Representations of Affine Hecke Algebras, I

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Introduction

Let K be a p-adic field with finite residue class field of q-elements. Let $\mathcal{G}$ be a connected split reductive group over K with connected center. Let $\mathcal{I}$ be an Iwahori subgroup of $\mathcal{G}$ and let $\mathcal{T}$ be the 'diagonal' subgroup of $\mathcal{I}$ (in a suitable sense). The group $N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$ (here $N_{\mathcal{G}}(\mathcal{T})$ is the normalizer of $\mathcal{T}$ in $\mathcal{G}$) is an extended affine Weyl group W (i.e. $W = \Omega \ltimes W'$ for certain abelian group Ω and for certain affine Weyl group W'). It is known that $\mathcal{G} = \bigcup_{w \in W} \mathcal{I}w\mathcal{I}$ and one can define an interesting associated ring structure on the free abelian group H_q with basis $\mathcal{I}w\mathcal{I}$, $w \in W$ (see [IM]). The ring H_q is an affine Hecke ring. We call $\mathbf{H}_q = H_q \otimes \mathbb{C}$ an affine Hecke algebra. According to Borel [Bo1] and Matsumoto [M], the category of admissible complex representations of G which have nonzero vectors fixed by $\mathcal{I}$ is equivalent to the category of finite dimensional representations (over $\mathbb{C}$) of $\mathbf{H}_q$. Thus an interesting part of the study of representations of p-adic groups can be reduced to that of affine Hecke algebras.

According to a conjecture of Langlands (see [La]) the irreducible complex representations of $\mathcal{G}$ should be essentially parametrized by the representations of the Galois group $\mathrm{Gal}(\bar{K}/K)$ into the complex dual group $\mathcal{G}^*(\mathbb{C})$ of $\mathcal{G}$ (in the sense of [La]): $\mathrm{Gal}(\bar{K}/K) \to \mathcal{G}^*(\mathbb{C})$.

Let Γ be the quotient group of $\mathrm{Gal}(\bar{K}/K)$ corresponding to the maximal tamely ramified extention of K. The group Γ has the generators F (Frobenius) and M (Monodromy); subject to the relation $FMF^{-1} = M^q$. According to the conjecture, the irreducible complex representations of $\mathcal{G}$ which have nonzero vectors fixed by the Iwahori group $\mathcal{I}$ should be essentially parametrized by the homomorphisms $\Gamma \to \mathcal{G}^*(\mathbb{C})$. More exactly, Langlands' original conjecture says that the representations should roughly be parametrized by the conjugacy classes of semisimple elements in $\mathcal{G}^*(\mathbb{C})$. A later refinement of the conjecture, due independently to Deligne and Langlands, added nilpotent elements in the picture.

Thus the representations considered should be essentially parametrized by the conjugacy classes of the pair (s, N) such that $\mathrm{Ad}(s)N = qN$, where s is a semisimple element of $\mathcal{G}^*(\mathbb{C})$, N is a nilpotent element in the Lie algebra $\mathbf{g}$ of $\mathcal{G}^*(\mathbb{C})$, and we say two pairs (s, N), (s', N') are conjugate if $s' = gsg^{-1}$, $N' = \mathrm{Ad}(g)N$ for some $g \in G$. For group $GL_n(K)$ this was proved by Berstein and Zelevinsky [BZ], [Z]. For general case, Lusztig (see [L4]) added a third ingredient to (s, N), namely an irreducible representation ρ of the group $A(s, N) = C_G(s) \cap C_G(N)/(C_G(s) \cap C_G(N))^0$ (here $G = \mathcal{G}^*(\mathbb{C})$ and $C_G(\cdot)$ denotes the centralizer in G) appearing in representation of the group $A(s, N)$ on the total complex coefficient homology group of $\mathcal{B}_N^s$, here $\mathcal{B}_N^s$ is the variety of Borel subalgebras of $\mathbf{g}$ containing N and fixed by $\mathrm{Ad}(s)$.

Now the category of admissible complex representations of $\mathcal{G}$ which have nonzero vectors fixed by $\mathcal{I}$ is equivalent to the category of finite dimensional representations (over $\mathbb{C}$) of the Hecke algebra $\mathbf{H}_q$ respect to the Iwahori group $\mathcal{I}$ (see [Bo1, M]). Therefore the conjecture can be stated as

(∗). The irreducible representations of $\mathbf{H}_q$ are naturally 1-1 correspondence with the conjugacy classes of triples (s, N, ρ) as above.

The conjecture (∗) was proved by Kazhdan and Lusztig in [KL4]. Actually they proved that (∗) is true when q is not a root of 1 (one can define $\mathbf{H}_q$ for arbitrary $q \in \mathbb{C}^*$). In [G1] Ginsburg also announced a proof, but the proof contains some errors since the main result is not correct as stated, pointed out by Kazhdan and Lusztig in [KL4]. However the work [G1] contains some very interesting ideas. Combine [KL4] and [G1] we can prove that (∗) is true for most roots of 1 (see chapter 4, actually we get more). In chapter 5 we shall show that for some roots of 1 (it is expected only for these roots, see [L17]) (∗) is not true.

Now we explain some details of the paper.

In chapter 1 we give the definitions of Coxeter groups and of Hecke algebras. We also recollect some definitions and results in [KL1] and [L6], which we shall need. In chapter 2 we give the definitions of extended affine Weyl groups and of affine Hecke algebras, and recall some results on cells in affine Weyl groups. Following Berstein, the center of an affine Hecke algebra is explicitly described. In chapter 3 we recall some work on Deligne-

Langlands conjecture for Hecke algebras by Ginsburg [G1-G2], Kazhdan and Lusztig [KL4]. We give some discussions to the standard modules (in the sense of [KL4]). For type A it is not difficult to determine the dimensions of standard modules. We also state two conjectures, one is concerned with the based rings of cells in affine Weyl groups, and another is for simple modules of affine Hecke algebras with two parameters, which is an analogue of the $(*)$. In chapter 4 we introduce an equivalence relations in $T \times \mathbb{C}^*$, where T is a maximal torus of a connected reductive group over $\mathbb{C}$. Combining some properties of the equivalence relation, results of Ginzburg and of Kazhdan & Lusztig in chapter 3, we prove that $(*)$ is true for most roots of 1. In chapter 5 we show that for some roots of 1 $(*)$ is not true by using some results in [Ka2] and in chapter 4. In chapter 6 we give some discussions to certain remarkable quotient algebras of $\mathbf{H}_q$. The chapters 7 and 8 are based on preprints "The based rings of cells in affine Weyl groups of type $\tilde{G}_2$, $\tilde{B}_2$", "Some simple modules of affine Hecke algebras", respectively. In chapter 7 we verify the conjecture in [L14] for cells in affine Weyl groups of type $\tilde{G}_2$, $\tilde{B}_2$. In chapetr 8 we show that the conjecture in [L14] is true for the second highest two-sided cell in an affine group. Once we know the structures of the based rings we can know the structures of the corresponding standard $\mathbf{H}_q$-modules. The explicit knowledge of based rings provides a way to compute the dimensions of simple $\mathbf{H}_q$-modules and their multiplicities in standard modules, also can be used to classify the simple $\mathbf{H}_q$-modules even though q is a root of 1. In chapter 7 we work out the dimensions of simple $\mathbf{H}_q$-modules for type $\tilde{A}_2$. An immediate consequence is that $\mathbf{H}_q \not\simeq \mathbf{H}_1 = \mathbb{C}[W]$ whenever $q \neq 1$ for type $\tilde{A}_2$. This leads to several questions.

I would like to thank Professor T.A. Springer for some helpful conversations.

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Contents

1. Hecke Algebras

In this chapter we give the definitions of extended Coxeter groups and of their Hecke algebras and some examples. Some definitions (such as these of Kazhdan-Lusztig polynomials and cells) and results from [KL1] and [L6] are recalled. We also show how to apply the definitions in [L6], which is a generalization of [KL1]. Several questions are proposed. We refer to [B, Hu] for more details about Coxeter groups and their Hecke algebras.

1.1. Basic definitions. A Coxeter group is a group W' which possesses a set $S = \{s_i\}_{i\in I}$ of generators subject to the relations

$$s_i^2 = 1, \qquad (s_is_j)^{m_{ij}} = 1 \ \ (i \neq j),$$

where $m_{ij} \in \{2, 3, 4, ..., \infty\}$. We also write m_{st} for m_{ij}, where $s = s_i,\ t = s_j$.

We call (W', S) a Coxeter system and S the set of distinguished generators or the set of simple reflections. Let l be the length function of W' and $\leq$ denote the usual partial order in W'.

In Lie theory we often need to consider extended Coxeter groups. If a group Ω acts on the Coxeter system (W', S), we then define a new group $W = \Omega \ltimes W'$ by $(\omega_1, w_1)(\omega_2, w_2) = (\omega_1\omega_2, \omega_2^{-1}(w_1)w_2)$. The group W is called an extended Coxeter group. The length function l can be extended to W by defining $l(\omega w) = l(w)$, and the partial order $\leq$ can be extended to W by defining $\omega w \leq \omega' u$ if and only if $\omega = \omega',\ w \leq u$, where $\omega', \omega \in \Omega,\ w, u \in W'$. We denote the extensions again by l and $\leq$, respectively.

Let $\mathbf{q}_s^{\frac{1}{2}},\ s \in S$ be indeterminates. We assume that $\mathbf{q}_s^{\frac{1}{2}} = \mathbf{q}_t^{\frac{1}{2}}$ if and only if s, t are conjugate in W. Let $\mathcal{A} = \mathbf{Z}[\mathbf{q}_s^{\frac{1}{2}}, \mathbf{q}_s^{-\frac{1}{2}}]_{s\in S}$ be the ring of Laurant polynomials in $\mathbf{q}_s^{\frac{1}{2}},\ s \in S$ with integer coefficients. The (generic) Hecke algebra $\mathcal{H}$ (over $\mathcal{A}$) of W is an associative $\mathcal{A}$-algebra. As an $\mathcal{A}$ module, $\mathcal{H}$ is free with a basis $T_w,\ w \in W$, and multiplication laws are

(1.1.1) $\quad (T_s - \mathbf{q}_s)(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_wT_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$

The generic Hecke algebra of W actually can be defined over $\mathbf{Z}[\mathbf{q}_s]_{s\in S}$, but it is convenient to define it over $\mathcal{A}$ for introducing Kazhdan-Lusztig polynomials and for defining cells in W.

Let $\mathcal{H}'$ be the subalgebra of $\mathcal{H}$ generated by T_s, $s \in S$. Then the algebra $\mathcal{H}$ is isomorphic to the "twisted" tensor product $\mathbf{Z}[\Omega] \otimes_{\mathbf{Z}} \mathcal{H}'$ by assigning $T_{\omega w} \to \omega \otimes T_w$, where $\mathbf{Z}[\Omega]$ is the group algebra over $\mathbf{Z}$, and the multiplication in $\mathbf{Z}[\Omega] \otimes_{\mathbf{Z}} \mathcal{H}'$ is given by

$$(\omega \otimes T_w)(\omega' \otimes T_u) = \omega\omega' \otimes T_{\omega'^{-1}(w)} T_u.$$

Note that $s, t \in S$ may be conjugate in W but not conjugate in W', thus $\mathcal{H}'$ may not be the generic Hecke algebra of W' in previous sense.

For an arbitrary $\mathcal{A}$-algebra $\mathcal{A}'$, The $\mathcal{A}'$-algebra $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{A}'$ is called an Hecke algebra.

Convention: We shall denote the images in $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{A}'$ of T_w, $w \in W$ by the same notations.

1.2. Two special choices of $\mathcal{A}'$ are particularly interesting.

(a). Let $\mathbf{q}^{\frac{1}{2}}$ be an indeterminate and let $A = \mathbf{Z}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$ be the ring Laurant polynomials in $\mathbf{q}^{\frac{1}{2}}$ with integer coefficients. Choose integers c_s, $s \in S$ such that $c_s = c_t$ whenever s, t are conjugate in W. There is a unique homomorphism of rings from $\mathcal{A}$ to A such that $\mathbf{q}_s^{\frac{1}{2}}$, $(s \in S)$ maps to $\mathbf{q}^{\frac{c_s}{2}}$. Thus A is an $\mathcal{A}$-algebra. The multiplication laws in the Hecke algebra $\mathcal{H} \otimes_{\mathcal{A}} A$ are (note the convention at the end of 1.1)

$$(1.2.1) \qquad (T_s - \mathbf{q}^{c_s})(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$$

(b). When all integers c_s are 1, we denote the Hecke algebra $\mathcal{H} \otimes_{\mathcal{A}} A$ by H. The multiplication laws in H are

$$(1.2.2) \qquad (T_s - \mathbf{q})(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$$

Sometimes H is also called the generic Hecke algebra of W (with one parameter). By now the Hecke algebra H and its various specializations $H \otimes_A A'$ are the most extensively studied Hecke algebras.

There is also a slight generalizations of the Hecke algebra $\mathcal{H}$. Let R be a commutative ring with 1. For any $s \in S$, choose $u_s,\ v_s \in R$ such that $u_s = u_t,\ v_s = v_t$ whenever $s,\ t$ are conjugate in W. Then there exists a unique associative R-algebra $\tilde{\mathcal{H}}$, which is a free R-module with a basis $T'_w,\ w \in W$ and multiplication is defined by

$$(1.2.3) \qquad T_s'^2 = u_s T_s' + v_s, \quad \text{if } s \in S; \qquad T'_w T'_u = T'_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$$

(see, e.g. [Hu]). Sometimes the R-algebra $\tilde{\mathcal{H}}$ is actually an Hecke algebra. Suppose that v_s has a square root $v_s^{\frac{1}{2}}$ and $v_s^{\frac{1}{2}}$ is invertible in R. Furthermore we assume that there exists an invertible element $u_s'^{\frac{1}{2}} \in R$ such that

$$(1.2.5) \qquad u_s'^{\frac{1}{2}} - u_s'^{-\frac{1}{2}} = u_s v_s^{-\frac{1}{2}}.$$

We set $T''_s = u_s'^{\frac{1}{2}} v_s^{-\frac{1}{2}} T'_s$, then $T_s''^2 = (u'_s - 1)T''_s + u'_s$. In this case the algebra $\tilde{\mathcal{H}}$ is an Hecke algebra in the sense of 1.1.

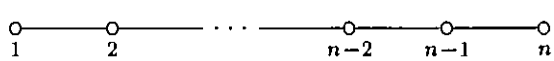
In Lie theory there are also other interesting algebras of Hecke type, see, e.g., [BM, Ca, MS].

1.3. Examples of Coxeter groups. It is convenient to represent a Coxeter system (W', S) by a graph Σ, the Coxeter graph of (W', S). The vertex set of Σ is one to one correspondence with S; a pair of vertices corresponding to s_i, s_j are jointed with an edge whenever $m_{ij} \geq 3$, and label such an edge with m_{ij} when $m_{ij} \geq 4$. Thus the graph Σ determines (W', S) up to isomorphism.

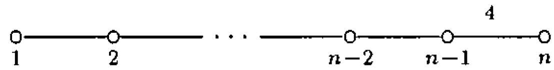
A Coxeter system (W', S) is called irreducible if for any $s, t \in S$ we can find a sequence $s = t_0, t_1, \cdots, t_k = t$ in S such that $m_{t_i, t_{i+1}} \geq 3$ (i.e. $t_i t_{i+1} \neq t_{i+1} t_i$), $0 \leq i \leq k-1$. We also call W' an irreducible Coxeter group when (W', S) is irreducible. Obviously, any Coxeter group is a direct product of some irreducible Coxeter groups.

The most important Coxeter groups in Lie theory are Weyl groups and affine Weyl groups. They are classified. The Coxeter graphs of irreducible Weyl groups and irreducible affine Weyl groups are as follows.

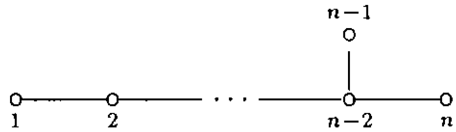
Type A_n $(n \geq 1)$.



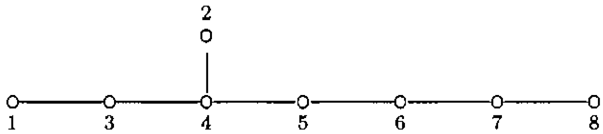
Type B_n ($n \geq 2$).



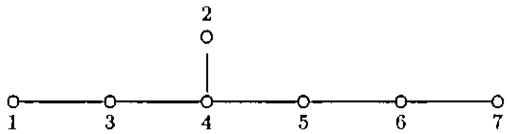
Type D_n ($n \geq 4$).



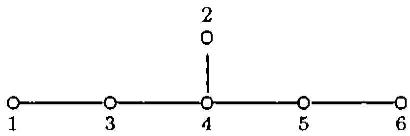
Type E_8.



Type E_7.

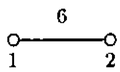


Type E_6.



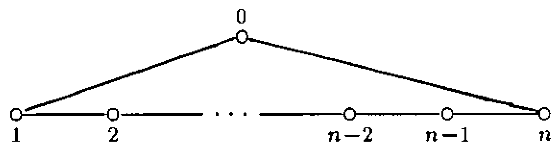
Type F_4.

Type G_2.

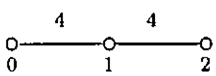


Type $\tilde{A}_1$.

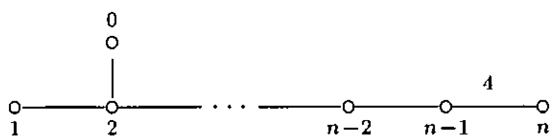
Type $\tilde{A}_n$ ($n \geq 2$).



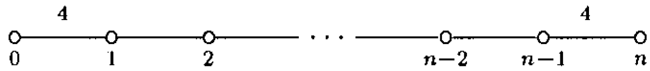
Type $\tilde{B}_2 = \tilde{C}_2$.



Type $\tilde{B}_n$ ($n \geq 3$).

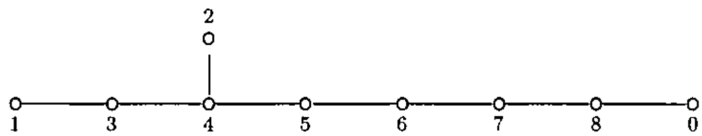


Type $\tilde{C}_n$ $(n \geq 3)$.

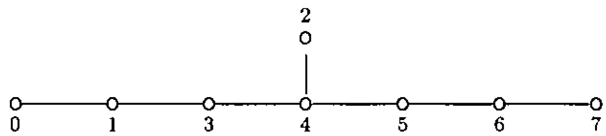


Type $\tilde{D}_n$ $(n \geq 4)$.

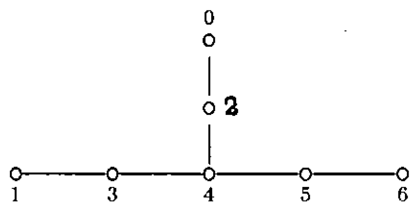
Type $\tilde{E}_8$.



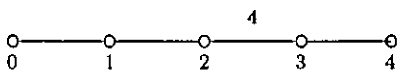
Type $\tilde{E}_7$.



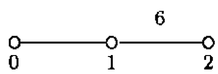
Type $\tilde{E}_6$.



Type $\tilde{F}_4$.



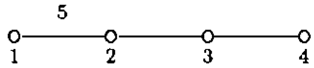
Type $\tilde{G}_2$.



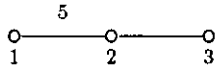
1.4. The Weyl group of type A_n is just the symmetric group $\mathfrak{S}_{n+1}$ of degree $n+1$. One may choose $\{(12),\ (23),\ \cdots,(n,n+1)\}$ as the set of simple reflections of $\mathfrak{S}_{n+1}$.

Except Weyl groups, the other irreducible finite Coxeter groups are dihedral groups $I_2(m)$ ($m = 5$ or $m > 6$, when $m = 3, 4, 6$, $I_2(m)$ are Weyl groups) and Coxeter group of type H_3 or H_4. Their Coxeter graphs are as follows.

Type H_4.



Type H_3.



Type $I_2(m)$. $\underset{1}{\circ}\overset{m}{\text{———}}\underset{2}{\circ}$

When (W', S) is crystallographic (i.e., $m_{ij} = 2, 3, 4, 6, \infty$ for arbitrary $s_i, s_j \in S$), W' can be realized as the Weyl group of certain Kac-Moody algebra (see [K]). Thus we have a Schubert variety $\mathcal{B}_w$ for each element $w \in W'$. This is a key to apply the powerful intersection cohomology theory to the Kazhdan-Lusztig theory.

1.5. Examples of Hecke algebras. (a). Let G be a Chevalley group over a finite field of q elements. Let B be a Borel subgroup of G and T the maximal torus in B. Then the group $W_0 = N_G(T)/T$ is a Weyl group. We have $G = \bigcup_{w \in W_0} BwB$. Let H be the free $\mathbf{Z}$-module generated by the double cosets BwB, $w \in W_0$. We denote T_w the double coset BwB regarding as an element in H. Define the multiplication in H by

$$T_w T_u = \sum_v m_{w,u,v} T_v, \tag{1.5.1}$$

where the structure constants $m_{w,u,v}$ are defined as the number of cosets of the form Bx in the set $Bw^{-1}Bv \cap BuB$:

$$m_{w,u,v} = |Bw^{-1}Bv \cap BuB/B|.$$

Then H is an associative ring with unit T_e, where e is the unit element in W_0. Moreover we have

$$(T_s - q)(T_s + 1) = 0, \quad \text{if } s \in S_0; \qquad T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u), \tag{1.5.2}$$

where S_0 is the set of simple reflections in W_0. (see [I]).

It is well known that $H \otimes_{\mathbf{Z}} \mathbb{C} \simeq \mathrm{End} 1_B^G$, where 1_B^G stands for the induced representation of the unit representation 1_B (over $\mathbb{C}$) of B (see [I, C2, Cu]). Thus part of the study of 1_B^G can be reduced to that of $H \otimes_{\mathbf{Z}} \mathbb{C}$.

(b). Let K be a p-adic field such that its residue field k contains q elements. Let G be a Chevalley group over the field K. Let B be an Iwahori subgroup of G. Let T

be the 'diagonal' subgroup of B (in a suitable sense). Then the group $W = N_G(T)/T$ is an extended affine Weyl group (i.e., there is a commutative subgroup Ω which acts on an affine Weyl group (W', S) such that $W \simeq \Omega \ltimes W'$, see 2.1 for definition). As the above example we have $G = \bigcup_{w \in W} BwB$. Let H be the free $\mathbf{Z}$-module generated by the double cosets BwB, $w \in W$. We denote T_w the double coset BwB regarding as an element in H. Define the multiplication in H by

$$T_w T_u = \sum_v m_{w,u,v} T_v, \tag{1.5.3}$$

where the structure constants $m_{w,u,v}$ are defined as the number of cosets of the form Bx in the set $Bw^{-1}Bv \cap BuB$:

$$m_{w,u,v} = |Bw^{-1}Bv \cap BuB/B|.$$

Then H is an associative ring with unit T_e, where e is the unit element in W. Moreover we have

$$(T_s - q)(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u), \tag{1.5.4}$$

where S is the set of simple reflections in W. (see [IM]).

It is known that the category of admissible complex representations of G which have nonzero vectors fixed by B is equivalent to the category of finite dimensional representations (over $\mathbb{C}$) of $H \otimes_{\mathbf{Z}} \mathbb{C}$ (see [Bo1, M]).

1.6. Kazhdan-Lusztig polynomials. The work [KL1] stimulates a lots of work and deeply increased our understanding to Coxeter groups and to Hecke algebräs. The key role is the Kazhdan-Lusztig polynomials. In this section we recall some definitions and results from [KL1].

We keep the notations in 1.1 and in 1.2 (b). Thus (W', S) is a Coxter system, $W = \Omega \ltimes W'$ is an extended Coxeter group and H is the generic Hecke algebra of W over $A = \mathbf{Z}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$.

Let $a \to \bar{a}$ be the involution of the ring A defined by $\bar{\mathbf{q}}^{\frac{1}{2}} = \mathbf{q}^{-\frac{1}{2}}$. This extends to an involution $h \to \bar{h}$ of the ring H defined by

$$\overline{\sum a_w T_w} = \sum \bar{a}_w T_{w^{-1}}^{-1} \quad (a_w \in A).$$

Note that T_w is invertible for any $w \in W$ since $T_s^{-1} = \mathbf{q}^{-1}T_s + (\mathbf{q}^{-1} - 1)$ for $s \in S$ and $T_\omega^{-1} = T_{\omega^{-1}}$ for $\omega \in \Omega$. Then (see [KL1, (1.1.c)]):

(a) For any $w \in W$, there is a unique element $C_w \in H$ such that

$$\bar{C}_w = C_w,$$

$$C_w = \mathbf{q}^{-\frac{l(w)}{2}} \sum_{y \leq w} P_{y,w} T_y,$$

where $P_{y,w} \in A$ is a polynomial in $\mathbf{q}$ of degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

The assertion (a) is equivalent to the following assertion.

(b) For any $w \in W$, there is a unique element $C'_w \in H$ such that $\bar{C}'_w = C'_w$ and $C'_w = \sum_{y \leq w} (-1)^{l(w)-l(y)} \mathbf{q}^{\frac{l(w)}{2}} \mathbf{q}^{-l(y)} \bar{P}_{y,w} T_y$, where $P_{y,w} \in A$ is a polynomial in $\mathbf{q}$ of degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

Note that our notations C_w, C'_w exchange these in [KL1] since we shall mainly use the elements C_w.

Obviously, C_w, $w \in W$ and C'_w, $w \in W$ are two A-bases of H. They are related by three involutions.

(c) Let j be the involution of the ring H given by

$$j(\sum a_w T_w) = \sum \bar{a}_w (-\mathbf{q})^{-l(w)} T_w,$$

then $C'_w = (-1)^{l(w)} j(C_w)$. (see [KL1]).

(d) Let Φ be the involution of the ring H defined by

$$\Phi(\mathbf{q}^{\frac{1}{2}}) = \mathbf{q}^{-\frac{1}{2}}, \ \Phi(T_w) = (-\mathbf{q})^{l(w)} T_{w^{-1}}^{-1},$$

then $C'_w = \Phi(C_w)$. (see [L11]).

(e) Let k be the involution of the A-algebra H given by

$$k(\sum a_w T_w) = \sum a_w (-\mathbf{q})^{l(w)} T_{w^{-1}}^{-1},$$

then $C'_w = (-1)^{l(w)} k(C_w)$.

The assertion (e) follows from that $jk = \bar{\cdot}$, $\bar{C}_w = C_w$, and $\bar{C}'_w = C'_w$. Note that k is an involution of A-algebra, in some cases this fact is useful in transforming the properties concerned with C'_w to C_w.

The polynomials $P_{y,w}$ are called Kazhdan-Lusztig polynomials. We have $P_{y,w} = \mu(y,w)\mathbf{q}^{\frac{1}{2}(l(w)-l(y)-1)}$+lower degree terms. we say that $y \prec w$ if $\mu(y,w) \neq 0$, we then set $\mu(w,y) = \mu(y,w)$.

1.7. Motivated by his definition of canonical bases of quantum groups (see [L19]), Lusztig gave another construction of the elements C_w, C'_w. Consider the $\mathbf{Z}[\mathbf{q}^{-\frac{1}{2}}]$-submodule $\mathcal{L}$ of H spanned by $\tilde{T}_w = \mathbf{q}^{-\frac{l(w)}{2}} T_w$, $w \in W$ and the $\mathbf{Z}[\mathbf{q}^{\frac{1}{2}}]$-submodule $\mathcal{L}'$ of H spanned by $\tilde{T}_w$, $w \in W$, then (see [L20])

(a) The restriction of $\pi : \mathcal{L} \to \mathcal{L}/\mathbf{q}^{-\frac{1}{2}}\mathcal{L}$ defines an isomorphism of $\mathbf{Z}$-modules $\pi_1 : \mathcal{L} \cap \bar{\mathcal{L}} \simeq \mathcal{L}/\mathbf{q}^{-\frac{1}{2}}\mathcal{L}$ and $\pi_1^{-1}(\pi(\tilde{T}_w)) = C_w$.

(b) The restriction of $\pi' : \mathcal{L}' \to \mathcal{L}'/\mathbf{q}^{\frac{1}{2}}\mathcal{L}'$ defines an isomorphism of $\mathbf{Z}$-modules π_1 : $\mathcal{L}' \cap \bar{\mathcal{L}}' \simeq \mathcal{L}'/\mathbf{q}^{\frac{1}{2}}\mathcal{L}'$ and $\pi_1'^{-1}(\pi'(\tilde{T}_w)) = C'_w$.

1.8. The elements C_w have the following properties (see [KL1]):

(a) For $s \in S$ we have

$$C_s C_w = \begin{cases} (\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}})C_w, & \text{if } sw \leq w \\ C_{sw} + \sum\limits_{\substack{y \prec w \\ sy \leq y}} \mu(y,w) C_y, & \text{if } sw \geq w. \end{cases}$$

$$C_w C_s = \begin{cases} (\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}})C_w, & \text{if } ws \leq w \\ C_{ws} + \sum\limits_{\substack{y \prec w \\ ys \leq y}} \mu(y,w) C_y, & \text{if } ws \geq w. \end{cases}$$

They are equivalent to the following recursion formulas of the Kazhdan-Lusztig polynomials.

(b) Assume that for $s, t \in S$ we have $sw > w, wt > w$, then

$$P_{y,sw} = \mathbf{q}^{1-a} P_{sy,w} + \mathbf{q}^a P_{y,w} - \sum_{\substack{z \\ y \leq z \prec w \\ sz < z}} \mu(z,w) \mathbf{q}^{\frac{l(w)-l(z)+1}{2}} P_{y,z}, \quad (y \leq sw)$$

where $a = 1$ if $sy < y$, $a = 0$ if $sy > y$; and $P_{y,sw} = P_{sy,sw}$.

$$P_{y,wt} = \mathbf{q}^{1-a} P_{sy,w} + \mathbf{q}^a P_{y,w} - \sum_{\substack{z \\ y \leq z \prec w \\ zt < z}} \mu(z,w) \mathbf{q}^{\frac{l(w)-l(z)+1}{2}} P_{y,z}, \quad (y \leq sw)$$

where $a = 1$ if $yt < y$, $a = 0$ if $yt > y$; and $P_{y,wt} = P_{yt,wt}$.

1.9. When (W', S) is a finite Coxeter group or a crystallographic group, it is known that the coefficients of $P_{y,w}$ are non-negative. This is proved in [KL2] when (W', S) is crystallographic. For H_3, H_4 it was done by Goresky [Go] and Alvis [A]. For dihedral groups I_m it is trivial since $P_{y,w} = 1$ for any $y \leq w$. It was conjectured in [KL1] that for arbitrary Coxeter group the Kazhdan-Lusztig polỳnomials have non-negative coefficients.

1.10. Question. (i). *It is known that the Kazhdan-Lusztig polynomials in crystallographic Coxeter groups are related to middle intersection cohomology of Schubert varieties. Now what polynomials are related to other intersection cohomology of Schubert varieties?*

(ii). *If we loose the restriction on the degree of $P_{y,w}$ to* $\deg P_{y,w} \leq (l(w) - l(y))$, *what happen for the Kazhdan-Lusztig polynomials and the elements C_w.*

1.11. Cell For any $w \in W$ we set $L(w) = \{s \in S \mid sw < w\}$, $R(w) = \{s \in S \mid ws < w\}$.

Let $w, u \in W'$, we say that $w \underset{L}{\leq} u$ (resp. $w \underset{R}{\leq} u$; $w \underset{LR}{\leq} u$) if there exists a sequence $w = w_0, w_1, \cdots, w_k = u$ in W' such that for each i, $1 \leq i \leq k$, we have $\mu(w_{i-1}, w_i) \neq 0$ and $L(w_{i-1} \not\subseteq L(w_i)$ (resp. $R(w_{i-1} \not\subseteq R(w_i)$; $L(w_{i-1} \not\subseteq L(w_i)$ or $R(w_{i-1} \not\subseteq R(w_i))$. Then for any $\omega, \omega' \in \Omega$ we say that $\omega w \underset{L}{\leq} \omega' u$ (resp. $w\omega \underset{R}{\leq} u\omega'$; $\omega w \underset{LR}{\leq} \omega' u$) if $w \underset{L}{\leq} u$ (resp. $w \underset{R}{\leq} u$; $w \underset{LR}{\leq} u$).

