UNIFORM LINEAR BOUND IN CHEVALLEY'S LEMMA

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Abstract. We obtain a uniform linear bound for the Chevalley function at a point in the source of an analytic mapping that is regular in the sense of Gabrielov. There is a version of Chevalley's lemma also along a fibre, or at a point of the image of a proper analytic mapping. We get a uniform linear bound for the Chevalley function for a closed Nash (or formally Nash) subanalytic set.

Contents

1. Introduction

Chevalley's lemma (1943) plays an important role in the solution of equations $f(x) = g(\varphi(x))$, where $y = \varphi(x)$ is an analytic mapping in several variables. Given $f(x)$ analytic (or, for example, $\mathcal{C}^\infty$ in the real case), the problem is to find conditions under which we can solve for $g(y)$ in the same class. Chevalley's lemma asserts that, given $x = a$ and $k \in \mathbb{N}$, there is a corresponding $l = l(k) < \infty$ such that the l-jet of a composite $g \circ \varphi$ at a determines the k-jet of g at $\varphi(a)$, modulo a formal relation among the components of φ at a. The "Chevalley function" of φ at a is the smallest $l(k)$.

In this article, we answer questions raised by works of Gabrielov, Izumi and Bierstone–Milman on finding bounds for the Chevalley function that are linear with respect to k or uniform with respect to a. Such bounds characterize important regularity or "tameness" properties of analytic mappings and their images [2], [3], [10], and measure loss of differentiability in classical problems on composite differentiable functions [3].

By way of comparison, the analogue of the Chevalley function for a linear analytic equation $f(x) = A(x) \cdot g(x)$ (where $A(x)$ is a matrix-valued analytic function and $f(x), g(x)$ are vector-valued) always has a linear bound, given by the exponent in the Artin-Rees lemma. Uniformity of the Artin-Rees exponent has been studied in [2], [5], [8].

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Let us now be more precise. Let $\varphi : M \to N$ denote an analytic mapping of analytic manifolds (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let $a \in M$. Let $\varphi_a^* : \mathcal{O}_{\varphi(a)} \to \mathcal{O}_a$ or $\hat{\varphi}_a^* : \widehat{\mathcal{O}}_{\varphi(a)} \to \widehat{\mathcal{O}}_a$ denote the induced homorphisms of analytic local rings or their completions, respectively. (We write $\mathcal{O}_a = \mathcal{O}_{M,a}$, and $\mathfrak{m}_a$ (or $\widehat{\mathfrak{m}}_a$) = maximal ideal of $\mathcal{O}_a$ (or $\widehat{\mathcal{O}}_a$).) According to Chevalley's lemma, there is an increasing function $l : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ denotes the nonnegative integers) such that

$$\hat{\varphi}_a^*(\widehat{\mathcal{O}}_{\varphi(a)}) \cap \widehat{\mathfrak{m}}_a^{l(k)+1} \ \subset\ \hat{\varphi}_a^*(\widehat{\mathfrak{m}}_{\varphi(a)}^{k+1})\ ;$$

i.e., if $F \in \widehat{\mathcal{O}}_{\varphi(a)}$ and $\hat{\varphi}_a^*(F)$ vanishes to order $l(k)$, then F vanishes to order k, modulo an element of $\operatorname{Ker}\hat{\varphi}_a^*$ ([4]; cf. Lemma 3.2 below). Let $l_{\varphi^*}(a,k)$ denote the least $l(k)$ satisfying Chevalley's lemma. We call $l_{\varphi^*}(a,k)$ the *Chevalley function* of $\hat{\varphi}_a^*$.

Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ denote local coordinate systems for M and N at a and $\varphi(a)$, respectively. The local rings $\mathcal{O}_a$ or $\widehat{\mathcal{O}}_a$ can be identified with the rings of convergent or formal power series $\mathbb{K}\{x\} = \mathbb{K}\{x_1, \ldots, x_m\}$ or $\mathbb{K}[\![x]\!] = \mathbb{K}[\![x_1, \ldots, x_m]\!]$, respectively. In the local coordinates, write $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$. Then $\operatorname{Ker}\hat{\varphi}_a^*$ is the *ideal of formal relations* $\{F(y) \in \mathbb{K}[\![y]\!] : F(\varphi_1(x), \ldots, \varphi_n(x)) = 0\}$ (and $\operatorname{Ker}\varphi_a^*$ is the analogous *ideal of analytic relations*). Chevalley's lemma is an analogue for such nonlinear relations of the Artin-Rees lemma. (See Remark 1.4.)

Let $r_a^1(\varphi)$ denote the generic rank of φ near a, and set

$$r_a^2(\varphi)\ :=\ \dim\frac{\widehat{\mathcal{O}}_{\varphi(a)}}{\operatorname{Ker}\hat{\varphi}_a^*}\ , \qquad r_a^3(\varphi)\ :=\ \dim\frac{\mathcal{O}_{\varphi(a)}}{\operatorname{Ker}\varphi_a^*}$$

(where dim denotes the Krull dimension). Then $r_a^1(\varphi) \le r_a^2(\varphi) \le r_a^3(\varphi)$. Gabrielov proved that if $r_a^1(\varphi) = r_a^2(\varphi)$, then $r_a^2(\varphi) = r_a^3(\varphi)$ [6]; i.e., if there are enough formal relations, then the ideal of formal relations is generated by convergent relations. The mapping φ is called *regular at a* if $r_a^1(\varphi) = r_a^3(\varphi)$. We say that φ is *regular* if it is regular at every point of M. Izumi [10] proved that φ is regular at a if and only if the Chevalley function of $\hat{\varphi}_a^*$ has a *linear (upper) bound*; i.e., there exist $\alpha, \beta \in \mathbb{N}$ such that

$$l_{\varphi^*}(a,k)\ \le\ \alpha k + \beta\ ,$$

for all $k \in \mathbb{N}$. On the other hand, Bierstone and Milman [2] proved that, if φ is regular, then $l_{\varphi^*}(a,k)$ has a *uniform bound*; i.e., for every compact $L \subset M$, there exists $l_L :\ \mathbb{N} \to \mathbb{N}$ such that

$$l_{\varphi^*}(a,k)\ \le\ l_L(k)\ ,$$

for all $a \in L$ and $k \in \mathbb{N}$. In this article, we prove that the Chevalley function associated to a regular mapping has a *uniform linear bound*:

Theorem 1.1. *Suppose that φ is regular. Then, for every compact $L \subset M$, there exist $\alpha_L, \beta_L \in \mathbb{N}$ such that*

$$l_{\varphi^*}(a,k)\ \le\ \alpha_L k + \beta_L\ ,$$

for all $a \in L$ and $k \in \mathbb{N}$.

Chevalley's lemma can be used also to compare two notions of order of vanishing of a real-analytic function at a point of a subanalytic set. Let X denote a closed subanalytic subset of $\mathbb{R}^n$. Let $b \in X$ and let $\mathcal{F}_b(X) \subset \mathbb{R}[\![y-b]\!]$ denote the formal local ideal of X at b. (See Lemma 3.6.) For all $F \in \widehat{\mathcal{O}}_b = \mathbb{R}[\![y-b]\!]$, we define

$$\begin{aligned} \mu_{X,b}(F) &:= \max\{l \in \mathbb{N} :\ |T_b^l F(y)| \le \mathrm{const}\, |y-b|^l,\ y \in X\}\ , \\ \nu_{X,b}(F) &:= \max\{l \in \mathbb{N} :\ F \in \widehat{\mathfrak{m}}_b^l + \mathcal{F}_b(X)\}\ , \end{aligned} \tag{1.1}$$

where $T_b^l F(y)$ denotes the Taylor polynomial of order l of F at b. Then there exists $l :\ \mathbb{N} \to \mathbb{N}$ such that, for all $k \in \mathbb{N}$, if $F \in \widehat{\mathcal{O}}_b$ and $\mu_{X,b}(F) > l(k)$, then $\nu_{X,b}(F) > k$. (See Section 3.) For each k, let $l_X(b,k)$ denote the least such $l(k)$. We call $l_X(b,k)$ the *Chevalley function* of X at b.

