb{N}\} \subset \mathcal{O}_\infty$ generates a copy of the compact operators $\mathbb{K}$ inside $\mathcal{O}_\infty$. Consider the $*$-homomorphism $\kappa : \mathcal{O}_\infty \hat{\otimes} \mathbb{K} \to \mathcal{O}_\infty$, which is defined as $a \otimes e_{ij} \mapsto s_i a s_j^*$. Due to the simplicity of all the C^*-algebras in sight, κ is injective. Let $\theta : \mathcal{O}_\infty \to \mathcal{O}_\infty \hat{\otimes} \mathbb{K}$ be simply the corner embedding, sending $a \mapsto a \otimes e_{11}$. The composite $\iota = \kappa\theta$ is given by $\iota(a) = s_1 a s_1^*$. This $*$-homomorphism is manifestly inner. □

Theorem 1.2. For any C^*-algebra A the comparison map

$$c_n(A\hat{\otimes}\mathcal{O}_\infty) : \mathrm{K}_n(A\hat{\otimes}\mathcal{O}_\infty) \to \mathrm{K}_n^{\mathrm{top}}(A\hat{\otimes}\mathcal{O}_\infty)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. Using the above Lemma 1.1, we start with the commutative diagram in $\mathtt{C}^*$

$$\begin{array}{ccccc} \mathcal{O}_\infty & & \xrightarrow{\quad\iota\quad} & & \mathcal{O}_\infty \\ & \searrow^{\theta} & & \nearrow^{\kappa} & \\ & & \mathcal{O}_\infty\hat{\otimes}\mathbb{K}, & & \end{array} \tag{2}$$

where the top horizontal arrow $\iota : \mathcal{O}_\infty \to \mathcal{O}_\infty$ is simply the inner endomorphism $a \mapsto s_1 a s_1^*$. Let us first assume that A is a unital C^*-algebra. After taking the minimal C^*-tensor product of the above diagram with any unital A we obtain

$$\begin{array}{ccccc} A\hat{\otimes}\mathcal{O}_\infty & & \xrightarrow{\quad \mathrm{id}\hat{\otimes}\iota \quad} & & A\hat{\otimes}\mathcal{O}_\infty \\ & \searrow^{R:=\mathrm{id}\hat{\otimes}\theta} & & \nearrow^{S:=\mathrm{id}\hat{\otimes}\kappa} & \\ & & A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}. & & \end{array} \tag{3}$$

It is known that if F is a matrix stable functor on $\mathtt{C}^*$ and f is an inner endomorphism in $\mathtt{C}^*$, then $F(f)$ is the identity map (see, for instance, Proposition 3.16. of [14]). Now applying the functors $\mathrm{K}_n(-)$, $\mathrm{K}_n^{\mathrm{top}}(-)$ and using the naturality of c_n, we get a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}_n(A\hat{\otimes}\mathcal{O}_\infty) & \xrightarrow{\mathrm{K}_n(R)} & \mathrm{K}_n(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}) & \xrightarrow{\mathrm{K}_n(S)} & \mathrm{K}_n(A\hat{\otimes}\mathcal{O}_\infty) \\ \downarrow{\scriptstyle c_n(A\hat{\otimes}\mathcal{O}_\infty)} & & \downarrow{\scriptstyle c_n(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K})} & & \downarrow{\scriptstyle c_n(A\hat{\otimes}\mathcal{O}_\infty)} \\ \mathrm{K}_n^{\mathrm{top}}(A\hat{\otimes}\mathcal{O}_\infty) & \xrightarrow{\mathrm{K}_n^{\mathrm{top}}(R)} & \mathrm{K}_n^{\mathrm{top}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}) & \xrightarrow{\mathrm{K}_n^{\mathrm{top}}(S)} & \mathrm{K}_n^{\mathrm{top}}(A\hat{\otimes}\mathcal{O}_\infty). \end{array} \tag{4}$$