For any $x, y \in W$ we write that $x \underset{L}{\sim} y$ (resp. $x \underset{R}{\sim} y$; $x \underset{LR}{\sim} y$) if $x \underset{L}{\leq} y \underset{L}{\leq} x$ (resp. $x \underset{R}{\leq} y \underset{R}{\leq} x$; $x \underset{LR}{\leq} y \underset{LR}{\leq} x$). The relations $\underset{L}{\leq}$, $\underset{R}{\leq}$, $\underset{LR}{\leq}$ are preorders in W. And the relations

$\underset{L}{\sim}, \underset{R}{\sim}, \underset{LR}{\sim}$ are equivalence relations in W; the corresponding equivalence classes are called left cells, right cells, two-sided cells of W, respectively. The preorder $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}$; $\underset{LR}{\leq}$) induces a partial order on the set of left (resp. right; two-sided) cells of W, we denote it again by $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}$; $\underset{LR}{\leq}$).

When $W = W'$ is a Weyl group, the definitions of left cell and two-sided cell coincide with the definitions given by Joseph [J1-J2]. The cells in Weyl groups were extensively investigated by Barbasch, Lusztig, Joseph, Vogan, etc., and play an important role in the representation theory of finite groups of Lie type (see [L7]) and in the theory of primitive ideals of universal enveloping algebras of semisimple Lie algebras.

For affine Weyl groups, the structure of left cells and two-sided cells are determined for type $\tilde{A}_n$ (see [Sh1, L8]), rank 2, 3 (see [L11, Bél, D]). Recently Shi found an algorithm, then he and his students determined the structure of cells in affine Weyl groups of type $\tilde{B}_4$, $\tilde{C}_4$, $\tilde{D}_4$ (see [Sh4]). For type $\tilde{D}_4$, see also [Ch]. In [L11-L14] Lusztig obtained a series of important results concerned with cells in affine Weyl groups.

1.12. ***a*-function** For an extended Coxeter group W, the function $a: W \to \mathbf{N}$ was introduced in [L11] and is a useful tool in cell theory and related topics.

Given $w, u \in W$, we write

$$C_w C_u = \sum_{v \in W} h_{w,u,v} C_v, \qquad h_{w,u,v} \in A.$$

For any $v \in W$, we define $a(v)$=the minimal non-negative integer i such that $\mathbf{q}^{\frac{i}{2}} h_{w,u,v} \in \mathbf{Z}[\mathbf{q}^{\frac{1}{2}}]$ for all $w, u \in W$. If such i doesnot exist, we set $a(v) = \infty$.

For a finite Coxeter group, the function a is always bounded. A non-trivial fact is that a is bounded for an affine Weyl group (see [L11]). In [L12], Lusztig obtained some interesting results under the assumption of a being bounded and of W' being crystallographic.

Assume that (W', S) is a crystallographic group, then all $h_{w,u,v}$ are Laurant polynomials in $\mathbf{q}^{\frac{1}{2}}$ with the same purity and have non-negative coefficients (see [L11]). It seems naturally to hope such property holds for arbitrary Coxeter groups.

Here are two questions.

1.13. Question. (i). *Find out all Coxeter groups whose a-function are bounded.*

(ii). *Find out a Coxeter group W' such that there exists some $w \in W'$ with $a(w) = \infty$.*

Generalized Cells

1.14. Lusztig generalized the definition of cells in [KL1] to the cases of simple reflections being given different weights (see [L6]). Strangely the interesting generalization is less developed. In the rest of the chapter we shall give some discussion to the generalization. We first recall the definition, then show how to apply the definition.

Let (W', S) be a Coxeter system and $W = \Omega \ltimes W'$ be an extended Coxeter group. Let $\varphi: W \to \Gamma$ be a map from W into an abelian group Γ such that $\varphi(\Omega) = \{e\}$ (e the unit element in Γ), $\varphi(s) = \varphi(t)$ whenever $s, t \in S$ are conjugate in W. We shall set $\varphi(w) = \mathbf{q}_w^{\frac{1}{2}}$, $(w \in W)$. Let H_φ be the Hecke algebra of W with respect to φ; this is an algebra over the group ring $\mathbf{Z}[\Gamma]$. As a $\mathbf{Z}[\Gamma]$ module, it is free with a basis T_w, $w \in W$. The multiplication is defined by

$$(1.14.1) \quad (T_s - \mathbf{q}_s)(T_s + 1) = 0, \quad \text{if } s \in S; \qquad T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u).$$

When $\mathbf{q}_s^{\frac{1}{2}} = \mathbf{q}_t^{\frac{1}{2}}$ if and only if s, t are conjugate in W and Γ is a free abelian group with a basis $\mathbf{q}_s^{\frac{1}{2}}$, $s \in S$, the algebra H_φ is canonically isomorphic to the algebra $\mathcal{H}$ in 1.1 if we identify $\mathbf{Z}[\Gamma]$ with $\mathcal{A}$. When $\mathbf{q}_s^{\frac{1}{2}} = \mathbf{q}_t^{\frac{1}{2}}$ for any $s, t \in S$, and Γ is a free abelian group generated by $\mathbf{q}_s^{\frac{1}{2}}$, the algebra H_φ is canonically isomorphic to the algebra H in 1.2 (b). Suitably choose the map $\varphi: W \to < \mathbf{q}_s^{\frac{1}{2}} >$ (the free abelian group generated by $\mathbf{q}_s^{\frac{1}{2}}$) we see that the Hecke algebra in 1.2 (a) is also canonically isomorphic to some H_φ.

It will be convenient to introduce a new basis $\tilde{T}_w = \mathbf{q}_w^{-\frac{1}{2}} T_w$, $(w \in W)$. We then have

$$(\tilde{T}_s + \mathbf{q}_s^{-\frac{1}{2}})(\tilde{T}_s - \mathbf{q}_s^{\frac{1}{2}}) = 0, \qquad (s \in S).$$

Let $a \to \bar{a}$ be the involution of the ring $\mathbf{Z}[\Gamma]$ which take γ to γ^{-1} for any $\gamma \in \Gamma$. This extends to an involution $h \to \bar{h}$ of the ring H_φ defined by

$$\overline{\sum a_w \tilde{T}_w} = \sum \bar{a}_w \tilde{T}_{w^{-1}}^{-1} \quad (a_w \in \mathbf{Z}[\Gamma]).$$

Note that T_w is invertible for any $w \in W$ since $\tilde{T}_s^{-1} = \tilde{T}_s + (\mathbf{q}_s^{-\frac{1}{2}} - \mathbf{q}_s^{\frac{1}{2}})$ for $s \in S$ and $T_\omega^{-1} = T_{\omega^{-1}}$ for $\omega \in \Omega$. We define the elements $R^*_{x,y} \in \mathbf{Z}[\Gamma]$, $(x, y \in W)$, by

$$\tilde{T}_{y^{-1}}^{-1} = \sum_x \bar{R}^*_{x,y} \tilde{T}_x.$$

It is easy to see that $R^*_{x,y} = 0$ unless $x \leq y$ in the standard partial order of W. Using the fact that $h \to \bar{h}$ is an involution, we see that

$$(1.14.2) \qquad \sum_{x \leq y \leq z} \bar{R}^*_{x,y} \bar{R}^*_{y,z} = \delta_{x,z},$$

for all $x \leq z$ in W. Note also that $\mathbf{q}_x^{-\frac{1}{2}} \mathbf{q}_y^{\frac{1}{2}} R^*_{x,y} \in \mathbf{Z}[\Gamma^2]$ (convention: $\Gamma^2 = \{\gamma^2 \mid \gamma \in \Gamma\}$).

(1.14.3). $R^*_{x,x} = 1$ for all $x \in W$.

(1.14.4). If $x < y$, $l(y) = l(x) + 1$, then x is obtained by dropping some $s \in S$ in a reduced expression of y, and we have $R^*_{x,y} = \mathbf{q}_s^{\frac{1}{2}} - \mathbf{q}_s^{-\frac{1}{2}}$.

(1.14.5). If $x < y$, $l(y) = l(x) + 2$, then x is obtained by dropping some $s, t \in S$ in a reduced expression of y, and we have $R^*_{x,y} = (\mathbf{q}_s^{\frac{1}{2}} - \mathbf{q}_s^{-\frac{1}{2}})(\mathbf{q}_t^{\frac{1}{2}} - \mathbf{q}_t^{-\frac{1}{2}})$.

We now assume that a total order in Γ is given which is compatible with the group structure on Γ. Let Γ_+ be the set of elements which are strictly positive (i.e. bigger than the unit element) for this total order and let $\Gamma_- = (\Gamma_+)^{-1}$. We shall assume that $\mathbf{q}_s^{\frac{1}{2}} \in \Gamma_+$ for all $s \in S$. We have (see 2. Proposition in [L6])

(a) Given $w \in W$, there is a unique element $C_w \in H_\varphi$ such that

$$\bar{C}_w = C_w,$$

$$C_w = \sum_{y \leq w} P^*_{y,w} \tilde{T}_y,$$

where $P^*_{y,w} \in \mathbf{Z}[\Gamma]$ is a $\mathbf{Z}$-linear combination of elements in Γ_- for $y < w$ and $P^*_{w,w} = 1$. Moreover $\mathbf{q}_y^{-\frac{1}{2}} \mathbf{q}_w^{\frac{1}{2}} P^*_{y,w} \in \mathbf{Z}[\Gamma^2]$.

(When φ is constant on S, this is the same as (1.1.c) of [KL1].)

We also can introduce the elements C'_w as 1.6(b) and show that C_w, C'_w are related by three involutions of H_φ as (c-e) in 1.6.

Now let $s \in S$, $w \in W$ be such that $w < sw$. For each y such that $sy < y < w$, we define an element

$$M^s_{y,w} \in \mathbf{Z}[\Gamma]$$

by the inductive condition

$$\sum_{\substack{y \leq z < w \\ sz < z}} P^*_{y,z} M^s_{z,w} - \mathbf{q}_s^{\frac{1}{2}} P^*_{y,w} \quad \text{is a combination of elements in } \Gamma_- \tag{1.14.6}$$

and by the symmetry condition

$$\bar{M}^s_{y,w} = M^s_{y,w}. \tag{1.14.7}$$

The condition (1.14.6) determines uniquely the coefficient of γ in $M^s_{y,w}$ for all $\gamma \in \Gamma - \Gamma_-$; the condition (1.14.7) determines other coefficients. We have $\mathbf{q}_s^{-\frac{1}{2}} \mathbf{q}_y^{-\frac{1}{2}} \mathbf{q}_w^{\frac{1}{2}} M^s_{y,w} \in \mathbf{Z}[\Gamma^2]$.

Let $s \in S$ and let $w \in W$, then (see 4. Proposition in [L6])

$$(\tilde{T}_s + \mathbf{q}_s^{-\frac{1}{2}}) C_w = C_{sw} + \sum_{\substack{z < w \\ sz < z}} M^s_{z,w} C_z, \qquad \text{if } w < sw \tag{1.14.8}$$

$$(\tilde{T}_s - \mathbf{q}_s^{\frac{1}{2}}) C_w = 0, \qquad \text{if } w > sw \tag{1.14.9}$$

Let j' be the anti-automorphism of the ring H_φ defined by $j'(\tilde{T}_w) = \tilde{T}_{w^{-1}}$ and $j'(a) = a$ for $a \in \mathbf{Z}[\Gamma]$. It is easy to see that $j'(C_w) = C_{w^{-1}}$. From (1.14.8-9) we can deduce

$$C_w(\tilde{T}_s + \mathbf{q}_s^{-\frac{1}{2}}) = C_{ws} + \sum_{\substack{z < w \\ zs < z}} M^s_{z^{-1},w^{-1}} C_z, \qquad \text{if } w < ws \tag{1.14.10}$$

$$C_w(\tilde{T}_s - \mathbf{q}_s^{\frac{1}{2}}) = 0, \qquad \text{if } w > ws \tag{1.14.11}$$

The identity (1.14.9) is equivalent to the following

$$P^*_{u,w} = \mathbf{q}_s^{-\frac{1}{2}} P^*_{su,w} \qquad \text{if } u < su \leq w,\ sw < w. \tag{1.14.12}$$

and the identity (1.14.11) is equivalent to the following

$$P^*_{u,w} = \mathbf{q}_s^{-\frac{1}{2}} P^*_{us,w} \qquad \text{if } u < us \leq w,\ ws < w. \tag{1.14.13}$$

(b) Let $y < w$ be such that $l(w) = l(y) + 1$. Then y is obtained by dropping a simple reflection s in a reduced expression of w. We have

(i). $P^*_{y,w} = \mathbf{q}_s^{-\frac{1}{2}}$.

(ii). Let t be a simple reflection such that $ty < y < w < tw$, then

$$M^t_{y,w} = \begin{cases} 0, & \text{if } \mathbf{q}_t^{\frac{1}{2}} < \mathbf{q}_s^{\frac{1}{2}}, \\ 1, & \text{if } \mathbf{q}_t^{\frac{1}{2}} = \mathbf{q}_s^{\frac{1}{2}}, \\ \mathbf{q}_s^{\frac{1}{2}}\mathbf{q}_t^{-\frac{1}{2}} + \mathbf{q}_s^{-\frac{1}{2}}\mathbf{q}_t^{\frac{1}{2}}, & \text{if } \mathbf{q}_t^{\frac{1}{2}} > \mathbf{q}_s^{\frac{1}{2}}. \end{cases}$$

1.15. Generalized cells. Let $w \in W$ be such that $w < sw$ for some $s \in S$. We shall write $z \underset{L,\varphi}{\leq} w$ if $z = \omega w$ ($\omega \in \Omega$) or sw or $M^s_{z,w} \neq 0$ (see (1.14.8)). We again use $\underset{L,\varphi}{\leq}$ for the preorder relation on W generated by the relation $z \underset{L,\varphi}{\leq} w$. The equivalence relation associated to the preorder $\underset{L,\varphi}{\leq}$ is denoted by $\underset{L,\varphi}{\sim}$ and the corresponding equivalence classes in W are called generalized left cells. Given $w, u \in W$, we say that $w \underset{LR,\varphi}{\leq} u$ if there is a sequence $w = w_0, w_1, ..., w_k = u$ of elements in W such that for $i = 0, 1, ..., n-1$, we have either $w_i \underset{L,\varphi}{\leq} w_{i+1}$ or $w_i^{-1} \underset{L,\varphi}{\leq} w_{i+1}^{-1}$. The equivalence relation associated to the preorder $\underset{LR,\varphi}{\leq}$ is denoted by $\underset{LR,\varphi}{\sim}$ and the corresponding equivalence classes in W are called generalized two-sided cells. From the definitions and (1.14.8-11) we get

(a) Let $h, h' \in H_\varphi$, $w \in W$. We write

$$hC_w = \sum_{u \in W} a_u C_u, \qquad a_u \in \mathbf{Z}[\Gamma],$$

$$hC_w h' = \sum_{u \in W} b_u C_u, \qquad b_u \in \mathbf{Z}[\Gamma].$$

Then $u \underset{L,\varphi}{\leq} w$ if $a_u \neq 0$, $u \underset{LR,\varphi}{\leq} w$ if $b_u \neq 0$.

For any $w \in W$, we denote I^L_w (resp. $\hat{I}^L_w$) the $\mathbf{Z}[\Gamma]$-submodule of H_φ spanned by the elements C_u, $u \underset{L,\varphi}{\leq} w$, (resp. by the elements C_u, $u \underset{L,\varphi}{\leq} w$, $u \underset{L,\varphi}{\not\sim} w$). we define similarly

I_w^{LR} (resp. $\hat{I}_w^{LR}$) the $\mathbf{Z}[\Gamma]$-submodule of H_φ spanned by the elements C_u, $u \underset{LR,\varphi}{\leq} w$, (resp. by the elements C_u, $u \underset{LR,\varphi}{\leq} w$, $u \underset{LR,\varphi}{\not\sim} w$).

It is clear from (a) that I_w^L, $\hat{I}_w^L$ are left ideals of H_φ and I_w^{LR}, $\hat{I}_w^{LR}$ are two-sided ideals of H_φ. Hence $I_w^L/\hat{I}_w^L$ is a left H_φ-module with a natural basis given by the images of C_u for u in the generalized left cell containing w; $I_w^{LR}/\hat{I}_w^{LR}$ is a two-sided H_φ-module with a natural basis given by the images of C_u for u in the generalized two-sided cell containing w.

1.16. Assume that Γ is a free abelian group with a basis $\mathbf{q}_s^{\frac{1}{2}}$, $s \in S$ (note that $\mathbf{q}_s^{\frac{1}{2}} = \mathbf{q}_t^{\frac{1}{2}}$ whenever $s, t \in S$ are conjugate in W). It is known that we can totally order the set $Q_S = \{\mathbf{q}_s^{\frac{1}{2}} \mid s \in S\}$. Assume given such a total order $\leq$ on Q_S. We define

$$\mathbf{q}_s^{-\frac{1}{2}} < \prod_{\substack{t<s \\ a_t \in \mathbf{Z}}} \mathbf{q}_t^{\frac{a_t}{2}} < \mathbf{q}_s^{\frac{1}{2}},$$

where only finite a_t are nonzero, and define $\mathbf{q}_s^{\frac{i}{2}} < \mathbf{q}_s^{\frac{j}{2}}$ if and only if $i < j$. It is easy to see that we define a total order on the group Γ which is compatible with the group structure. Thus the definition and results in 1.14-15 can be applied to the Hecke algebra H_φ.

If we have $\mathrm{Card}\{\mathbf{q}_s^{\frac{1}{2}} \mid s \in S\} \leq \mathrm{Card}\mathbf{R}$ ($\mathbf{R}$ the field of real numbers), where Card denotes the cardinal of a set, we can find more natural total orders on Γ which are compatible with the group structure. Actually we can find many injective homomorphisms of abelian groups $\tau: \Gamma \to \mathbf{R}$ such that $\tau(\mathbf{q}_s^{\frac{1}{2}}) > 0$ for any $s \in S$. The total order in $\tau(\Gamma)$ gives rise to an total order on Γ which is compatible with the group structure on Γ.

1.17. Proposition. *Assume that W is a finite Coxeter group and let S be the set of simple reflections in W. Let w_0 be the longest element in W. We have*

(i). *Let $h \in H_\varphi$ be such that $\tilde{T}_s h = \mathbf{q}_s^{\frac{1}{2}} h$ for any $s \in S$, then $h = a\sum_{w\in W} T_w$ for some $a \in \mathbf{Z}[\Gamma]$.*

(ii). $C_{w_0} = \mathbf{q}_{w_0}^{-\frac{1}{2}} \sum_{w\in W} T_w$.

(iii). *The set $\{w_0\}$ is a generalized two-sided cell of W.*

(iv). *The set $\{e\}$ is a generalized two-sided cell of W.*

Proof. (i). First we have $\tilde{T}_s \sum_{w\in W} T_w = \mathbf{q}_s^{\frac{1}{2}} \sum_{w\in W} T_w$. Let

$$h = a \sum_{w\in W} T_w + \sum_{\substack{w\in W\\ w<w_0}} a_w T_w, \qquad a, a_w \in \mathbf{Z}[\Gamma]$$

If $a_w \neq 0$ for some $w < w_0$. Choose one w such that $l(w)$ is maximal. Since $w < w_0$, we can find some $s \in S$ such that $w < sw$. Thus in the expression

$$\tilde{T}_s h = \mathbf{q}_s^{\frac{1}{2}} a \sum_{w\in W} T_w + \sum_{w\in W} b_w T_w, \qquad b_w \in \mathbf{Z}[\Gamma]$$

We have $b_{sw} = a_w \neq 0$, it contradict our assumptions on h and on w. Thus all $a_w = 0$ whenever $w < w_0$.

(ii). By (1.14.9) and (i) we see that $C_{w_0} = a \sum_{w\in W} T_w$ for some $a \in \mathbf{Z}[\Gamma]$. Note that $R^*_{x,y} = 0$ unless $x \le y$ and that $R^*_{y,y} = 1$, from $\bar{C}_{w_0} = C_{w_0}$ we get $a = \mathbf{q}_{w_0}^{-\frac{1}{2}}$. The assertion is proved. One also can use (1.14.12) to prove (ii).

(iii). For any $s \in S$ we have $\tilde{T}_s C_{w_0} = C_{w_0}\tilde{T}_s = \mathbf{q}_s^{\frac{1}{2}} C_{w_0}$. By the definitions of generalized cells we get the assertion.

(iv). It follows from (1.14.8), (1.14.10) and the definition of generalized two-sided cell.

1.18. Example. Let (W, S) be a dihedral group of type $I_2(2m)$ (see 1.4). Let $s, t \in S$, then $(st)^{2m} = 1$. Let $\varphi: W \to \Gamma$, H_φ be as in 1.14. We assume that $\mathbf{q}_t^{\frac{1}{2}} > \mathbf{q}_s^{\frac{1}{2}}$. We shall write $\mathbf{u}$, $\mathbf{v}$ instead of $\mathbf{q}_s^{\frac{1}{2}}$, $\mathbf{q}_t^{\frac{1}{2}}$ respectively. We have

(a) Let $y, w \in W$ be such that $sy < y < w < sw$, then $M^s_{y,w} = 0$.

Proof. We have $w = t \cdot w_1$, $y = s \cdot y_1$ for some $w_1, y_1 \in W$. (Convention: $x = x_1 \cdot x_2$ means that $x = x_1x_2$ and $l(x) = l(x_1) + l(x_2)$.) Using (1.14.12) we get $P^*_{y,w} = \mathbf{v}^{-1} P^*_{ty,w}$. Note that $\mathbf{v} > \mathbf{u}$, we see that $\mathbf{u}P^*_{y,w} = \mathbf{u}\mathbf{v}^{-1}P^*_{ty,w}$ is a $\mathbf{Z}$-combination of elements in Γ_-. Using induction on $l(w) - l(y)$ and using the definition of $M^s_{y,w}$ we obtain (a).

We consider the simplest example (see [L6]): type B_2. Then $(st)^4 = 1$. We have

$$P^*_{t,tst} = \mathbf{v}^{-1}(\mathbf{u}^{-1} - \mathbf{u}), \qquad P^*_{e,tst} = \mathbf{v}^{-2}(\mathbf{u}^{-1} - \mathbf{u}),$$

$$P^*_{s,sts} = \mathbf{v}^{-1}(\mathbf{u}^{-1} + \mathbf{u}), \qquad P^*_{e,sts} = \mathbf{u}^{-1}\mathbf{v}^{-1}(\mathbf{u}^{-1} + \mathbf{u}),$$

and $P^*_{y,w} = \mathbf{q}_y^{\frac{1}{2}}\mathbf{q}_w^{-\frac{1}{2}}$ for all other pairs $y \leq w$. (In Particular, $P^*_{y,w}$ may have negative coefficients). We have

$$M^t_{ts,sts} = M^t_{t,st} = \mathbf{u}\mathbf{v}^{-1} + \mathbf{u}^{-1}\mathbf{v}.$$

The generalized left cells are $\{e\}$, $\{s\}$, $\{t, st\}, \{tst\}$, $\{ts, sts\}$, $\{stst\}$. The corresponding H_φ-modules $I^L_w/\hat{I}^L_w$ (with scalars extended to an algebraic closure of $\mathbf{Q}(\mathbf{q}^{\frac{1}{2}})$) are all irreducible. (this is in contrast with the situation in [KL1].) The generalized two-sided cells are $\{e\}$, $\{s\}$, $\{tst\}, \{t, st, ts, sts\}$, $\{stst\}$.

The second simplest example is type G_2. Then $(st)^6 = 1$. We have

$$\begin{aligned}
&P^*_{t,tst} = P^*_{tst,tstst} = \mathbf{v}^{-1}(\mathbf{u}^{-1} - \mathbf{u}),\\
&P^*_{e,tst} = P^*_{ts,tstst} = P^*_{st,tstst} = \mathbf{v}^{-2}(\mathbf{u}^{-1} - \mathbf{u}),\\
&P^*_{t,tstst} = \mathbf{v}^{-2}(\mathbf{u}^{-2} - 1 + \mathbf{u}^2), \qquad\qquad P^*_{e,tstst} = \mathbf{v}^{-3}(\mathbf{u}^{-2} - 1 + \mathbf{u}^2),\\
&P^*_{s,sts} = P^*_{sts,ststs} = \mathbf{v}^{-1}(\mathbf{u}^{-1} + \mathbf{u}),\\
&P^*_{e,sts} = P^*_{st,ststs} = P^*_{ts,ststs} = \mathbf{u}^{-1}\mathbf{v}^{-1}(\mathbf{u}^{-1} + \mathbf{u}),\\
&P^*_{t,ststs} = \mathbf{u}^{-2}\mathbf{v}^{-1}(\mathbf{u}^{-1} + \mathbf{u}), \qquad P^*_{s,ststs} = \mathbf{u}^{-1}\mathbf{v}^{-2}(\mathbf{u}^{-1} + \mathbf{u}),\\
&P^*_{e,ststs} = \mathbf{u}^{-2}\mathbf{v}^{-2}(\mathbf{u}^{-1} + \mathbf{u}),
\end{aligned}$$

and $P^*_{y,w} = \mathbf{q}_y^{\frac{1}{2}}\mathbf{q}_w^{-\frac{1}{2}}$ for all other pairs $y \leq w$. We have

$$M^t_{tsts,ststs} = M^t_{tst,stst} = M^t_{ts,sts} = M^t_{t,st} = \mathbf{u}\mathbf{v}^{-1} + \mathbf{u}^{-1}\mathbf{v},$$

$$M^t_{t,stst} = M^t_{ts,ststs} = 1.$$

The generalized left cells are

$$\{e\},\ \{s\},\ \{tstst\},\ \{ststst\},\ \{t, st, tst, stst\}, \{ts, sts, tsts, ststs\}.$$

The first four corresponding H_φ-modules $I^L_w/\hat{I}^L_w$ (with scalars extended to an algebraic closure of $\mathbf{Q}(\mathbf{q}^{\frac{1}{2}})$) are all irreducible; the last two corresponding H_φ-modules are not irreducible. (This is in contrast with the situation in [KL1], also in contrast with the case type B_2.) The generalized two-sided cells are

$$\{e\},\ \{s\},\ \{t, st, ts, tst, sts, tsts, tsts, ststs\},\ \{tstst\},\ \{ststst\}.$$

For type B_n, the generized left cells provide all simple modules of the corresponding Hecke algebras over an algebraic closed field of characteristic 0 (see [L6]). In general, the generalized should provide more simple modules for finite dimensional Hecke algebras. The work [L11-L14] showed that the cells in affine Weyl groups are interesting to the representations of affine Hecke algebras of one parameters. The generalized cells in affine Weyl groups should be interesting to the representations of affine Hecke algebras with unequal parameters. It is likely that any generalized left (resp. two-sided) cell of an extended Coxeter group contain in a left (resp. two-sided) cell of the sense 1.11.

Finally we state

1.19. Conjecture. *Let W be an extended Coxeter group. Let φ, H_φ as in 1.14-15. Then the generized left cells and generalized two-sided cells in W defined in 1.15 only depend on the order relations among $\mathbf{q}_s^{\frac{1}{2}}$, $s \in S$, not depend on the parameters $\mathbf{q}_s^{\frac{1}{2}}$, $s \in S$.*

2. Affine Weyl groups and Affine Hecke Algebras

In this chapter we consider some elementary properties of extended affine Weyl groups and of their Hecke algebras. We are mainly interested in the structure of the center of affine Hecke algebras and cells in affine Weyl group.

2.1. Extended affine Weyl groups. Let G be a connected reductive group over $\mathbb{C}$ and T a maximal torus of G. Let $N_G(T)$ be the normalizer of T in G. Then $W_0 = N_G(T)/T$ is a Weyl group, which acts on the character group $X = \mathrm{Hom}(T, \mathbb{C}^*)$ of T. Consider the semi-direct product $W = W_0 \ltimes X$. Let $R \subset X$ be the root system of W_0, which spans the root lattice P in X. The group P is W_0-stable and the subgroup $W' = W_0 \ltimes P$ of W is an affine Weyl group. Moreover W' is a normal subgroup of W. Let S be the set of simple reflections in W'. There exists an abelian subgroup Ω of W such that $\omega S \omega^{-1} = S$ for any $\omega \in \Omega$ and $W = \Omega \ltimes W'$. (Ω is isomorphic to the center of G, also isomorphic to the quotient group X/P.) Thus W is an extended Coxeter group in the sense of 1.1. We shall call W an extended affine Weyl group (In some bibliographys W is also called a modified affine Weyl group, see, e.g. [Ca]). The Hecke algebras of W are called affine Hecke algebras (this slightly different the usual definition, when G is adjoint, our definition agrees with the usual one).

Let R^+, R^- be the set of positive roots and the set of negative roots, respectively. Let Δ be the set of simple roots of R. Let $R^\vee \subset \mathrm{Hom}(\mathbb{C}^*, T)$ be the dual root system of R. For any $\alpha \in R$ we shall denote $\alpha^\vee \in R^\vee$ the dual of α. Then the length of an element wx ($w \in W_0$, $x \in X$) is given by the following formula (see [IM])

$$l(wx) = \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^-}} |< x, \alpha^\vee > + 1| + \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^+}} |< x, \alpha^\vee >|. \tag{2.1.1}$$

Let

$$X^+ = \{x \in X \mid l(wx) = l(w) + l(x) \text{ for any } w \in W_0\} = \{x \in X \mid < x, \alpha^\vee > \geq 0\}$$

be the set of dominant weights. By the above formula we have

(2.1.2) For any $x, y \in X^+$, $l(xy) = l(x) + l(y)$.

2.2. The center of generic affine Hecke algebras. Let H be the generic Hecke algebra of W over $A = \mathbf{Z}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$ with the standard basis T_w, $w \in W$. Following Berstein (see [L5]) we introduced another basis and describe the structure of the center H.

For any $x \in X$, we can find $y, z \in X^+$ such that $x = yz^{-1}$. Let

$$\theta_x = \mathbf{q}^{-\frac{l(y)}{2}} T_y (\mathbf{q}^{-\frac{l(z)}{2}} T_z)^{-1}.$$

According to (2.1.2) we know the definition of θ_x is independent of the choice of y, z. Moreover we have

(2.2.1) $\qquad \theta_x \theta_{x'} = \theta_{xx'}, \quad$ for any $x, x' \in X$.

Let O_x be the conjugacy class of x in W and let $z_x = \sum_{x' \in O_x} \theta_{x'}$. Then (Berstein, see [L5])

(a) The elements $T_w \theta_x$, $w \in W_0$, $x \in X$ form A-basis of H.

(b) The elements $\theta_x T_w$, $w \in W_0$, $x \in X$ form A-basis of H.

(c) The center of H is a free A-module and the elements z_x, $x \in X^+$ form A-basis of the center of H.

For $x \in X^+$, denote $d(x', x)$ the dimension of the x'-weight space $V(x)_{x'}$ of $V(x)$, where $V(x)$ is the irreducible representation of G with highest weight x. We set $U_x = \sum_{x' \in X^+} d(x', x) z_{x'}$, $x \in X^+$. Then we have

(d) The elements U_x, $x \in X^+$ form A-basis of the center of H.

We shall denote S_0 the set of simple reflections in W_0. For any $r \in S_0$, we also write α_r for the corresponding simple root. We have

(e) $T_r \theta_{r(x)} = \theta_x T_r - (\mathbf{q} - 1)\theta_x (1 + \theta_{\alpha_r}^{-1} + \cdots \theta_{\alpha_r}^{1-n})$, for any $r \in S_0$, $x \in X$ with $< x, \alpha_r^\vee >= n \geq 1$.

(f) $T_r \theta_x = \theta_x T_r \qquad$ for any $r \in S_0$, $x \in X$ with $< x, \alpha_r^\vee >= 0$.

A special case of the formula in (e) is

(g) $T_r \theta_{r(x)} = \theta_x T_r - (\mathbf{q} - 1)\theta_x$, for any $r \in S_0$, $x \in X$ with $< x, \alpha_r^\vee >= 1$.