Theorem 1.2. *Suppose that X is a Nash (or formally Nash) subanalytic subset of $\mathbb{R}^n$. Then the Chevalley function of X has a* uniform linear bound*; i.e., for every compact $K \subset X$, there exists $\alpha_K, \beta_K \in \mathbb{N}$ such that*

$$l_X(b,k) \ \le\ \alpha_K k + \beta_K\ ,$$

for all $b \in K$ and $k \in \mathbb{N}$.

Theorems 1.1 and 1.2 are the main new results in this article. They answer questions raised in [3, 1.28].

The closed *Nash subanalytic* subsets X of $\mathbb{R}^n$ are the images of regular proper real-analytic mappings $\varphi :\ M \to \mathbb{R}^n$. In particular, a closed semianalytic set is Nash. A closed subanalytic subset X of $\mathbb{R}^n$ is *formally Nash* if, for every $b \in X$, there is a closed Nash subanalytic subset Y of X such that $\mathcal{F}_b(X) = \mathcal{F}_b(Y)$ [3]. Unlike the situation of Theorem 1.1, the converse of Theorem 1.2 is false [3, Example 12.8].

The main theorem of [3] (Theorem 1.13) asserts that, if X is a closed subanalytic subset of $\mathbb{R}^n$, then the existence of a uniform bound for $l_X(b,k)$ is equivalent to several other natural analytic and algebro-geometric conditions; for example, semicoherence [3, Definition 1.2], stratification by the diagram of initial exponents of the ideal $\mathcal{F}_b(X)$, $b \in X$ [3, Theorem 8.1], and a $\mathcal{C}^\infty$ composite function property [3, §1.5]. A uniform bound for the Chevalley function measures loss of differentiability in a $\mathcal{C}^r$ version of the composite function theorem. We use the techniques of [3] to prove Theorems 1.1 and 1.2 here.

Wang [12, Theorem 1.1] used [9, Theorem 1.2] to prove that the Chevalley function associated to a regular proper real-analytic mapping $\varphi\colon M \to \mathbb{R}^n$ has a uniform linear bound if and only if $X = \varphi(M)$ has a *uniform linear product estimate*; i.e., for every compact $K \subset X$, there exist $\alpha_K, \beta_K \in \mathbb{N}$ such that, for all $b \in K$ and $F, G \in \widehat{\mathcal{O}}_b$,

$$\nu_{X_i,b}(F \cdot G) \ \le\ \alpha_K(\nu_{X_i,b}(F) + \nu_{X_i,b}(G)) + \beta_K\ ,$$

where $X_b = \bigcup_i X_i$ is a decomposition of the germ X_b into finitely many irreducible subanalytic components. We therefore obtain the following from Theorem 1.1:

Theorem 1.3. *A closed Nash subanalytic subset of $\mathbb{R}^n$ admits a uniform linear product estimate.*

Remark 1.4. The Artin-Rees lemma can be viewed as a version of Chevalley's lemma for linear relations over a Noetherian ring R: Suppose that $\Psi :\ E \to G$ is a homomorphism of finitely-generated modules over R, and let $F \subset G$ denote the image of Ψ. Let $\mathfrak{m}$ be a maximal ideal of R. Then $F \cap \mathfrak{m}^l G \subset \mathfrak{m}^k F$ if and only if $\Psi^{-1}(\mathfrak{m}^l G) \subset \operatorname{Ker}\Psi + \mathfrak{m}^k E$. The Artin-Rees lemma says that there exists $\beta \in \mathbb{N}$

such that $F \cap \mathfrak{m}^{k+\beta} G = \mathfrak{m}^k (F \cap \mathfrak{m}^\beta G)$, for all k. In particular, there is always a *linear Artin-Rees exponent* $l(k) = k + \beta$. Uniform versions of the Artin-Rees lemma were proved in [2, Theorem 7.4], [5], [8]. A uniform Artin-Rees exponent for a homomorphism of $\mathcal{O}_M$-modules, where M is a real-analytic manifold, measures loss of differentiability in Malgrange division, in the same way that a uniform bound for the Chevalley function relates to composite differentiable functions. (See [2].)

2. Techniques

2.1. **Linear algebra lemma.** Let R denote a commutative ring with identity, and let E and F be R-modules. If $B \in \mathrm{Hom}_R(E, F)$ and $r \in \mathbb{N}$, $r \geq 1$, we define

$$\mathrm{ad}^r B \ \in \ \mathrm{Hom}_R\left(F, \mathrm{Hom}_R\left(\bigwedge^r E, \bigwedge^{r+1} F\right)\right)$$

by the formula

$$(\mathrm{ad}^r B)(\omega)(\eta_1 \wedge \cdots \wedge \eta_r) \ = \ \omega \wedge B\eta_1 \wedge \cdots \wedge B\eta_r \ ,$$

where $\omega \in F$ and $\eta_1, \ldots, \eta_r \in E$. ($\mathrm{ad}^0 B := \mathrm{id}_F$, the identity mapping of F.) Clearly, if $r > \mathrm{rk}\, B$ then $\mathrm{ad}^r B = 0$, and if $r = \mathrm{rk}\, B$ then $\mathrm{ad}^r B \cdot B = 0$. ($\mathrm{rk}\, B$ means the smallest r such that $\bigwedge^s B = 0$ for all $s > r$.) If R is a field, then $\mathrm{rk}\, B = \dim \mathrm{Im}\, B$, so we get:

Lemma 2.1 ([1, §6])**.** *Let E and F be finite-dimensional vector spaces over a field $\mathbb{K}$. If $B \colon E \to F$ is a linear transformation and $r = \mathrm{rk}\, B$, then*

$$\mathrm{Im}\, B \ = \ \mathrm{Ker}\, \mathrm{ad}^r B \ .$$

In particular, if A is another linear transformation with target F, then $A\xi + B\eta = 0$ (for some η) if and only if $\xi \in \mathrm{Ker}\, \mathrm{ad}^r B \cdot A$.

2.2. **The diagram of initial exponents.** Let A be a commutative ring with identity. Consider the total ordering of $\mathbb{N}^n$ given by the lexicographic ordering of $(n+1)$-tuples $(|\beta|, \beta_1, \ldots, \beta_n)$, where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ and $|\beta| = \beta_1 + \cdots + \beta_n$. For any formal power series $F(Y) = \sum_{\beta \in \mathbb{N}^n} F_\beta Y^\beta \in A[\![Y]\!] = A[\![Y_1, \ldots, Y_n]\!]$, we define the *support* $\mathrm{supp}\, F := \{\beta \in \mathbb{N}^n \colon F_\beta \neq 0\}$ and the *initial exponent* $\exp F := \min \mathrm{supp}\, F$. ($\exp F := \infty$ if $F = 0$.)