Since $S \circ R$ is the inner endomorphism $\mathrm{id}\hat{\otimes}\iota : A\hat{\otimes}\mathcal{O}_\infty \to A\hat{\otimes}\mathcal{O}_\infty$, we conclude that $\mathrm{K}_n(S) \circ \mathrm{K}_n(R)$ is the identity map due to the matrix stability of algebraic K-theory on the category of unital C^*-algebras. Moreover, $\mathrm{K}_n^{\mathrm{top}}(S) \circ \mathrm{K}_n^{\mathrm{top}}(R)$ is also the identity map due to the matrix stability of $\mathrm{K}_n^{\mathrm{top}}(-)$. The assertion for unital A now follows by a simple diagram chase. Indeed, it is easily seen that $\mathrm{K}_n(R)$ must be injective and $\mathrm{K}_n^{\mathrm{top}}(S)$ must be surjective. Since $A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ is stable, we conclude that $c_n(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K})$ is an isomorphism. Thus $c_n(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}) \circ \mathrm{K}_n(R)$ is injective whence so is $c_n(A\hat{\otimes}\mathcal{O}_\infty)$ (the left vertical one). Similarly, $\mathrm{K}_n^{\mathrm{top}}(S) \circ c_n(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K})$ is surjective whence so is $c_n(A\hat{\otimes}\mathcal{O}_\infty)$ (the right vertical one).
The proof for nonunital A follows by a simple excision argument (see, for example, Proposition 3.1 below). □

Remark 1.3. The above result is actually not new. Indeed, in an unpublished manuscript [8] Cortiñas–Phillips have shown that the comparison map $c_n(A) : \mathrm{K}_n(A) \to \mathrm{K}_n^{\mathrm{top}}(A)$ is an isomorphism for any purely infinite C^*-algebra A. Notice that the C^*-algebra $A\hat{\otimes}\mathcal{O}_\infty$ is purely infinite for any $A \in \mathtt{C}^*$ [21]. Although the above result is weaker, the arguments are much simpler.

2. Algebraic K-theory of certain $\mathcal{O}_\infty$-stable C^*-algebras

As an application we now explicitly compute the algebraic K-theory groups of certain $\mathcal{O}_\infty$-stable C^*-algebras. It must be noted that complete calculation of the algebraic K-theory groups of an arbitrary ring is an extremely difficult task in general.

2.1. **Certain semigroup C^*-algebras.** A recent result of Li asserts that for a countable integral domain R with vanishing Jacobson radical (which is, in addition, not a field) the left regular $ax+b$-semigroup C^*-algebra $C^*_\lambda(R \rtimes R^\times)$ is $\mathcal{O}_\infty$-absorbing, i.e., $C^*_\lambda(R \rtimes R^\times)\hat{\otimes}\mathcal{O}_\infty \cong C^*_\lambda(R \rtimes R^\times)$ (see Theorem 1.3 of [24]).

Now we focus on the main object of our interest, namely, the left regular $ax+b$-semigroup C^*-algebra of the ring of integers R of a number field K. It is shown in [13] that

$$\mathrm{K}_*^{\mathrm{top}}(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{[X]\in G\backslash\mathcal{I}} \mathrm{K}_*^{\mathrm{top}}(C^*(G_X)),$$

where $\mathcal{I}$ is the set of fractional ideal of R, $G = K \rtimes K^\times$, and G_X is the stabilizer of X under the G-action on $\mathcal{I}$. The orbit space $G \setminus \mathcal{I}$ can be identified with the ideal class group of K. As a consequence of Theorem 1.2 we obtain

Theorem 2.1. The algebraic K-theory of the $ax + b$-semigroup C^*-algebra of the ring of integers R of a number field K is 2-periodic and explicitly given by

$$\mathrm{K}_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{[X]\in G\backslash\mathcal{I}} \mathrm{K}_*^{\mathrm{top}}(C^*(G_X)).$$

2.2. **$\mathcal{O}_\infty$-stabilized noncommutative tori.** We recall some basic material before stating our result. A good reference for generalities on noncommutative tori is Rieffel's survey [34]. For any real-valued skew bilinear form θ on $\mathbb{Z}^n$ ($n \geqslant 2$) the C^*-algebra of the noncommutative n-torus A^n_θ can be defined as the universal C^*-algebra generated by unitaries $U_x \in \mathbb{Z}^n$ subject to the relation

$$U_xU_y = \exp(\pi i\theta(x, y))U_{x+y} \qquad \forall x, y \in \mathbb{Z}^n.$$