The formula in (g) is equivalent to

(h) $\theta_{r(x)} = \mathbf{q}T_r^{-1}\theta_x T_r^{-1}$, for any $r \in S_0$, $x \in X$ with $< x, \alpha_r^\vee >= 1$.

For any $q \in \mathbb{C}^*$, we can regard $\mathbb{C}$ as an A-algebra by specializing $\mathbf{q}^{\frac{1}{2}}$ to a square root $q^{\frac{1}{2}}$. Then we consider the tensor product $H \otimes_A \mathbb{C}$, this is a $\mathbb{C}$-algebra, we denote it by $\mathbf{H}_q$. We shall denote the images in $\mathbf{H}_q$ of T_w, C_w, $U_x, \ldots$, by the same notations.

2.3. Assume that G has a simply connected derived group, it is equivalent to that X^+ contains all fundamental weights concerned with the root system R. For each simple root $\alpha \in R$, we denote x_α the corresponding fundmental weight in X^+. Given $w \in W_0$, we set

$$x_w = w(\prod_{\substack{\alpha\in\Delta \\ w(\alpha)\in R^-}} x_\alpha).$$

It is known that (see [St2])

(a) $A[X]$ is a free $A[X]^{W_0}$-module with a basis x_w, $w \in W_0$.

Using 2.2(a-c) we see that

(b) H is a free $\mathcal{Z}(H)$-module with a basis $T_w\theta_{x_u}$, $w, u \in W_0$, where $\mathcal{Z}(H)$ is the center of H.

The image in $\mathbf{H}_q$ of $\mathcal{Z}(H)$ is in the center of $\mathcal{Z}(\mathbf{H}_q)$ of $\mathbf{H}_q$. By (b) we see that (cf. [M])

(c) Any simple $\mathbf{H}_q$-module has dimension $\leq |W_0|$.

However, for each $q \in \mathbb{C}^*$, there are a lots of simple $\mathbf{H}_q$-modules with dimension $|W_0|$ (see [M, Ka1], see also 3.11). So we have

2.4. Proposition. *Let G be as in 2.3. Then the elements U_x, $x \in X^+$ is a $\mathbb{C}$-basis of the center of $\mathbf{H}_q$* . (Note our convention on notations at the end of 2.2.)

2.5. Proposition. *Assume that G is simply connected, simple algebraic group over $\mathbb{C}$. We have*

(i). *If R is not of type D_{2k} $(2k = n)$, then there exists a fundamental weight which we denote by x_0 such that H is generated by T_r $(r \in S_0)$ and θ_{x_0}.*

(ii). *If R is of type D_{2k} $(2k = n)$, then there exist two fundamental weights which we denote by x_0, x_0' such that H is generated by T_r $(r \in S_0)$ and $\theta_{x_0}, \theta_{x_0'}$.*

Proof. We number the simple reflections $r_1, r_2, ..., r_n$ in S_0 and the fundamental weights $x_1, x_2, ..., x_n$ in X according to the Coxeter graphs in 1.3. In case (i) we choose x_0 to be the following:

x_n for type $A_n,\ B_n,\ D_n,\ E_n,\ F_4$;

x_1 for type $C_n,\ G_2$.

In case (ii) we choose $x_0 = x_n, x_0' = x_{n-1}$.

We claim such choices satisfy our requirement. We take type A_n as an example to prove the claim. Let $\underline{H}$ be the subalgebra of H generated by T_r $(r \in S_0)$ and $\theta_{x_0} = \theta_{x_n}$. It is enough to prove that the elements $\theta_{x_1}^{\pm 1}, \theta_{x_2}^{\pm 1}, ..., \theta_{x_n}^{\pm 1}$ are in $\underline{H}$. We shall write $T_i,\ T_i'$ for $T_{r_i},\ T_{r_i}^{-1}$, respectively. We have

$$r_n(x_n) = x_{n-1}x_n^{-1}, \quad r_1(x_1) = x_2 x_1^{-1}, \tag{2.5.1}$$

$$r_i(x_i x_{i+1}^{-1}) = x_{i-1}x_i^{-1}, \qquad \text{for any } 1 < i < n, \tag{2.5.2}$$

Using 2.2(h) and (2.5.1) we get

$$\theta_{x_{n-1}}\theta_{x_n}^{-1} = \mathbf{q}T_n'\theta_{x_n}T_n'. \tag{2.5.3}$$

Using 2.2(h) and (2.5.2) repeatedly we obtain

(2.5.4)

$$\theta_{x_{n-2}}\theta_{x_{n-1}}^{-1} = \mathbf{q}T_{n-1}'\theta_{n-1}\theta_{x_n}^{-1}T_{n-1}', \quad ..., \quad \theta_{x_2}\theta_{x_3}^{-1} = \mathbf{q}T_3'\theta_3\theta_{x_4}^{-1}T_3', \quad \theta_{x_1}\theta_{x_2}^{-1} = \mathbf{q}T_2'\theta_2\theta_{x_3}^{-1}T_2'.$$

Again using 2.2(h) and (2.5.1) we get

$$\theta_{x_1}^{-1} = \mathbf{q}T_1'\theta_{x_1}\theta_{x_2}^{-1}T_1'. \tag{2.5.5}$$

By assumption we have $\theta_{x_n} \in \underline{H}$. Combine this and (2.5.3-5) we see that $\theta_{x_1}^{\pm 1}, \theta_{x_2}^{\pm 1}, ..., \theta_{x_n}^{\pm 1}$ are in $\underline{H}$. The proposition is proved for type A_n. For other types, the arguments are similar.

2.6. Cells in W. In [L11-L14] Lusztig proved a number of results concerned with the cells in affine Weyl groups. Here are some of them except those given other references.

(a) The number of left cells of W is finite. Each left cell of W contains a unique element in $\mathcal{D} = \{w \in W_a \mid 2\deg P_{e,w} = \ell(w) - a(w)\}$, where e is the unit element of W.

(b) The set $\mathcal{D}$ is a finite set of involutions in W'.

(c) $a(w) \le a(w_0) = |R^+| = \nu$, where w_0 is the longest element of W_0.

(d) The set $c_0 = \{w \in W \mid a(w) = \nu\}$ is a two-sided cell of W. The two-sided cell c_0 contains $|W_0|$ left (resp. right) cells (see [Bé2, Sh2-Sh3]). c_0 is the lowest two-sided cell (concerned with the partial order $\underset{LR}{\le}$) in the set Cell(W) of two-sided in W. Assume that X contains all fundamental weights, then

$$c_0 = \{xw_0y \in W \mid l(xw_0y) = l(x) + l(w_0) + l(y)\}.$$

We would like to state a conjecture concerned with the number of left cells in a two sided cell c. For any subset I of S, let

$$\Gamma_{c,I} = \{\text{left cell } \Gamma \text{ in } c \mid R(\Gamma) = I\}$$

(set $R(\Gamma) = R(w)$ for any $w \in \Gamma$, this is well defined, see [KL1]), we conjecture that $\#\Gamma_{c,I} \le \#\Gamma_{c_0,I}$. Lusztig has a conjecture (see [As]) concerned with the number of left cells in a two-sided cell which needs the following result.

(e) There exists a natural bijection between the set Cell(W) of two-sided cells in W and the set of nilpotent G-orbits in $\mathbf{g}$ which preserves the partial orders.

(f) For $y, w \in W$, we have $a(w) \ge a(y)$ whenever $y \underset{LR}{\le} w$. Moreover, if $y \underset{LR}{\le} w$ (resp. $y \underset{L}{\le} w$, $y \underset{R}{\le} w$) and $a(y) = a(w)$, then $y \underset{LR}{\sim} w$ (resp. $y \underset{L}{\sim} w$, $y \underset{R}{\sim} w$). In particular, a is constant on a two-sided cell of W.

(g) $x \underset{L}{\le} w$, (resp. $x \underset{R}{\le} w$; $x \underset{LR}{\le} w$) if and only there exists h (resp. h'; h, h') in H such that $a_x \ne 0$ (resp. $b_x \ne 0$; $c_x \ne 0$), where a_x(resp. $b_x; c_x$) is defined by

$$hC_w = \sum_{u\in W} a_u C_u, \quad a_u \in A,$$

$$(\text{resp.} \quad C_w h' = \sum_{u\in W} b_u C_u, \quad b_u \in A; \quad hC_w h' = \sum_{u\in W} c_u C_u, \quad c_u \in A).$$

Given a two-sided cell c of W, let $H^{\leq c}$ (resp. $H^{<c}$ be the free A-module spanned by C_w, $w \in W$ such that $w \underset{LR}{\leq} u$ (resp. $w \underset{LR}{\leq} u$, $w \not\sim u$) for some $u \in c$. Let $\mathbf{H}_q{}^{\leq c}$ be the image in $\mathbf{H}_q$ of $H^{\leq c}$. By (g) we see that

(h) $H^{\leq c}$ is a two-sided ideal of H and $\mathbf{H}_q{}^{\leq c}$ is a two-sided cell of $\mathbf{H}_q$. We also call them the cell ideals. We shall write $H_{\geq c}$ for the quotient $H/H^{<c}$.

Note that $C_{w_0} = \mathbf{q}^{-\frac{l(w_0)}{2}} \sum_{w \in W_0} T_w$ and that $T_s C_{w_0} = C_{w_0} T_s = \mathbf{q} C_{w_0}$ for any $s \in S_0$.

(i) $H^{\leq c_0}$ is spanned by $\theta_x C_{w_0} \theta_y$, $x, y \in X$. If X contains all fundamental weights of R, then the elements $\theta_{x_w} C_{w_0} \theta_{x_u} U_x$, $w, u \in W_0$, $x \in X^+$, is an A-basis of $H^{\leq c_0}$.

Proof. By 2.2(a-b) we get the first assertion. The second assertion follows from 2.3(b) and [X1, 2.9].

2.7. The based rings of two-sided cells. For $w, u, v \in W$, we define the integer $\gamma_{w,u,v}$ by the condition $\mathbf{q}^{\frac{a(v)}{2}} h_{w,u,v} - \gamma_{w,u,v} \in \mathbf{q}^{\frac{1}{2}} \mathbf{Z}[\mathbf{q}^{\frac{1}{2}}]$ (see 1.12 for the definition of $h_{w,u,v}$). Since (W', S) is crystallographic, by 1.12 we know that $\gamma_{w,u,v}$ is non-negative (see [L11]). We have

(a) $\gamma_{w,u,z} \neq 0 \Rightarrow w \underset{L}{\sim} u^{-1}$, $u \underset{L}{\sim} z$, $w \underset{R}{\sim} z$. In particular, $w \underset{LR}{\sim} u \underset{LR}{\sim} z$ if $\gamma_{w,u,z} \neq 0$.

(b) Let $d \in \mathcal{D}$, then $\gamma_{w,d,u} \neq 0 \Leftrightarrow w = u$ and $w \underset{L}{\sim} d$, $\gamma_{w,d,w} = 1$. Moreover, $\gamma_{w,d,w} = \gamma_{d,w^{-1},w^{-1}} = \gamma_{w,w^{-1},d} = 1$.

Let $J_{\mathbf{Z}}$ be the free $\mathbf{Z}$-module with a basis t_w, $w \in W$. In [L12] Lusztig proved that $t_w t_u = \sum_{v \in W} \gamma_{w,u,v} t_v$ defines an associative ring structure on $J_{\mathbf{Z}}$. The ring $J_{\mathbf{Z}}$ is called the based ring of W. The unit in $J_{\mathbf{Z}}$ is $\sum_{d \in \mathcal{D}} t_d$. For each two-sided cell c, the subspace $J_{\mathbf{Z},c}$ of $J_{\mathbf{Z}}$ spanned by t_w, $w \in c$, is a two-sided ideal of $J_{\mathbf{Z}}$ by (a). $J_{\mathbf{Z},c}$ is in fact an associative ring with unit $\sum_{d \in \mathcal{D} \cap c} t_d$, which is called the based ring of the two-sided cell c. We have $J_{\mathbf{Z}} = \bigoplus_c J_{\mathbf{Z},c}$, i.e. $J_{\mathbf{Z}}$ is the direct sum of algebras $J_{\mathbf{Z},c}$, where c runs over the set Cell(W) of two-sided cells of W. The rings $J_{\mathbf{Z}}$, $J_{\mathbf{Z},c}$ turn out very interesting. The following result establishes the connection between the Hecke algebra H and the based ring $J_{\mathbf{Z}}$.

(c) The A-linear map $\phi: H \to J_{\mathbf{Z}} \otimes_{\mathbf{Z}} A$ defined by

$$\phi(C_w) = \sum_{\substack{u \in W \\ d \in \mathcal{D} \\ a(d)=a(u)}} h_{w,d,u} t_u, \qquad w \in W$$

is a homomorphism of A-algebras with 1. Moreover ϕ is injective and ϕ maps the center of H into the center of $J_{\mathbf{Z}} \otimes_{\mathbf{Z}} A$ and $J_{\mathbf{Z}} \otimes_{\mathbf{Z}} A$ is finitely generated over $\phi(\mathcal{Z}(H))$. (see [L13]).

The $\mathbb{C}$-algebra $\mathbf{J} = J_{\mathbf{Z}} \otimes \mathbb{C}$ is called the asymototic Hecke algebra of W. The homomorphism ϕ in (c) induces a homomorphism of $\mathbb{C}$-algebras $\phi_q : \mathbf{H}_q \to \mathbf{J}$. We have (see [L13])

(d) For any $q \in \mathbb{C}^*$, the homomorphism ϕ_q is injective and ϕ_q maps the center of $\mathbf{H}_q$ into the center of $\mathbf{J}$, and $\mathbf{J}$ is finitely generated over $\phi_q(\mathcal{Z}(\mathbf{H}_q))$.

We conjecture that the center of $\mathbf{J}$ is spanned by various $\phi_q(\mathcal{Z}(\mathbf{H}_q))$, $q \in \mathbb{C}^*$.

Affine Hecke algebras with two parameters

2.8. We keep the notations in 2.1. We shall consider the affine Hecke algebras with unequal parameters. We shall assume that G is simple and G is one of the types: B_n, C_n, F_4, G_2. Thus there exist simple reflections in S which are not conjugate in W.

Let R' be the set of long roots in R when R is of type B_n, be the set of short roots in R when R is of other types. Let $R'' = R - R'$. Let $R'^+ = R' \cap R^+$(resp. $R''^+ = R' \cap R^+$) be the set of positive roots in R' (resp. R''). Define two function l', l'' : $W \to \mathbf{N}$ as follows: for $w \in W_0$, $x \in X$, we set

(2.8.1) $$l'(wx) = \sum_{\substack{\alpha \in R'^+ \\ w(\alpha) \in R^-}} |< x, \alpha^\vee > +1| + \sum_{\substack{\alpha \in R'^+ \\ w(\alpha) \in R^+}} |< x, \alpha^\vee >|.$$

(2.8.2) $$l''(wx) = \sum_{\substack{\alpha \in R''^+ \\ w(\alpha) \in R^-}} |< x, \alpha^\vee > +1| + \sum_{\substack{\alpha \in R''^+ \\ w(\alpha) \in R^+}} |< x, \alpha^\vee >|.$$

In some sense the functions l', l'' are length functions of W corresponding to roots of different lengths. Namely, Let

$$S' = \{s \in S \mid s \text{ is the simple reflection respect to some roots in } R'\},$$

$$S'' = \{s \in S \mid s \text{ is the simple reflection respect to some roots in } R''\},$$

for any reduced expression $t_1 t_2 \cdots t_k$ of an element u in W', Let

$$S'_u = \{t_i \mid t_i \in S',\ 1 \le i \le k\}, \qquad S''_u = \{t_i \mid t_i \in S'',\ 1 \le i \le k\},$$

then $l'(u) = \#S'_u$, $l''(u) = \#S''_u$. Obviously we have
(2.8.3). Let $u \in W$, then $l(u) = l'(u) + l''(u)$. (see (2.1.1) for the formula of $l(u)$).

Let $\mathbf{u}, \mathbf{v}$ be two indeterminates and let $B = \mathbf{Z}[\mathbf{u}^{\pm 1}, \mathbf{v}^{\pm 1}]$ be the Laurant polynomials in $\mathbf{u}, \mathbf{v}$ with integer coefficients. We define the Hecke algebra $\tilde{H}$ (over B) of W with parameters $\mathbf{u}^2, \mathbf{v}^2$ as follows: $\tilde{H}$ is a free B-module with a B-basis T_w, $w \in W$ and multiplication laws

(2.8.4) $$T_w T_u = T_{wu}, \quad \text{if } l(wu) = l(w) + l(u),$$

$$(T_s - \mathbf{u}^2)(T_s + 1) = 0, \quad \text{if } s \in S'; \qquad (T_s - \mathbf{v}^2)(T_s + 1) = 0, \quad \text{if } s \in S''.$$

When G is not adjoint, or is not type B_n, the algebra $\tilde{H}$ is essentially the algebra $\mathcal{H}$ in 1.1. When G is adjoint of type B_n, then $W = W'$ is an affine Weyl group of type $\tilde{C}_n$. We number the simple reflections in S according to the Coxeter graph in 1.3, then s_n, s_0 are not conjugate in W. So the algebra $\mathcal{H}$ for W has three parameters. In this paper we donot consider such Hecke algebra although the properties for $\tilde{H}$ have their counterparts for the algebra $\mathcal{H}$ (see [L18]). It seems necessary to consider the Hecke algebra $\mathcal{H}$ separatedly.

2.9. The center of $\tilde{H}$. Following [L18] we construct the center of $\tilde{H}$, which essentially is similar to the case in [L18].

As in 2.1, let

$$X^+ = \{x \in X \mid l(wx) = l(w) + l(x) \text{ for any } w \in W_0\} = \{x \in X \mid < x, \alpha^\vee > \ge 0\}$$

be the set of dominant weights. By the formulas (2.8.1-2) we have

(2.9.1) For any $x, y \in X^+$, $l'(xy) = l'(x) + l'(y)$, $l''(xy) = l''(x) + l''(y)$.

For any $x \in X$, we can find $y, z \in X^+$ such that $x = yz^{-1}$. Let

$$\theta_x = \mathbf{u}^{-l'(y)}\mathbf{v}^{-l''(y)}T_y(\mathbf{u}^{-l'(z)}\mathbf{v}^{-l''(z)}T_z)^{-1}.$$

According to (2.9.1) we know the definition of θ_x is independent of the choice of y, z. Moreover we have

(2.9.2) $\qquad \theta_x\theta_{x'} = \theta_{xx'}, \quad$ for any $x, x' \in X$.

Let O_x be the conjugacy class of x in W and let $z_x = \sum_{x' \in O_x} \theta_{x'}$. Then using the method in [L18] one can prove the following result without difficulty.

(a) The elements $T_w\theta_x,\ w \in W_0,\ x \in X$ form B-basis of $\tilde{H}$.

(b) The elements $\theta_x T_w,\ w \in W_0,\ x \in X$ form B-basis of $\tilde{H}$.

(c) The center of $\tilde{H}$ is a free B-module and the elements $z_x,\ x \in X^+$ form A-basis of the center of $\tilde{H}$.

For $x \in X^+$, we set $U_x = \sum_{x' \in X^+} d(x', x) z_{x'}$, $x \in X^+$, see 2.2 for the definition of $d(x', x)$. Then we have

(d) The elements $U_x,\ x \in X^+$ form B-basis of the center of $\tilde{H}$.

Let $S_0 = S \cap W_0,\ S'_0 = S' \cap W_0,\ S''_0 = S'' \cap W_0$.

(e) $T_r\theta_{r(x)} = \theta_x T_r - (\mathbf{u}^2 - 1)\theta_x(1 + \theta_{\alpha_r}^{-1} + \cdots \theta_{\alpha_r}^{1-n})$, for any $r \in S'_0$, $x \in X$ with $< x, \alpha_r^\vee >= n$.

$T_r\theta_{r(x)} = \theta_x T_r - (\mathbf{v}^2 - 1)\theta_x(1 + \theta_{\alpha_r}^{-1} + \cdots \theta_{\alpha_r}^{1-n})$, for any $r \in S''_0$, $x \in X$ with $< x, \alpha_r^\vee >= n$.

(f) $T_r\theta_x = \theta_x T_r \qquad$ for any $r \in S_0$, $x \in X$ with $< x, \alpha_r^\vee >= 0$.

A special case of the formulas in (e) are

(g) $T_r\theta_{r(x)} = \theta_x T_r - (\mathbf{u}^2 - 1)\theta_x$, for any $r \in S'_0$, $x \in X$ with $< x, \alpha_r^\vee >= 1$.

$T_r\theta_{r(x)} = \theta_x T_r - (\mathbf{v}^2 - 1)\theta_x$, for any $r \in S''_0$, $x \in X$ with $< x, \alpha_r^\vee >= 1$.

The formulas in (g) are equivalent to

(h) $\theta_{r(x)} = \mathbf{u}^2 T_r^{-1}\theta_x T_r^{-1}$, for any $r \in S'_0$, $x \in X$ with $< x, \alpha_r^\vee >= 1$.

$\theta_{r(x)} = \mathbf{v}^2 T_r^{-1}\theta_x T_r^{-1}$, for any $r \in S''_0$, $x \in X$ with $< x, \alpha_r^\vee >= 1$.

We shall denote $\tilde{H}^\circ$ the $B' = \mathbf{Z}[\mathbf{u}^{\pm 2}, \mathbf{v}^{\pm 2}]$ subalgebra of $\tilde{H}$ generated by T_s, $(s \in S_0)$, θ_x $(x \in X)$.

For any $a, b \in \mathbb{C}^*$, we can regard $\mathbb{C}$ as an B-algebra by specializing $\mathbf{u}, \mathbf{v}$ to $a^{\frac{1}{2}}, b^{\frac{1}{2}}$ respectively. Then we consider the tensor product $\tilde{H} \otimes_B \mathbb{C}$, this is a $\mathbb{C}$-algebra, we denote it by $\tilde{\mathbf{H}}_{a,b}$ (it is easy to see that $\tilde{\mathbf{H}}_{a,b}$ only depends on a, b, not depends the choices of the square roots of a, b, actually we have $\tilde{\mathbf{H}}_{a,b} = \tilde{H}^\circ \otimes_{B'} \mathbb{C}$). We shall denote the images in $\tilde{\mathbf{H}}_{a,b}$ of T_w, θ_x, $U_x, \ldots$ by the same notations.

2.10. For $w \in W$, we define $x_w \in X$ as in 2.3. Using 2.9(a-c) and 2.3(a) we see that

(a) $\tilde{H}$ is a free $\mathcal{Z}(\tilde{H})$-module with a basis $T_w \theta_{x_u}$, $w, u \in W_0$, where $\mathcal{Z}(\tilde{H})$ is the center of $\tilde{H}$.

By (a) we see that (see [M])

(b) Any simple $\tilde{\mathbf{H}}_{a,b}$-module has dimension $\leq |W_0|$.

But, for any $a, b \in \mathbb{C}^*$, there are a lots of simple $\tilde{\mathbf{H}}_{a,b}$-modules with dimension $|W_0|$ (see [M, Ka1]). So we have

2.11. Proposition. *Assume that X contains all fundamental weights of R, then the elements U_x, $x \in X^+$ is a $\mathbb{C}$-basis of the center of* $\mathbf{H}_q$. (Note our convention on notations at the end of 2.9.

Similar to 2.5 we have

2.12. Proposition. *Assume that G is simply connected, simple algebraic group over $\mathbb{C}$, and is one of the types B_n, C_n, F_4, G_2, the there exists a fundamental weight which we denote by x_0 such that $\tilde{H}$ is generated by T_r $(r \in S_0)$ and θ_{x_0}.*

2.13. Assume that we have a total order on the abelian group $\{\mathbf{u}^{\frac{i}{2}} \mathbf{v}^{\frac{j}{2}} \mid i, j \in \mathbf{Z}\}$ which is compatible with the multiplication. If $\mathbf{u}^{\frac{1}{2}} > 1$ and $\mathbf{v}^{\frac{1}{2}} > 1$, then we can defined the generalized cells of W through the algebra $\tilde{H}$ as in 1.15. We conjecture that the set c_0 in 2.6(d) is always a generalized two-sided cell in the sense of 1.15.

3. Kazhdan-Lusztig Classification on Simple Modules of Affine Hecke Algebras

In this chapter we We first recall some results of Ginsburg [G1-G2], Kazhdan and Lusztig [KL4], which we shall need. We then give some discussions to the standard modules (in the sense of [KL4]). For type A it is not difficult to determine the dimensions of standard modules. In general one seems can determine the dimensions of standard modules through Green functions. A conjecture concerned with the based rings of two-sided cells in affine Weyl groups is stated. We also stated a conjecture concerned with classification of simple modules of affine Hecke algebras of two parameters, which form analogue of the $(*)$ in the introduction of the paper.

Throughout this chapter all varieties and algebraic groups are over $\mathbb{C}$ except specified indications. We shall use algebraic (equivalent) K-theory instead of topological (equivalent) K-theory. See [T] for comparing between them.

For an algebraic group G and a G-variety M, we shall denote $K_G(M)$ the Grothendieck group of the category of G-equivalent coherent sheaves on M and $\mathbf{K}_G(M) = K_G(M)\otimes\mathbb{C}$ its complexification. The K-group $\mathbf{K}_G(M)$ has a natural $\mathbf{R}_G = \mathbf{K}_G(point)$ module structure. When $G = \{1\}$ is the unit group, we shall omit the subscript G of these K-groups.

we shall use $\mathcal{O}_M$ for the structure sheaf of M.

3.1. Convolution in K-theory. Following Ginzburg(see [G2]) we consider the convolution in K-theory. In section 3.1-3.3 we use the account in [GV].

Let G be an algebraic group over $\mathbb{C}$ and $M_1,\ M_2,\ M_3$ be smooth quasi-projective G-varieties. So the varieties $M_1\times M_2\times M_3,\ M_1\times M_2,\ M_2\times M_3,\ M_1\times M_3$ are also G-varieties and G acts on them diagonally. Then the natural projections $p_{ij}:\ M_1\times M_2\times M_3\to M_i\times M_j$ commute with G-actions, i.e., they are G-equivalent morphism. Let Z be a G-stable closed subvariety of $M_1\times M_2$ and $\tilde{Z}$ be a G-stable closed subvariety of $M_2\times M_3$, Assume that the map

$$(3.1.1)\qquad p_{13}:\ p_{12}^{-1}(Z)\cap p_{23}^{-1}(\tilde{Z})\to M_1\times M_3 \quad \text{is proper.}$$

Then its image is a G-stable closed subvariety of $M_1 \times M_3$ and is called the composition of Z and $\tilde{Z}$. we denote it by $Z \circ \tilde{Z}$. Following Ginzburg we define the convolution map (see [G2])

$$(3.1.2) \qquad * : K_G(Z) \otimes K_G\tilde{Z}) \to K_G(Z \circ \tilde{Z})$$

as follows. Let $[\mathcal{F}] \in K_G(Z)$ and $[\tilde{\mathcal{F}}] \in K_G(\tilde{Z})$ be the classes of certain coherent sheaves $\mathcal{F}$ on Z and $\tilde{\mathcal{F}}$ on $\tilde{Z}$. Set

$$(3.1.3) \qquad \mathcal{F} * \tilde{\mathcal{F}} = (Rp_{13})_*(p_{12}^*\mathcal{F} \overset{L}{\underset{\mathcal{O}_{M_1\times M_2\times M_3}}{\otimes}} p_{23}^*\tilde{\mathcal{F}}).$$

In this formula the upper star stands for the pullback morphism, well-defined for smooth maps (see, e.g. [Fu]), for example, $p_{12}^*\mathcal{F} \boxtimes \mathcal{O}_{M_3}$. To define $\overset{L}{\otimes}$, on $M_1 \times M_2 \times M_3$ we choose a finite resolutions $\mathbf{F}_{12}^{\cdot}$ of $p_{12}^*\mathcal{F}$ and a finite resolutions $\mathbf{F}_{23}^{\cdot}$ of $p_{23}^*\tilde{\mathcal{F}}$ by G-equivariant locally free sheaves, which exist since $M_1 \times M_2 \times M_3$ is smooth and quasi-projective (see [T]). The simple complex $\mathbf{F}_{12} \otimes \mathbf{F}_{23}$ associated to the tensor product of the resolutions represents the tensor product $\overset{L}{\otimes}$ in a derived category. This complex is exact off $p_{12}^{-1}(Z) \cap p_{23}^{-1}(\tilde{Z})$. Hence, its derived direct image $\mathcal{F} * \tilde{\mathcal{F}} = (Rp_{13})_*(\mathbf{F}_{12}^{\cdot} \otimes \mathbf{F}_{23}^{\cdot})$ is a complex of sheaves on $M_1 \times M_3$ whose cohomolgy sheaves are coherent sheaves on $M_1 \times M_3$ since (3.1.1); moreover, these cohomology sheaves are supported on $Z \circ \tilde{Z}$. We let $[\mathcal{F}] * [\tilde{\mathcal{F}}] \in K_G(Z \circ \tilde{Z})$ be the alternating sum of these cohomology sheaves. The definition of $*$ doesnot depend on the choices involved. Furthermore, the convolution is associated in a natural way.

3.2. From now on we assume G is a reductive group and s a semisimple element in G. Evaluating a character of the group G at the element s gives rise to an algebra homomorphism: $\mathbf{R}_G \to \mathbb{C}$. Let $\mathbb{C}_s$ denote the 1-dimensional $\mathbf{R}_G$-module arising from the homomorphism. It is known that each simple $\mathbf{R}_G$-module is isomorphic to some $\mathbb{C}_s$, and $\mathbb{C}_s$ is isomorphic to $\mathbb{C}_t$ if and only if s, t are conjugate in G.

Let M be a smooth G-variety and let M^s be the s-fixed points subvariety. By the slice theorem [Lu], M^s is smooth. Let $\mathcal{N}_s^*$ denote the conormal sheaf at the subvariety $M^s \hookrightarrow M$. The s-action on M induces a natural s-action on $\mathcal{N}_s^*$. We set

$$\lambda(M^s) = \sum(-1)^k \mathrm{tr}(s, \overset{k}{\wedge}\mathcal{N}_s^*) \in \mathbf{K}(M^s).$$

Here $\mathbf{K}(M^s)$ is the ordinary K-group; and for a vector bundle E with semisimple s-action on the fibres, we use the notation $\mathrm{tr}(s, E) = \sum a_k \cdot E_k$, where a_k are the eigenvalues of the action s and E_k stands for the subbundle of E corresponding to the eigenvalue a_k. The element $\lambda(M^s)$ is invertible in $\mathbf{K}(M^s)$ since all the eigenvalues of the action s are nonzero.

The inclusion of varieties $i: M^s \hookrightarrow M$ gives rise to the direct image functor $i_!$: $\mathbf{K}(M^s) \to \mathbf{K}(M)$ and to the inverse image functor $i^!: \mathbf{K}(M) \to \mathbf{K}(M^s)$. The later is defined by the formula

$$i^!\mathcal{F} = \sum (-1)^k \mathrm{Tor}^k_{\mathcal{O}_M}(\mathcal{F}, \lambda(M^s)^{-1}).$$

The morphism $i^!$ clearly factors through the quotient $\mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(M)$. One has the following localization theorem (see [T])

(a) (i). $i^! \circ i_! = \mathrm{Id}_{\mathbf{K}(M^s)}$. In particular, the morphism below is surjective:

$$i^!: \mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(M) \twoheadrightarrow \mathbf{K}(M^s)$$

(ii). This morphism form isomorphism provided the group G is abelian.

Keep the notations G and s. Let M_1, M_2, M_3 be smooth G-varieties and Let Z be a closed G-subvariety of $M_1 \times M_2$ and let $j: Z \hookrightarrow M_1 \times M_2$ be the inclusion. Define a morphism $\mathbf{r}_s: \mathbf{K}_G(Z) \to \mathbf{K}(Z^s)$ by the formula

$$\mathcal{F} \longmapsto \mathbf{r}_s(\mathcal{F}) = \sum (-1)^k \mathrm{Tor}^k_{\mathcal{O}_{M_1 \times M_2}}(\lambda(M_1^s)^{-1} \times \mathcal{O}_{M_2^s}, j_!\mathcal{F}). \tag{3.2.1}$$

The Tor groups on the right hand side are supported on Z^s, since $Z^s = Z \cap (M_1 \times M_2)$. The assignment $\mathcal{F} \longmapsto \mathbf{r}_s(\mathcal{F})$ factors through $\mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(Z)$. So we get a morphism $\mathbf{r}_s: \mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(Z) \to \mathbf{K}(Z^s)$. Let $\tilde{Z}$ be a closed G-subvariety $M_2 \times M_3$. Similarly we have a morphism $\mathbf{r}_s: \mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(\tilde{Z}) \to \mathbf{K}(\tilde{Z}^s)$.