Let I be an ideal in $A[\![Y]\!]$. The *diagram of initial exponents* of I is defined as

$$\mathfrak{N}(I) \ := \ \{\exp F \colon F \in I \setminus \{0\}\} \ .$$

Clearly, $\mathfrak{N}(I) + \mathbb{N}^n = \mathfrak{N}(I)$.

Suppose that A is a field $\mathbb{K}$. Then, by the formal division theorem of Hironaka [7] (see [2, Theorem 6.2]),

$$\mathbb{K}[\![Y]\!] \ = \ I \oplus \mathbb{K}[\![Y]\!]^{\mathfrak{N}(I)}, \tag{2.1}$$

where $\mathbb{K}[\![Y]\!]^{\mathfrak{N}}$ is defined as $\{F \in \mathbb{K}[\![Y]\!] \colon \mathrm{supp}\, F \subset \mathbb{N}^n \setminus \mathfrak{N}\}$, for any $\mathfrak{N} \in \mathbb{N}^n$ such that $\mathfrak{N} + \mathbb{N}^n = \mathfrak{N}$.

2.3. **Fibred product.** Let M denote an analytic manifold over $\mathbb{K}$, and let $s \in \mathbb{N}$, $s \geq 1$. Let $\varphi\colon\ M \to N$ be an analytic mapping. We denote by M^s_φ the s-fold *fibred product* of M with itself *over* N; i.e.,

$$M^s_\varphi \ := \ \{\underline{a} = (a^1, \ldots, a^s) \in M^s\colon\ \varphi(a^1) = \cdots = \varphi(a^s)\} \ ;$$

M^s_φ is a closed analytic subset of M^s. There is a natural mapping $\underline{\varphi} = \underline{\varphi}^s\colon$ $M^s_\varphi \to N$ given by $\underline{\varphi}(\underline{a}) = \varphi(a^1)$; i.e., for each $i = 1, \ldots, s$, $\underline{\varphi} = \varphi \circ \rho^i$, where $\rho^i\colon\ M^s_\varphi \ni (x^1, \ldots, x^s) \mapsto x^i \in M$.

Suppose that $\mathbb{K} = \mathbb{R}$. Let E be a closed subanalytic subset of M, and let $\varphi\colon\ E \to \mathbb{R}^n$ be a continuous subanalytic mapping. Then the fibred product E^s_φ is a closed subanalytic subset of M^s, and the canonical mapping $\underline{\varphi} = \underline{\varphi}^s\colon\ E^s_\varphi \to \mathbb{R}^n$ is subanalytic.

Let $\mathring{E}^s_\varphi$ denote the subset of E^s_φ consisting of points $\underline{x} = (x^1, \ldots, x^s) \in E^s_\varphi$ such that each x^i lies in a distinct connected component of the fibre $\varphi^{-1}(\underline{\varphi}(\underline{x}))$. If φ is proper, then $\mathring{E}^s_\varphi$ is a subanalytic subset of M^s [3, §7].

2.4. **Jets.** Let N denote an analytic manifold (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and let $b \in N$. Let $l \in \mathbb{N}$ and let $J^l(b)$ denote $\widehat{\mathcal{O}}_b/\widehat{\mathfrak{m}}_b^{l+1}$. If $F \in \widehat{\mathcal{O}}_b$, then $J^lF(b)$ denotes the image of F in $J^l(b)$. Let M be an analytic manifold, and let $\varphi\colon M \to N$ be an analytic mapping. If $a \in \varphi^{-1}(b)$, then the homomorphism $\hat{\varphi}^*_a\colon \widehat{\mathcal{O}}_b \to \widehat{\mathcal{O}}_a$ induces a linear transformation $J^l\varphi(a)\colon J^l(b) \to J^l(a)$.

Suppose that $N = \mathbb{K}^n$. Let $y = (y_1, \ldots, y_n)$ denote the affine coordinates of $\mathbb{K}^n$. Taylor series expansion induces an identification of $\widehat{\mathcal{O}}_b$ with the ring of formal power series $\mathbb{K}[\![y-b]\!] = \mathbb{K}[\![y_1-b_1, \ldots, y_n-b_n]\!]$ (we write $F(y) = \sum_{\beta\in\mathbb{N}^n} F_\beta(y-b)^\beta$), and hence an identification of $J^l(b)$ with $\mathbb{K}^q$, $q = \binom{n+l}{l}$, with respect to which $J^lF(b) = (D^\beta F(b))_{|\beta|\leq l}$, where D^β denotes $1/\beta!$ times the formal derivative of order $\beta \in \mathbb{N}$.

Using a system of coordinates $x = (x_1, \ldots, x_m)$ for M in a neighbourhood of a, we can identify $J^l(a)$ with $\mathbb{K}^p$, $p = \binom{m+l}{l}$. Then

$$J^l\varphi(a)\colon\ (F_\beta)_{|\beta|\leq l} \ \mapsto \ ((\hat{\varphi}^*_a(F))_\alpha)_{|\alpha|\leq l} \ = \ \left(\sum_{|\beta|\leq l} F_\beta L^\beta_\alpha(a)\right)_{|\alpha|\leq l} ,$$

where $L^\beta_\alpha(a) = (\partial^{|\alpha|}\varphi^\beta/\partial x^\alpha)(a)/\alpha!$ and $\varphi^\beta = \varphi_1^{\beta_1}\ldots\varphi_n^{\beta_n}$ $(\varphi = (\varphi_1, \ldots, \varphi_n))$.

Set $J^l_b := J^l(b) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_b = \bigoplus_{|\beta|\leq l} \mathbb{K}[\![y-b]\!]$. We put $J^l_bF(y) := (D^\beta F(y))_{|\beta|\leq l} \in J^l_b$. (Evaluating at b transforms J^l_bF to $J^lF(b)$.) The ring homomorphism $\hat{\varphi}^*_a\colon \widehat{\mathcal{O}}_b \to \widehat{\mathcal{O}}_a$ induces a homomorphism of $\mathbb{K}[\![x-a]\!]$-modules,

$$\begin{array}{cccc} J^l_a\varphi\colon & J^l(b)\otimes_{\mathbb{K}}\widehat{\mathcal{O}}_a & \longrightarrow & J^l(a)\otimes_{\mathbb{K}}\widehat{\mathcal{O}}_a \\ & \| & & \| \\ & \bigoplus_{|\beta|\leq l} \mathbb{K}[\![x-a]\!] & & \bigoplus_{|\alpha|\leq l} \mathbb{K}[\![x-a]\!] \end{array}$$

such that, if $F \in \widehat{\mathcal{O}}_b$, then

$$J^l_a\varphi\left((\hat{\varphi}^*_a(D^\beta F))_{|\beta|\leq l}\right) \ = \ (D^\alpha(\hat{\varphi}^*_a(F)))_{|\alpha|\leq l}.$$

By evaluation at a, $J^l_a\varphi$ induces $J^l\varphi(a)\colon J^l(b) \to J^l(a)$. $J^l_a\varphi$ identifies with the matrix (with rows indexed by $\alpha \in \mathbb{N}^m$, $|\alpha| \leq l$, and columns indexed by $\beta \in \mathbb{N}^n$, $|\beta| \leq l$)

whose entries are the Taylor expansions at a of the $D^\alpha\varphi^\beta = (\partial^{|\alpha|}\varphi^\beta/\partial x^\alpha)/\alpha!$, $|\alpha| \le l$, $|\beta| \le l$.