Using the Pimnser–Voiculescu exact sequence one can compute the $\mathrm{K}^{\mathrm{top}}$-theory of A^n_θ as an abelian group, namely,

$$\mathrm{K}_0^{\mathrm{top}}(A^n_\theta) \simeq \mathbb{Z}^{2^{n-1}} \quad \text{and} \quad \mathrm{K}_1^{\mathrm{top}}(A^n_\theta) \simeq \mathbb{Z}^{2^{n-1}}. \tag{5}$$

Theorem 2.2. The algebraic K-theory of the $\mathcal{O}_\infty$-stabilized noncommutative n-torus A^n_θ is 2-periodic and explicitly given by

$$\mathrm{K}_0(A^n_\theta\hat{\otimes}\mathcal{O}_\infty) \simeq \mathbb{Z}^{2^{n-1}} \quad \text{and} \quad \mathrm{K}_1(A^n_\theta\hat{\otimes}\mathcal{O}_\infty) \simeq \mathbb{Z}^{2^{n-1}}.$$

Proof. By Theorem 1.2 one has an isomorphism $\mathrm{K}_*(A^n_\theta\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{K}_*^{\mathrm{top}}(A^n_\theta\hat{\otimes}\mathcal{O}_\infty)$. Using the Künneth Theorem one now computes that $\mathrm{K}_*^{\mathrm{top}}(A^n_\theta\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{K}_*^{\mathrm{top}}(A^n_\theta)$. Observe that $\mathrm{K}_0^{\mathrm{top}}(\mathcal{O}_\infty) \simeq \mathbb{Z}$ and $\mathrm{K}_1^{\mathrm{top}}(\mathcal{O}_\infty) \simeq 0$ and all C^*-algebras in sight belong to the UCT-class. Now use Equation (5). ☐

Remark 2.3. It follows from [33] that for irrational θ the projections in A_θ^n generate all of $\mathrm{K}_0(A_\theta^n \hat{\otimes} \mathcal{O}_\infty)$ and

$$\mathrm{K}_1(A_\theta^n \hat{\otimes} \mathcal{O}_\infty) \cong \mathrm{K}_1^{\mathrm{top}}(A_\theta^n \hat{\otimes} \mathcal{O}_\infty) \cong \mathrm{K}_1^{\mathrm{top}}(A_\theta^n) \xleftarrow{\sim} UA_\theta^n / U^0 A_\theta^n.$$

Here UA_θ^n denotes the group of unitary elements in A_θ^n and $U^0 A_\theta^n$ denotes the connected component of the identity element of UA_θ^n. Thus one obtains a good description of the elements of the algebraic K-theory groups in low degrees in terms of projections and unitaries.

3. The generalized homology theory $\mathbf{K}(-\hat{\otimes}\mathcal{O}_\infty)$

A functor $F : \mathtt{C}^* \longrightarrow \mathtt{hSp}$ is called *homotopy invariant* if it sends the evaluation at t map $\mathrm{ev}_t : A[0,1] \to A$ to an isomorphism in $\mathtt{hSp}$ for all $A \in \mathtt{C}^*$. Such a functor is called *excisive* if for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathtt{C}^*$ the induced diagram $F(A) \to F(B) \to F(C) \to \Sigma F(A)$ is an exact triangle in $\mathtt{hSp}$. A homotopy invariant excisive functor $F : \mathtt{C}^* \longrightarrow \mathtt{hSp}$ is called an $\mathtt{hSp}$-valued *generalized homology theory* on $\mathtt{C}^*$.

It is known that the algebraic K-theory functor acquires special properties after compact stabilization. We are going to show that the same is true after $\mathcal{O}_\infty$-stabilization.

Proposition 3.1. The functor $\mathbf{K}(-\hat{\otimes}\mathcal{O}_\infty) : \mathtt{C}^* \longrightarrow \mathtt{hSp}$ is an excisive functor.