(b) Bivariant fixed-point theorem. Assume that $Z, \tilde{Z}$ satisfy (3.1.1), then the following diagram

$$\begin{array}{ccc}
(\mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(Z)) \otimes (\mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(\tilde{Z})) & \longrightarrow & \mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(Z \circ \tilde{Z}) \\
\mathbf{r}_s \downarrow & & \downarrow \mathbf{r}_s \\
\mathbf{K}(Z^s) \otimes \mathbf{K}(\tilde{Z}^s) & \longrightarrow & \mathbf{K}(Z^s \circ \tilde{Z}^s)
\end{array}$$

commutes. That is, the convolution commutes with the the morphism $\mathbf{r}_s$.

The K-theoretic convolution has its counterpart in homology. For $Z \subset M_1 \times M_2$, the complex coeffcients Borel-Moore homology group $H_i(Z)$ may be defined (via Poincaré duality) as the relative cohomology $H^{m-i}(M_1 \times M_2, M_1 \times M_2 \backslash Z)$, where $m = \dim M_1 \times M_2$. The cup product in cohomology gives rise to a cap product on Borel-Moore homology, which replaces the functor $\overset{L}{\otimes}$ in K-theory. One defines a homology counterpart of (3.1.2) as a map

$$* : \; H_i(Z) \otimes H_j(\tilde{Z}) \to H_{i+j-d}(Z \circ \tilde{Z}), \quad d = \dim M_2,$$

given by the formula: $c * \tilde{c} = (p_{13})_*(p_{12}^* c \cap p_{23}^* \tilde{c})$.

Associate any element $\mathcal{F} \in \mathbf{K}(Z)$ to its *Chern character* class $ch(\mathcal{F}) \in H_*(Z)$ (see [FM]). Further, let $\mathrm{Td}(M_2) \in H_*(M_2)$ denote the *Todd class* of the manifold M_2. Define a morphism $\mathbf{c} : \; \mathbf{K}(Z) \to H_*(Z)$ by the formula

$$\mathbf{c}(\mathcal{F}) = pr_2^* \mathrm{Td}(M_2) \cdot ch(\mathcal{F}),$$

where pr_2 is the second projection $M_1 \times M_2 \to M_2$.

(c) Bivariant Riemann-Roch theorem. The morphism $\mathbf{c}$ commutes with the convolution. That is, the diagram

$$\begin{array}{ccc} (\mathbf{K}(Z)) \otimes \mathbf{K}(\tilde{Z}) & \longrightarrow & \mathbf{K}(Z \circ \tilde{Z}) \\ {\scriptstyle \mathbf{c}}\downarrow & & \downarrow{\scriptstyle \mathbf{c}} \\ H_*(Z) \otimes H_*(\tilde{Z}) & \longrightarrow & H_*(Z \circ \tilde{Z}) \end{array}$$

commutes.

Combining (c) and (d) (applied to Z^s), we obtain the following result.

(d) The composition morphism $\mathbf{c} \circ \mathbf{r}_s : \; \mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(Z) \to H_*(Z^s)$ commutes with convolution.

3.3. Convolution algebras. Let G be an algebraic group, M a smooth quasi-projective G-variety. Let $\pi : \; M \to N$ be a G-equivariant proper morphism. Set

$$Z = M \times_N M = \{(m, m') \in M \times M \mid \pi(m) = \pi(m')\} \subset M \times M.$$

We view Z as a G-equivariant correspondence in $M \times M$. Clearly we have $Z \circ Z = Z$. Thus the convolution makes $\mathbf{K}_G(Z)$ an associative $\mathbf{R}_G$-algebra. Observe further that the diagonal of $M \times M$ is contained in Z and the class of the structure sheaf of the diagonal is the unit of the algebra $\mathbf{K}_G(Z)$. Similarly, the convolution makes the Borel-Moore homology group $H_*(Z)$ into a finite dimensional $\mathbb{C}$-algebra whose unit is the fundamental class of the diagonal.

3.4. Examples of convolutions algebras. Assume that M is a projective variety and $N = \{point\}$. Let $\pi : M \to N$ be the unique morphism from M to N. Then we have $Z = M \times M$. We assume that G is reductive (possibly disconnected). For any semisimple element $s \in G$, by 3.2(d) we have an algebra homomorphism

$$\mathbf{c} \circ \mathbf{r}_s : \mathbb{C}_s \otimes_{\mathbf{R}_G} \mathbf{K}_G(M \times M) \to H_*(M^s \times M^s) \tag{3.4.1}$$

Now the algebra $H_*(M^s \times M^s)$ and the group $A(s) = C_G(s)/C_G(s)^0$ naturally act on the homology group $H_*(M^s)$, and the actions commute. Let ρ be a simple $A(s)$-module which appears in the homology group $H_*(M^s)$. Let

$$E_{s,\rho} = (\rho^* \otimes H_*(M^s))^{A(s)} = \mathrm{Hom}_{A(s)}(\rho, H_*(M^s)),$$

where ρ^* is the dual of ρ. The space $E_{s,\rho}$ is in fact a simple $H_*(M^s \times M^s)$-module and by varying ρ one gets each simple $H_*(M^s \times M^s)$-module exactly once. Furthermore, we can regard $E_{s,\rho}$ as a $\mathbf{K}_G(M \times M)$-module, via $\mathbf{c} \circ \mathbf{r}_s$; this is a simple $\mathbf{K}_G(M \times M)$-module and $(s, \rho) \to E_{s,\rho}$ defines a bijection between the set of pairs (s, ρ) as above (up to G-conjugacy) and the set of isomorphism classes of simple $\mathbf{K}_G(M \times M)$-modules.

Let $i : M \times M \to M \times M$ be the G-equivariant morphism defined by $i(m, m') = (m', m)$. For any G-equivariant coherent sheaf $\mathcal{F}$, we shall write $\tilde{\mathcal{F}}$ for $i^!(\mathcal{F}^*)$, here $\mathcal{F}^*$ is the dual sheaf fo $\mathcal{F}$. We write E_s for $E_{s,\rho}$ when ρ is the unit representation of $C_G(s)/C_G(s)^0$.

When both G and M are finite, the algebra $\mathbf{K}_G(Z)$ is one defined in [L15]. When G is reductive and M is finite, the algebra $\mathbf{K}_G(Z)$ is one defined in [L14]. When G acts trivially on a finite set M, then the convolution algebra $\mathbf{K}_G(M \times M)$ is isomorphic to the algebra $M_{k\times k}(\mathbf{R}_G)$, the $k \times k$ matrix ring over $\mathbf{R}_G$, where $k = |M|$.

Another examples is that an affine Hecke algebra can be realized as a convolution algebra, this is a key of Kazhdan & Lusztig's work on classification of simple modules of affine Hecke algebra (see [KL4]). We shall recall their work in next section.

Assume that Y is a finite G-variety, i.e., a finite G-set. Any coherent sheaf on Y is a vector bundle (v.b.) on Y. If V form irreducible G-v.b. on Y, then the set $\{y \in Y \mid V_y \neq 0\}$ is a single G-orbit $\mathcal{O}$ in Y, and for any $y \in \mathcal{O}$, the obvious representation of the isotropy group G_y on V_y is irreducible; this gives a bijection between the set of irreducible G-v.b. on Y (up to isomorphism) and the set of pairs (y, ρ) where $y \in Y$, ρ form irreducible (algebraic) representation of G_y, modulo the obvious action of G. In chapters 7 and 8 we often write ρ_y for the irreducible G-v.b. on Y corresponding to the pair (y, ρ).

3.5. Geometric realization of affine Hecke algebra. In this section we shall assume that G is a connected algebraic group with simply connected derived group. Let $\mathbf{g}$ be the Lie algebra of G, $\mathcal{N}$ be the variety of all nilpotent elements of $\mathbf{g}$, and let $\mathcal{B}$ be the variety of all Borel subalgebras of $\mathbf{g}$. Let $\tilde{\mathcal{N}} = \{(N, \mathbf{b}) \in \mathcal{N} \times \mathcal{B} \mid N \in \mathbf{b}\}$ and let $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution by projecting $(N, \mathbf{b})$ to N. Let

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \simeq \{(N, \mathbf{b}, \mathbf{b}') \mid N \in \mathbf{b} \cap \mathbf{b}' \text{ is nilpotent}, \ \mathbf{b}, \mathbf{b}' \in \mathcal{B}\}$$

be the Steinberg variety. The Steinberg variety can obtained in another way. The group G acts on $\mathbf{g}$ through adjoint action, so G acts on the variety $\mathcal{B}$. Let G acts on $\mathcal{B} \times \mathcal{B}$ diagonally, then the number of G-orbits in $\mathcal{B} \times \mathcal{B}$ is $|W_0|$. Let $T^*(\mathcal{B} \times \mathcal{B})$ be the cotangent bundel of $\mathcal{B} \times \mathcal{B}$, then the union of conormal bundels of all G-orbits in $\mathcal{B} \times \mathcal{B}$ is isomorphic to Z.

Convention: For any $g \in G$, $x \in \mathbf{g}$, $\mathbf{b} \in \mathcal{B}$, we shall write $g.x$, $g.\mathbf{b}$ instead of $\mathrm{Ad}g(x)$, $\mathrm{Ad}g(\mathbf{b})$, respectively.

Let $G \times \mathbb{C}^*$ act on $\tilde{\mathcal{N}}$ by

$$(g, q) : \ (N, \mathbf{b}) \longmapsto (g.q^{-1}N, g.\mathbf{b}). \tag{3.5.1}$$

Then $G \times \mathbb{C}^*$ acts on Z by

$$(g, q) : \ (N, \mathbf{b}, \mathbf{b}') \longmapsto (g.q^{-1}N, g.\mathbf{b}, g.\mathbf{b}'). \tag{3.5.2}$$

We keep the notations in 2.1-2.2. Let $\mathbf{H} = H \otimes_A \mathbb{C}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$. we shall identify $\mathbf{A} = \mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$ with $\mathbf{R}_{\mathbb{C}^*}$ by regarding $\mathbf{q}$ as the indentity representation $\mathbb{C}^* \to \mathbb{C}^*$. Let $\dot{\mathbf{H}}$ be the $\mathbf{A}$-subalgebra of $\mathbf{H}$ generated T_w, θ_x, $w \in W$, $x \in X$. We shall indetify the center of $\dot{\mathbf{H}}$ with the ring $\mathbf{R}_{G\times\mathbb{C}^*}$ by regarding U_x, $x \in X^+$ as the the irreducible module of highest weight x. Then (see [KL4], see also [G2])

(a) There exists an $\mathbf{R}_{G\times\mathbb{C}^*}$-algebra isomorphism between the affine Hecke algebra $\dot{\mathbf{H}}$ and the convolution algebra $\mathbf{K}_{G\times\mathbb{C}^*}(Z)$.

3.6. Standard modules. Let G be as in 3.5. Given a semisimple element $(s, q) \in G\times\mathbb{C}^*$, let

$$\mathbf{g}_{s,q} = \{\xi \in \mathbf{g} \mid g.\xi = q\xi\},$$

$$\mathcal{N}_{s,q} = \{\xi \in \mathcal{N} \mid g.\xi = q\xi\}.$$

For any $N \in \mathcal{N}_{s,q}$, consider the variety

$$\mathcal{B}_N^s = \{\mathbf{b} \in \mathcal{B} \mid N \in \mathbf{b} \text{ and } s.\mathbf{b} = \mathbf{b}\}.$$

Obviously the group $C_G(s) \cap C_G(N)$ acts on the variety $\mathcal{B}_N^s$. So the group $A(s, N) = C_G(s) \cap C_G(N)/(C_G(s) \cap C_G(N))^0$ acts on the homology group $H_*(\mathcal{B}_N^s)$. Let $A(s, N)^\vee$ be the set of isomorphism classes of the irreducible $A(s, N)$-modules which appear in $H_*(\mathcal{B}_N^s)$. It is shown in [KL4] that

(a) There exists an $\dot{\mathbf{H}}$-module structure on $\mathbf{K}(\mathcal{B}_N^s) = H_*(\mathcal{B}_N^s)$ such that
   (i). The action commutes with the action of $A(s, N)^\vee$.
   (ii). $\mathbf{q}$ acts on it by scalar q and $U_x, x \in X^+$ acts on it by scalar $tr(s, V(x))$, where $V(x)$ is the irreducible G-module with highest weight x.

For any $\rho \in A(s, N)^\vee$, let

$$M_{s,N,q,\rho} = \mathrm{Hom}_{A(s,N)}(\rho, \mathbf{K}(\mathcal{B}_N^s)) = (\mathbf{K}(\mathcal{B}_N^s) \otimes \rho^*)^{A(s,N)},$$

where ρ^* is the dual module of ρ. By (a) we see that $M_{s,N,q,\rho}$ form $\dot{\mathbf{H}}$-module. The module $M_{s,N,q,\rho}$ is called a standard module. We say that two quadruples (s, N, q, ρ), (s', N', q', ρ')

are G-conjugate if there exists some $g \in G$ such that $s' = gsg^{-1}$, $N' = g.N$, $q = q'$, $\rho' = g(\rho)$ (note that we have a natural bijection $g : A(s,N)^\vee \to A(s',N')^\vee$ when first two conditions hold). Now we can state the main results of [KL4].

(b) Let L be a simple module of $\dot{\mathbf{H}}$ such that $\mathbf{q}$ acts on it by scalar q, then L is a quotient module of some standard module $M_{s,N,q,\rho}$.

(c) Two standard module $M_{s,N,q,\rho}$, $M_{s',N',q',\rho'}$ are isomorphic if and only (s,N,q,ρ), (s',N',q',ρ') are G-conjugate.

(d) When q is not a root of 1, then each standard module $M_{s,N,q,\rho}$ has a unique quotient module, denoted by $L_{s,N,q,\rho}$. Moreover, $L_{s,N,q,\rho}$, $L_{s',N',q,\rho'}$ are isomorphic if and only (s,N,q,ρ), (s',N',q,ρ') are G-conjugate.

In [L17] Lusztig conjecture (d) remains true provided that $\sum_{w\in W_0} q^{l(w)} \neq 0$.

For any $q \in \mathbb{C}^*$, regard $\mathbb{C}$ as an $\mathbf{A}$-algebra (resp. $\mathbb{C}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$-algebra) by specifying $\mathbf{q}$ to q (resp. $\mathbf{q}^{\frac{1}{2}}$ to a square root of q), then consider the Hecke algebra

$$\mathbf{H}_q = \dot{\mathbf{H}} \otimes_{\mathbf{A}} \mathbb{C} = \mathbf{H} \otimes_{\mathbb{C}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]} \mathbb{C}.$$

For any semisimple element s in G, let $\mathbf{I}_{s,q}$ be the ideal of $\mathbf{H}_q$ generated by $U_x - tr(s, V(x))$, $x \in X^+$ and let $\mathbf{H}_{s,q}$ be the quotient algebra $\mathbf{H}_q/\mathbf{I}_{s,q}$ of $\mathbf{H}_q$. Then it is easy to see that $\dot{\mathbf{H}}$ acts on the standard module $M_{s,N,q,\rho}$ factoring through the algebra $\mathbf{H}_{s,q}$. The following result is due to Ginzburg, which can be deduced from 3.2(d) and 3.5(a).

(e) The algebra $\mathbf{H}_{s,q} = \mathbb{C}_{s,q} \otimes \mathbf{K}_{G\times\mathbb{C}^*}(Z)$ is isomorphic to the convolution algebra $H_*(Z^{s,q})$.

The above results yield naturally two questions.

3.7. Question. (i). *Determine the dimensions of the standard modules $M_{s,N,q,\rho}$ and the simple modules $L_{s,N,q,\rho}$ when q is not a root of 1.*

(ii). *Classify the simple modules of $\mathbf{H}_q$ when q is a root of 1.*

For the question 3.7 (ii) in next chapter we will show that 3.6(d) is true for most roots of 1 by Combining 3.6(d-e). It is difficult to get the dimensions of simple $\mathbf{H}_q$-modules. But we can say a few words for the dimensions of standard modules.

3.8. Demensions of certain standard modules. The following results can used to calculate the dimensions of certain standard modules.

(a) $H_{odd}(\mathcal{B}_N^s) = 0$ and $H_{even}(\mathcal{B}_N^s)$ is isomorphic to Chow group of $\mathcal{B}_N^s$. (see [CLP]).

(b) Assume a connected diagonalizable algebraic group D acts on a variety M, then $\chi(M) = \chi(M^D)$, where $\chi(\cdot)$ denotes the Euler number (see [BB], I am grateful to R.V. Gurjar for providing the reference).

Notations are as in 3.5. For any semisimple element $(s,q) \in G \times \mathbb{C}^*$, we always have $s.0 = 0 = q0$. Since G has simply connected derived group, so $C_G(s)$ is connected. Thus $A(s,0)^\vee$ only contains the unit representation, denoted by 1. Moreover we have $M_{s,0,q,1} = \mathbf{K}(\mathcal{B}^s) = H_*(\mathcal{B}^s)$. By (a) we know that

(3.8.1) $\quad \dim M_{s,0,q,1} = \chi(\mathcal{B}^s)$.

Let T_s be a maximal torus of G containing s. Then T_s acts on $\mathcal{B}^s$. Using (b) we see that $\chi(\mathcal{B}^s) = \chi(\mathcal{B}^{T_s})$. It is well known that $\mathcal{B}^{T_s}$ is a finite set of $|W_0|$ elements. Thus we get

(3.8.2) $\quad \dim M_{s,0,q,1} = |W_0|$.

One also can obtain (3.8.2) by using 3.9(c) and results in [X1].

Now we assume that $G = SL_n(\mathbb{C})$. The results (a-b) are also sufficient to determine the dimensions of standard modules of $\mathbf{H}_q$ in this case. Let $(s,q) \in G \times \mathbb{C}^*$ be a semisimple element, it is harmless to assume that $(s,q) \in T \times \mathbb{C}^*$, where T is the subgroup of G consisting of diagonal matrices in G. Let $N \in \mathcal{N}_{s,q}$. Note that $\mathbf{g} = sl_n(\mathbb{C})$, we see that the sizes of Jordan blocks of N determines a partition of n, denoted by λ. It is known that $C_G(s) \cap C_G(N)$ is connected. Thus $A(s,N)^\vee$ only contains the unit representation, also denoted by 1. Moreover we have $M_{s,N,q,1} = \mathbf{K}(\mathcal{B}_N^s) = H_*(\mathcal{B}_N^s)$. By (a) we know that

(3.8.3) $\quad \dim M_{s,N,q,1} = \chi(\mathcal{B}_N^s)$.

A direct calculation shows that the pair (s,N) is conjugate to certain pair (s',N') such that $s' \in T$, $N' = \mathrm{diag}(N_1', N_2', ..., N_k')$, where each N_i' is a Jordan block. We may choose all N_i' to be upper triangular matrices. It is no harm to assume that $(s,N) = (s',N')$.

It is known (also easy to check) that $T_N = T \cap C_G(N)$ is a maximal torus in the group $C_G(N)$. Since $T_N \subset C_G(s) \cap C_G(N)$, T_N acts on the variety $\mathcal{B}_N^s$. Using (b) we know that $\chi(\mathcal{B}_N^s) = \chi((\mathcal{B}_N^s)^{T_N})$. Let $t \in T_N$ be a regular element in $C_G(N)$, then $g = t\exp N$ is a regular element in G. According to [St1] the variety $(\mathcal{B}_N^s)^{T_N} = \mathcal{B}^g$ is finite and

(3.8.4) *The cardinal* $|(\mathcal{B}_N^s)^{T_N}|$ *of* $(\mathcal{B}_N^s)^{T_N}$ *is the number of elements* $w \in W_0$ *such that* $w.N \in \mathbf{b}$, *where* $\mathbf{b}$ *is the Borel subalgebra of* $\mathbf{g}$ *consisting of all upper triangular matrices in* $\mathbf{g}$.

Let $\mu = (\mu_1, \mu_2, ..., \mu_k)$ be the dual partition, by (3.8.4) we get

$$(3.8.5) \qquad \dim M_{s,N,q,1} = \frac{n!}{\mu_1!\mu_2!\cdots\mu_k!}.$$

Note that $\dim M_{s,N,q,1}$ is the number of left cells contain in the two-sided cell of W corresponding to the nilpotent G-orbit containing N.

For other types we can get similar results when $C_G(N)$ is connected.

Finally we state a result concerned with the relations between $M_{s,N,q,\rho}$ and $M_{s,0,q,1}$.

(c). *The injection* $\mathcal{B}_N^s \hookrightarrow \mathcal{B}_N$ *induces an* $\mathbf{H}_q$*-module injection* $M_{s,N,q,\rho} \hookrightarrow M_{s,0,q,1}$.

For type A this was proved in [HS]. In general it follows from the results in [CLP].

For type A, we can get a little more. Let $s, t \in T$, $N \in \mathcal{N}_{s,q}$, $N' \in \mathcal{N}_{t,q}$. Assume that $\mathcal{B}_{N'}^t \subseteq \mathcal{B}_N^s$, then the inclusion induces an $\mathbf{H}_q$-module injection: $M_{t,N',q,1} \hookrightarrow M_{s,N,q,1}$. The proof is similar to that in [HS]. This fact should be useful in calculating the multiplicities of simple modules in standard modules. The multiplicities should have a nice combinational description.

3.9. The asymototic Hecke algebra J. The work [L12-L14] show that the asymototic Hecke algebra $\mathbf{J} = J_{\mathbf{Z}} \otimes \mathbb{C}$ of W is interesting in representation theory of affine Hecke algebras. Let $\phi_q : \mathbf{H}_q \to \mathbf{J}$ be the homomorphism induced from the homomorphism ϕ in 2.7(c). Thus any $\mathbf{J}$-module E gives rise to an $\mathbf{H}_q$-mdoule E_q via ϕ_q. We shall need the following result.

(a) The involution of $\dot{\mathbf{H}}$ defined by $T_s \to -\mathbf{q}T_s^{-1}$, $(s \in S \cap W_0)$, $\theta_x \to \theta_x^{-1}$ is just the involution of $\dot{\mathbf{H}}$ obtained from k in 1.6(e).

From 2.7(d) we know that (see [L13])

(b) The algebra $\mathbf{J}$ is finitely generated over its center. In particular, each simple $\mathbf{J}$-module is of finite dimension over $\mathbb{C}$.

We recall that there is a bijection between Cell(W) and the set of nilpotent G-orbits in $\mathbf{g}$ which preverse the partial orders. For $c \in$Cell(W), we denote $\mathcal{N}_c$ the corresponding nilpotent G-orbit.

For any two-sided cell c of W, let $\mathbf{J}_c = J_{\mathbf{Z},c} \otimes \mathbb{C}$. Then $\mathbf{J} = \bigoplus_{c \in \mathrm{Cell}(W)} \mathbf{J}_c$. For each simple module E of $\mathbf{J}$, there exists a unique two-sided cell c of W such that $\mathbf{J}_c E \neq 0$, we call c the two-sided cell attached to E and denote it by c_E. Similarly, for each simple module L of $\mathbf{H}_q$, there exists a unique two-sided cell c of W such that

$$C_w L = 0 \quad \text{when } w \notin c \text{ and } w \underset{LR}{\leq} u \text{ for some } u \in c,$$

$$C_u L \neq 0 \quad \text{for some } u \in c,$$

we call c the two-sided cell attached to L and denote it by c_L.

(c) Let E be a simple $\mathbf{J}$-module. Then E_q is isomorphic to certain standard module $M_{s,N,q,\rho}$, $N \in \mathcal{N}_{c_E}$. Each standard $\mathbf{H}_q$-module can be obtain in this way. (Here we need (a) and results in [L14]).

(d) Let E be as in (c). For each simple constituent L' of E_q, we have $c_E \underset{LR}{\leq} c_{L'}$.

(e) For each simple $\mathbf{H}_q$-module L, there exists some simple $\mathbf{J}$-module E such that

(i). $c_E = c_L$.

(ii). L is a simple quotient of E_q.

(iii). For any other simple constituent L' of E_q, we have $c_L \underset{LR}{\leq} c_{L'}$, $c_L \neq c_{L'}$.

(f) Assume that q is not a root of 1 or q=1. Then for each simple $\mathbf{J}$-module E, E_q has a unique simple constituent L such that $c_L = c_E$, which is the unique quotient of E_q, denote it by L_E. The map $E \to L_E$ defines a bijection between the set of isomorphism classes of simple $\mathbf{J}$-modules and the set of isomorphism classes of simple $\mathbf{H}_q$-modules.

We shall denote $Y_{q,c}$ the set of isomorphism classes of simple $\mathbf{H}_q$-modules to which the attached two-sided cell is c. For a semisimple element s in G, we denote $Y_{s,q}$ the set of isomorphism classes of simple $\mathbf{H}_q$-modules on which U_x $(x \in X^+)$ acts by scalar $tr(s, V(x))$. Set $Y_{s,q,c} = Y_{q,c} \cap Y_{s,q}$.

The results (c-e) have several consequences. We need a simple fact. For each semisimple element $(s,q) \in G \times \mathbb{C}^*$, there exists a unique nilpotent G-orbit $\mathbf{n}_{s,q}$ such that $\bar{\mathbf{n}}_{s,q} \supseteq \mathcal{N}_{s,q}$ and for other nilpotent orbit $\mathbf{n}$ if $\bar{\mathbf{n}} \supseteq \mathcal{N}_{s,q}$, then $\bar{\mathbf{n}} \supseteq \mathbf{n}_{s,q}$.

3.10. Corollary. *For arbitrary $N \in \mathbf{n}_{s,q}$, $\rho \in A(s,N)^\vee$, the module $M_{s,N,q,\rho}$ is simple.*

We conjecture that $M_{s,N,q,1}$ is simple if and only if $N \in \mathbf{n}_{s,q}$. When $\rho \neq 1$, the module $M_{s,N,q,\rho}$ may be simple for $N \notin \mathbf{n}_{s,q}$, see the dimension of E_4 in the part (G) of 8.3.

3.11. Corollary. *Assume that $\mathcal{N}_{s,q} = \{0\}$, then $M_{s,0,q,1}$ is simple and $Y_{s,q} = \{M_{s,0,q,1}\}$.*

Note that $\dim M_{s,0,q,1} = |W_0|$.

3.12. Corollary. *Assume that $\mathbf{g}_{s,q} \subseteq \mathbf{g}_\alpha \oplus \mathbf{g}_{-\alpha}$ and $A(s,N)^\vee = \{1\}$ for some $0 \neq N \in \mathcal{N}_{s,q}$. (Note that N, N' asr conjugate under $C_G(s)$ for any $N, N' \in \mathcal{N}_{s,q}$ in our case). Then $|Y_{s,q}| \leq 2$, and*

(3.12.1). *$\dim L_{s,N,q,1} = |W_0|/2$, where $0 \neq N \in \mathcal{N}_{s,q}$.*

(3.12.2). *If $|Y_{s,q}| = 2$, we have $\dim L_{s,0,q,1} = |W_0|/2$.*

3.13. Let $K(\mathbf{H}_q)$ (resp. $K(\mathbf{J})$) be the Grothendieck group of $\mathbf{H}_q$-mdoules (resp. $\mathbf{J}$-modules) of finite dimensions over $\mathbb{C}$. Then by 3.9(e) we see that $E \to E_q$ defines a surjection $(\phi_q)_*: K(\mathbf{J}) \to K(\mathbf{H}_q)$ (see [L13]).

For a two-sided cell c of W, we denote $K(\mathbf{H}_q)_c$ the subgroup of $K(\mathbf{H}_q)$ spanned by those simple $\mathbf{H}_q$-modules L with $c_L = c$. Then we have $K(\mathbf{H}_q) = \bigoplus_c K(\mathbf{H}_q)_c$, where c runs over the set of two-sided cells in W. It is obvious that $K(\mathbf{J}) = \bigoplus_c K(\mathbf{J}_c)$, where the definition of $K(\mathbf{J}_c)$ is similar to that of $K(\mathbf{J})$. By the definition of ϕ_q we see that $(\phi_q)_*$ is compatible with the filtrations

$$K(\mathbf{J})_{\geq c} = \bigoplus_{\substack{c \leq c' \\ LR}} K(\mathbf{J}_{c'}), \qquad K(\mathbf{H}_q)_{\geq c} = \bigoplus_{\substack{c \leq c' \\ LR}} K(\mathbf{H}_q)_{c'}$$

of $K(\mathbf{J})$, $K(\mathbf{H}_q)$, hence $(\phi_q)_*$ induces a surjection $(\phi_q)_{*,c}: K(\mathbf{J}_c) \to K(\mathbf{H}_q)_c$. $(\phi_q)_{*,c}$ maps the $\mathbf{J}_c$-module E to the sum of simple constituents M of E_q with $c_M = c$, where E_q is the $\mathbf{H}_q$-module obtaining from E and the homomorphism $\phi_{q,c}: \mathbf{H}_q \to \mathbf{J} \to \mathbf{J}_c$.

For any $\mathbf{J}_c$-module E, we also denote $(\phi_q)_{*,c}(E)$ the direct sum of simple constituents of E_q to which the attached two-sided cell is c. When E is a simple $\mathbf{J}_c$-mdole, I hope that $(\phi_q)_{*,c}(E)$ is either 0 or a simple $\mathbf{H}_q$-module. Furthermore, I hope that $(\phi_q)_{*,c}(E_1) \simeq (\phi_q)_{*,c}(E_2)$ as $\mathbf{H}_q$-modules if and only if $E_1 \simeq E_2$ as $\mathbf{J}_c$-modules when E_1, E_2 are simple $\mathbf{J}_c$-modules and $(\phi_q)_{*,c}(E_1) \neq 0$. If it is true, then the set $\{(\phi_q)_{*,c}(E) \mid c$ two-sided cell of W, E a simple $\mathbf{J}_c$-module (up to isomorphism)$\}$-$\{0\}$ is a complete set of simple $\mathbf{H}_q$-modules, i.e. any simple $\mathbf{H}_q$-mdoule isomorphic to some $(\phi_q)_{*,c}(E)$ and any two modules in the above set are not isomorphic. Then we get the classification of simple $\mathbf{H}_q$-modules. When q is not a root of 1 or $q = 1$, the above idea is valid (see [L13]).

In chapter 7 we apply the above idea to classify the simple $\mathbf{H}_q$-modules under the assumption that W is of type $\tilde{G}_2$ or $\tilde{B}_2$.

Lusztig conjectured in [L17] that $(\phi_q)_*$ is an isomorphism if and only if $\sum_{w \in W} q^{l(w)} \neq 0$. When $(\phi_q)_*$ is an isomorphism, so is $(\phi_q)_{*,c}$, in [L13] Lusztig show that the set $\{(\phi_q)_{*,c}(E) \mid c$ two-sided cell of W, E a simple $\mathbf{J}_c$-module (up to isomorphism)$\}$ is a complete set of simple $\mathbf{H}_q$-modules, Lusztig also show that $(\phi_q)_*$ is an isomorphism when q is not a root of 1 or $q = 1$ ([L13]).

3.14. A conjecture of Lusztig concerned with the structure of $\mathbf{J}_c$. In this section we shall assume that G is simply connected simple algebraic group. In [L14] Lusztig give a nice conjecture concerned with the structure of the ring $\mathbf{J}_c$. Now we state his conjecture.

For each two-sided cell c of W, we have the corresponding nilpotent G-orbit $\mathcal{N}_c$ in $\mathbf{g}$. Choose an element $N \in \mathcal{N}_c$ and let F_c be a maximal reductive subgroup of $C_G(N)$, Lusztig conjectured that there exists a finite F_c-set Y and a bijection $\pi: c \xrightarrow{\sim}$ set of irreducible F_c-v.b. on $Y \times Y$ (up to isomorphism) such that $t_w \to \pi(w)$ defines a $\mathbb{C}$-algebra isomorphism (preserving the unit element) between $\mathbf{J}_c$ and $\mathbf{K}_{F_c}(Y \times Y)$ and $\pi(w^{-1}) = \widetilde{\pi(w)}$, $w \in c$. When $c = c_0$ is the lowest two-sided cell, the conjecture was verified in [X1]. In chapter 7 we show that the conjecture is true when W is of type $\tilde{G}_2$, $\tilde{B}_2$. When W is of type $\tilde{A}_2$,

$\tilde{A}_1$, we know the conjecture is valid (see [X2]).