Let $\underline{a} = (a^1, \dots, a^s) \in M^s_\varphi$ and let $b = \underline{\varphi}(\underline{a})$. For each $i = 1, \dots, s$, the homomorphism $J^l_b = J^l(b) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_b \to J^l(a^i) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{a^i} = J^l_{a^i}$ over $\hat{\varphi}^*_{a^i}$, as defined above (using a coordinate system $x^i = (x^i_1, \dots, x^i_m)$ for M in a neighbourhood of a^i), followed by the canonical homomorphism $J^l(a^i) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{a^i} \to J^l(a^i) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}}$ over $(\hat{\rho^i})^*_{\underline{a}} \colon \widehat{\mathcal{O}}_{a^i} \to \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}}$, induces an $\widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}}$-homomorphism $J^l(b) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}} \to J^l(a^i) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}}$. We thus obtain an $\widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}}$-homomorphism

$$\begin{array}{cccc} J^l_{\underline{a}}\varphi\colon & J^l(b) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}} & \longrightarrow & \bigoplus_{i=1}^s J^l(a^i) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}} \\ & \| & & \| \\ & \bigoplus_{|\beta|\le l} \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}} & & \bigoplus_{i=1}^s \bigoplus_{|\alpha|\le l} \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}} \end{array} .$$

For any (germ at $\underline{a}$ of an) analytic subspace L of M^s_φ, we also write

$$J^l_{\underline{a}}\varphi\colon\; J^l(b) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{L,\underline{a}} \;\to\; \bigoplus_{i=1}^s J^l(a^i) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{L,\underline{a}} \tag{2.2}$$

for the induced $\widehat{\mathcal{O}}_{L,\underline{a}}$-homomorphism. Evaluation at $\underline{a}$ transforms $J^l_{\underline{a}}\varphi$ to

$$J^l\varphi(\underline{a}) = (J^l\varphi(a^1), \dots, J^l\varphi(a^s))\colon\; J^l(b) \to \bigoplus_{i=1}^s J^l(a^i). \tag{2.3}$$

3. Ideals of relations and Chevalley functions

Let M denote an analytic manifold (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and let $\varphi = (\varphi_1, \dots, \varphi_n)\colon M \to \mathbb{K}^n$ be an analytic mapping. If $a \in M$, let $\mathcal{R}_a$ denote the ideal of formal relations $\operatorname{Ker}\hat{\varphi}^*_a$.

Remark 3.1. $\mathcal{R}_a$ is constant on connected components of the fibres of φ [3, Lemma 5.1].

Let s be a positive integer, and let $\underline{a} = (a^1, \dots, a^s) \in M^s_\varphi$. Put

$$\mathcal{R}_{\underline{a}} \;:=\; \bigcap_{i=1}^s \mathcal{R}_{a^i} \;=\; \bigcap_{i=1}^s \operatorname{Ker}\hat{\varphi}^*_{a^i} \;\subset\; \widehat{\mathcal{O}}_{\underline{\varphi}(\underline{a})} \;. \tag{3.1}$$

If $k \in \mathbb{N}$, we also write

$$\mathcal{R}^k(\underline{a}) \;:=\; \frac{\mathcal{R}_{\underline{a}} + \widehat{\mathfrak{m}}^{k+1}_{\underline{\varphi}(\underline{a})}}{\widehat{\mathfrak{m}}^{k+1}_{\underline{\varphi}(\underline{a})}} \;\subset\; J^k(\underline{\varphi}(\underline{a}))\;.$$

If $b \in \mathbb{K}^n$, let $\pi^k(b)\colon \widehat{\mathcal{O}}_b \to J^k(b)$ denote the canonical projection. For $l \ge k$, let $\pi^{lk}(b)\colon J^l(b) \to J^k(b)$ be the projection. Set

$$E^l(\underline{a}) \;:=\; \operatorname{Ker} J^l\varphi(\underline{a}), \quad\text{and}\quad E^{lk}(\underline{a}) \;:=\; \pi^{lk}(\underline{\varphi}(\underline{a})).E^l(\underline{a})\;.$$

3.1. **Chevalley's lemma.**

Lemma 3.2 ([2, Lemma 8.2.2]; cf. [4, § II, Lemma 7]). *Let $\underline{a} \in M^s_\varphi$, $\underline{a} = (a^1, \ldots, a^s)$. For all $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that $\mathcal{R}^k(\underline{a}) = E^{lk}(\underline{a})$; i.e., such that if $F \in \widehat{\mathcal{O}}_{\underline{\varphi(a)}}$ and $\hat{\varphi}^*_{a^i}(F) \in \widehat{\mathfrak{m}}^{l+1}_{a^i}$, $i = 1, \ldots, s$, then $F \in \mathcal{R}_{\underline{a}} + \widehat{\mathfrak{m}}^{k+1}_{\underline{\varphi(a)}}$.*

We write $l(\underline{a}, k) = l_{\varphi^*}(\underline{a}, k)$ for the least l satisfying the conclusion of the lemma.

Proof of Lemma 3.2. If $k \le l_1 \le l_2$, then

$$\mathcal{R}^k(\underline{a}) \ \subset \ E^{l_2,k}(\underline{a}) \ \subset \ E^{l_1,k}(\underline{a}) \ ,$$

and the projection $\pi^{l_2,l_1}(\underline{\varphi(a)})$ maps $\bigcap_{l \ge l_2} E^{ll_2}(\underline{a})$ onto $\bigcap_{l \ge l_1} E^{ll_1}(\underline{a})$. It follows that $\mathcal{R}^k(\underline{a}) = \bigcap_{l \ge k} E^{lk}(\underline{a})$. Since $\dim J^k(\underline{\varphi(a)}) < \infty$, there exists $l \in \mathbb{N}$ such that $\mathcal{R}^k(\underline{a}) = E^{lk}(\underline{a})$. □

3.2. **Generic Chevalley function.**

Let $\underline{a} \in M^s_\varphi$ and $k \in \mathbb{N}$. Set

$$H_{\underline{a}}(k) \ := \ \dim_{\mathbb{K}} \frac{J^k(\underline{\varphi(a)})}{\mathcal{R}^k(\underline{a})} \ , \qquad d^{lk}(\underline{a}) \ := \ \dim_{\mathbb{K}} \frac{J^k(\underline{\varphi(a)})}{E^{lk}(\underline{a})} \ , \text{ if } l \ge k$$

($H_{\underline{a}}$ is the *Hilbert-Samuel function* of $\widehat{\mathcal{O}}_{\underline{\varphi(a)}}/\mathcal{R}_{\underline{a}}$).

Remark 3.3. $d^{lk}(\underline{a}) \le H_{\underline{a}}(k)$ since $\mathcal{R}^k(\underline{a}) \subset E^{lk}(\underline{a})$. $\mathcal{R}^k(\underline{a}) = E^{lk}(\underline{a})$ (and $d^{lk}(\underline{a}) = H_{\underline{a}}(k)$) if and only if $l \ge l(\underline{a}, k)$.

Lemma 3.4 ([2, Lemma 8.3.3]). *Let L be a* subanalytic leaf *in M^s_φ (i.e., a connected subanalytic subset of M^s_φ which is an analytic submanifold of M^s; see Remark 4.4). Then there is a residual subset D of L such that, if $\underline{a}, \underline{a}' \in D$, then $H_{\underline{a}}(k) = H_{\underline{a}'}(k)$ and $l(\underline{a}, k) = l(\underline{a}', k)$, for all $k \in \mathbb{N}$.*

Definition 3.5. We define the *generic Chevalley function* of L as $l(L, k) := l(\underline{a}, k)$ ($k \in \mathbb{N}$), where $\underline{a} \in D$.