Proof. It follows from the Suslin–Wodzicki Theorem [39, 40] that the functor $\mathbf{K}$ is excisive on $\mathtt{C}^*$. Since $\mathcal{O}_\infty$ is a nuclear C^*-algebra, the functor $-\hat{\otimes}\mathcal{O}_\infty$ preserves exactness in $\mathtt{C}^*$ whence $\mathbf{K}(-\hat{\otimes}\mathcal{O}_\infty)$ is excisive. □

Recall that a functor $F : \mathtt{C}^* \longrightarrow \mathtt{hSp}$ is called *split exact* if it sends a split exact sequence in $\mathtt{C}^*$ to a direct sum diagram in the additive category $\mathtt{hSp}$. It follows from the above proposition that the functors $\mathbf{K}(-)$ as well as $\mathbf{K}(-\hat{\otimes}\mathcal{O}_\infty)$ are split exact.

Proposition 3.2. The functor $\mathbf{K}(-\hat{\otimes}\mathcal{O}_\infty) : \mathtt{C}^* \longrightarrow \mathtt{hSp}$ is a homotopy invariant functor.

Proof. In the following commutative diagram

$$\begin{array}{ccc} \mathrm{K}_n(A\hat{\otimes}\mathcal{O}_\infty[0,1]) & \xrightarrow{\mathrm{ev}_*} & \mathrm{K}_n(A\hat{\otimes}\mathcal{O}_\infty) \\ \downarrow & & \downarrow \\ \mathrm{K}_n^{\mathrm{top}}(A\hat{\otimes}\mathcal{O}_\infty[0,1]) & \xrightarrow{\mathrm{ev}_*} & \mathrm{K}_n^{\mathrm{top}}(A\hat{\otimes}\mathcal{O}_\infty) \end{array}$$

both vertical arrows (Theorem 1.2) and the bottom horizontal arrow (homotopy invariance of $\mathrm{K}^{\mathrm{top}}$-theory) are known to be isomorphisms. This proves that the map $\mathbf{K}(A\hat{\otimes}\mathcal{O}_\infty[0,1]) \to \mathbf{K}(A\hat{\otimes}\mathcal{O}_\infty)$ of Ω-spectra induces an isomorphism on homotopy groups. Thus it is an isomorphism in $\mathtt{hSp}$. □

Theorem 3.3. The functor $\mathbf{K}(-\hat{\otimes}\mathcal{O}_\infty) : \mathtt{C}^* \longrightarrow \mathtt{hSp}$ is a generalized homology theory. Moreover, the functor $\mathbf{K}(-\hat{\otimes}\mathcal{O}_\infty)$ is a model for topological K-theory.

Proof. The first assertion is a consequence of Propositions 3.1 and 3.2. It follows from Theorem 1.2 that the natural comparison map induces an isomorphism $\pi_n(\mathbf{K}(A\hat{\otimes}\mathcal{O}_\infty)) \cong \mathrm{K}_n^{\mathrm{top}}(A\hat{\otimes}\mathcal{O}_\infty)$; since A and $A\hat{\otimes}\mathcal{O}_\infty$ are KK-equivalent, there are natural identifications

$$\pi_n(\mathbf{K}(A\hat{\otimes}\mathcal{O}_\infty)) \cong \mathrm{K}_n^{\mathrm{top}}(A\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{K}_n^{\mathrm{top}}(A)$$

for all $n \in \mathbb{Z}$. The second assertion is now clear. □

Remark 3.4. For any unital C^*-algebra A by applying the Waldhausen K-theory machine [42] on $A\hat{\otimes}\mathcal{O}_\infty$ one actually obtains a (highly structured) symmetric spectrum model for the topological K-theory of A [18]. The functor $\mathbf{K}(-\hat{\otimes}\mathcal{O}_\infty)$ is also C^*-stable.