Let c, N, F_c be as above. Note that $\mathcal{B}_N$ is an F_c-variety. Perhaps Lusztig's original ideal is the following conjecture.

3.15. Conjecture. *There exists a bijection $\pi: c \xrightarrow{\sim}$ set of irreducible F_c-v.b. on $\mathcal{B}_N \times \mathcal{B}_N$ (up to isomorphism) such that*

(i). $t_w \to \pi(w)$ *defines a $\mathbb{C}$-algebra isomorphism (preserving the unit element) between $\mathbf{J}_c$ and the convolution $\mathbf{K}_{F_c}(\mathcal{B}_N \times \mathcal{B}_N)$.*

(ii). $\pi(w^{-1}) = \widetilde{\pi(w)}$, $w \in c$.

(iii). *The homomorphism ϕ in 2.7(c) has a natural geometric interpretation.*

Affine Hecke algebras with two parameters

3.16. In the rest part of the chapter we shall consider the representations of affine Hecke algebras of two parameters. We mainly follow the line in [L9]. We shall assume that G is simple, simply connected algebraic group. We keep the notations in 2.8-2.9.

Let $\tilde{G} = G \times \mathbb{C}^* \times \mathbb{C}^*$. Then $\tilde{G}$ acts on $\mathcal{B}$ as follows: G acts on $\mathcal{B}$ through adjoint action, $\mathbb{C}^* \times \mathbb{C}^*$ acts trivially. We have

$$K_{\tilde{G}}(\mathcal{B}) = K_G(\mathcal{B}) \otimes R_{\mathbb{C}^* \times \mathbb{C}^*} = K_G(\mathcal{B}) \otimes \mathbf{Z}[\mathbf{u}^{\pm 2}, \mathbf{v}^{\pm 2}],$$

where $\mathbf{u}^2, \mathbf{v}^2$ are the generators of $R_{\mathbb{C}^* \times \mathbb{C}^*}$ corresponding to the obvious projections: $p_1, p_2 : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$, $R_{\mathbb{C}^* \times \mathbb{C}^*} = K_{\mathbb{C}^* \times \mathbb{C}^*}(point)$ is the Grothendieck group of the rational representations of $\mathbb{C}^* \times \mathbb{C}^*$.

For each $s \in S_0$, we denote $\mathcal{P}_s$ the variety of all parabolic subalgebras of $\mathbf{g}$ corresponding to s and let $\pi_s : \mathcal{B} \to \mathcal{P}_s$ be the natural map. There is a unique endomorphism

$$T_s : K_{\tilde{G}}(\mathcal{B}) \to K_{\tilde{G}}(\mathcal{B}) \tag{3.16.1}$$

with the following property: if E is a $\tilde{G}$-equivariant algebraic vector bundle on $\mathcal{B}$, then

$$E + T_s E = (\pi_s^*(\pi_s)_*(E^*) - \pi_s^*(\pi_s)_*(E^* \otimes \Omega_s^1))^*, \tag{3.16.2}$$

where Ω_s^1 is the line bundle on $\mathcal{B}$ of holomorphic differential 1-forms along the fibres of π_s, G acts on Ω_s^1 in an obvious way, let $\mathbb{C}^*$ acts on each fibre of Ω_s^1 by scalar multiplication, and then $\mathbb{C}^* \times \mathbb{C}^*$ acts each fibre of Ω_s^1 through the projection p_1 if $s \in S_0'$ and through the projection p_2 if $s \in S_0''$. Here $(\pi_s)_*(E^*)$ is the alternating sum of the higher direct images of E^* under π_s in the category of coherent sheaves (we use E'^* for the dual of a vector bundle E'); these higher direct images are again $\tilde{G}$-equivariant algebraic vector bundle on $\mathcal{P}_s$ (see [Fu]), hence their alternating sums defines an element in $K_{\tilde{G}}(\mathcal{P}_s)$.

For any element $x \in X$, we define an endomorphism

$$\theta_x : K_{\tilde{G}}(\mathcal{B}) \to K_{\tilde{G}}(\mathcal{B}) \tag{3.16.3}$$

by

$$\theta_x E = E \otimes L_x, \tag{3.16.4}$$

where L_x is the line bundle on $\mathcal{B}$ associated to the weight $x : T \to \mathbb{C}^*$, it is a $\tilde{G}$-equivariant bundle with the obvious action of G and with trivial action of $\mathbb{C}^* \times \mathbb{C}^*$.

3.17. Proposition. *The endomorphisms T_s, θ_x of $K_{\tilde{G}}(\mathcal{B})$ defined in 3.16 give rise to a left $\tilde{H}^\circ$-module structure on $K_{\tilde{G}}(\mathcal{B})$. (The action of $\mathbf{Z}[\mathbf{u}^{\pm 2}, \mathbf{v}^{\pm 2}] \subset \tilde{H}^\circ$ is defined to be the same as the restriction to $R_{\mathbb{C}^* \times \mathbb{C}^*}$ of the action of $R_{\tilde{G}}$, note that $K_{\tilde{G}}(\mathcal{B})$ is naturally a $R_{\tilde{G}}$ module.) This $\tilde{H}^\circ$-module structure commutes with the $R_{\tilde{G}}$-module structure on $K_{\tilde{G}}(\mathcal{B})$.*

Proof. We identify $K_{\tilde{G}}(\mathcal{B})$ with $B'[X]$ and identify $R_{\tilde{G}}$ with $B'[X]^{W_0}$, where $B' = \mathbf{Z}[\mathbf{u}^{\pm 2}, \mathbf{v}^{\pm 2}]$. Then the canonical ring hommomorphism $R_{\tilde{G}} \to K_{\tilde{G}}(\mathcal{B})$ becomes the inclusion $B'[X]^{W_0} \hookrightarrow B'[X]$. Under these identifications, the endomorphisms in 3.16 become B'-linear maps and satisfy

$$T_s(x) = \frac{s(x)\alpha_s - x\alpha_s}{\alpha_s - 1} + \mathbf{u}^2 \frac{x\alpha_s - s(x)}{\alpha_s - 1}, \quad s \in S_0', \quad x \in X, \tag{3.17.1}$$

$$T_s(x) = \frac{s(x)\alpha_s - x\alpha_s}{\alpha_s - 1} + \mathbf{v}^2 \frac{x\alpha_s - s(x)}{\alpha_s - 1}, \quad s \in S_0'', \quad x \in X, \tag{3.17.2}$$

$$\theta_y(x) = xy. \tag{3.17.3}$$

Let $\mathcal{I}$ be the left ideal of $\tilde{H}^\circ$ generated by $C = \sum_{w\in W} T_w$, then $\theta_x C$, $x \in X$ is a B' basis of $\mathcal{I}$. Under the natural map $B'[X] \to \mathcal{I}$, $x \to \theta_x C$, the actions (3.17.1-2) (resp. (3.17.3) become left multiplications by T_s (resp. θ_x) for the left $\tilde{H}^\circ$-module structure of the left ideal $\mathcal{I}$. The proposition is proved.

3.18. Motivated by a conjecture in [L4] we formulate a conjecture, which is an analogue of the (*) in the introduction. We need some notations.

We set

(3.18.1) $$f_{W_0}(\mathbf{u}^2, \mathbf{v}^2) = \sum_{w\in W} \mathbf{u}^{2l'(w)}\mathbf{v}^{2l''(w)}.$$

Then

(3.18.2)
$$f_{W_0}(\mathbf{u}^2, \mathbf{v}^2) = \prod_{i=1}^{n}(1 + \mathbf{u}^2 + \cdots + \mathbf{u}^{2(i-1)})(1 + \mathbf{u}^{2(i-1)}\mathbf{v}^2), \quad \text{if } G \text{ is of type } B_n \text{ or } C_n,$$

(3.18.3) $$f_{W_0}(\mathbf{u}^2, \mathbf{v}^2) = (1 + \mathbf{u}^2)(1 + \mathbf{u}^2 + \mathbf{u}^4)(1 + \mathbf{v}^2)(1 + \mathbf{v}^2 + \mathbf{v}^4)\cdot$$
$$(1 + \mathbf{u}^2\mathbf{v}^4)(1 + \mathbf{u}^4\mathbf{v}^2)(1 + \mathbf{u}^2\mathbf{v}^2)(1 + \mathbf{u}^4\mathbf{v}^4)(1 + \mathbf{u}^6\mathbf{v}^6), \quad \text{if } G \text{ is of type } F_4,$$

(3.18.4) $$f_{W_0}(\mathbf{u}^2, \mathbf{v}^2) = (1 + \mathbf{u}^2)(1 + \mathbf{v}^2)(1 + \mathbf{u}^2\mathbf{v}^2 + \mathbf{u}^4\mathbf{v}^4), \quad \text{if } G \text{ is of type } G_2,$$

Let T be a maxiaml torus of G and $\mathbf{t}$ its Lie algebra. We have

(3.18.5) $$\mathbf{g} = \mathbf{t} \oplus (\bigoplus_{\alpha\in R} \mathbf{g}_\alpha).$$

For any $s \in T$, $(a, b) \in \mathbb{C}^* \times \mathbb{C}^*$, we set

(3.18.6) $$\mathbf{g}_{s,a,b} = \{0\} \cup ((\bigoplus_{\substack{\alpha\in R' \\ \alpha(s)=a}} \mathbf{g}_\alpha) \oplus (\bigoplus_{\substack{\alpha\in R'' \\ \alpha(s)=b}} \mathbf{g}_\alpha)),$$

(3.18.7) $$\mathcal{N}_{s,a,b} = \mathcal{N} \cap \mathbf{g}_{s,a,b}.$$

For any $N \in \mathcal{N}_{s,a,b}$, Let $A(s, N)$, $A(s, N)^\vee$ be as in 3.6.

3.19. Conjecture. *If $f_{W_0}(a, b) \neq 0$, then there is a natural one-to-one correspondence between the set of isomorphism classes of simple $\tilde{\mathbf{H}}_{a,b}$-modules and the set of G-conjugacy classes of the triples (s, N, ρ), where $s \in T$, $N \in \mathcal{N}_{s,a,b}$, $\rho \in A(s, N)^\vee$. When $f_{W_0}(a, b) = 0$, no such natural correspondence exists.*

4. An Equivalence Relation in $T \times \mathbb{C}^*$

Let G be a connected reductive group and T a maximal torus of G. Motivated by the result 3.6(e) (due to Ginzburg) we introduce an equivalence relation in the set $T \times \mathbb{C}^*$. We prove some properties of the equivalence relation. Combine these properties, 3.6(d) (due to Kazhdan and Lusztig) and 3.6(e) we can prove that the conjecture $(*)$ in the introduction of the paper is true for most roots of 1. The main results are Theorem 4.5 and Theorem 4.6. For type A_n, our results also confirm a conjecture of Zelevinsky [Z, 8.7].

4.1. The equivalence relation. For any $(s,q) \in T \times \mathbb{C}^*$ we write

$$R_{s,q} = \{\alpha \in R \mid \alpha(s) = q\}. \tag{4.1.1}$$

Given two semisimple elements (s,q) and (t,r) in $T \times \mathbb{C}^*$, we shall write $(s,q) \sim (t,r)$ if $R_{s,q} = R_{t,r}$ and $R_{s,1} = R_{t,1}$.

The condition $R_{s,q} = R_{t,r}$ is equivalent to that $\mathbf{g}_{s,q} = \mathbf{g}_{t,r}$ (see chapter 3 for the definition of notations). It is also equivalent to that $\mathcal{N}_{s,q} = \mathcal{N}_{t,r}$. When G has simply connected derived group, then the condition $R_{s,1} = R_{t,1}$ is equivalent to that $C_G(s) = C_G(t)$. Obviously the relation $\sim$ in $T \times \mathbb{C}^*$ is an equivalence relation. We have

(a). *The number of equivalence classes in $T \times \mathbb{C}^*$ respect to $\sim$ is finite.*

Proof. The set R of roots is a finite set. From the definition it is easy to see that the number is less than $2^{2|R|}$.

(b). *Assume that we have a surjective homomorphism $f: G \to G'$ such that* $\ker f$ *contains in the center of G, then $s \sim t$ if and only if $f(s) \sim f(t)$.*

Proof. Let $r \in T$. It is known that r contains in the center of G if and only if $\alpha(r) = 1$ for any $\alpha \in R$. The assertion then follows from the definition of $\sim$.

4.2. Proposition. *Assume that G has simply connected derived group. Let (s,q) and (t,r) be two semisimple elements in $T \times \mathbb{C}^*$ such that $(s,q) \sim (t,r)$. Then* (i). *For any* $N \in \mathcal{N}_{s,q} = \mathcal{N}_{t,r}$, *we have* $\mathcal{B}_N^s = \mathcal{B}_N^t$.

(ii). $Z^{s,q} = Z^{t,r}$ and $\mathcal{N}_{s,q} = \mathcal{N}_{t,r}$.

(iii). $A(s,N) = A(t,N)$ *and* $A(s,N)^\vee = A(t,N)^\vee$.

Proof. Since $C_G(s) = C_G(t)$, according to Steinberg (see [St1]) we have $\mathcal{B}^s = \mathcal{B}^t$. Thus (i) follows from $\mathcal{B}_N^s = \mathcal{B}^s \cap \mathcal{B}_N$, $\mathcal{B}_N^t = \mathcal{B}^t \cap \mathcal{B}_N$. (ii) follows from (i). The third assertion is obvious by definition and (ii).

4.3. Proposition. *Keep the set up and notations in 3.5-3.6. Assume that* $(s,q) \sim (t,r)$, *then*

(i). $\mathbf{H}_{s,q} \simeq \mathbf{H}_{t,r}$.

(ii). *For any* $\rho \in A(s,N)^\vee = A(t,N)^\vee$, *the standard module* $M_{s,N,q,\rho}$ *has a unique quotient if and only if* $M_{t,N,r,\rho}$ *has a unique quotient.*

(iii). *Let* $N' \in \mathcal{N}_{s,q} = \mathcal{N}_{t,r}$ *be another nilpotent element and* $\rho' \in A(s,N')^\vee$. *Assume the standard modules* $M_{s,N,q,\rho}$, $M_{t,N,r,\rho}$, $M_{s,N',q,\rho'}$, $M_{t,N',r,\rho'}$ *all possess a unique simple quotient module respectively, denote them by* $L_{s,N,q,\rho}$, $L_{t,N,r,\rho}$, $L_{s,N',q,\rho'}$, $L_{t,N',r,\rho'}$, *respectively. then* $L_{s,N,q,\rho} \simeq L_{s,N',q,\rho'}$ *if and only if* $L_{t,N,r,\rho} \simeq L_{t,N',r,\rho'}$.

Proof. The assertion (i) follows from 3.6(e) and 4.2 (ii). The other assertions follow from these facts: the definitions of standard modules, 3.6(d), (i) and 4.2.

The proposition is proved.

4.4. For simplicity in the rest of this chapter we shall assume that G is simply connected, simple except specified indications. Let

$$\mathbb{C}^*_{W_0} = \{q \in \mathbb{C}^* \mid \text{the order of } q > e_n + 1\}, \tag{4.4.1}$$

where e_n is the maximal exponent of W_0. We shall write $o(q)$ for the order of q.

4.5. Theorem. (i). *For any* $(s,q) \in T \times \mathbb{C}^*_{W_0}$ *and* $r \in \mathbb{C}^*_{W_0}$, *there exists* $t \in T$ *such that* $(s,q) \sim (t,r)$. *In particular we have*

(ii). *For any element* $(s,q) \in T \times \mathbb{C}^*_{W_0}$, *there exists* $(t,r) \in T \times \mathbb{C}^*_{W_0}$ *such that* r *is not a root of 1 and* $(s,q) \sim (t,r)$.

We shall prove the theorem case by case. Combining 4.3, 4.5 and 3.6(d) we get

4.6. Theorem. *Let G is simply connected, simple algebraic group. Assume that $o(q) > e_n + 1$, then*

(i). *Each standard module $M_{s,N,q,\rho}$ has a unique quotient module, denoted by $L_{s,N,q,\rho}$.*

(ii). *$L_{s,N,q,\rho}$, $L_{s',N',q,\rho'}$ are isomorphic if and only if (s, N, q, ρ), (s', N', q, ρ') are G-conjugate.*

(iii). *Any simple $\mathbf{H}_q$-module is isomorphic to some $L_{s,N,q,\rho}$.*

4.7. We call an element $(s, q) \in T \times \mathbb{C}^*$ is good if there exists an element $(t, r) \in T \times \mathbb{C}^*_{W_0}$ such that $(s, q) \sim (t, r)$, is semi-good if $(s, q) \sim (t, r)$ for some $(t, r) \in T \times \mathbb{C}^*$ with $f_{W_0}(r) \neq 0$, is bad if (s, q) is not semi-good, where

$$f_{W_0} = \sum_{w \in W_0} \mathbf{q}^{l(w)} \in \mathbf{A}.$$

Example: Suppose that G is simple and of rank n. Let $s \in T$ be such that $\alpha(s) = q$ for all $\alpha \in \Delta$ and assume that $o(q) = e_n$, the biggest exponent of W_0. If G is not of type A_n, then $e_n - 1$ is not an exponent, (s, q) is semi-good but not good.

We also prove the following result through case by case analysis. The result will be needed in next chapter.

4.8. Theorem. *Let $(s, q) \in T \times \mathbb{C}^*$, then (s, q) is bad if and only if $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$.*

4.9. Even if (s, q) is bad, the variety $\mathcal{N}_{s,q}$ is possible to be irreducible. For example, let $G = SL_3(\mathbb{C})$, and s =diag$(-1, 1 - 1) \in G$, then $(s, -1)$ is bad but $\mathcal{N}_{s,-1}$ is irreducible (cf. [KL4, 5.15] and 5.8 in next chapter).

We shall need a result of Lusztig in [L17], which can be verified directly when G is a classical group. For each $w \in W_0$, we choose an element $\dot{w} \in N_G(T)$ such that its image in W_0 is w. Note that $\mathbf{t}$ (the Lie algebra of T) is W_0 stable. Let $f_w(\mathbf{q}) = det(\mathbf{q} - w)$ be the eigenpolynomial of w on the space $\mathbf{t}$.

(a) Any element g in $\dot{w}T$ is semisimple.

Proof. Consider the adjoint representation Ad: $G \to GL(\mathbf{g})$. It is easy to check that the image Ad(g) of g is semisimple. It is known that the kernel of Ad is the center of G. So g is semisimple.

A result of Lusztig in [L17] can be expressed as

(b) Assume that $q \neq 1$, then $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ if and only if s is conjugate to some element in $\dot{w}T$ such that $f_w(q) = 0$.

We need several more notations: $\mathbf{g}^+$ stands for $\bigoplus_{\alpha \in R^+} \mathbf{g}_\alpha$, T_{reg} denote the set of regular elements in the maximal torus T. Now we begin our proof of 4.5 and 4.8 through case by case analysis.

4.10. Type A_n: We have $G = SL_{n+1}(\mathbb{C})$. We choose the maximal torus T to be the set of the diagonal matrices in G.

Let $\alpha_{ij}, x_k \in X = \mathrm{Hom}(T, \mathbb{C}^*)$, $1 \le i < j \le n+1, 1 \le k \le n$ be defined as follows.

$$\alpha_{ij} : \mathrm{diag}\ (a_1, a_2, \cdots, a_{n+1}) \to a_i a_j^{-1},$$

$$x_k : \mathrm{diag}\ (a_1, a_2, \cdots, a_{n+1}) \to a_1 a_2 \cdots a_k.$$

Then let $R^+ = \{\alpha_{ij} \mid 1 \le i < j \le n+1\}$. Thus the set of simple roots is $\Delta = \{\alpha_{i,i+1} \mid 1 \le i \le n\}$, and $x_1, x_2, ..., x_n$ are the fundamental weights.

The normalizer $N_G(T)$ of T in G is generated by $T, P_i(-1)P_{ij} \in G$ $(1 \le i \neq j \le n+1)$, where $P_i(-1)$ is the matrix obtained by multiplying the i-th row of the identity matrix I_{n+1} with -1, the P_{ij} is the matrix obtained by exchanging the i-th row and the j-th row of the matrix I_{n+1}. The Weyl group $W_0 = N_G(T)/T$ is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$ of degree $n+1$.

Given an element $(s, q) \in T \times \mathbb{C}^*, q \neq 1$. Obviously through an element of the Weyl group W_0, s is conjugate to certain element $D \in T$ of the following form:

$$D = \mathrm{diag}(D_1, D_2, ..., D_k), \tag{4.10.1}$$

where

$$D_i = \begin{pmatrix} d_i q^{m_i} I_{r_{i,i}} & 0 & \dots & 0 & 0 \\ 0 & d_i q^{m_i - 1} I_{r_{i,i-1}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_i q I_{r_{i,1}} & 0 \\ 0 & 0 & \dots & 0 & d_i I_{r_{i,0}} \end{pmatrix}, \qquad 1 \le i \le k,$$

all $r_{i,j}$ are positive integers, all m_i are non-negative integers, $d_i \in \mathbb{C}^*$, moreover, $\max\{m_1, m_2, ..., m_k\} < o(q)$, and

$$d_i q^m (d_j q^{m'})^{-1} \neq 1, q, q^{-1}, \quad \text{for any } 1 \leq i \neq j \leq k, \quad 0 \leq m \leq m_i, \quad 0 \leq m' \leq m_j.$$

Let (s, q) be as above. We have

(a). $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ *if and only if* $m_i + 1 = o(q)$ *for some* $1 \leq i \leq k$.

(b). *If* $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$, *then for arbitrary* $r \in \mathbb{C}^*$ *with* $o(r) - 1 > \max\{m_1, m_2, ..., m_k\}$, *we can find some* $t \in T$ *such that* $(s, q) \sim (t, r)$.

Proof. It is harmless to assume that $s = D$ (notations as above).

(a). If $o(q) - 1 > m_i$ for each $1 \leq i \leq k$, then we have $\mathbf{g}_{s,q} \subset \mathbf{g}^+$, so $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$. Suppose that $m_i + 1 = o(q)$ for some i, then the exist positive roots β_1, β_2, ..., β_m, $1 \leq m \leq m_i$, such that $\beta_1 + \cdots + \beta_j \in R^+$, $1 \leq j \leq m$ and

$$\mathbf{g}'_{s,q} = \mathbf{g}_{\beta_1} + \mathbf{g}_{\beta_2} + \cdots + \mathbf{g}_{\beta_m} + \mathbf{g}_{-\beta_1-\beta_2-\cdots-\beta_m} \in \mathbf{g}_{s,q}.$$

Note that $\mathbf{g}'_{s,q}$ contains semisimple elements, so $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$.

(b). Assume that $o(q) - 1 > \max\{m_1, m_2, ..., m_k\}$. Choose $a_i \in \mathbb{C}^*$, $1 \leq i \leq k$, be such that $a_i r^m (a_j r^{m'})^{-1} > \max\{1, |r|, |r|^{-1}\}$ for arbitrary $1 \leq i < j \leq k$, $1 \leq m \leq m_i$, $1 \leq m' \leq m_j$, and be such that $E = \operatorname{diag}(E_1, E_2, ..., E_k) \in G$, where

$$E_i = \begin{pmatrix} a_i r^{m_i} I_{r_{i,i}} & 0 & \dots & 0 & 0 \\ 0 & a_i r^{m_i - 1} I_{r_{i,i-1}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_i r I_{r_{i,1}} & 0 \\ 0 & 0 & \dots & 0 & a_i I_{r_{i,0}} \end{pmatrix}, \qquad 1 \leq i \leq k.$$

Then we have $(D, q) \sim (E, r)$.

The assertions (a-b) are proved.

(c). *Let* $s \in T$. *If* q *is a primitive* $(n+1)$*-th root of 1, we have*

(i). $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ *if and only if* s *is conjugate to the element*

$$diag(q^{\frac{n}{2}}, q^{\frac{n-3}{2}}, ..., q^{\frac{3-n}{2}}, q^{-\frac{n}{2}})$$

. (ii). *If* $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$, *then for any* $r \in \mathbb{C}^*_{W_0}$, *we can find* $t \in T$ *such that* $(s,q) \sim (t,r)$.

Proof. It follows from (a-b).

(d). *Let* $(s,q) \in T \times \mathbb{C}^*$ *with* $q \neq 1$. *Assume that* $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$, *then there exists a sequence* $t_1, t_2, ..., t_k, ...$ *in* T_{reg} *such that*

$$\mathbf{g}_{t_k,q} \neq \mathcal{N}_{t_k,q}, \quad \text{and} \quad \lim_{k\to\infty} t_k = s.$$

(In this paper all limits are respect to the complex topology.)

Proof. By the proof of (a) we see that s is conjugate to certain element $D = \mathrm{diag}(dq^m, dq^{m-1}, ..., dq, d, a_1, a_2, ..., a_{n-m}) \in T$, where $m+1$ is the order of q. Note that $q^{i-j} \neq 1$ for any $0 \leq i \neq j \leq m$. Choose positive numbers $b_1, b_2, ..., b_k, ...$, in the interval $(1, +\infty)$ such that $\lim_{k\to\infty} b_k = 1$ and such that

$$a_i b_k^{n-m-2i+1} a_j^{-1} b_k^{2j-1-n+m} \neq 1, \ \ 1 \leq i \neq j \leq n-m;$$

$$a_i b_k^{n-m-2i+1} d^{-1} q^{-l} \neq 1, \ \ 0 \leq l \leq m.$$

Let

$$t_k = \mathrm{diag}(dq^m, dq^{m-1}, ..., dq, d, a_1 b_k^{n-m-1}, a_2 b_k^{n-m-3}, ..., a_{n-m} b_k^{m-n+1}) \in T,$$

then the sequence $t_1, t_2, ..., t_k, ...$ satisfies our requirement. The assertion is proved.

4.11. Type B_n. Since $\alpha(s) = 1$ for all $\alpha \in R$ whenever s is in the center of G. So we will consider the special orthogonal group $SO_{2n+1}(\mathbb{C})$ instead of the spin group $Spin_{2n+1}(\mathbb{C})$. But the results are also valid for $Spin_{2n+1}(\mathbb{C})$ for the above reason (see also 4.1(b)).

The group

$$G = SO_{2n+1}(\mathbb{C}) = \{g \in SL_{2n+1}(\mathbb{C}) \mid \tilde{g} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix} g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix} \},$$

where $\tilde{g}$ is the transpose of g. We choose the maximal torus T to be the set of the diagonal matrices in G. Then

$$T = \{\mathrm{diag}(1, a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \mid a_1, a_2, ..., a_n \in \mathbb{C}^*\}.$$

The normalizer $N_G(T)$ of T in G is generated by $T; P_{ij}P_{i+n,j+n},\ P_{i,i+n} \in G\ (2 \leq i \neq j \leq n+1)$, where P_{ij} is the matrix obtained by exchanging the i-th row and the j-th row of the matrix I_{2n+1}. The Weyl group $W_0 = N_G(T)/T$ is isomorphic to the semi-direct product $(\mathbf{Z}/2\mathbf{Z})^n \ltimes \mathfrak{S}_n$.

Let $\alpha_{ij}, \beta_{ij}, \gamma_i \in X = \mathrm{Hom}(T, \mathbb{C}^*),\ 1 \leq i < j \leq n+1, 1 \leq k \leq n$ be defined as follows.

$$\alpha_{ij} : \mathrm{diag}\ (1, a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \to a_i a_j^{-1},$$

$$\beta_{ij} : \mathrm{diag}\ (1, a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \to a_i a_j,$$

$$\gamma_i : \mathrm{diag}\ (1, a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \to a_i.$$

Then let $R^+ = \{\alpha_{ij},\ \beta_{ij},\ \gamma_k |\ 1 \leq i < j \leq n,\ 1 \leq k \leq n\}$. Thus the set of simple roots is $\Delta = \{\alpha_{i,i+1},\ \gamma_n \mid 1 \leq i \leq n-1\}$.

Give an element $(s, q) \in T \times \mathbb{C}^*, q \neq 1$. Obviously through an element of the Weyl group W_0, s is conjugate to certain element $D \in T$ of the following form:

$$D = \mathrm{diag}(1, D_1, D_2, ..., D_k, D_1^{-1}, D_2^{-1}, ..., D_k^{-1}), \tag{4.11.1}$$

where

$$D_i = \begin{pmatrix} d_i q^{m_i} I_{r_{i,i}} & 0 & \dots & 0 & 0 \\ 0 & d_i q^{m_i - 1} I_{r_{i,i-1}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_i q I_{r_{i,1}} & 0 \\ 0 & 0 & \dots & 0 & d_i I_{r_{i,0}} \end{pmatrix}, \qquad 1 \leq i \leq k,$$

all $r_{i,j}$ are positive integers, all m_i are non-negative integers, $d_i \in \mathbb{C}^*$, moreover $\max\{m_1, m_2, ..., m_k\} < o(q)$, and

$$d_i q^m (d_j q^{m'})^{\pm 1} \neq 1, q, q^{-1}, \text{ for any } 1 \leq i \neq j \leq k\ \ 0 \leq m \leq m_i,\ 0 \leq m' \leq m_j.$$

(a). *$\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ if and only if at least one of the following conditions is satisfied.*

(i). *There is some i such that $m_i + 1 = o(q)$.*

(ii). *$o(q)$ is even and there are some i, m $(0 \leq m \leq m_i)$ such that $d_i q^m = q$ and $2m_i - 2m + 2 \geq o(q)$.*

Proof. We may prove the assertions as the case of type A_n.

(b). *If* $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$, *then there exists a sequence* $t_1, t_2, ..., t_k, ...$ *in* T_{reg} *such that*

$$\mathbf{g}_{t_k,q} \neq \mathcal{N}_{t_k,q}, \quad \text{and} \quad \lim_{k\to\infty} t_k = s.$$

Proof. Assume that $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$. By (a) we see that s is conjugate to certain element $D = \mathrm{diag}(1, D_1, D_2, D_1^{-1}, D_2^{-1}) \in T$, such that

$$D_1 = \mathrm{diag}(dq^m, dq^{m-1}, ..., dq, d), \qquad \text{for some } d \in \mathbb{C}^* \text{ if } o(q) = m+1 \leq n,$$

or

$$D_1 = \mathrm{diag}(q^m, q^{m-1}, ..., q), \qquad \text{if } o(q) = 2m \leq 2n,$$

$$D_2 = \mathrm{diag}(a_1, a_2, ..., a_k) \in GL_k(\mathbb{C}) \qquad \text{for some } k \in \mathbf{N}.$$

We then can prove (b) as the case of type A_n.

(c). *Let* D *be as (4.11.1), then the following two conditions are equivalent.*

(i). $\mathbf{g}_{D,q} = \mathcal{N}_{D,q}$ *but* $\mathcal{N}_{D,q} \not\subset \mathbf{g}^+$.

(ii). $o(q) = 2n' - 1$ *for some* n', $\frac{n}{2} < n' \leq n$, *and there exists some* i *and* m $(0 \leq m < m_i)$ *such that* $d_i q^m = q$, $2m_i - 2m + 2 > o(q)$.

The proof is straight.

(d). (i). *If* $o(q) > 2n$, , *then we have* $\mathbf{g}_{D,q} = \mathcal{N}_{D,q} \subset \mathbf{g}^+$.

(ii). *Assume that* $\mathbf{g}_{D,q} = \mathcal{N}_{D,q} \subset \mathbf{g}^+$, *then for any* $r \in \mathbb{C}^*_{W_0}$, *there exists* $E \in T$ *such that* $(D,q) \sim (E,r)$.

Part (i) is trivial. The proof of part (ii) is similar to type A_n although a little more care is needed.

(e). *Let* $s \in T$. *Assume that* q *is a primitive* $2n$*-th root of 1, then*

(i). $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ *if and only if* (s,q) *is conjugate to the following element*

$$D = \mathrm{diag}(1, q^n, q^{n-1}, \cdots, q^2, q, q^{-n}, q^{1-n}, \cdots, q^{-2}, q^{-1}).$$

(ii). *If* $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$, *then for any* $r \in \mathbb{C}^*_{W_0}$, *we can find* $t \in T$ *such that* $(s,q) \sim (t,r)$.