Proof of Lemma 3.4. For $\underline{a} \in M^s_\varphi$ and $l \ge k$, write $J^l\varphi(\underline{a})$ (2.3) (using local coordinates for M^s as in §2.4, in a neighbourhood of a point of $\overline{L}$) as a block matrix

$$\begin{aligned} J^l\varphi(\underline{a}) \ &= \ (S^{lk}(\underline{a}), T^{lk}(\underline{a})) \\ &= \ \begin{pmatrix} J^k\varphi(\underline{a}) & 0 \\ * & * \end{pmatrix} \end{aligned}$$

corresponding to the decomposition of vectors $\xi = (\xi_\beta)_{\beta \in \mathbb{N}^n, |\beta| \le l}$ in the source as $\xi = (\xi^k, \zeta^{lk})$, where $\xi^k = (\xi_\beta)_{|\beta| \le k}$ and $\zeta^{lk} = (\xi_\beta)_{k < |\beta| \le l}$. Then

$$E^{lk}(\underline{a}) \ = \ \{\eta = (\eta_\beta)_{|\beta| \le k} \colon \ S^{lk}(\underline{a}) \cdot \eta \in \operatorname{Im} T^{lk}(\underline{a})\} \ .$$

Thus, by Lemma 2.1

$$E^{lk}(\underline{a}) \ = \ \operatorname{Ker} \Theta^{lk}(\underline{a}), \quad \text{and} \quad d^{lk}(\underline{a}) \ = \ \operatorname{rk} \Theta^{lk}(\underline{a}) \ ,$$

where

$$\Theta^{lk}(\underline{a}) \ := \ \operatorname{ad}^{r^{lk}(\underline{a})} T^{lk}(\underline{a}) \cdot S^{lk}(\underline{a}) \ , \quad r^{lk}(\underline{a}) \ := \ \operatorname{rk} T^{lk}(\underline{a}) \ .$$

Set

$$r^{lk}(L) \ := \ \max_{\underline{a} \in L} r^{lk}(\underline{a}), \quad \text{and} \quad d^{lk}_L(\underline{a}) \ := \ \operatorname{rk} \Theta^{lk}_L(\underline{a}), \quad \underline{a} \in L \ ,$$

where

$$\Theta_L^{lk}(\underline{a}) \ := \ \mathrm{ad}^{r^{lk}(L)} T^{lk}(\underline{a}) \cdot S^{lk}(\underline{a})$$

(so that $\Theta_L^{lk}(\underline{a}) = 0$ if $r^{lk}(\underline{a}) < r^{lk}(L)$). Let $Y^{lk} := \{\underline{a} \in L\colon\ r^{lk}(\underline{a}) < r^{lk}(L)\}$. Set

$$d^{lk}(L) \ := \ \max_{\underline{a}\in L} d_L^{lk}(\underline{a}) \ .$$

Clearly, $d_L^{lk}(\underline{a}) = 0$ if $\underline{a} \in Y^{lk}$, and $d_L^{lk}(\underline{a}) = d^{lk}(\underline{a})$ if $\underline{a} \in L \setminus Y^{lk}$. Also set

$$Z^{lk} \ := \ Y^{lk} \cup \left\{\underline{a} \in L\colon d_L^{lk}(\underline{a}) < d^{lk}(L)\right\} \ .$$

Then Y^{lk} and Z^{lk} are proper closed analytic subsets of L. For all $\underline{a} \in L \setminus Z^{lk}$, $r^{lk}(\underline{a}) = r^{lk}(L)$ and $d^{lk}(\underline{a}) = d_L^{lk}(\underline{a}) = d^{lk}(L)$. Put

$$D^k \ := \ L \setminus \bigcup_{l>k} Z^{lk} \ , \quad D \ := \ \bigcap_{k\geq 1} D^k \ . \tag{3.2}$$

By the Baire Category Theorem, the D^k (and hence also D) are residual subsets of L.

Fix $k \in \mathbb{N}$. If $\underline{a} \in D^k$, then $d^{lk}(\underline{a}) = d^{lk}(L)$, for all $l > k$. If, in addition, $l \geq l(\underline{a}, k)$, then $H_{\underline{a}}(k) = d^{lk}(L)$, by Remark 3.3. If $\underline{a}, \underline{a}' \in D^k$, then, choosing $l \geq l(\underline{a}, k)$ and $\geq l(\underline{a}', k)$, we get $H_{\underline{a}}(k) = H_{\underline{a}'}(k)$. For the second assertion of the lemma, suppose that $l \geq l(\underline{a}, k)$. Then $H_{\underline{a}'}(k) = H_{\underline{a}}(k) = d^{lk}(\underline{a}) = d^{lk}(L) = d^{lk}(\underline{a}')$, so that $l \geq l(\underline{a}', k)$, by Remark 3.3. In the same way, $l \geq l(\underline{a}', k)$ implies that $l \geq l(\underline{a}, k)$. $\square$

3.3. **Chevalley function of a subanalytic set.** Let N denote a real-analytic manifold, and let X be a closed subanalytic subset of N. If $b \in X$, then $\mathcal{F}_b(X)$ or $\mathcal{R}_b \subset \widehat{\mathcal{O}}_b$ denotes the *formal local ideal* of X at b, in the sense of the following simple lemma:

Lemma 3.6. *Let $b \in X$. The following three definitions of $\mathcal{F}_b(X)$ are equivalent:*

1. *Let M be a real-analytic manifold and let $\varphi\colon M \to N$ be a proper real-analytic mapping such that $X = \varphi(M)$. Then $\mathcal{F}_b(X) = \bigcap_{a\in\varphi^{-1}(b)} \ker \hat{\varphi}_a^*$.*
2. *$\mathcal{F}_b(X) = \{F \in \widehat{\mathcal{O}}_b\colon\ (F \circ \gamma)(t) \equiv 0$ for every real-analytic arc $\gamma(t)$ in X such that $\gamma(0) = b\}$.*
3. *$\mathcal{F}_b(X) = \{F \in \widehat{\mathcal{O}}_b\colon\ T_b^k F(y) = o(|y - b|^k)$, where $y \in X$, for all $k \in \mathbb{N}\}$. Here $T_b^k F(y)$ denotes the Taylor polynomial of order k of F at b, in any local coordinate system.*

Assume that $N = \mathbb{R}^n$, with coordinates $y = (y_1, \ldots, y_n)$. Let $b \in X$. Recall (1.1).

Remark 3.7. $\nu_{X,b}(F) \leq \mu_{X,b}(F)$: Suppose that $F \in \widehat{\mathfrak{m}}_b^l + \mathcal{F}_b(X)$; say $F = G + H$, where $G \in \widehat{\mathfrak{m}}_b^l$ and $H \in \mathcal{F}_b(X)$. Then $|T_b^l G(y)| \leq c|y - b|^l$ and $T_b^l H(y) = o(|y - b|^l)$, $y \in X$, by Lemma 3.6. Hence $|T_b^l F(y)| \leq \mathrm{const}|y - b|^l$ on X.