4. K-regularity of $\mathcal{O}_\infty$-stable C^*-algebras

Let F be any functor on $\mathtt{C}^*$. A C^*-algebra A is called *F-regular* if the canonical inclusion $A \to A[t_1, \cdots, t_n]$ induces an isomorphism $F(A) \overset{\sim}{\to} F(A[t_1, \cdots, t_n])$ for all $n \in \mathbb{N}$. This map has a one-sided inverse induced by the evaluation map ev_0. Rosenberg conjectured that any C^*-algebra A is K_0-regular. Using the techniques developed to prove the Karoubi conjectures [17], it is shown in Theorem 3.4 of [37] that the conjecture is true if A is stable. In fact, the Theorem in ibid. asserts that a stable C^*-algebra is K_m-regular for all $m \in \mathbb{Z}$. A C^*-algebra is called K-*regular* if it is K_m-regular for all $m \in \mathbb{Z}$.

Theorem 4.1. The C^*-algebras $A\hat{\otimes}\mathcal{O}_\infty$ are K-regular for all $A \in \mathtt{C}^*$.

Proof. For the sake of better readability let us set $A_\infty := A\hat{\otimes}\mathcal{O}_\infty$ and $B^n := B[t_1, \cdots, t_n]$ for any $A, B \in \mathtt{C}^*$. Using excision we may assume that A is unital. Arguing as in the proof of Theorem 1.2 we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}_m(A_\infty) & \longrightarrow & \mathrm{K}_m(A_\infty\hat{\otimes}\mathbb{K}) & \longrightarrow & \mathrm{K}_m(A_\infty) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}_m((A_\infty)^n) & \longrightarrow & \mathrm{K}_m((A_\infty\hat{\otimes}\mathbb{K})^n) & \longrightarrow & \mathrm{K}_m((A_\infty)^n). \end{array} \tag{6}$$

Due to the stability of $A_\infty\hat{\otimes}\mathbb{K}$ the middle vertical arrow is an isomorphism. Moreover, the compositions of the top and the bottom horizontal arrows are again isomorphisms due to the matrix stability of the functor $\mathrm{K}_m(-)$ for unital algebras. Observe that the composite $*$-homomorphisms $A_\infty \to A_\infty\hat{\otimes}\mathbb{K} \to A_\infty$ and $(A_\infty)^n \to (A_\infty\hat{\otimes}\mathbb{K})^n \to (A_\infty)^n$ are still inner. Now a similar diagram chase as before enables one to conclude that the left vertical arrow must be an isomorphism. ☐

Remark 4.2. Purely infinite simple C^*-algebras like $\mathcal{O}_\infty$ can be regarded as *maximally noncommutative*. Rather surprisingly, one needs fairly sophisticated techniques to establish the K-regularity of commutative C^*-algebras (see [37, 9]).

5. Topological $\mathbb{T}$-duality and noncommutative motives

Let us recall very briefly axiomatic topological $\mathbb{T}$-duality from [4]. Let B be a topological base space. Consider the category of *pairs* (E, h), where $\pi : E \to B$ is a principal S^1-bundle over B and $h \in \mathrm{H}^3(E, \mathbb{Z})$. Two such pairs (E_1, h_1) and (E_2, h_2) are isomorphic if there is an isomorphism $F : E_1 \to E_2$ of principal bundles such that $F^*h_2 = h_1$. Two pairs (E_1, h_1) and (E_2, h_2) are said to be $\mathbb{T}$-*dual* if there is a Thom class $\mathbf{Th}$ for $S(V)$ such that $h_1 = i_1^*\mathbf{Th}$ and $h_2 = i_2^*\mathbf{Th}$. Here $S(V)$ is the sphere bundle of $V := E_1 \times_{S^1} \mathbb{C} \oplus E_2 \times_{S^1} \mathbb{C}$ and $i_k : E_k \to S(V)$ are the canonical maps for $k = 1, 2$. In ibid. Bunke–Schick showed that $B \mapsto \{\text{isom. classes of pairs over } B\}$ as a functor on topological spaces is representable. The representing space $\mathbf{E}$ supports a universal pair and any pair on B can be obtained via a pullback along some map $B \to \mathbf{E}$ (defined up to homotopy). Using the explicit construction of the universal object and the $\mathbb{T}$-dual of the universal pair the authors were able to prove the existence and uniqueness of $\mathbb{T}$-duality for S^1-bundles. If (E_1, h_1) and (E_2, h_2) are $\mathbb{T}$-dual, then there is an isomorphism of twisted K-theories:

$$\mathrm{K}^{\mathrm{od}}(E_1, h_1) \simeq \mathrm{K}^{\mathrm{ev}}(E_2, h_2) \quad \text{and} \quad \mathrm{K}^{\mathrm{ev}}(E_1, h_1) \simeq \mathrm{K}^{\mathrm{od}}(E_2, h_2). \tag{7}$$