Proof. It follows from (a) and (d).

4.12. Type C_n. We consider

$$G = Sp_{2n}(\mathbb{C}) = \{g \in SL_{2n}(\mathbb{C}) \mid \tilde{g}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\},$$

where $\tilde{g}$ is the transpose of g. We choose the maximal torus T to be the set of the diagonal matrices in G. Then

$$T = \{\mathrm{diag}(a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \mid a_1, a_2, ..., a_n \in \mathbb{C}^*\}.$$

The normalizer $N_G(T)$ of T in G is generated by $T; P_{ij}P_{i+n,j+n}, P_{i,i+n} \in G$ $(1 \leq i \neq j \leq n)$, where P_{ij} is the matrix obtained by exchanging the i-th row and the j-th row of the matrix I_{2n}. The Weyl group $W_0 = N_G(T)/T$ is isomorphic to the semi-direct product $(\mathbf{Z}/2\mathbf{Z})^n \ltimes \mathfrak{S}_n$.

Let $\alpha_{ij}, \beta_{ij}, \gamma_i \in X = \mathrm{Hom}(T, \mathbb{C}^*)$, $1 \leq i < j \leq n+1, 1 \leq k \leq n$ be dfeined as follows.

$$\alpha_{ij} : \mathrm{diag}\ (a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \to a_i a_j^{-1},$$

$$\beta_{ij} : \mathrm{diag}\ (a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \to a_i a_j,$$

$$\gamma_i : \mathrm{diag}\ (1, a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \to a_i^2.$$

Then let $R^+ = \{\alpha_{ij},\ \beta_{ij},\ \gamma_k \mid 1 \leq i < j \leq n,\ 1 \leq k \leq n\}$. Thus the set of simple roots is $\Delta = \{\alpha_{i,i+1},\ \gamma_n \mid 1 \leq i \leq n-1\}$.

Give an element $(s, q) \in T \times \mathbb{C}^*, q \neq 1$. Obviously through an element of the Weyl group W_0, s is conjugate to certain element $D \in T$ of the following form:

$$D = \mathrm{diag}(D_1, D_2, ..., D_k, D_1^{-1}, D_2^{-1}, ..., D_k^{-1}), \tag{4.12.1}$$

where

$$D_i = \begin{pmatrix} d_i q^{m_i} I_{r_{i,i}} & 0 & \dots & 0 & 0 \\ 0 & d_i q^{m_i - 1} I_{r_{i,i-1}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_i q I_{r_{i,1}} & 0 \\ 0 & 0 & \dots & 0 & d_i I_{r_{i,0}} \end{pmatrix}, \qquad 1 \leq i \leq k,$$

all $r_{i,j}$ are positive integers, all m_i are non-negative integers, $d_i \in \mathbb{C}^*$, moreover $\max\{m_1, m_2, ..., m_k\} < o(q)$, and

$$d_i q^m (d_j q^{m'})^{\pm 1} \neq 1, q, q^{-1}, \quad \text{for any } \ 1 \leq i \neq j \leq k, 0 \leq m \leq m_i, \ 0 \leq m' \leq m_j.$$

(a). *$\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ if and only if at least one of the following conditions is satisfied.*

(i). *There is some i such that $m_i + 1 = o(q)$.*

(ii). *$o(q)$ is even and there are some i, m ($0 \leq m \leq m_i$) such that $d_i q^m = q^{\frac{1}{2}}$ and $2m_i - 2m + 2 \geq o(q)$.*

Proof. We may prove the lemma as the case of type A_n.

(b). *If $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$, then there exists a sequence $t_1, t_2, ..., t_k, ...$ in T_{reg} such that*

$$\mathbf{g}_{t_k,q} \neq \mathcal{N}_{t_k,q}, \quad \text{and} \quad \lim_{k\to\infty} t_k = s.$$

Proof. Assume that $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$. By (a) we see that s is conjugate to certain element $D = \mathrm{diag}(1, D_1, D_2, D_1^{-1}, D_2^{-1}) \in T$, such that

$$D_1 = \mathrm{diag}(dq^m, dq^{m-1}, ..., dq, d), \qquad \text{for some } d \in \mathbb{C}^* \text{ if } o(q) = m + 1 \leq n,$$

$$D_1 = \mathrm{diag}(q^{\frac{2m-1}{2}}, q^{\frac{2m-3}{2}}, ..., q^{\frac{3}{2}}, q^{\frac{1}{2}}), \qquad \text{if } o(q) = 2m \leq 2n,$$

$$D_2 = \mathrm{diag}(a_1, a_2, ..., a_k) \in GL_k(\mathbb{C}) \qquad \text{for some } k \in \mathbf{N}.$$

We then can prove (b) as the case of type A_n.

(c). *Let D be as (4.12.1), then the following two conditions are equivalent.*

(i). *$\mathbf{g}_{D,q} = \mathcal{N}_{D,q}$ but $\mathcal{N}_{D,q} \not\subset \mathbf{g}^+$.*

(ii). *$o(q) = 2n' - 1$ for some n', $\frac{n}{2} < n' \leq n$, and there exists some i and m ($0 \leq m \leq m_i$) such that $d_i q^m = q^{\frac{1}{2}}$, $2m_i - 2m + 2 > o(q)$.*

The proof is straight.

(d). (i). *If $o(q) > 2n$, then we have $\mathbf{g}_{D,q} = \mathcal{N}_{D,q} \subset \mathbf{g}^+$.*

(ii). *Assume that* $\mathbf{g}_{D,q} = \mathcal{N}_{D,q} \subset \mathbf{g}^+$, *then for any* $r \in \mathbb{C}^*_{W_0}$, *there exists* $E \in T$ *such that* $(D,q) \sim (E,r)$.

Part (i) is trivial. Part (ii) is similar to case of type A_n.

(e). *Let* $s \in T$. *Assume that* q *is a primitive* $2n$*-th root of 1, then*

(i). $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ *if and only if* (s,q) *is conjugate to the following element*

$$D = \mathrm{diag}(q^{\frac{2n-1}{2}}, q^{\frac{2n-3}{2}}, \cdots, q^{\frac{3}{2}}, q^{\frac{1}{2}}, q^{\frac{1-2n}{2}}, q^{\frac{3-2n}{2}}, \cdots, q^{-\frac{3}{2}}, q^{-\frac{1}{2}}).$$

(ii). *If* $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$, *then for any* $r \in \mathbb{C}^*_{W_0}$, *we can find* $t \in T$ *such that* $(s,q) \sim (t,r)$.

Proof. It follows from (a) and (d).

4.13. Type D_n. We consider

$$G = SO_{2n}(\mathbb{C}) = \{g \in SL_{2n}(\mathbb{C}) \mid \tilde{g}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\},$$

where $\tilde{g}$ is the transpose of g. We choose the maximal torus T to be the set of the diagonal matrices in G. Then

$$T = \{\mathrm{diag}(a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \mid a_1, a_2, ..., a_n \in \mathbb{C}^*\}.$$

The normalizer $N_G(T)$ of T in G is generated by T; $P_{ij}P_{i+n,j+n}, P_{i,i+n}P_{j,j+n} \in G$ $(1 \leq i \neq j \leq n)$ where P_{ij} is the matrix obtained by exchanging the i-th row and the j-th row of the matrix I_{2n}. The Weyl group $W_0 = N_G(T)/T$ is isomorphic to the semi-direct product $(\mathbf{Z}/2\mathbf{Z})^{n-1} \ltimes \mathfrak{S}_n$.

Let $\alpha_{ij}, \beta_{ij}, \gamma_i \in X = \mathrm{Hom}(T, \mathbb{C}^*)$, $1 \leq i < j \leq n+1, 1 \leq k \leq n$ be dfeined as follows.

$$\alpha_{ij} : \mathrm{diag}\ (a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \to a_i a_j^{-1},$$

$$\beta_{ij} : \mathrm{diag}\ (a_1, a_2, ..., a_n, a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \to a_i a_j.$$

Then let $R^+ = \{\alpha_{ij},\ \beta_{ij} \mid 1 \leq i < j \leq n,\ 1 \leq k \leq n\}$. Thus the set of simple roots is $\Delta = \{\alpha_{i,i+1},\ \beta_{n-1,n} \mid 1 \leq i \leq n-1\}$.

Give an element $(s,q) \in T \times \mathbb{C}^*, q \neq 1$. Obviously through an element of the Weyl group W_0, s is conjugate to certain element $D \in T$ of the following form:

$$D = \mathrm{diag}(D_1, D_2, ..., D_k, D_1^{-1}, D_2^{-1}, ..., D_k^{-1}), \tag{4.13.1}$$

where

$$D_i = \begin{pmatrix} d_i q^{m_i} I_{r_{i,i}} & 0 & \dots & 0 & 0 \\ 0 & d_i q^{m_i - 1} I_{r_{i,i-1}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_i q I_{r_{i,1}} & 0 \\ 0 & 0 & \dots & 0 & d_i I_{r_{i,0}} \end{pmatrix}, \qquad 1 \leq i \leq k,$$

all $r_{i,j}$ are positive integers, all m_i are non-negative integers, $d_i \in \mathbb{C}^*$, moreover $\max\{m_1, m_2, ..., m_k\} < o(q)$ and

$$d_i q^m (d_j q^{m'})^{\pm 1} \neq 1, q, q^{-1}, \quad \text{for any } 1 \leq i \neq j \leq k,\ 0 \leq m \leq m_i,\ 0 \leq m' \leq m_j. \tag{4.13.2}$$

or

$$d_i q^m (d_j q^{m'})^{\pm 1} \neq 1, q, q^{-1}, \quad \text{if } i \neq j, \text{ and } i \notin \{k-1, k\} \text{ or } j \notin \{k-1, k\}, \tag{4.13.3}$$

$0 \leq m \leq m_i$, $0 \leq m' \leq m_j$, and $D_k = (d_k)$, $d_{k-1} q^m d_k^{-1} \neq 1, q$ for all $0 \leq m \leq m_{k-1}$, $d_{k-1} q^l d_k = q$ for some $0 \leq l \leq m_{k-1}$. We also require that m_{k-1} is as big as possible.

(a). *$\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ if and only if at least one of the following conditions is satisfied.*

(i).. *There is some i such that $m_i + 1 = o(q)$.*

(ii). *$o(q)$ is even and there are some i, m $(0 \leq m \leq m_i)$ such that $d_i q^m = 1$ and $2m_i - 2m + 2 \geq o(q)$.*

Proof. We may prove the lemma as the case of type A_n. The results in [C1] and 4.9(b) are helpful in the proof.

(b). *If $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$, then there exists a sequence $t_1, t_2, ..., t_k, ...$ in T_{reg} such that*

$$\mathbf{g}_{t_k,q} \neq \mathcal{N}_{t_k,q}, \quad \text{and} \quad \lim_{k \to \infty} t_k = s.$$

Proof. Assume that $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$. By (d1) we see that s is conjugate to certain element $D = \mathrm{diag}(D_1, D_2, D_1^{-1}, D_2^{-1}) \in T$, such that

$$D_1 = \mathrm{diag}(dq^m, dq^{m-1}, ..., dq, d), \qquad \text{for some } d \in \mathbb{C}^* \text{ if } o(q) = m+1 \leq n,$$

$$D_1 = \mathrm{diag}(q^m, q^{m-1}, ..., q, 1), \qquad \text{if } o(q) = 2m-2 < 2n,$$

$$D_2 = \mathrm{diag}(a_1, a_2, ..., a_k) \in GL_k(\mathbb{C}) \qquad \text{for some } k \in \mathbf{N}.$$

We then can prove (b) as the case of type A_n.

(c). *Let D be as (4.14.1), then the following two conditions are equivalent.*

(i). $\mathbf{g}_{D,q} = \mathcal{N}_{D,q}$ *but* $\mathcal{N}_{D,q} \not\subset \mathbf{g}^+$.

(ii). $o(q) = 2n' - 1$ *for some* n', $\frac{n}{2} < n' \leq n-1$, *and there exists some* i *and* m $(0 \leq m < m_i)$ *such that* $d_i q^m = 1$, $2m_i - 2m + 2 > o(q)$.

The proof is straight.

(d). (i). *If* $o(q) > 2n-2$, , *then we have* $\mathbf{g}_{D,q} = \mathcal{N}_{D,q} \subset \mathbf{g}^+$.

(ii). *Assume that* $\mathbf{g}_{D,q} = \mathcal{N}_{D,q} \subset \mathbf{g}^+$, *then for any* $r \in \mathbb{C}^*_{W_0}$, *there exists* $E \in T$ *such that* $(D,q) \sim (E,r)$.

Part (i) is trivial. Part (ii) is similar to case of type A_n but more tedious.

(e). *Let* $s \in T$. *Assume that* q *is a primitive* $(2n-2)$*-th root of 1, then*

(i). $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ *if and only if* s *is conjugate to the following element*

$$\mathrm{diag}(q^{n-1}, q^{n-2}, ..., q, 1, q^{1-n}, q^{2-n}, ..., q^{-1}, 1).$$

Proof. It follows from (d1).

(ii). *If* $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$, *then for any* $r \in \mathbb{C}^*_{W_0}$, *we can find* $t \in T$ *such that* $(s,q) \sim (t,r)$.

Proof. One can prove the assertions using (a) and (d).

4.14. Exceptioal types. Let G be a simple algebraic group of adjoint type, then

(4.14.1) $T \simeq \mathrm{Hom}(P, \mathbb{C}^*)$, where T is a maximal torus in G and $P = \mathrm{Hom}(T, \mathbb{C}^*)$ is the character group of T. Note that P is also the root lattice.

There are no simple realizations for algebraic groups of exceptional types, so for us the property (4.14.1) is important. To use it we need explicit structure of the root systems of exceptional types. We adopt the approach in [OV]. For type F_4, the approach is the same as in [B].

Type E_6: Let $\varepsilon_1, \varepsilon_2, ..., \varepsilon_6$ be vectors in $\mathbf{R}^6$ satisfying $\sum \varepsilon_i = 0$ and

$$(\varepsilon_i, \varepsilon_i) = 5/6, \qquad (\varepsilon_i, \varepsilon_j) = -1/6 \quad \text{for} \quad i \neq j.$$

Let $\varepsilon \in \mathbf{R}^6$ be such that $(\varepsilon, \varepsilon_i) = 0$ for all i and $(\varepsilon, \varepsilon) = 1/2$.

Type E_7, E_8, G_2: Let $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n+1}$ ($n =$ rank) be vectors in $\mathbf{R}^{n+1}$ satisfying $\sum \varepsilon_i = 0$ and

$$(\varepsilon_i, \varepsilon_i) = n/(n+1), \qquad (\varepsilon_i, \varepsilon_j) = -1/(n+1) \quad \text{for} \quad i \neq j.$$

Type F_4: Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ be an orthonormal basis of $\mathbf{R}^4$.

Then we have

(a) Type E_6. The roots are: $\varepsilon_i - \varepsilon_j$, $\pm 2\varepsilon$, $\varepsilon_i + \varepsilon_j + \varepsilon_k \pm \varepsilon$. We choose $\varepsilon_i - \varepsilon_{i+1}$, $\varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon$ as the set of simple roots.

(b) Type E_7. The roots are $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l$. We choose $\varepsilon_i - \varepsilon_{i+1}$ $(i < 7)$, $\varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8$ as the set of simple roots.

(c) Type E_8. The roots are $\varepsilon_i - \varepsilon_j$, $\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$. We choose $\varepsilon_i - \varepsilon_{i+1}$ $(i < 8)$, $\varepsilon_6 + \varepsilon_7 + \varepsilon_8$ as the set of simple roots.

(d) Type F_4. The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $\pm\varepsilon_i$, $(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2$. We choose $(\varepsilon_1 - \varepsilon_2 - \varepsilon - 3 - \varepsilon_4)/2$, ε_4, $\varepsilon_3 - \varepsilon_4$, $\varepsilon_2 - \varepsilon_3$ as the set of simple roots.

(e) Type G_2. The root are: $\varepsilon_i - \varepsilon_j$, $\pm\varepsilon_i$. We choose ε_2, $\varepsilon_3 - \varepsilon_2$ as the set of simple roots.

It is convenient to present the formulas for f_{W_0}. We have

Type	f_{W_0}
E_8	$\dfrac{(\mathbf{q}^{30}-1)(\mathbf{q}^{24}-1)(\mathbf{q}^{20}-1)(\mathbf{q}^{18}-1)(\mathbf{q}^{14}-1)(\mathbf{q}^{12}-1)(\mathbf{q}^{8}-1)(\mathbf{q}^{2}-1)}{(\mathbf{q}-1)^8}$
E_7	$\dfrac{(\mathbf{q}^{18}-1)(\mathbf{q}^{14}-1)(\mathbf{q}^{12}-1)(\mathbf{q}^{10}-1)(\mathbf{q}^{8}-1)(\mathbf{q}^{6}-1)(\mathbf{q}^{2}-1)}{(\mathbf{q}-1)^7}$
E_6	$\dfrac{(\mathbf{q}^{12}-1)(\mathbf{q}^{9}-1)(\mathbf{q}^{8}-1)(\mathbf{q}^{6}-1)(\mathbf{q}^{5}-1)(\mathbf{q}^{2}-1)}{(\mathbf{q}-1)^6}$
F_4	$\dfrac{(\mathbf{q}^{12}-1)(\mathbf{q}^{8}-1)(\mathbf{q}^{6}-1)(\mathbf{q}^{2}-1)}{(\mathbf{q}-1)^4}$
G_2	$\dfrac{(\mathbf{q}^{6}-1)(\mathbf{q}^{2}-1)}{(\mathbf{q}-1)^2}$

Now using (4.14.1), (a-e) and 4.9(b) we can prove the following results through lengthy case by case analysis.

Assume that G is a simple algebraic group of exceptional type. We have

(f). *If* $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$, *then*

(i). *There exists a sequence* $t_1, t_2, ..., t_k, ...$ *in* T_{reg} *such that*

$$\mathbf{g}_{t_k,q} \neq \mathcal{N}_{t_k,q}, \quad \text{and} \quad \lim_{k\to\infty} t_k = s.$$

or

(ii). *There exists a sequence* $t_1, t_2, ..., t_k, ...$ *in* T_{reg} *such that for any* $w \in W_0$ *we have*

$$\lim_{k\to\infty} \prod_{\alpha\in R+} \frac{1-qw(\alpha)(t_k)}{1-w(\alpha)(t_k)} = 0$$

and $\lim_{k\to\infty} t_k = s$.

Proof. We shall freely use the results on the conjugacy classes in Weyl groups in [C1]. We number the simple roots according to the Coxeter graphs in 1.3. By means of the adjoint representations and using 4.9(b) and the results in [C1] we see that $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ $(s \in T)$ if and only if s is conjugate to an element $t \in T$ satisfys

Type E_8:

$$\alpha_8(t) = -1, \qquad q = -1$$

$$\alpha_8(t) = \alpha_7(t) = q, \qquad o(q) = 3$$

$$\alpha_8(t) = \alpha_7(t) = \alpha_6(t) = q, \qquad o(q) = 4$$

$$\alpha_i(t) = q, \quad i = 5, 6, 7, 8 \qquad o(q) = 5$$

$$\alpha_i(t) = q, \quad 4 \le i \le 8 \qquad o(q) = 6$$

$$\alpha_i(t) = q, \quad 2 \le i \le 5 \qquad o(q) = 6$$

$$\alpha_i(t) = q, \quad 3 \le i \le 8 \qquad o(q) = 7$$

$$\alpha_i(t) = q, \quad 1 \le i \le 5 \qquad o(q) = 8$$

$$\alpha_i(t) = q, \quad 3 \le i \le 8 \text{ or } i = 1 \qquad o(q) = 8$$

$$\alpha_i(t) = q, \quad 3 \le i \le 8 \text{ or } i = 1 \text{ and } \alpha_2(t) = q^3 \qquad o(q) = 9$$

$$\alpha_i(t) = q, \quad 1 \le i \le 6 \qquad o(q) = 9$$

$$\alpha_i(t) = q, \quad 2 \le i \le 7 \qquad o(q) = 10$$

$$\alpha_i(t) = q, \quad 1 \le i \le 6 \qquad o(q) = 12$$

$$\alpha_i(t) = q, \quad 2 \le i \le 8 \qquad o(q) = 12$$

$$\alpha_i(t) = q, \quad 1 \le i \le 7 \qquad o(q) = 14$$

$$\alpha_i(t) = q, \quad 2 \le i \le 8 \qquad o(q) = 14$$

$$\text{and } \alpha_1(t) = \pm 1 \qquad o(q) = 14$$

$$\alpha_i(t) = q, \quad 1 \le i \le 7 \qquad o(q) = 18$$

$$\alpha_i(t) = q, \quad 1 \le i \le 8 \qquad o(q) = 20$$

$$\alpha_i(t) = q, \quad 1 \le i \le 8 \qquad o(q) = 24$$

$$\alpha_i(t) = q, \quad 1 \le i \le 8 \qquad o(q) = 30$$

Type E_7:

$$\alpha_7(t) = -1, \qquad q = -1$$

$$\alpha_7(t) = \alpha_6(t) = q, \qquad o(q) = 3$$

$$\alpha_7(t) = \alpha_6(t) = \alpha_5(t) = q, \qquad o(q) = 4$$

$$\alpha_i(t) = q, \quad i = 4, 5, 6, 7 \qquad o(q) = 5$$

$$\alpha_i(t) = q, \quad 3 \le i \le 7 \qquad o(q) = 6$$

$$\alpha_i(t) = q, \quad 2 \le i \le 5 \qquad o(q) = 6$$

$$\alpha_i(t) = q, \quad 3 \le i \le 7 \text{ or } i = 1 \qquad o(q) = 7$$

$$\alpha_i(t) = q, \quad 3 \le i \le 7 \text{ or } i = 1 \text{ and } \alpha_2(t) = 1 \qquad o(q) = 8$$

$$\alpha_i(t) = q, \quad 1 \le i \le 5 \qquad o(q) = 8$$

$$\alpha_i(t) = q, \quad 1 \le i \le 6 \qquad o(q) = 9$$

$$\alpha_i(t) = q, \quad 2 \le i \le 7 \qquad o(q) = 10$$

$$\alpha_i(t) = q, \quad 1 \le i \le 6 \qquad o(q) = 12$$

$$\alpha_i(t) = q, \quad 1 \le i \le 7 \qquad o(q) = 14$$

$$\alpha_i(t) = q, \quad 1 \le i \le 7 \qquad o(q) = 18$$

Type E_6:

$$\alpha_6(t) = -1, \qquad q = -1$$

$$\alpha_6(t) = \alpha_5(t) = q, \qquad o(q) = 3$$

$$\alpha_6(t) = \alpha_5(t) = \alpha_4(t) = q, \qquad o(q) = 4$$

$$\alpha_i(t) = q, \quad i = 3, 4, 5, 6 \qquad o(q) = 5$$

$$\alpha_i(t) = q, \quad 3 \le i \le 6 \text{ or } i = 1 \qquad o(q) = 6$$

$$\alpha_i(t) = q, \quad 2 \le i \le 5 \qquad o(q) = 6$$

$$\alpha_i(t) = q, \quad 1 \le i \le 5 \qquad o(q) = 8$$

$$\alpha_i(t) = q, \quad 1 \le i \le 6 \qquad o(q) = 9$$

$$\alpha_i(t) = q, \quad 1 \le i \le 6 \qquad o(q) = 12$$

Type F_4:

$$
\begin{array}{rl}
\alpha_1(t) = -1 \text{ or } \alpha_4 = -1, & q = -1 \\
\alpha_1(t) = \alpha_2(t) = q \text{ or } \alpha_3(t) = \alpha_4(t) = q, & o(q) = 3 \\
\alpha_2(t) = \alpha_3(t) = q, & o(q) = 4 \\
\alpha_1(t) = \alpha_2(t) = q,\ \alpha_3(t)\alpha_4(t) = \pm 1, & o(q) = 4 \\
\alpha_i(t) = q,\ \ i = 3, 4, 5, 6 & o(q) = 5 \\
\alpha_i(t) = q,\ \ 1 \leq i \leq 3 \text{ or } i = 1 & o(q) = 6 \\
\alpha_i(t) = q,\ \ 2 \leq i \leq 4 & o(q) = 6 \\
\alpha_i(t) = q,\ \ 1 \leq i \leq 4 & o(q) = 8 \\
\alpha_i(t) = q,\ \ 1 \leq i \leq 4 & o(q) = 12
\end{array}
$$

Type G_2:

$$
\begin{array}{rl}
\alpha_1(t) = -1 \text{ or } \alpha_2(t) = -1, & q = -1 \\
\alpha_1(t) = \alpha_2(q) = q, & o(q) = 8 \\
\alpha_1(t) = \alpha_2(t) = q, & o(q) = 12
\end{array}
$$

It is sufficient to prove (f) for those $t \in T$ satisfying the conditions in the tables. We use type G_2 as an example to prove it. We identify an element $r \in T$ with the pair $(\alpha_1(r), \alpha_2(r))$.

If $q = -1$ and $\alpha_1(t) = -1$, we choose a sequence $a_1, a_2, ..., a_k, ...$ of real positive numbers such that $\lim_{k\to\infty} a_k = 1$ and such that all $t_k = (-1, a_k\alpha_2(t)) \in T_{reg}$ (it is possible by a simple calculation). Then $\mathbf{g}_{t_k,-1} \neq \mathcal{N}_{t_k,-1}$ and $\lim_{k\to\infty} t_k = t$. Similarly we deal with the case $\alpha_2(t) = -1$.

If $o(q) = 3$, then $\alpha_1(t) = \alpha_2(t) = q$. we choose a sequence $a_1, a_2, ..., a_k, ...$ of real positive numbers in the open interval (0,1) such that $\lim_{k\to\infty} a_k = 1$. Then all $t_k = (a_k q, a_k q) \in T_{reg}$ and $\lim_{k\to\infty} t_k = t$. For arbitrary $w \in W_0$, one may check that

$$\lim_{k\to\infty} \prod_{\alpha\in R^+} \frac{1 - qw(\alpha)(t_k)}{1 - w(\alpha)(t_k)} = 0.$$

If $o(q) = 6$, then $\alpha_1(t) = \alpha_2(t) = q$. The element t is regular and for any $w \in W_0$ we have

$$\prod_{\alpha \in R^+} \frac{1 - qw(\alpha)(t)}{1 - w(\alpha)(t)} = 0.$$

We can deal with other types in a similar way. This complete the proof.

(g). (i). *If $o(q) > e_n + 1$, , then we have $\mathbf{g}_{w(s),q} = \mathcal{N}_{w(s),q} \subset \mathbf{g}^+$ for some $w \in W_0$.*

(ii). *Assume that $\mathbf{g}_{s,q} = \mathcal{N}_{s,q} \subset \mathbf{g}^+$, then for any $r \in \mathbb{C}^*_{W_0}$, there exists $t \in T$ such that $(s, q) \sim (t, r)$.*

Proof. (i). It is equivalent to prove that $w^{-1}(R_{s,q}) \subset R^+$ for some $w \in W_0$. Let $R^+_{s,q} = R_{s,q} \cap R^+$, $R^-_{s,q} = R_{s,q} \cap R^-$. If $R_{s,q} = R^+_{s,q}$, nothing need to argue since we can choose $w = e$, the unit in W_0. Now assume that $R_{s,q} \neq R^+_{s,q}$. Note that $o(q) > e_n + 1$, we see that the subgroup of the root lattice P generated by $R^+_{s,q}$ doesnot contain any element of $R^-_{s,q}$. Choose $\beta \in R^-_{s,q}$, let w_1 be the reflection respect to β. Then $w_1(\beta) \in R^+$, $w_1(R^+_{s,q}) \subset R^+$. Thus $|w_1(R^+_{s,q})| = |R^+_{w_1(s),q}| > |R^+_{s,q}|$. We now can use induction on $|R^-_{s,q}|$ since $R_{w_1(s),q} = R_{s,q}$.

(ii). We can prove the assertion case by case. We omit the tedious proof.

(h). *Let $s \in T$. Assume that q is a primitive $(e_n + 1)$-th root of 1, then*

(i). *$\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ if and only if s is conjugate to the an element t such that $\alpha(t) = q$ for any simple root α.*

(ii). *If $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$, then for any $r \in \mathbb{C}^*_{W_0}$, we can find $t \in T$ such that $(s, q) \sim (t, r)$.*

Proof. (i). It follows from the proof of part (i) of (f). Using the proof of (f) and the proof of part (i) of (g) we see that if $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$, then we can find $w \in W_0$ such that $\mathbf{g}_{w(s),q} = \mathcal{N}_{w(s),q} \subset \mathbf{g}^+$. Then (ii) can be proved case by case.

(j). *Let $s \in T$. If $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$, then we can find $(t, r) \in T \times \mathbb{C}^*$ such that $f_{W_0}(r) \neq 0$ and $(s, q) \sim (t, r)$.*

Proof. Use the table and case by case analysis. We omit the details.

4.15. Now we can see that 4.6 and 4.8 follow from the results in 4.10-4.14.

It would be interesting to find a necessary and sufficient condition for the natural isomorphism between $\mathbf{H}_{s,q}$ and $\mathbf{H}_{t,r}$.

4.15. Conjecture. *Assume that G has a simply connected derived group, then $\mathbf{H}_{s,q} \simeq \mathbf{H}_{t,r}$ if $\mathbf{g}_{s,q} = \mathbf{g}_{t,r}$.*

5. The Lowest Two-Sided Cell

Notations are as in chapter 3. It is known that $\mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}\times\mathcal{B})$ may be regarded as an ideal of the algebra $\mathbf{K}_{G\times\mathbf{C}^*}(Z)$. In this chapter we will give an explicit description for the ideal. Another purpose is to show that 3.6(d) is not true when $f_{W_0}(q)=0$. For simplicity we assume that G is simply connected, simple algebraic group. All these are done by using the knowledge concerned with the lowest cell

$$c_0=\{w\in W\,|\,a(w)=l(w_0)\}$$

The two-sided cell c_0 corresponds to the nilpotent G-orbit $\{0\}$ under Lusztig's bijection between the set Cell(W) of two-sided cell of W and the set of nilpotent G-orbits in $\mathbf{g}$.

5.1. The ideal $\mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}\times\mathcal{B})$ of $\mathbf{K}_{G\times\mathbf{C}^*}(Z)$. For arbitrary nilpotent G-orbit $\mathcal{C}$, let

$$Z_{\bar{\mathcal{C}}}=\{(N,\mathbf{b},\mathbf{b}')\in Z\mid N\in\bar{\mathcal{C}}\},$$

where $\bar{\mathcal{C}}$ is the closure of $\mathcal{C}$. The variety $Z_{\bar{\mathcal{C}}}$ is $G\times\mathbb{C}^*$-stable. It is known that the inclusion $Z_{\bar{\mathcal{C}}}\hookrightarrow Z$ induces an injection

$$\mathbf{K}_{G\times\mathbf{C}^*}(Z_{\bar{\mathcal{C}}})\hookrightarrow\mathbf{K}_{G\times\mathbf{C}^*}(Z)$$

and the image is an ideal (we denote it again by $\mathbf{K}_{G\times\mathbf{C}^*}(Z_{\overline{\mathcal{C}}})$) of the convolution algebra $\mathbf{K}_{G\times\mathbf{C}^*}(Z)$ (see [KL4]). It is conjectured the ideal is closely related to the two-sided cell corresponding to the nilpotent G-orbit $\mathcal{C}$ (see [Du, p.32; G4]). We shall give an explicit description to the ideal when $\mathcal{C}$ is the class $\{0\}$ (see Theorem 5.4).

We shall identify $\mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B})$ wiht $\mathbf{A}[X]$. Let $\mathbf{A}[X]^{W_0}$ be the W_0-invariant set of $\mathbf{A}[X]$. It is known that $\mathbf{A}[X]^{W_0}=\mathbf{R}_{G\times\mathbf{C}^*}=\mathbf{A}\otimes_{\mathbf{C}}\mathbf{R}_G$. We have (see [KL4])

(a) The external tensor product in K-theory defines an isomorphism

$$\boxtimes:\ \mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}\times\mathcal{B})\simeq\mathbf{A}[X]\underset{\mathbf{A}[X]^{W_0}}{\otimes}\mathbf{A}[X]$$

as $\mathbf{A}[X]^{W_0}$-modules.