Definition 3.8 (*Chevalley functions*)**.** Let $b \in X$ and let $k \in \mathbb{N}$. Set

$$l_X(b, k) \ := \ \min\{l \in \mathbb{N}\colon\ \text{If } F \in \widehat{\mathcal{O}}_b \text{ and } \mu_{X,b}(F) > l, \text{ then } \nu_{X,b}(F) > k\} \ .$$

Let $\varphi\colon M \to N$ be a proper real-analytic mapping such that $X = \varphi(M)$. Set

$$l_{\varphi^*}(b, k) \ := \ \min\{l \in \mathbb{N}\colon\ \text{If } F \in \widehat{\mathcal{O}}_b \text{ and } \nu_{M,a}(\hat{\varphi}_a^*(F)) > l \text{ for all } a \in \varphi^{-1}(b), \text{ then } \nu_{X,b}(F) > k\} \ .$$

Remark 3.9. Suppose that $b = \underline{\varphi}(\underline{a})$, where $\underline{a} = (a^1, \ldots, a^s) \in M^s_\varphi$, $s \geq 1$. By Lemma 3.2, $l_{\varphi^*}(\underline{a}, k) < \infty$. If $\underline{a}$ includes a point a^i in every connected component of $\varphi^{-1}(b)$, then $\bigcap_{i=1}^s \operatorname{Ker} \hat{\varphi}^*_{a^i} = \mathcal{F}_b(X)$ (by Remark 3.1 and Lemma 3.6), so that $l_{\varphi^*}(b, k) \leq l_{\varphi^*}(\underline{a}, k)$.

Lemma 3.10 (see [3, Lemma 6.5])**.** *Let $\varphi\colon M \to N$ be a proper real-analytic mapping such that $X = \varphi(M)$. Then $l_X(b, \cdot) \leq l_{\varphi^*}(b, \cdot)$ for all $b \in X$.*

4. Proofs of the main theorems

Let $\varphi\colon M \to \mathbb{K}^n$ be an analytic mapping from a manifold M (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let s be a positive integer. Let $\underline{a} = (a^1, \ldots, a^s) \in M^s_\varphi$, and let $b = \underline{\varphi}(\underline{a})$.

Remark 4.1. By (2.1), the Chevalley functions $l_{\varphi^*}(\underline{a}, k)$ and $l_{\varphi^*}(b, k)$ (Definitions 3.8) can be defined using power series that are supported outside the diagram of initial exponents: Set $\mathfrak{N}_{\underline{a}} := \mathfrak{N}(\mathcal{R}_{\underline{a}})$ and $\mathfrak{N}_b := \mathfrak{N}(\mathcal{R}_b)$ (cf. 3.1 and Lemma 3.6). Then

$$\begin{aligned}
l_{\varphi^*}(\underline{a}, k) \;&=\; \min\{l \in \mathbb{N}\colon \text{ If } F \in \widehat{\mathcal{O}}_b^{\mathfrak{N}_{\underline{a}}} \text{ and } \hat{\varphi}^*_{a^i}(F) \in \widehat{\mathfrak{m}}_{a^i}^{l+1},\ i = 1, \ldots, s,\\
&\qquad\qquad \text{then } F \in \mathcal{R}_{\underline{a}} + \widehat{\mathfrak{m}}_b^{k+1}\}\ ,\\
l_{\varphi^*}(b, k) \;&=\; \min\{l \in \mathbb{N}\colon \text{ If } F \in \widehat{\mathcal{O}}_b^{\mathfrak{N}_b} \text{ and } \hat{\varphi}^*_{a}(F) \in \widehat{\mathfrak{m}}_{a}^{l+1},\ \text{for all } a \in \varphi^{-1}(b),\\
&\qquad\qquad \text{then } F \in \mathcal{R}_b + \widehat{\mathfrak{m}}_b^{k+1}\}\ .
\end{aligned}$$

(In the latter, we assume that φ is a proper real-analytic mapping.)

If $l \in \mathbb{N}$, set $J^l(b)^{\mathfrak{N}_{\underline{a}}} := \{\xi = (\xi_\beta)_{|\beta| \leq l} \in J^l(b)\colon \xi_\beta = 0 \text{ if } \beta \in \mathfrak{N}_{\underline{a}}\}$. Consider the linear mapping

$$\Phi^l(\underline{a})\colon\ J^l(b)^{\mathfrak{N}_{\underline{a}}} \to \bigoplus_{i=1}^s J^l(a^i)$$

obtained by restriction of $J^l\underline{\varphi}(\underline{a})\colon J^l(b) \to \bigoplus J^l(a^i)$ (2.3). Given $k \leq l$, write $\Phi^l(\underline{a})$ as a block matrix

$$\Phi^l(\underline{a}) \;=\; (A^{lk}(\underline{a}), B^{lk}(\underline{a}))\ ,$$

where $A^{lk}(\underline{a})$ is given by the restriction of $\Phi^l(\underline{a})$ to $J^k(b)^{\mathfrak{N}_{\underline{a}}}$.

Remark 4.2. If $\xi \in J^l(b)^{\mathfrak{N}_{\underline{a}}}$, write $\xi = (\eta, \zeta)$ corresponding to this block decomposition. Then $l \geq l_{\varphi^*}(\underline{a}, k)$ if and only if $A^{lk}(\underline{a})\eta + B^{lk}(\underline{a})\zeta = 0$ implies $\eta = 0$ [3, Lemma 8.13].

Lemma 4.3 ((cf. [3, Prop. 8.15])**.** *Let $s \geq 1$ and consider $\underline{\varphi} = \underline{\varphi}^s\colon M^s_\varphi \to \mathbb{R}^n$. Let L be a relatively compact subanalytic leaf in M^s_φ (cf. Lemma 3.4) such that $\mathfrak{N}_{\underline{a}} = \mathfrak{N}(\mathcal{R}_{\underline{a}})$ is constant on L. Let $l(k) = l(L, k)$ denote the generic Chevalley function of L. Then there exists $p \in \mathbb{N}$ such that $l_{\varphi^*}(\underline{a}, k) \leq l(k) + p$, for all $\underline{a} \in L$ and $k \in \mathbb{N}$.*

Proof. Set $\mathfrak{N} = \mathfrak{N}_{\underline{a}}$, $\underline{a} \in L$. We can assume that $\overline{L}$ lies in a coordinate chart for M^s as in §2.4. Let $k \in \mathbb{N}$ and let $l = l(k)$. Let $\underline{a} = (a^1, \ldots, a^s) \in L$, and set $b = \underline{\varphi}(\underline{a})$. Consider the linear mapping $\Phi^l(\underline{a}) = (A^{lk}(\underline{a}), B^{lk}(\underline{a}))\colon J^l(b)^{\mathfrak{N}} \to \bigoplus_{i=1}^s J^l(a^i)$ as above. The $\widehat{\mathcal{O}}_{L,\underline{a}}$-homomorphism $J^l_{\underline{a}}\underline{\varphi}\colon J^l(b) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{L,\underline{a}} \to \bigoplus_{i=1}^s J^l(a^i) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{L,\underline{a}}$ (2.2) induces an $\widehat{\mathcal{O}}_{L,\underline{a}}$-homomorphism

$$\Phi^l_{\underline{a}} \;=\; (A^{lk}_{\underline{a}}, B^{lk}_{\underline{a}})\colon\ J^l(b)^{\mathfrak{N}} \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{L,\underline{a}} \;\to\; \bigoplus_{i=1}^s J^l(a^i) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{L,\underline{a}}\ ;$$

evaluating at $\underline{a}$ transforms $\Phi^l_{\underline{a}}$ to $\Phi^l(\underline{a}) = (A^{lk}(\underline{a}), B^{lk}(\underline{a}))$.