The theory of topological $\mathbb{T}$-duality is not limited to S^1-bundles. However, for higher torus bundles the theory becomes more subtle [3, 5] and sometimes necessitates the use of noncommutative geometry [30]; moreover, C^*-algebras appear quite naturally in the context of twisted K-theory [36]. The readers may refer to [38] for a recent survey on the interaction between C^*-algebras, noncommutative geometry, and $\mathbb{T}$-duality.

We denote the category of separable C^*-algebras by $\mathtt{SC}^*$ and the bivariant K-theory category by $\mathtt{KK}$. There is a canonical functor $\iota : \mathtt{SC}^* \longrightarrow \mathtt{KK}$, which is identity on objects and admits a univeral characterization [16, 11]. Recall from [20] that there is a category of noncommutative motives $\mathtt{Hmo}_0$, whose objects are k-linear DG categories ($k = \mathbb{C}$ for our purposes). The theory of noncommutative motives is an active area of research with interesting applications to K-theory [41] as well as a wide variety of other mathematics [22, 29]. Building upon an earlier work of Quillen [32] the author constructed a functorial passage $\mathtt{HPf}_{\mathtt{dg}}$ from separable C^*-algebras to noncommutative motives and proved the following two results (amongst others) in [26]:

Theorem 5.1. There is a dotted functor below making the following diagram of categories commute (up to a natural isomorphism):

$$\begin{array}{ccc} \mathtt{SC}^* & \xrightarrow{A \mapsto A\hat{\otimes}\mathbb{K}} & \mathtt{SC}^* \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mathtt{HPf}_{\mathtt{dg}}} \\ \mathtt{KK} & \dashrightarrow & \mathtt{Hmo}_0. \end{array}$$

Theorem 5.2. For any $A \in \mathtt{SC}^*$ the homotopy groups of the nonconnective K-theory spectrum of $\mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathbb{K})$ are naturally isomorphic to the topological K-theory groups of A.

Remark 5.3. In [26] the author phrased the results in terms of $\mathtt{NCC}_{\mathtt{dg}}$, which was called the category of noncommutative DG correspondences. The category $\mathtt{NCC}_{\mathtt{dg}}$ is equivalent to $\mathtt{Hmo}_0$. Moreover, in Theorem 3.7 of ibid. actually the connective version of Theorem 5.2 was proven. However, the extension to the nonconnective version is straightforward.

The crucial insight of Rosenberg in [36] was that certain $\mathbb{K}$-bundles on locally compact spaces can be used to model twisted K-theory. More precisely, given any pair (E, h) with E locally compact one can construct a noncommutative stable C^*-algebra $\mathrm{CT}(E, h)$, whose topological K-theory is the twisted K-theory of the pair (E, h). This formalism extends to certain infinite dimensional spaces through the use of σ-C^*-algebras [25]. In [1, 2] the authors extended the formalism of $\mathbb{T}$-duality to C^*-algebras and showed that under favourable circumstances if B and B' are $\mathbb{T}$-dual C^*-algebras, then there is an invertible element in $\mathrm{KK}_0(B, \Sigma B')$ that implements the twisted K-theory isomorphism (as in (7)). Thanks to Theorem 5.1 we conclude that if two stable C^*-algebras B and B' are $\mathbb{T}$-dual, such that there is an invertible element $\alpha \in \mathrm{KK}_0(B, \Sigma B')$, then their noncommutative motives $\mathtt{HPf}_{\mathtt{dg}}(B)$ and $\mathtt{HPf}_{\mathtt{dg}}(B')$ are isomorphic in $\mathtt{Hmo}_0$. Moreover, Theorem 5.2 says that the invertible element implements the twisted K-theory isomorphism.