We shall identify $\mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}\times\mathcal{B})$ with $\mathbf{A}[X] \underset{\mathbf{A}[X]^{W_0}}{\otimes} \mathbf{A}[X]$, and regard them as an ideal of the algebra $\mathbf{K}_{G\times\mathbf{C}^*}(Z)$.

(b) There exist a unique left $\dot{\mathbf{H}}$-module structure (denoted $h\circ\xi$) on $\mathbf{A}[X]$ such that

$$T_s\circ x = \frac{s(x)-x\alpha_s}{\alpha_s-1}+\mathbf{q}\frac{x\alpha_s-s(x)\alpha_s^{-1}}{\alpha_s-1}, \qquad (s\in S_0,\ x\in X),$$

$$\theta_{x_1}\circ x = x_1x, \qquad (x_1,x\in X).$$

The action $\circ$ is a $\mathbf{q}$-analogue of the usual 'dot' action of W on $\mathbf{A}[X]$.

(c) In $\mathbf{K}_{G\times\mathbf{C}^*}(Z)=\dot{\mathbf{H}}$ we have

$$h(x\boxtimes y)=h\circ x\boxtimes y, \qquad (x\boxtimes y)h = x\boxtimes h\circ y, \qquad h\in\mathbf{K}_{G\times\mathbf{C}^*}(Z),\ x,y\in X.$$

5.2. Lemma. *There is a unique left $\dot{\mathbf{H}}$-module structure (denoted $h*\xi$) on $\mathbf{A}[X]$ extending the obvious $\mathbf{A}$ action and such that*

$$T_s * x = \frac{\alpha_s s(x)-x\alpha_s}{\alpha_s-1}+\mathbf{q}\frac{x\alpha_s-s(x)}{\alpha_s-1}, \qquad (s\in S_0,\ x\in X),$$

$$\theta_{x_1} * x = x_1x, \qquad (x_1,x\in X).$$

Proof. We use Kato's trick to prove it. Let $\mathbf{I}$ be the left ideal of $\dot{\mathbf{H}}$ generated by $\sum_{w\in W_0}T_w$. Then the $\mathbf{A}$-linear map $\mathbf{A}[X]\to\mathbf{I}$ defined by $x\to\theta_x\sum_{w\in W_0}T_w$ is an isomporphism by 2.2(b). One checks easily that under this isomprphism the action $*$ becomes the left multiplication on $\mathbf{I}$. The lemma is proved.

It is easy to see that the action $*$ is a $\mathbf{q}$-analogue of usual action of W on $\mathbf{A}[X]$.

Let δ be the half of the sum of all positive roots in R. We have

5.3. Lemma. *For $h\in\dot{\mathbf{H}}, x\in X$, we have*

$$h\circ x = (\theta_\delta^{-1}h\theta_\delta)*x.$$

The proof is straight.

5.4. Theorem. *Let $\dot{\mathbf{H}}_{c_0}$ be the two-sided ideal of $\dot{\mathbf{H}}$ generated by $\sum_{w\in W_0}T_w$. It has natural $\dot{\mathbf{H}}$-bimodule structure through left and right multiplications. The map*

$$\mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}\times\mathcal{B})\simeq\mathbf{A}[X]\otimes_{\mathbf{A}[X]^{W_0}}\mathbf{A}[X]\to\dot{\mathbf{H}}_{c_0}$$

defined by $x \boxtimes y \to \theta_\delta \theta_x \sum_{w\in W_0} T_w \theta_y \theta_\delta$ *is an* $\dot{\mathbf{H}}$*-bimodule isomprphism.*

Proof. It follows from 5.1(c) and (5.2-3). The map is obviously surjective. Using 2.6(i) we see the map is injective.

5.5. Now we shall classify the simple $\mathbf{H}_q$-modules attached to c_0. Recall the concept of attached two-sided cell in 3.9. For any semisimple element s in G. It is known (see [X2]) that at most one simple $\mathbf{H}_q$-module (up to isomorphism) attached to c_0 such that U_x acts on it by scalar $tr(s, V(x))$. That is $|Y_{s,q,c_0}| \le 1$. We shall give a necessary and sufficient condition for $|Y_{s,q,c_0}| = 0$. We need some preparations.

Let $C = \sum_{w\in W_0} T_w$. For any $x \in X$, we write

$$W_x = \{w \in W_0 \mid w(x) = x\},$$

and

$$f_{W_x} = \sum_{w\in W_x} \mathbf{q}^{l(w)}.$$

We shall need a result of Kato [Ka2] (see also [Gu]).

(a) If $x \in X^+$, then

$$C\theta_x C = f_{W_x} \sum_{w\in W_0} w(\theta_x \prod_{\alpha\in R^+} \frac{1-q\theta_\alpha}{1-\theta_\alpha})C.$$

We shall write $M_{s,q}$ instead of the standard module $M_{s,0,q,1}$. It is known that (see [KL4])

$$M_{s,q} \simeq \mathbb{C}_{s,q} \otimes_{\mathbf{R}_{G\times\mathbf{C}^*}} \mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}), \tag{5.5.1}$$

where $\dot{\mathbf{H}}$ acts on $\mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}) = \mathbf{A}[X]$ by $\circ$ (see 5.1(b)).

5.6. Lemma. *Let* $\mathbf{I}$ *be the left ideal of* $\mathbf{H}_q$ *generated by* $C = \sum_{w\in W_0} T_w$*, and let* $\mathbf{I}_s$ *be the left ideal of* $\mathbf{H}_q$ *generated by* $(U_x - tr(s, V(x))C$*, then the quotient* $\mathbf{I}/\mathbf{I}_s$ *is just the standard module* $M_{s,q}$*.*

Proof. Using (5.3) and (5.5.1) we see that $M_{s,q} \simeq \mathbb{C}_{s,q} \otimes_{\mathbf{R}_{G\times\mathbf{C}^*}} \mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B})$, where $\dot{\mathbf{H}}$ acts on $\mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}) = \mathbf{A}[X]$ by $*$ (see (5.2)). By the definition of $*$ we see that

$$\mathbf{I}/\mathbf{I}_s \simeq \mathbb{C}_{s,q} \otimes_{\mathbf{R}_{G\times\mathbf{C}^*}} \mathbf{K}_{G\times\mathbf{C}^*}(\mathcal{B}).$$

The lemma is proved.

5.7. It is proved in [X2] that

(a) $Y_{s,q,c_0} = \emptyset$ if and only if $CM_{s,q} = 0$.

According to (5.6) we know that $CM_{s,q} = 0$ is equivalent to

(b) $C\theta_x C \in \mathbf{I}_s$ for any $x \in X$.

Note that any element in X is conjugate to an element in X^+ by an element in W_0. Using 2.2(h) we see that (b) is equivalent to

(c) $C\theta_x C \in \mathbf{I}_s$ for any $x \in X^+$.

This implies that

(e) If $f_{W_0}(q) \neq 0$, then $|Y_{s,q,c_0}| = 1$.

5.8. Theorem. (i). *$Y_{s,q,c_0} = \emptyset$ if and only if $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$ (i.e., $\mathbf{g}_{s,q}$ contains semisimple elements).*

(ii). *If $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$, then for any simple constituent L of $M_{s,0,q,1}$ we can find a nonzero nilpotent element $N \in \mathcal{N}_{s,q}$ and $\rho \in A(s,N)^\vee$ such that L is a quotient module of $M_{s,N,q,\rho}$. In particular, 3.6(d) is not true when $f_{W_0}(q) = 0$.*

Proof. (i). Suppose that $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$.

According to 5.7(a-c) it is sufficient to prove that $C\theta_x C \in \mathbf{I}_s$ for any $x \in X^+$. By 5.5(a) this is equivalent to prove that

$$f_{W_x}(q) \sum_{w \in W_0} x(w^{-1}(s)) \prod_{\alpha \in R^+} \frac{1 - q\alpha}{1 - \alpha}(w^{-1}(s)) = 0. \tag{5.8.1}$$

Note that

$$\sum_{w \in W_0} w(x \prod_{\alpha \in R^+} \frac{1 - q\alpha}{1 - \alpha}) \in \mathbf{A}[X]^{W_0}$$

is a holomorphic function on T. It is easy to check that when $\mathbf{g}_{s,q} \neq \mathcal{N}_{s,q}$, for any $w \in W_0$ we have

$$\prod_{\alpha \in R^+} (1 - q\alpha(w(s))) = 0.$$

When G is of classical type, we can find a sequence $t_1, t_2, ..., t_k, ...$ in T_{reg} such that $\lim_{k\to\infty} t_k = s$, thus for any $w \in W_0$ we have

$$\lim_{k\to\infty} \prod_{\alpha\in R^+} \frac{1 - q\alpha(w(t_k))}{1 - \alpha(w(t_k))} = 0.$$

This implies that (5.8.1), in particular, $C\theta_x C \in \mathbf{I}_s$. Similarly using results in 4.14 we see that $C\theta_x C \in \mathbf{I}_s$ when G is of exceptional type. One direction is proved.

Now assume that $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$. Choose $(t, r) \in T \times \mathbb{C}^*$ be such that $(s, q) \sim (t, r)$ and $f_{W_0}(r) \neq 0$ (see 4.8). By (f), 4.3 we see that $|Y_{s,q,c_0}| = 1$.

(ii). By (i) we know that $c_L \neq c_0$. Note that the nilpotent G-orbit corresponds to c_0 is $\{0\}$. Using 3.9(c-d) and 2.6(e) we get (ii).

The theorem is proved.

5.9. There are several interesting special cases. We always have $f_{W_0}(-1) = 0$.

A. Assume that $s \in T$, $q = -1$, then the following conditions are equivalent.

(a) $|Y_{s,q,c_0}| = 1$.

(b) The standard module $M_{s,q}$ is simple.

(c) $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$.

(d) $\mathbf{g}_{s,q} = \mathcal{N}_{s,q} = \{0\}$.

(e) There is no $\alpha \in R$ such that $\alpha(s) = -1$.

(f) $tr(s, V(\delta)) \neq 0$.

By the theorem 5.8 we see that (a)$\Leftrightarrow$(c). Obviously we have (d)$\Leftrightarrow$(e). Since the character of $V(\delta)$ is $\prod_{\alpha\in R^+} \delta^{-1}(1+\alpha)$ (see [Ko]), (e)$\Leftrightarrow$(f). If there is some α such that $\alpha(s) = -1$, then $\mathbf{g}_\alpha + \mathbf{g}_{-\alpha} \subseteq \mathbf{g}_{s,q}$, but the space $\mathbf{g}_\alpha + \mathbf{g}_{-\alpha}$ contains semisimple elements, thus (c)$\Rightarrow$(e). We also have (e)$\Rightarrow$(d)$\Rightarrow$(c). By 3.11 we know that (d)$\Rightarrow$(b). Note that $\dim \mathbf{H}_{s,q} = |W_0|^2$ and $\dim M_{s,q} = |W_0|$, again using 3.11 we see that (b)$\Rightarrow$(d). We have proved these conditions are equivalent.

According to [X2] we know that (f) is equivalent to the following

(g) In $\mathbf{H}_{-1}$ we have $C_{w_0} C_\delta = C_{w_0\delta}$.

B. Let $q \in \mathbb{C}^*$. We assume that $f_{W_0}(q) = 0$ but for any fundamental weight $x \in X^+$ we have $f_{W_x}(q) \neq 0$. It is proved in [X2] there is a unique (up to conjugacy) semisimple element $s \in G$ such that $Y_{s,q,c_0} = \emptyset$. Let $w \in W_0$ be such that $f_w(q) = 0$, then by 4.9(b) and 5.8, for any element $t \in \dot{w}T$ we have $Y_{t,q,c_0} = \emptyset$. This implies that the elements in $\dot{w}T$ are conjugate. When q is an $(e_n + 1)$-th primitive root of 1, then $f_w(q) = 0$ if and only if w is a Coxeter element. The assertion that the elements in $\dot{w}T$ are conjugate if w is a Coxeter element was proved in [St1]. When G is of classical type, $f_w(q) = 0$ but $f_{W_x}(q) \neq 0$ (for any fundamental weight $x \in X^+$) imply that q is an $(e_n + 1)$-th primitive root of 1. Thus $f_w(q) = 0$ if and only if w is a Coxeter element. When G is of exceptional type, $f_w(q) = 0$ but $f_{W_x}(q) \neq 0$ (for any fundamental weight $x \in X^+$) doesnot imply that q is an $(e_n + 1)$-th primitive root of 1. Thus $f_w(q) = 0$ is possible for a non-Coxeter-element. For exceptional types we list the conjugacy classes of these elements w such that $f_w(q) = 0$ but $f_{W_x}(q) \neq 0$ for any fundamental weight $x \in X^+$. The associated type to the conjugacy class of the element are the same as in [C1].

Type E_8. E_8, $E_8(a_1)$, $E_8(a_2)$, $E_8(a_5)$.

Type E_7. E_7, $E_7(a_1)$.

Type E_6. E_6, $E_6(a_1)$.

Type F_4. F_4, B_4.

Type G_2. G_2, A_2.

C. Let $s \in T$ be such that $\alpha(s) = q$ for any simple root in $\alpha \in R$. We have $Y_{s,q,c_0} = \emptyset$ whenever $f_{W_0}(q) = 0$. This also can be proved by using 2.5.

D. If q =1, the simple module in Y_{s,q,c_0} has a simple realization we explain now. We may assume that $s \in T$. let L_s be the vector space over $\mathbb{C}$-vector with a basis $\{w(s) \mid w \in W_0\}$. This is a unique $\mathbf{H}_1 = \mathbb{C}[W]$ module structure $\cdot$ on L_s such that $u \cdot w(s) = uw(s)$ if $u \in W_0$ and $\theta_x \cdot w(s) = x(w(s))$ for $x \in X$. L_s is obviously a simple $\mathbf{H}_1$-module and $C_{w_0} L_s \neq 0$. So $L_s \in Y_{s,q,c_0}$. Note that $\dim L_s = |W_0|$ if and only if $w(s) \neq s$ if $e \neq w \in W_0$, i.e. s is a regular semisimple element.

6. Quotient Algebras

Given $q \in \mathbb{C}^*$. The problem of classifying simple $\mathbf{H}_q$-modules is equivalent to the problem of classifying simple $\mathbf{H}_{s,q}$-modules for all semisimple elements s in G. Note that the standard module $M_{s,N,q,\rho}$ is actually an $\mathbf{H}_{s,q}$-module. Thus it is also interesting to study the algebras $\mathbf{H}_{s,q}$. Ginsburg gave a nice geometric realization for the algebras $\mathbf{H}_{s,q}$. In chapter 4 we have showed that the isomorphism classes of these algebras $\mathbf{H}_{s,q}$ are finite. It would be interesting to classify the algebras $\mathbf{H}_{s,q}$, $(s,q) \in G \times \mathbb{C}^*$ semisimple. In this chapter we give some discussions to the algebras. In the same way we discuss the algebra $\tilde{\mathbf{H}}_{a,b}$ and its quotient algebras $\tilde{\mathbf{H}}_{s,a,b}$.

6.1. Completions. For arbitrary $x_1, x_2 \in X^+$, in $\mathbf{H}_q$ we have

$$U_{x_1}U_{x_2} = U_{x_1+x_2} + \sum_{y<x_1+x_2} a_y U_y, \qquad a_y \in \mathbb{C}.$$

This implies that

$$\bigcap_{k=1}^{\infty} \mathbf{I}_{s,q}^k = \emptyset. \tag{6.1.1}$$

where $s \in G$ is a semisimple element and $\mathbf{I}_{s,q}$ is the two-sided ideal of $\mathbf{H}_q$ generated by $U_x - tr(s, V(x))$, $x \in X^+$.

We have the following natural inverse limit system

$$\cdots \to \mathbf{H}_q/\mathbf{I}_{s,q}^k \to \cdots \to \mathbf{H}_q/\mathbf{I}_{s,q}^2 \to \mathbf{H}_q/\mathbf{I}_{s,q}.$$

By (6.1.1) we see that the completion $\hat{\mathbf{H}}_{s,q}$ of $\mathbf{H}_q$ with respect to the ideal $\mathbf{I}_{s,q}$ is just the inverse limit $\varprojlim \mathbf{H}_q/\mathbf{I}_{s,q}^k$. Let $\mathrm{Mod}(\mathbf{H}_q)$ be the category of finite dimensional $\mathbf{H}_q$-modules (over $\mathbb{C}$). Let $\mathrm{Mod}(\hat{\mathbf{H}}_{s,q})$ be category of finite dimensional $\hat{\mathbf{H}}_{s,q}$-modules (over $\mathbb{C}$).

We have the following

6.2. Theorem. *The category* $\mathrm{Mod}(\mathbf{H}_q)$ *of finite dimensional (over* $\mathbb{C}$*)* $\mathbf{H}_q$*-modules is the direct sum of the categories* $\mathrm{Mod}(\hat{\mathbf{H}}_{s,q})$*, where the direct sum is over the set* $\mathcal{S}$ *of representatives of semisimple conjugacy classes of* G.

Proof. First, each finite dimensional $\hat{\mathbf{H}}_{s,q}$-module has a natural $\mathbf{H}_q$-module structure.

Let M be a finite dimensional $\mathbf{H}_q$-module. For $s \in \mathcal{S}$ semisimple, we write

$$M_s = \{m \in M \mid (U_x - tr(s, V(x)))^k m = 0 \text{ for all } x \in X^+ \text{ and for some integer } k > 0\}.$$

The space M_s is an $\mathbf{H}_q$-module. We have

$$M = \bigoplus_{s \in \mathcal{S}} M_s.$$

Moreover, if $s, t \in \mathcal{S}$ are different, we have

$$\mathrm{Hom}_{\mathbf{H}_q}(M_s, M_t) = 0.$$

Since M is finite dimensional, M_s is actually an $\hat{\mathbf{H}}_{s,q}$-module. The theorem is proved.

Note that $\mathbf{H}_{s,q}$ is also a quotient algebra of $\hat{\mathbf{H}}_{s,q}$.

6.3. Let E be a simple $\mathbf{H}_q$-module, then $E \in Y_{s,q}$ for some semisimple element $s \in \mathcal{S}$. We regard E as an $\mathbf{H}_{s,q}$-module in a natural way. Then obviously the set of isomorphism classes of simple $\mathbf{H}_{s,q}$-modules is $Y_{s,q}$. By a theorem of Betti (see [St2]) we know that $\dim \mathbf{H}_{s,q} = |W_0|^2$. Thus $\sum_{E \in Y_{s,q}} (\dim E)^2 \leq |W_0|^2$. The equality holds if and only if $\mathbf{H}_{s,q}$ is semisimple. In particular, we see that any simple $\mathbf{H}_q$-module has dimension $\leq |W_0|$ and the equality holds if and only if $\mathbf{H}_{s,q}$ is simple. According to Ginzburg [G3] we know that the algebra $\mathbf{H}_{s,q}$ is either simple or non-semisimple. Thus the equality $\sum_{E \in Y_{s,q}} (\dim E)^2 = |W_0|^2$ holds if and only if $\mathbf{H}_{s,q}$ is simple.

When $\mathbf{H}_{s,q}$ is simple, we have $\#Y_{s,q} = 1$. Moreover the natural map $\mathbf{H}_q{}^{\leq c_0} \to \mathbf{H}_{s,q}$ is a surjective map. Let $E \in Y_{s,q}$, then we have $\dim E = |W_0|$ and $c_E = c_0$. Let $\mathbf{H}_{q,W_0}$ be the subalgebra of $\mathbf{H}_{s,q}$ generated by the images of T_w, $w \in W_0$. The algebra $\mathbf{H}_{q,W_0}$ is isomorphic to the Hecke algebra of W_0 over $\mathbb{C}$ with parameter q. It is easy to see that as an $\mathbf{H}_{q,W_0}$-module, E is isomorphic to the left regular module of $\mathbf{H}_{q,W_0}$. In general $\#Y_{s,q} = 1$ does not imply that $\mathbf{H}_{s,q}$ is simple. We summarize the discussions as follows.

6.4. Proposition. *Let E be simple $\mathbf{H}_q$-module, then the following conditions are equivalent.*

(i). $\mathit{dim}E = |W_0|$.

(ii). $\mathbf{H}_{s,q}$ *is a simple algebra.*

(iii). *As an* $\mathbf{H}_{q,W_0}$*-module,* E *is isomorphic to the (left) regular module of* $\mathbf{H}_{q,W_0}$.

(iv). $\mathcal{N}_{s,q} = \{0\}$.

(v). $\mathbf{H}_q^{\leq c_0} \to \mathbf{H}_{s,q}$ *is a surjective map.*

6.5. Conjecture. (i). *Let* $(s,q),(t,r) \in T \times \mathbb{C}^*$ *be semisimple, then* $\hat{\mathbf{H}}_{s,q} \simeq \hat{\mathbf{H}}_{t,r}$ *if* $\mathcal{N}_{s,q} = \mathcal{N}_{t,r}$.

(ii). *Consider the homomorphism* $\phi_{q,c_0} : \mathbf{H}_q \longrightarrow \mathbf{J}_{c_0}$. *It induces a homomorphism* $\phi_{s,q,c_0} : \mathbf{H}_{s,q} \to \mathbf{J}_{c_0}/\phi_{q,c_0}(\mathbf{I}_{s,q})$. *The ideal* $\ker\phi_{s,q,c_0}$ *of* $\mathbf{H}_{s,q}$ *should be nilpotent. Note that* $\mathbf{J}_{c_0}/\phi_{q,c_0}(\mathbf{I}_{s,q})$ *is a simple* $\mathbb{C}$*-algebra of dimension* $|W_0|^2$ *(see [X1]).*

6.6. One may consider the algebra $\tilde{\mathbf{H}}_{a,b}$, $(a,b) \in \mathbb{C}^* \times \mathbb{C}^*$ in a similar way. Given $s \in G$ semisimple, let $\tilde{\mathbf{I}}_{s,a,b}$ be the two-sided ideal of $\tilde{\mathbf{H}}_{a,b}$ generated by $U_x - tr(s, V(x))$, $x \in X^+$. For the same reason of (6.1.1) we know that the completion $\hat{\tilde{\mathbf{H}}}_{s,a,b}$ of $\tilde{\mathbf{H}}_{a,b}$ with respect to the ideal $\tilde{\mathbf{I}}_{s,a,b}$ is just the inverse limit $\varprojlim \tilde{\mathbf{H}}_{a,b}/\tilde{\mathbf{I}}^k_{s,a,b}$. Let $\mathrm{Mod}(\tilde{\mathbf{H}}_{a,b})$ be the category of finite dimensional $\tilde{\mathbf{H}}_{a,b}$-modules (over $\mathbb{C}$). Let $\mathrm{Mod}(\hat{\tilde{\mathbf{H}}}_{s,a,b})$ be category of finite dimensional $\hat{\tilde{\mathbf{H}}}_{s,a,b}$-modules (over $\mathbb{C}$). Similar to 6.2 we have the following

(a). *The category* $\mathrm{Mod}(\tilde{\mathbf{H}}_{a,b})$ *of finite dimensional (over* $\mathbb{C}$*)* $\tilde{\mathbf{H}}_{a,b}$*-modules is the direct sum of the categories* $\mathrm{Mod}(\hat{\tilde{\mathbf{H}}}_{s,a,b})$*, where the direct sum is over the set* $\mathcal{S}$ *of representatives of semisimple conjugacy classes of* G.

Note that $\tilde{\mathbf{H}}_{s,a,b} = \tilde{\mathbf{H}}_{a,b}/\tilde{\mathbf{I}}_{s,a,b}$ is a quotient algebra of $\hat{\tilde{\mathbf{H}}}_{s,a,b}$.

Let E be a simple $\tilde{\mathbf{H}}_{a,b}$-module, then $E \in Y_{s,a,b}$ for some semisimple element $s \in \mathcal{S}$ ($Y_{s,a,b}$ is the set of isomorphism classes of simple $\tilde{\mathbf{H}}_{a,b}$-modules on which $\tilde{\mathbf{I}}_{s,a,b}$ act by scalar 0). We may regard E as an $\tilde{\mathbf{H}}_{s,a,b}$-module in a natural way. Then obviously the set of isomorphism classes of simple $\tilde{\mathbf{H}}_{s,a,b}$-modules is $Y_{s,a,b}$. According to theorem of Betti (see [St2]) we know that $\dim\tilde{\mathbf{H}}_{s,a,b} = |W_0|^2$. Thus $\sum_{E \in Y_{s,a,b}} (\dim E)^2 \leq |W_0|^2$. The equality holds if and only if $\mathbf{H}_{s,q}$ is semisimple. In particular, we see that any simple $\mathbf{H}_q$-module has dimension $\leq |W_0|$ and the equality holds if and only if $\tilde{\mathbf{H}}_{s,a,b}$ is simple.

Let $s \in T$, then (see [Ka1])

(b) . $\tilde{\mathbf{H}}_{s,a,b}$ is simple if and only if $\mathcal{N}_{s,a,b} = \{0\}$ (see (3.18.7) for the notation).

6.7. Conjecture. *We keep the notations in 6.6. Let* $(s,a,b),(t,c,d)$ *be two elements in* $T \times \mathbb{C}^* \times \mathbb{C}^*$. *Assume that* $\mathcal{N}_{s,a,b} = \mathcal{N}_{t,c,d}$, *then*

(i). $\tilde{\mathbf{H}}_{s,a,b} \simeq \tilde{\mathbf{H}}_{t,c,d}$.

(ii). $\hat{\tilde{\mathbf{H}}}_{s,a,b} \simeq \hat{\tilde{\mathbf{H}}}_{t,c,d}$.

7. The Based Rings of Cells in Affine Weyl Groups of Type $\widetilde{G_2}$, $\widetilde{B_2}$ $\widetilde{A_2}$

The work [L12-14] and [X2] show that the base rings of two-sided cells in affine Weyl groups are interesting to understand the classification of simple modules of the corresponding Hecke algebras H_q ($q \in \mathbb{C}^*$) even if q is a root of 1.

In this chapter we determine the based rings of cells in affine Weyl groups of type $\tilde{G}_2$, $\tilde{B}_2$ (see 7.2), which confirm the conjecture in [L14] (see also 3.14). Then we apply the results to classify the simple modules of the corresponding Hecke algebra H_q ($q \in \mathbb{C}^*$).

The explicit descriptions about the based rings enable us to understand the structure of standard modules in a concrete way, so that provide a way to compute the dimensions of simple modules of $\mathbf{H}_q$. As an example we work out the case of type $\tilde{A}_2$ (see 7.7). An immediate consequence is that $\mathbf{H}_q \not\simeq \mathbf{H}_1 = \mathbb{C}[W]$ whenever $q \neq 1$, here $\mathbf{H}_q$ is an affine Hecke algebra of type $\tilde{A}_2$. This result leads to several questions (see 7.7). This chapter is based on the preprint [X3], only section 7.7 is added.

The based ring $\mathbf{J}_c$

7.1. We refer to 3.14 for Lusztig's conjecture concerned with based rings of cells in affine Weyl groups, the conjecture is a guideline to determine the structure of the based rings. Except special indications, until section 7.5, G is always a simple, simply connected algebraic group over $\mathbb{C}$ of type G_2 or B_2, then W is of type $\tilde{G}_2$ or $\tilde{B}_2$ and $S = \{r_0, r_1, r_2\}$. The cells in W have been described in [L11] explicitly.

In the case $\tilde{G}_2$, $W = W' = W_0 \ltimes P$. We assume that $(r_0r_1)^3 = (r_1r_2)^6 = (r_0r_2)^2 = e$. W has five two-sided cells: $c_e = \{w \in W \mid a(w) = 0\} = \{e\}$, $c_1 = \{w \in W \mid a(w) = 1\}$, $c_2 = \{w \in W \mid a(w) = 2\}$, $c_3 = \{w \in W \mid a(w) = 3\}$, $c_0 = \{w \in W \mid a(w) = 6\}$. (see 1.12 for the definition of the function $a: W \to \mathbf{N}$.)

In the case $\tilde{B}_2$, $W = \Omega \ltimes W'$, $\Omega = \{e, \omega\}$. We assume that $\omega r_0 = r_2\omega$, $\omega r_1 = r_1\omega$, $\omega r_2 = \omega r_0$. We have four two-sided cells: $c_e = \{w \in W \mid a(w) = 0\} = \{e, \omega\} = \Omega$, $c_1 = \{w \in W \mid a(w) = 1\}$, $c_2 = \{w \in W \mid a(w) = 2\}$, $c_0 = \{w \in W \mid a(w) = 4\}$.

One of the main result of this chapter is the following.

Theorem 7.2. *We keep the set up in 7.1. For each two-sided cell c of W, there exists a finite F_c-set Y (see 3.4 for the term) and a bijection*

$$\pi: c \xrightarrow{\sim} \text{set of irreducible } F_c\text{-v.b. on } Y \times Y \text{ (up to isomorphism)}$$

with the following properties:

(i) *The $\mathbb{C}$-linear map $\mathbf{J}_c \to \mathbf{K}_{F_c}(Y \times Y)$, $t_w \to \pi(w)$ is an algebra isomorphism (preserving the unit element).*

(ii) $\pi(w^{-1}) = \widetilde{\pi(w)}$ $(w \in c)$.

When Γ is a left cell in c_1, $\mathbf{J}_{\Gamma\cap\Gamma^{-1}}$ has been described in [L15], here $\mathbf{J}_{\Gamma\cap\Gamma^{-1}}$ is the $\mathbb{C}$-subspace of $\mathbf{J}_{c_1}$ spanned by elements t_w, $w \in \Gamma \cap \Gamma^{-1}$.

Proof. We prove the theorem case by case. Before our proof, we make the following convention: we often write $i_1 i_2 \dots i_n$ instead of w when $r_{i_1} r_{i_2} \dots r_{i_n}$ is a reduced expression of w.

(A). When $c = c_e$, F_c is the center of G, we take Y to be a one point set and F_c acts on Y trivially, and the theorem then is trivial. When $c = c_0$, the lowest two-sided cell of W, then $F_c = G$ and the theorem is proved in [X1]. Thus we only need to verify the theorem for the two-sided cell c of W with $c \neq c_e$ or c_0. In (B-D), G is assumed to be of type G_2 and then W is of type $\tilde{G}_2$.

(B). Case $c = c_1$, then $c = \{2, 212, 21212, 21, 2121, 210, 21210, 1, 121, 12121, 12, 1212, 10, 1210, 121210, 0, 01210, 0121210, 01, 0121, 012121, 012, 01212\}$ (note the convention in the beginning of the proof), and $F_c = \mathfrak{S}_3$, the symmetric group of three letters. Let $Y = \{i \mid 1 \leq i \leq 5\}$ be the F-set such that as F-sets we have $\{1\} \simeq \{2\} \simeq \mathfrak{S}_3/\mathfrak{S}_3$, $\{3,4,5\} \simeq \mathfrak{S}_3/\mathfrak{S}_2$. We assume that $\mathfrak{S}_2$ leaves stable on 3.

For $\mathfrak{S}_3$ we have three simple representations: the unit representation 1, the sign representation ε, and the unique simple representation σ of degree 2. We use the notations 1, ε for their restriction on $\mathfrak{S}_2$ again. One may verify that the following bijection (note the convention in 3.4)

$$\pi: c \xrightarrow{\sim} \text{set of irreducible } F_c\text{-v.b. on } Y \times Y \text{ (up to isomorphism)},$$

$$\begin{array}{rrr}
0 \to 1_{(1,1)}, & 01210 \to \sigma_{(1,1)}, & 0121210 \to \varepsilon_{(1,1)}, \\
1 \to 1_{(2,2)}, & 121 \to \sigma_{(2,2)}, & 12121 \to \varepsilon_{(2,2)}, \\
01 \to 1_{(1,2)}, & 0121 \to \sigma_{(1,2)}, & 012121 \to \varepsilon_{(1,2)}, \\
10 \to 1_{(2,1)}, & 1210 \to \sigma_{(2,1)}, & 121210 \to \varepsilon_{(2,1)}, \\
2 \to 1_{(3,3)}, & 212 \to 1_{(3,4)}, & 21212 \to \varepsilon_{(3,3)}, \\
012 \to 1_{(1,3)}, & 01212 \to \varepsilon_{(1,3)}, & \\
12 \to 1_{(2,3)}, & 1212 \to \varepsilon_{(2,3)}, & \\
210 \to 1_{(3,1)}, & 21210 \to \varepsilon_{(3,1)}, & \\
21 \to 1_{(3,2)}, & 2121 \to \varepsilon_{(3,2)}, &
\end{array}$$

is just what we need.