Let $r = \operatorname{rk} B^{lk}_{\underline{a}} =$ generic rank of $B^{lk}(\underline{x})$, $\underline{x} \in L$. Let $\Theta_{\underline{a}} = \operatorname{ad}^r B^{lk}_{\underline{a}} \cdot A^{lk}_{\underline{a}}$. Then $\operatorname{Ker} \Theta_{\underline{a}} = 0$ (i.e., $\operatorname{Ker} \Theta(\underline{x}) = 0$ generically on L, where $\Theta(\underline{x}) = \operatorname{ad}^r B^{lk}(\underline{x}) \cdot A^{lk}(\underline{x})$, by Remark 4.2). Let $d = \operatorname{rk} \Theta_{\underline{a}}$. Then there is a nonzero minor $\delta_{\underline{a}} \in \mathcal{O}_{L,\underline{a}}$ of $\Theta_{\underline{a}}$ of order d; $\delta_{\underline{a}}$ is induced by a minor $\delta(\underline{x})$ of order d of $\Theta(\underline{x})$, $\underline{x} \in L$, such that $\delta(\underline{x}) \neq 0$ on a residual subset of L. Since δ is a restriction to L of an analytic function defined in a neighbourhood of $\overline{L}$, the order of $\delta_{\underline{x}}$, $\underline{x} \in L$, is bounded on L; say, $\delta_{\underline{x}} \leq p$.

We claim that $l_{\varphi^*}(\underline{a}, k) \leq l(k) + p$ for all $\underline{a} \in L$: Let $\underline{a} = (a^1, \ldots, a^s) \in L$, and let $b = \underline{\varphi}(\underline{a})$. Let $l = l(k)$ and $l' = l + p$. Suppose that $F \in \widehat{\mathcal{O}}_b^{\mathfrak{N}}$ and $\hat{\varphi}^*_{a^i}(F) \in \widehat{\mathfrak{m}}_{a^i}^{l'+1}$, $i = 1, \ldots, s$. Let $\hat{\xi}_{\underline{a}} = (\hat{\eta}_{\underline{a}}, \hat{\zeta}_{\underline{a}})$ denote the element of $J^l(b)^{\mathfrak{N}} \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_{L,\underline{a}}$ induced by $J^l_b F \in J^l(b) \otimes_{\mathbb{K}} \widehat{\mathcal{O}}_b$ via the pull-back. Then each component of $A^{lk}_{\underline{a}} \hat{\eta}_{\underline{a}} + B^{lk}_{\underline{a}} \hat{\zeta}_{\underline{a}}$ belongs to $\widehat{\mathfrak{m}}_{L,\underline{a}}^{l'+1-l}$ (as we see by taking formal derivatives of order $\leq l$ of the $\hat{\varphi}^*_{a^i}(F)$). It follows that each component of $\Theta_{\underline{a}} \hat{\eta}_{\underline{a}}$ and therefore (by Cramer's rule) each component of $\delta_{\underline{a}} \cdot \hat{\eta}_{\underline{a}}$ belongs to $\widehat{\mathfrak{m}}_{L,\underline{a}}^{l'+1-l}$. Thus, each component of $\hat{\eta}_{\underline{a}}$ lies in $\widehat{\mathfrak{m}}_{L,\underline{a}}^{l'+1-l-p} = \widehat{\mathfrak{m}}_{L,\underline{a}}$; i.e., $\hat{\eta}_{\underline{a}}(\underline{a}) = 0$, so that F vanishes to order k at $b = \underline{\varphi}(\underline{a})$. □

Proof of Theorem 1.1. By [2, Theorems A,C], there is a locally finite partition of M into relatively compact subanalytic leaves L such that the diagram of initial exponents $\mathfrak{N}_a = \mathfrak{N}(\mathcal{R}_a)$ is constant on each L. Given L, let $l(L, k)$ denote the generic Chevalley function. (In particular, $l(L, k) = l_{\varphi^*}(a, k)$, for all a in a residual subset of L.) Since φ is regular, there exist α_L, γ_L such that $l(L, k) \leq \alpha_L k + \gamma_L$, for all $k \in \mathbb{N}$ (by [10]). By Lemma 4.3 (in the case $s = 1$), there exists $p_L \in \mathbb{N}$ such that $l_{\varphi^*}(a, k) \leq \alpha_L k + \gamma_L + p_L$, for all $a \in L$ and all k. The result follows. □

Remark 4.4. In the case $\mathbb{K} = \mathbb{C}$, we define "subanalytic leaf" using the underlying real structure. If φ is regular, then the diagram $\mathfrak{N}_a$ is, in fact, an upper-semicontinuous function of a, with respect to the $\mathbb{K}$-analytic Zariski topology of M (and a natural total ordering of $\{\mathfrak{N} \in \mathbb{N}^n : \mathfrak{N} + \mathbb{N}^n = \mathfrak{N}\}$) [2, Theorem C], but we do not need the more precise result here.

Lemma 4.5. *Let $s \geq 1$ and let $\underline{a} = (a^1, \ldots, a^n) \in M^s_\varphi$. Suppose that φ is regular at $a^1, \ldots, a^n$. Then there exist $\alpha, \beta \in \mathbb{R}$ such that $l_{\varphi^*}(\underline{a}, k) \leq \alpha k + \beta$, for all $k \in \mathbb{N}$.*

Proof. Let $b = \underline{\varphi}(\underline{a})$. For each $i = 1, \ldots, s$, since φ is regular at a^i, there exist α^i, β^i such that

$$l_{\varphi^*}(a^i, k) \ \leq \ \alpha^i k + \beta^i, \quad \text{for all } k \ . \tag{4.1}$$

Of course, $\bigcap_{i=1}^s \operatorname{Ker} \hat{\varphi}^*_{a^i}$ is the kernel of the homomorphism $\widehat{\mathcal{O}}_b \to \bigoplus_{i=1}^s \widehat{\mathcal{O}}_b / \ker \hat{\varphi}^*_{a^i}$. By the Artin-Rees lemma (cf. Remark 1.4), there exists $\lambda \in \mathbb{N}$ such that, if $F \in \widehat{\mathfrak{m}}_b^{k+\lambda} + \ker \hat{\varphi}^*_{a^i}$, $i = 1, \ldots, s$, then

$$F \ \in \ \widehat{\mathfrak{m}}_b^k + \bigcap_{i=1}^s \operatorname{Ker} \hat{\varphi}^*_{a^i} \ . \tag{4.2}$$

Now let $F \in \widehat{\mathcal{O}}_b$ and suppose that $\hat{\varphi}^*_{a^i}(F) \in \widehat{\mathfrak{m}}_{a^i}^{\alpha^i(\lambda+k)+\beta^i+1}$, $i = 1, \ldots, s$. Then $F \in \widehat{\mathfrak{m}}_b^{\lambda+k+1} + \operatorname{Ker} \hat{\varphi}^*_{a^i}$, $i = 1, \ldots, s$, by (4.1), so that $F \in \widehat{\mathfrak{m}}_b^{k+1} + \bigcap_{i=1}^s \operatorname{Ker} \hat{\varphi}^*_{a^i}$, by (4.2). In other words, $l_{\varphi^*}(\underline{a}, k) \leq \alpha k + \beta$, where $\alpha = \max \alpha^i$ and $\beta = \lambda \max \alpha^i + \max \beta^i$. □

Proof of Theorem 1.2. Suppose that $\varphi\colon M \to \mathbb{R}^n$ is a real-analytic mapping, where M is compact. Let $X = \varphi(M)$. Let $s \geq 1$, $\underline{a} \in M^s_\varphi$, $b = \underline{\varphi}(\underline{a})$. If $\underline{a} = (a^1, \dots, a^s)$ includes a point a^i in every connected component of $\varphi^{-1}(b)$, then

$$l_X(b,k) \ \leq \ l_{\varphi^*}(\underline{a}, k) \ , \tag{4.3}$$

by Remark 3.9 and Lemma 3.10.