Theorem 5.4. If A and A' are isomorphic in $\mathtt{KK}$, then the noncommutative motives of $A\hat{\otimes}\mathcal{O}_\infty$ and $A'\hat{\otimes}\mathcal{O}_\infty$ are isomorphic in $\mathtt{Hmo}_0$.

Proof. Let $\alpha \in \mathrm{KK}_0(A, A')$ be an invertible element. Consider once again the commutative diagram

(8)
$$\begin{array}{ccc} A\hat{\otimes}\mathcal{O}_\infty & \xrightarrow{\mathrm{id}_A\hat{\otimes}\iota} & A\hat{\otimes}\mathcal{O}_\infty \\ & {\scriptstyle R:=\mathrm{id}_A\hat{\otimes}\theta}\searrow \quad \nearrow{\scriptstyle S:=\mathrm{id}_A\hat{\otimes}\kappa} & \\ & A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}. & \end{array}$$

Then from Theorem 5.1 one obtains a commutative diagram in $\mathtt{Hmo}_0$

(9)
$$\begin{array}{ccccc} \mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty) & \xrightarrow{\mathtt{HPf}_{\mathtt{dg}}(R)} & \mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}) & \xrightarrow{\mathtt{HPf}_{\mathtt{dg}}(S)} & \mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty) \\ & & \downarrow {\scriptstyle \beta=\mathtt{HPf}_{\mathtt{dg}}(\alpha\hat{\otimes}\mathrm{id}_{\mathcal{O}_\infty}\hat{\otimes}\mathrm{id}_{\mathbb{K}})} & & \\ \mathtt{HPf}_{\mathtt{dg}}(A'\hat{\otimes}\mathcal{O}_\infty) & \xrightarrow{\mathtt{HPf}_{\mathtt{dg}}(R')} & \mathtt{HPf}_{\mathtt{dg}}(A'\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}) & \xrightarrow{\mathtt{HPf}_{\mathtt{dg}}(S')} & \mathtt{HPf}_{\mathtt{dg}}(A'\hat{\otimes}\mathcal{O}_\infty). \end{array}$$

where R' and S' are defined in the obvious manner (replace A by A' in diagram 8). Since α is invertible, so are $\alpha\hat{\otimes}\mathrm{id}_{\mathcal{O}_\infty}$ and $\alpha\hat{\otimes}\mathrm{id}_{\mathcal{O}_\infty}\hat{\otimes}\mathrm{id}_{\mathbb{K}}$. Therefore, the middle vertical arrow β is an isomorphism. Observe that $S \circ R$ is a morphism in $\mathtt{SC}^*$. It was shown in Lemma 2.3 of [26] that the functor $\mathtt{HPf}_{\mathtt{dg}}(-)$ is matrix stable on $\mathtt{SC}^*$, whence by Lemma 1.1

$$\mathtt{HPf}_{\mathtt{dg}}(S) \circ \mathtt{HPf}_{\mathtt{dg}}(R) = \mathrm{id}_{\mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty)} \quad \text{and} \quad \mathtt{HPf}_{\mathtt{dg}}(S') \circ \mathtt{HPf}_{\mathtt{dg}}(R') = \mathrm{id}_{\mathtt{HPf}_{\mathtt{dg}}(A'\hat{\otimes}\mathcal{O}_\infty)}.$$

Thus the maps $\mathtt{HPf}_{\mathtt{dg}}(R)$ and $\mathtt{HPf}_{\mathtt{dg}}(R')$ possess left inverses. An inspection of diagram 9 reveals that it suffices to show that they also possess right inverses. The composite $*$-homomorphism $\mathbb{K} \overset{i}{\hookrightarrow} \mathcal{O}_\infty \overset{\theta}{\to} \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ defines an invertible element $\theta\circ i = \gamma \in \mathrm{KK}_0(\mathbb{K}, \mathcal{O}_\infty\hat{\otimes}\mathbb{K})$. Consequently, $\mathrm{id}_A\hat{\otimes}\gamma \in \mathrm{KK}_0(A\hat{\otimes}\mathbb{K}, A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K})$ is an invertible element. By Theorem 5.1 $\mathrm{id}_A\hat{\otimes}\gamma\hat{\otimes}\mathrm{id}_{\mathbb{K}} = (\mathrm{id}_A\hat{\otimes}\theta\hat{\otimes}\mathrm{id}_{\mathbb{K}}) \circ (\mathrm{id}_A\hat{\otimes}i\hat{\otimes}\mathrm{id}_{\mathbb{K}})$ induces an isomorphism

$$\mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathbb{K}\hat{\otimes}\mathbb{K}) \overset{\sim}{\to} \mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}\hat{\otimes}\mathbb{K}).$$

Let us set $I = \mathrm{id}_A\hat{\otimes}i$, so that $\mathtt{HPf}_{\mathtt{dg}}(R\hat{\otimes}\mathrm{id}_{\mathbb{K}}) \circ \mathtt{HPf}_{\mathtt{dg}}(I\hat{\otimes}\mathrm{id}_{\mathbb{K}})$ is the above isomorphism. Now consider the following commutative diagram

$$\begin{array}{ccc} A\hat{\otimes}\mathbb{K} & \longrightarrow & A\hat{\otimes}\mathbb{K}\hat{\otimes}\mathbb{K} \\ \downarrow {\scriptstyle I} & & \downarrow {\scriptstyle I\hat{\otimes}\mathrm{id}_{\mathbb{K}}} \\ A\hat{\otimes}\mathcal{O}_\infty & \longrightarrow & A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K} \\ \downarrow {\scriptstyle R} & & \downarrow {\scriptstyle R\hat{\otimes}\mathrm{id}_{\mathbb{K}}} \\ A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K} & \longrightarrow & A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}\hat{\otimes}\mathbb{K}. \end{array}$$

Here all the horizontal arrows are corner embeddings. Now the top and the bottom horizontal arrows are homotopic to isomorphisms. Since $\mathtt{HPf}_{\mathtt{dg}}(-)$ is homotopy invariant on stable C^*-algebras, it sends the top and the bottom horizontal arrows to isomorphisms. We already know that it sends $(R\hat{\otimes}\mathrm{id}_{\mathbb{K}})\circ(I\hat{\otimes}\mathrm{id}_{\mathbb{K}})$ to an isomorphism. It follows that $\mathtt{HPf}_{\mathtt{dg}}(R)$ has a right inverse. Similarly, one can prove that $\mathtt{HPf}_{\mathtt{dg}}(R')$ has a right inverse. □

Corollary 5.5. The functor $\mathtt{HPf}_{\mathtt{dg}}(-\hat{\otimes}\mathcal{O}_\infty)$ is C^*-stable.

Proof. For any separable C^*-algebra A the corner embedding $A \to A\hat{\otimes}\mathbb{K}$ is $\mathtt{KK}$-invertible. □

Now we prove the $\mathcal{O}_\infty$-analogue of Theorem 5.2.

Theorem 5.6. For any $A \in \mathtt{SC}^*$ the homotopy groups of the nonconnective K-theory spectrum of $\mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty)$ are naturally isomorphic to the topological K-theory groups of A.

Proof. By the above Corollary the nonconnective K-theory spectra of $\mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty)$ and $\mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K})$ are weakly equivalent. By Theorem 5.1 the homotopy groups of the nonconnective K-theory spectrum of $\mathtt{HPf}_{\mathtt{dg}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K})$ are isomorphic to the topological K-theory groups of $A\hat{\otimes}\mathcal{O}_\infty$, which are in turn isomorphic to those of A. □

Remark 5.7. Since noncommutative motives are univeral additive invariants, an isomorphism of noncommutative motives is more fundamental than that of twisted K-theories.

Remark 5.8. The above results provide a connection between the Dixmier–Douady theory via $\mathcal{O}_\infty\hat{\otimes}\mathbb{K}$-bundles due to Dadarlat–Pennig [15] and noncommutative motives.

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