Let L be a relatively compact subanalytic leaf in M^s_φ, such that $\mathfrak{N}_{\underline{a}} = \mathfrak{N}(\mathcal{R}_{\underline{a}})$ is constant on L. Suppose that φ is regular at a^i, for all $\underline{a} = (a^1, \dots, a^s) \in L$ and $i = 1, \dots, s$. Let $l(L,k)$ denote the generic Chevalley function of L. By Lemma 4.5, there exist α, β such that $l(L,k) \leq \alpha k + \beta$. Therefore, by Lemma 4.3, there exist α_L, β_L such that

$$l_{\varphi^*}(\underline{a}, k) \ \leq \ \alpha_L k + \beta_L, \quad \text{for all } \ \underline{a} \in L \ . \tag{4.4}$$

To prove the theorem, we can assume that X is compact. Let φ be a mapping as above, such that $X = \varphi(M)$. We consider first the case that X is Nash. Then we can assume that φ is regular. Let s denote a bound on the number of connected components of a fibre $\varphi^{-1}(b)$, for all $b \in X$. Then there is a finite partition of M^s_φ into relatively compact subanalytic leaves L, such that $\mathfrak{N}_{\underline{a}} = \mathfrak{N}(\mathcal{R}_{\underline{a}})$ is constant on every L. By (4.3) and (4.4), for each L, there exist α_L, β_L such that $l_X(b,k) \leq \alpha_L k + \beta_L$, for all $b \in \underline{\varphi}(L)$ and all k. Therefore, $l_X(b,k)$ has a uniform linear bound.

Finally, we consider X formally Nash. Let $\mathrm{NR}(\varphi) \subset M$ denote the set of points at which φ is not regular. Then $\mathrm{NR}(\varphi)$ is a nowhere-dense closed analytic subset of M ([11, Theorem 1]). For each positive integer s, set

$$\mathrm{NR}(\underline{\varphi}^s) \ := \ M^s_\varphi \cap \bigcup_{i=1}^{s} \{\underline{a} = (a^i, \dots, a^s) \in M^s \colon a^i \in \mathrm{NR}(\varphi)\} \ ;$$

then $\mathrm{NR}(\underline{\varphi}^s)$ is a closed analytic subset of M^s_φ.

If $b \in X$ and a, a' belong to the same connected component of $\varphi^{-1}(b)$, then φ is regular at a if and only if φ is regular at a' (cf. Remark 3.1). Let t be a bound on the number of connected components of a fibre $\varphi^{-1}(b)$, for all $b \in X$. For each $s \leq t$, define $X_s := \{b \in X \colon \varphi^{-1}(b)$ has precisely s regular components$\}$ and $Y_s := \{b \in X \colon \varphi^{-1}(b)$ has at least s regular components$\}$. Then $X_s = Y_s \setminus Y_{s+1}$, and

$$Y_s \ = \ \underline{\varphi}^s(\mathring{M}^s_\varphi \setminus \mathrm{NR}(\underline{\varphi}^s)) \ ;$$

in particular, all the X_s and Y_s are subanalytic (cf. §3.2).

The hypothesis of the theorem implies:

(1) $X = \bigcup_{s=1}^t X_s$;

(2) If $b \in X_s$ and $\underline{a} \in (\underline{\varphi}^s)^{-1}(b) \bigcap (\mathring{M}^s_\varphi \setminus \mathrm{NR}(\underline{\varphi}^s))$, then $\mathcal{R}_{\underline{a}} = \mathcal{R}_b$.

((2) follows from the fact that $\mathcal{F}_b(X) = \mathcal{F}_b(Y_b)$, where Y_b is some closed Nash subanalytic subset of X, and (1) from the fact that the latter condition holds for all $b \in X$.)

By [11, Theorem 2], for each s, there is a finite stratification $\mathcal{L}_s$ of M^s_φ compatible with $\mathrm{NR}(\underline{\varphi}^s)$ such that $\mathfrak{N}_{\underline{a}} = \mathfrak{N}(\mathcal{R}_{\underline{a}})$ is constant on every stratum $L \subset \mathring{M}^s_\varphi \setminus \mathrm{NR}(\underline{\varphi}^s)$, $L \in \mathcal{L}_s$. Clearly,

$$X_s \ = \ \bigcup_{\substack{L \in \mathcal{L}_s \\ L \subset \mathring{M}^s_\varphi \setminus \mathrm{NR}(\underline{\varphi}^s)}} \underline{\varphi}^s\left(L \cap \mathring{M}^s_\varphi\right) \cap X_s \ ;$$

hence

$$X = \bigcup_{s=1}^{t} \bigcup_{\substack{L \in \mathcal{L}_s \\ L \subset M_\varphi^s \setminus \mathrm{NR}(\underline{\varphi}^s)}} \underline{\varphi}^s \left(L \cap \mathring{M}_\varphi^s \right) .$$

Again by (4.3) and (4.4), for each L, there exist α_L, β_L such that $l_X(b,k) \le \alpha_L k + \beta_L$, for all $b \in \underline{\varphi}(L)$ and all k. The result follows. $\square$

References

1. E. Bierstone and P.D. Milman, *Composite differentiable functions*, Ann. of Math. (2) **116** (1982), 541–558.
2. E. Bierstone and P.D. Milman, *Relations among analytic functions I*, Ann. Inst. Fourier (Grenoble) **37:1** (1987), 187–239; *II*, **37:2** (1987), 49–77.
3. E. Bierstone and P.D. Milman, *Geometric and differential properties of subanalytic sets*, Ann. of Math. (2) **147** (1998), 731–785.
4. C. Chevalley, *On the theory of local rings*, Ann. of Math. (2) **44** (1943), 690–708.
5. A.J. Duncan and L. O'Carroll, *A full uniform Artin-Rees theorem*, J. Reine Agnew. Math. **394** (1989), 203–207.
6. A.M. Gabrielov, *Formal relations between analytic functions*, Math. USSR Izv. **7** (1973), 1056–1088.
7. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II*, Ann. of Math. (2) **79** (1964), 109–326.
8. C. Huneke, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992), 203–223.
9. S. Izumi, *Linear complementary inequalities for orders of germs of analytic functions*, Invent. Math. **65** (1982), 459–471.
10. S. Izumi, *Gabrielov's rank condition is equivalent to an inequality of reduced orders*, Math. Ann. **276** (1986), 81–89.
11. W. Pawłucki, *On Gabrielov's regularity condition for analytic mappings*, Duke Math. J. **65** (1992), 299–311.
12. T. Wang, *Linear Chevalley estimates*, Trans. Amer. Math. Soc. **347** (1995), 4877–4898.

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