Let $Y' = \{z_i \mid 1 \le i \le 7\}$ be the F_c-set such that as F_c-sets we have $\{z_1\} \simeq \mathfrak{S}_3/\mathfrak{S}_3$, $\{z_2, z_3, z_4\} \simeq \{z_5, z_6, z_7\} \simeq \mathfrak{S}_3/\mathfrak{S}_2$. One may check that there exists a bijection between c and the set of isomorphism classes of irreducible F-v.b. on $Y' \times Y'$ with the properties (i) and (ii) in Theorem 7.2.

(C). Case $c = c_2$, then $F_c = SL_2(\mathbb{C})$ and

$$c = \{w(i,j,k) \mid 1 \le i,j \le 6,\ k \ge 0\}$$

where $(w(i,j,k) = w_i r_0 r_2 (r_1 r_2 r_1 r_2 r_0)^k w_j^{-1}$, $w_1 = e$, $w_2 = r_1$, $w_3 = r_2 r_1$, $w_4 = r_1 r_2 r_1$, $w_5 = r_2 r_1 r_2 r_1$, $w_6 = r_0 r_1 r_2 r_1)$. We write $w(k)$ for $w(1,1,k)$. We have

(a) $\mathcal{D} \cap c = \{w(i,i,0) \mid 1 \le i \le 6\} = \{w_i r_0 r_2 w_i^{-1} \mid 1 \le i \le 6\}$. (see 2.6 for the definition of $\mathcal{D}$.)

(b) $w(i,j,k) \underset{L}{\sim} w(m,n,k')$ if and only if $j = n$,
$w(i,j,k) \underset{R}{\sim} w(m,n,k')$ if and only if $i = m$.

Let $Y = \{1,2,3,4,5,6\}$ and let F_c acts on Y trivially. Then the map $\pi\colon w(i,j,k) \to V(k)_{(i,j)}$ defines a bijection between the two-sided cell c and the set of isomorphism classes of irreducible F_c-v.b. on $Y \times Y$, where $V(k)$ is the irreducible representation of F_c with highest weight k. We claim the bijection is what we need. In fact, 7.2(ii) is obvious. π

gives rise to a $\mathbb{C}$-linear map $\pi: \mathbf{J}_c \to \mathbf{K}_{F_c}(Y \times Y)$ which preserves the unit element. To complete the proof we need to check the following equality.

(c) $$t_{w(i,j,k)}t_{w(m,n,k')} = \pi(w(i,j,k))\pi(w(m,n,k'))$$

When $j \neq m$, using (b), 2.7(a) and the definition of π, we see that (c) is true since both sides in (c) are 0. Now we assume that $j = m$, then (c) is equivalent to the following.

(d) $$t_{w(i,j,k)}t_{w(j,n,k')} = \sum_{\substack{k'' \in \mathbf{N} \\ |k-k'| \leq k'' \leq k+k'}} t_{w(i,n,k'')}.$$

We say that (d) deduces from the following two assertions.

(e) $\gamma_{w(i,j,k),w(j,n,k'),w(i,n,k'')} = \gamma_{w(k),w(k'),w(k'')}$.

(f) $t_{w(1)}t_{w(k')} = t_{w(k'+1)} + t_{w(k'-1)}$ (we assume that $t_{w(-1)} = 0$).

In fact, according to (a) and 2.7(b), we have $t_{w(0)}t_{w(k')} = t_{w(k')} = t_{w(k')}t_{w(0)}$. Using induction on k and using (f), we get

(g) $$t_{w(k)}t_{w(k')} = \sum_{\substack{k'' \in \mathbf{N} \\ |k-k'| \leq k'' \leq k+k'}} t_{w(k'')}$$

Combine (e), (g) and 2.7(a) and we see that (d) holds.

Now we return to (e) and (f). First we prove (e). Consider the algebra $H_{\geq c} = H/H^{<c}$ (see 2.6(h)). We write C_w for its image in $H_{\geq c}$ again. Then in $H_{\geq c}$ there exists h_i, h'_j such that

(h) $$C_{w(i,j,k)} = h_i C_{w(k)} h'_j.$$

By (h) and the definition of γ we obtain

(i) $$\gamma_{w(i,j,k),w(j,n,k'),w(i,n,k'')} = \gamma_{w(1,j,k),w(j,1,k'),w(k'')}.$$

Using [X2, 2.6], we see that there exists $h \in H_{\geq c}$ such that

(j) $C_{w(k)} = hC_{02}$ (note the convention in the beginning of the proof).

By (h) and (j) we know that

(k) $$C_{w(k)}h'_j = C_{w(1,j,k)} = hC_{w(1,j,0)}.$$

From (b), 2.6(f-g), we have in $H_{\geq c}$

(ℓ) $$C_{w(1,j,0)}C_{w(j,1,k')} = \sum_{m\in\mathbf{N}} a_m C_{w(m)},\ a_m \in A.$$

According to (a), (b) and 2.7(a), we have the following

(m) When $m \neq k'$, we have $\mathbf{q}a_m \in \mathbf{q}^{\frac{1}{2}}\mathbb{C}[\mathbf{q}^{\frac{1}{2}}]$ and $\mathbf{q}^{\frac{1}{2}}a_{k'} \in 1+\mathbf{q}^{\frac{1}{2}}\mathbb{C}[\mathbf{q}^{\frac{1}{2}}]$.

By (k) and (ℓ) we get

(n) $C_{w(1,j,k)}C_{w(j,1,k')} = \sum_{m\in\mathbf{N}} a_m hC_{w(m)}$

It is easy to see (1.8(a))

(o) $$C_{02}C_{w(m)} = C_{w(m)}C_{02} = (\mathbf{q}^{\frac{1}{2}}+\mathbf{q}^{-\frac{1}{2}})^2 C_{w(m)}.$$

From (o), (n), (j), (b), 2.6(f-g), we obtain

(p) $$\begin{aligned} C_{w(1,j,k)}C_{w(j,1,k')} &= (\mathbf{q}^{\frac{1}{2}}+\mathbf{q}^{-\frac{1}{2}})^{-2}\sum_{m\in\mathbf{N}} a_m C_{w(k)}C_{w(m)} \\ &= (\mathbf{q}^{\frac{1}{2}}+\mathbf{q}^{-\frac{1}{2}})^{-2}\sum_{m,k''\in\mathbf{N}} a_m h_{w(k),w(m),w(k'')}C_{w(k'')} \end{aligned}$$

Using (o) and (j) again, we see that $h_{w(k),w(m),w(k'')} = (\mathbf{q}^{\frac{1}{2}}+\mathbf{q}^{-\frac{1}{2}})^2 \cdot a_{k,m,k''}$. Since $h_{w(k),w(m),w(k'')}$ is polynomial in $\mathbf{q}^{\frac{1}{2}}+\mathbf{q}^{-\frac{1}{2}}$ and its degree ≤ 2 by definition of c, we know that $a_{k,m,k''} \in \mathbf{Z}$ (in fact, $a_{k,m,k''} \in \mathbf{N}$ by the positivity of $h_{w(k),w(m),w(k'')}$ (cf. [L11]).) Thus we have

(q) $$C_{w(1,j,k)}C_{w(j,1,k')} = \sum_{m,k''\in\mathbf{N}} a_m a_{k,m,k''} C_{w(k'')}$$

Combine (m) and (q) and we get

(r) $$\gamma_{w(1,j,k),w(j,1,k'),w(k'')} = \gamma_{w(k),w(k'),w(k'')}.$$

Then (e) follows from (i) and (r).

Now we prove (f). In $H_{\geq c}$ we have

(s) $$C_{w(1)} = (C_0C_2C_1C_2C_1 - 2(\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}}))C_{02} = C_{0212120}$$

Using (o) and (s), we get

(t) $$C_{w(1)}C_{w(k')} = (\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}})^2(C_1C_2C_1C_2C_1 - 2(\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}}))C_{w(k')}.$$

From (t) and 1.8(a), it is not difficult to see that

(u) $$C_{w(1)}C_{w(k')} = (\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}})^2C_0C_{2121w(k')} = (\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}})^2C_0C_{w(5,1,k')}.$$

Using 1.8(a) again, we have in $H_{\geq c}$

(v) $$C_0C_{w(5,1,k')} = C_{w(k'+1)} + \sum_{w(m)\prec w(5,1,k')} \mu(w(m), w(5,1,k'))C_{w(m)}.$$

By [L11, 10.4.3] we see that $\mu(w(m), w(5,1,k')) = \mu(w(5,1,m), w(k'))$. According to 1.8(f), we see that $\mu(w(5,1,m), w(k')) \neq 0$ if and only if $r_0w(5,1,m) = w(k')$, i.e. $m = k'-1$; moreover, $\mu(w(5,1,k'-1), w(k') = 1$. Hence $C_{w(1)}C_{w(k')} = (\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}})^2$ $(C_{w(k'+1)} + C_{w(k'-1)})$ and (f) follows.

(D). Case $c = c_3$, then $F_c = SL_2(\mathbb{C})$ and

$$c = \{u(i,j,k) \mid 1 \leq i,j \leq 6,\ k \geq 0\},$$

where $u(i,j,k) = u_ir_0r_1r_0(r_2r_1r_0)^ku_j^{-1}$, $u_1 = e$, $u_2 = r_2$, $u_3 = r_1r_2$, $u_4 = r_2r_1r_2$, $u_5 = r_1r_2r_1r_2$, $u_6 = r_0r_1r_2r_1r_2$. We have

(a) $\mathcal{D} \cap c = \{u(i,i,0) \mid 1 \leq i \leq 6\}$

(b) $u(i,j,k) \underset{L}{\sim} u(m,n,k')$ if and only if $j = n$.
$u(i,j,k) \underset{R}{\sim} u(m,n,k')$ if and nly if $i = m$.

Let $Y = \{1,2,3,4,5,6\}$, and let F_c acts on Y trivially, then the bijection $\pi: c \xrightarrow{\sim}$ the set of isomorphism classes of irreducible F_c-v.b. on $Y \times Y$ defined by $u(i,j,k) \to V(k)_{(i,j)}$ satisfies 7.2(i) and 7.2(ii). The proof is similar to the case c_2 in (C).

From now on we assume that G is of type B_2. Then W is of type $\widetilde{B}_2$.

(E). Case $c = c_2$, then $F_c = \mathbf{Z}/2 \times SL_2(\mathbb{C})$, and

$$c = \{\omega^p v(i,j,k) \mid 1 \le i,j \le 4,\ k \ge 0,\ p = 0,1\}$$

where $v(i,j,k) = v_i r_0 r_2 (r_1 r_2 r_0)^k v_j^{-1}$, $v_1 = e$, $v_2 = r_1$, $v_3 = r_2 r_1$, $v_4 = r_0 r_1$. We have

(a) $\mathcal{D} \cap c = \{v(i,i,0) \mid 1 \le i \le 4\}$.

(b) $\omega^p v(i,j,k) \underset{L}{\sim} \omega^{p'} v(m,n,k')$ if and only if $j = n$,
$\omega^p v(i,j,k) \underset{R}{\sim} \omega^{p'} v(m,n,k')$ if and only if $i = m$.

Let $Y = \{1,2,3,4\}$ and let F_c acts on Y trivially. As the same way in (C), we know that the bijection $\pi\colon c \xrightarrow{\sim}$ the set of isomorphism classes of irreducible F_c-v.b. on $Y \times Y$ defined by $\omega^p v(i,j,k) \to (\varepsilon^p, V(k))_{(i,j)}$ satisfies 7.2(i) and 7.2(ii), where ε is the sign representation of $\mathbf{Z}/2$.

(F). Case $c = c_1$, then $F_c = \mathbf{Z}/2 \ltimes \mathbb{C}^*$, where $\mathbf{Z}/2$ acts on $\mathbb{C}^*$ by $z \to z^{-1}$;

$$c = \{\omega^p (r_0 r_1 \omega)^k \omega^{p'}, \omega^p (r_0 r_1 \omega)^{k'} r_0 \omega^{p'}, \omega^p (r_1 r_0 \omega)^k \omega^{p'},$$
$$\omega^p (r_1 r_0 \omega)^{k'} r_1 \omega^{p'}, \omega^p r_0 r_1 r_0 \omega^{p'} \mid k > 0,\ k' \ge 0,\ p,p' = 0,1\}$$

We shall regard $\mathbf{q}^i$ ($i \in \mathbf{Z}$) as the simple representation of $\mathbb{C}^*$ defined by $z \to z^i$. Let $\sigma(k)$ ($k > 0$) be the simple representation of F_c such that the restriction to $\mathbb{C}^*$ of σ_k is the direct sum of $\mathbf{q}^k$ and $\mathbf{q}^{-k}$. The sign representation ε of $\mathbf{Z}/2$ gives rise to a simple representation of F_c via the natural homomorphism $F_c \to \mathbf{Z}/2$, we denote it again by ε. Let $\sigma(0) = 1$ be the unit representation of F_c.

Let $Y = \{1,2,3,4\}$ be the F_c-set such that as F_c-sets we have $\{1\} \simeq \{2\} \simeq F_c/F_c$ and $\{3,4\} \simeq F_c/F_c^0$. As in (C), we can check that the following bijection π has the properties 7.2(i) and 7.2(ii).

$\pi\colon c \xrightarrow{\sim}$ the set of irreducible F_c-v.b. on $Y \times Y$ (up to isomorphism),

$$r_0 r_1 r_0 \to \varepsilon_{(1,1)}, \quad \omega r_0 r_1 r_0 \to \varepsilon_{(2,1)}, \quad r_0 r_1 r_0 \omega \to \varepsilon_{(1,2)}, \quad r_2 r_1 r_2 \to \varepsilon_{(2,2)},$$

$$(r_0r_1\omega)^k r_0 \to \sigma(k)_{(1,1)}, \qquad (r_0r_1\omega)^k r_0\omega \to \sigma(k)_{(1,2)},$$
$$\omega(r_0r_1\omega)^k r_0 \to \sigma(k)_{(2,1)}, \qquad \omega(r_0r_1\omega)^k r_0\omega \to \sigma(k)_{(2,2)},$$
$$(r_1r_0\omega)^k r_1 \to \mathbf{q}^{k}_{(3,3)}, \qquad \omega(r_1r_0\omega)^k r_1\omega \to \mathbf{q}^{-k}_{(3,3)},$$
$$(r_1r_0\omega)^k r_1\omega \to \mathbf{q}^{k-1}_{(3,4)}, \qquad \omega(r_1r_0\omega)^k r_1 \to \mathbf{q}^{-1-k}_{(3,4)},$$
$$(r_0r_1\omega)^{k'} \to \mathbf{q}^{k'}_{(1,3)}, \qquad (r_0r_1\omega)^{k'}\omega \to \mathbf{q}^{1-k'}_{(1,3)},$$
$$\omega(r_1r_0\omega)^{k'}\omega \to \mathbf{q}^{-k'}_{(3,1)}, \qquad (r_1r_0\omega)^{k'}\omega \to \mathbf{q}^{k'-1}_{(3,1)},$$
$$\omega(r_0r_1\omega)^{k'}\omega \to \mathbf{q}^{1-k'}_{(2,3)}, \qquad \omega(r_0r_1\omega)^{k'} \to \mathbf{q}^{k'}_{(2,3)},$$
$$(r_1r_0\omega)^{k'} \to \mathbf{q}^{k'-1}_{3,2)}, \qquad \omega(r_1r_0\omega)^{k'} \to \mathbf{q}^{-k'}_{(3,2)},$$

where $k \geq 0,\ k' > 0$ are integers.

Application to Representation

7.3 We shall apply the idea in 3.13 to classify the simple $\mathbf{H}_q$-modules under the assumption that W is of type $\tilde{G}_2$ or $\tilde{B}_2$. When W is of type $\tilde{A}_1, \tilde{A}_2$, see [X2]. The results in [X2] and in this chapter show that the map $(\phi_q)_{*,c}$ (see 3.13 for its definition) is a good way to understand the classification of simple $\mathbf{H}_q$-modules even if q is a root of 1. Our second main result in the chapter is the following. We refer to chapter 3, especially 3.13, for notations.

Theorem 7.4. *Let W be an extended affine Weyl group type $\tilde{G}_2$ or $\tilde{B}_2$ as in 7.1.*

(i) *Assume that E is a simple $\mathbf{J}$-module, then E_q has at most one simple constituent L such that $c_L = c_E$.*

(ii) *Given two simple $\mathbf{J}$-modules E, E' such that E_q has a simple constituent L with $c_L = c_E$ and E'_q has a simple constituent L' with $c_{L'} = c_{E'}$. Then $L \simeq L'$ if and only if $E \simeq E'$.*

(iii) *The set $\Lambda = \{(\phi_q)_{*,c}(E) \mid c$ a two-sided cell of W, E a simple $\mathbf{J}_c$-module (up to isomorphism)$\} - \{0\}$ is a basis of $K(\mathbf{H}_q)$.*

(iv) *$(\phi_q)_*$ is an isomorphism if and only if $\sum_{w \in W_0} q^{l(w)} \neq 0$.*

Proof. Since $(\phi_q)_*$ is surjective (see 3.13), we know that (iii) follows from (i) and (ii).

Each $\mathbf{J}_c$-module E gives rise to an $\mathbf{H}_q$-module E_q via the homomorphism $\phi_{q,c}\colon \mathbf{H}_q \to \mathbf{J} \to \mathbf{J}_c$. Then the assertion (i) and (ii) are equivalent to the following assertion:

(▼) Given a simple $\mathbf{J}_c$-module E, E_q has at most one simple constituent L such that $c_L = c$. Suppose that the simple $\mathbf{J}_c$-module E (resp. E'), E_q (resp. E'_q) has a simple constituent L (resp. L') such that $c_L = c$ (resp. $c_{L'} = c$), then $L \simeq L'$ if and only if $E \simeq E'$.

We prove (▼) case by case.

(A). When $c = c_e$, then $(\phi_q)_{*,c}$ is an isomorphism, what we need is trivial. When $c = c_0$, in [X2] we have shown that Λ_{c_0} is a complete set of irreducible H_q-modules L with $c_L = c_0$ and that $(\phi_q)_{*,c}$ is an isomorphism if and only if $\sum\limits_{w\in W_0} q^{l(w)} \neq 0$. In (B-D) we assume that W is of type $\tilde{G}_2$.

(B). Case $c = c_1$. $\mathbf{J}_c$ has four simple modules E_1, E_2, E_3, E_4. $\dim E_1 = \dim E_2 = 3$. $\dim E_3 = 2$, $\dim E_4 = 1$. When $q + 1 \neq 0$, one verifies that $E_{i,q}$ $(1 \leq i \leq 4)$ has a unique simple constituent L_i such that $c_{L_i} = c_1$ and $L_i \not\simeq L_j$ if $i \neq j$. L_i $(1 \leq i \leq 4)$ in fact is a quotient of $E_{i,q}$. When $q + 1 = 0$, one can check that $E_{i,q}$ $(1 \leq i \leq 3)$ has a unique simple constituent L_i such that $c_{L_i} = c_1$ and $L_i \not\simeq L_j$ if $i \neq j$. We also have $(\phi_q)_{*,c}(E_4) = 0$.

(C). Case $c = c_2$. We have $\mathbf{J}_c \simeq M_{6\times 6}(\mathbf{R}_{F_c})$ (the 6×6 matrix ring over $\mathbf{R}_{F_c}$, where $\mathbf{R}_{F_c} = \mathbb{C}$ tensor with the representations ring of F_c, see 7.2 and 3.4). For each semisimple conjugacy class s of $F_c = SL_2(\mathbb{C})$, we have a simple representation ψ_s of $\mathbf{J}_c$:

$$\psi_s\colon \mathbf{J}_c \simeq M_{6\times 6}(\mathbf{R}_{F_c}) \to M_{6\times 6}(\mathbb{C}),$$
$$(m_{ij}) \to (tr(s, m_{ij})).$$

Any simple representation of $\mathbf{J}_c$ is isomorphic to some ψ_s. Let E_s be a simple $\mathbf{J}_c$-module which provides the representation ψ_s.

$E_{s,q}$ in fact is an $\mathbf{H}_q^{\geq c}$-module, where $\mathbf{H}_q^{\geq c}$ is the quotient algebra of $\mathbf{H}_q$ modulo the two-sided ideal generated by C_w, $w \underset{LR}{\leq} r_0 r_2$ but $w \notin c$. We denote the image in $\mathbf{H}_q^{\geq c}$ of C_w again by C_w. By (h) and (j) in the part (C) of the proof of 7.2, we see that the two-sided ideal of $\mathbf{H}_q^{\geq c}$ spanned by C_w, $w \underset{LR}{\sim} r_0r_2$, is generated by C_{02}. Hence for any simple constituent L of $E_{s,q}$, $c_L = c$ if and only if $C_{02}L \neq 0$. We have

(a)
$$\phi_{q_{0,c}}(C_{02}) = \begin{pmatrix} [2]^2 & [2] & 0 & V(1) & [2]V(1) & [2]V(1) \\ & & & 0 & & \end{pmatrix}$$

(b)
$$\phi_{q_{0,c}}(C_{02121}) = \begin{pmatrix} V(1) & [2]V(1) & V(1)+[2] & [2]^2 & [2] & [2] \\ & & & 0 & & \end{pmatrix}$$

where $[2] = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$.

By (a) we know that $E_{s,q}$ has at most one simple constituent to which the attached two-sided cell is c. In fact, when $D = C_{02}E_{s,q} = 0$, $E_{s,q}$ obviously has no such simple constituent. When $D \neq 0$, from (a) we see that $\dim D = 1$. Let N be the $\mathbf{H}_q$-submodule of $E_{s,q}$ generated by D, then N has a maximal submodule N_0 which does not contain D, so $C_{02}N_0 = 0$. Note that $C_{02}(E_{s,q}/N) = 0$, we know that $E_{s,q}$ has at most one simple constituent L such that $c_L = c$.

If $D \neq 0$, then either $[2] \neq 0$ or $\varphi(s) = tr(s, V(1)) \neq 0$, using (a) and (b) we see that either $C_{02}D = D$ or $C_{02121}D = D$. Thus we have either $C_{02}(N/N_0) \neq 0$ or $C_{02121}(N/N_0) \neq 0$. That is to say, $E_{s,q}$ has a simple constituent $L_s = N/N_0$ with $c_{L_s} = c$.

From (b) we see that the eigenpolynomial of C_{02121} on L_s is $(\lambda - \varphi(s))\lambda^{b(s)}$, where $b(s) = \dim L_s - 1$. Since $s \to \varphi(s)$ defines a bijection between the set of semisimple conjugacy classes and $\mathbb{C}$, we know that when $C_{02}E_{s,q} \neq 0$, $C_{02}E_{t,q} \neq 0$, then $L_s \simeq L_t$ if and only if $s = t$, i.e. $E_s = E_t$.

The ($\blacktriangledown$) is proved for the two-sided cell c_2 in W.

We also showed that $(\phi_q)_{*,c}$ is an isomorphism when $q + 1 \neq 0$ and $(\phi_q)_{*,c}(E_{s,q}) = 0$ if and only if $q + 1 = 0$, $\varphi(s) = 0$. It is known that only one semisimple conjugacy class s in $SL_2(\mathbb{C})$ such that $\varphi(s) = 0$.

(D). Case $c = c_3$. We have $\mathbf{J}_c \simeq M_{6\times 6}(\mathbf{R}_{F_c})$ (see 7.2) and

$\phi_{q,c}(C_{010}) =$

(c) $$\begin{pmatrix} [2]^3-[2] & [2]V(1)+[2]^2 & [2]^2V(1)+[2] & V(1)^2+[2]V(1) & [2]V(2) & [2]^2V(2) \\ & & & 0 & & \end{pmatrix}$$

(d) $$C_{0102}C_{010} = [2]V(1) + [2]^2$$

(e) $$C_{010212}C_{101} = C_{010210210} + [2]C_{010210} + C_{010}$$

(f) $$C_{010210212}C_{010} = C_{010210210210} + [2]C_{010210210} + 2C_{010210} + [2]C_{010}$$

Note that $F_c = SL_2(\mathbb{C})$. For a semisimple conjugacy class s of F_c, let ψ_s, E_s be as in (C). Via $\phi_{q,c}$, E_s gives rise to an $\mathbf{H}_q$-module $E'_{s,q}$. As the same way as in (C), we know that $E'_{s,q}$ has at most one simple constituent such that to which the attached two-sided cell is $c = c_3$. When $C_{010}E'_{s,q} = 0$, $E'_{s,q}$ has no such simple constituent. Moreover, if $C_{010}E'_{s,q} \neq 0$, then $E'_{s,q}$ has a unique simple constituent L'_s such that $c_{L'_s} = c$. let $b'(s) = \dim L'_s - 1$, from (c-f) we know that the eigenpolynomials of C_{0102}, C_{010212}, $C_{010210212}$ on L'_s are $(\lambda - [2]\varphi(s) - [2]^2)\lambda^{b'(s)}$, $(\lambda - \varphi(s)^2 - [2]\varphi(s))\lambda^{b'(s)}$, $(\lambda - \varphi(s)^3 - [2]\varphi(s)^2)\lambda^{b'(s)}$, respectively. Thus if $C_{010}E'_{s,q} \neq 0$, $C_{010}E'_{t,q} \neq 0$, then $L'_s \simeq L'_t$ if and only if $s = t$, i.e. $E_s = E_t$. We have proved (▼) for $c = c_3$.

It is obvious that $C_{010}E'_{s,q} = 0$ if and only if $q + 1 = 0$, $\varphi(s) = 0$ or $q^2 + q + 1 = 0$, $\varphi(s) + [2] = 0$. Thus $(\phi_q)_{*,c}$ is an isomorphism if $[2]([2]^2 - 1) \neq 0$. If $[2] = 0$ or $[2]^2 = 1$, there exists a unique semisimple conjugacy class s in G such that $(\phi_q)_{*,c}(E_s) = 0$.

In (E-F) we assume that W is of type $\tilde{B}_2$.

(E). Case $c = c_2$, then $\mathbf{J}_c \simeq M_{4\times 4}(\mathbf{R}_{F_c})$, $F_c = \mathbf{Z}/2 \times SL_2(\mathbb{C})$ (see 7.2). We have

$$\text{(g)} \qquad \phi_{q,c}(C_{02}) = \begin{pmatrix} [2]^2 & V(1)+[2] & [2]V(1) & [2]V(1) \\ & & 0 & \end{pmatrix}$$

$$\text{(h)} \qquad \phi_{q,c}(C_{021}) = \begin{pmatrix} V(1)+[2] & [2]V(1)+[2]^2 & V(1)+[2] & V(1)+[2] \\ & & 0 & \end{pmatrix}$$

$$\text{(i)} \qquad \phi_{q,c}(C_\omega) = \begin{pmatrix} \varepsilon & & & 0 \\ & \varepsilon & & \\ & & \varepsilon & \\ 0 & & & \varepsilon \end{pmatrix}$$

where ε is the sign representation of $\mathbf{Z}/2$ and we also write ε instead of $(\varepsilon, V(0))$.

For each semisimple conjugacy class s of F_c, let ψ_s be the simple representation of $\mathbf{J}_c$ defined by

$$\psi_s(m_{ij}) = (tr(s, m_{ij}) \in M_{4\times 4}(\mathbb{C}).$$

Let E_s be a simple $\mathbf{J}_c$-module which provides ψ_s. Each simple $\mathbf{J}_c$-module is isomoprhic to some E_s. Via $\phi_{q,c}$, E_s gives rise to an $\mathbf{H}_q$-module $E_{s,q}$.

As the same way as in (C), we know that $E_{s,q}$ has no simple constituent such that to which the attached two-sided cell is c when $C_{02}E_{s,q} = 0$, and $E_{s,q}$ has exactly one simple constituent L_s such that $c_{L_s} = c$ when $C_{02}E_{s,q} \neq 0$. Moreover, the eigenpolynomials of C_ω, C_{021} on L_s are $(\lambda - \varphi'(s))^{b(s)+1}$, $(\lambda - \varphi(s) - [2])\lambda^{b(s)}$, respectively, where $\varphi'(s) = tr(s, \varepsilon)$, $\varphi(s) = tr(s, V(1))$, $b(s) = \dim L_s - 1$, so if $C_{02}E_{s,q} \neq 0$, $C_{02}E_{t,q} \neq 0$, then $L_s \simeq L_t$ if and only if $s = t$. We have proved the ($\blacktriangledown$) for the two-sided cell $c = c_2$.

It is obvious that $C_{02}E_{s,q} = 0$ if and only if $q+1 = 0$, $\varphi(s) = 0$. Hence $(\phi_q)_{*,c}$ is an isomorphism when $q+1 \neq 0$, and there exist two semisimple conjugacy classes s_1, s_2 such that $(\phi_q)_{*,c}(E_{s_i}) = 0$ $(i = 1, 2)$ when $q+1 = 0$.

(F). Case $c = c_1$. We have $F_c = \mathbf{Z}/2 \ltimes \mathbb{C}^*$, $\mathbf{Z}/2$ acts on $\mathbb{C}^*$ by $z \to z^{-1}$. Any element in F_c is semisimple. Let $e \neq \alpha \in \mathbf{Z}/2$, then $\alpha z = z^{-1}\alpha$, and the set $s(\alpha) = \{\alpha z \mid z \in \mathbb{C}^*\}$ is the conjugacy class containing α. For any $z \in \mathbb{C}^*$, let $s(z)$ be the conjugacy class containing

z. Then $s(1)=\{1\}$, $s(-1)=\{-1\}$, $s(z)=\{z,z^{-1}\}$ if $z^2\neq 1$. By 7.2 and 3.4 we know that $\{E_{s(\alpha)}, E_{s(z)}\,(z\in\mathbb{C}^*), E_{s(1),\varepsilon}, E_{s(-1),\varepsilon}\}$ is a complete set of simple $\mathbf{J}_c$-modules, where ε is the sign representation of $\mathbf{Z}/2$.

It is not difficult to see that $E_{s(\alpha),q}$ is a simple $\mathbf{H}_q$-module and $C_0E_{s(\alpha),q}\neq 0$ when $q+1\neq 0$, $(\phi_q)_{*,c}(E_{s(\alpha)})=0$ when $q+1=0$. We always have $C_1E_{s(\alpha),q}=0$.

One verifies that each $E_{s(z),q}$ $(z\in\mathbb{C}^*)$ has exactly one simple constituent to which the attached two-sided cell is c, we denote it by $L_{s(z)}$, $L_{s(z)}$ in fact is a quotient module of $E_{s(z),q}$. the eigenpolynomial of $C_{r_1r_0\omega}$ on $L_{s(z)}$ is $(\lambda-z-z^{-1})\lambda^{b(z)}$, where $b(z)=\dim L_{s(z)}-1$. So $L_{s(z)}\simeq L_{s(z')}$ if and only if $s(z)=s(z')$ when $z,z'\in\mathbb{C}^*$. One checks that for any $z\in\mathbb{C}^*$, $C_1L_{s(z)}\neq 0$ if $q+1\neq 0$, thus $L_{s(z)}\not\simeq E_{s(\alpha),q}$ when $q+1\neq 0$ for any $z\in\mathbb{C}^*$.

We have $\dim E_{s(i),\varepsilon,q}=1$, $i=\pm 1$, C_0, C_2, C_1, C_ω acts on $E_{s(i),\varepsilon,q}$ by scalars 0, 0, $[2]$, i, respectively. So $(\phi_q)_{*,c}(E_{s(i),\varepsilon})=0$ if $q+1=0$ and $(\phi_q)_{*,c}(E_{s(i),\varepsilon})=E_{s(i),\varepsilon,q}=L_{i,\varepsilon}$ if $q+1\neq 0$. Now assume that $q+1\neq 0$. Obviously we have $L_{1,\varepsilon}\not\simeq L_{-1,\varepsilon}$, $L_{i,\varepsilon}\not\simeq E_{s(\alpha),q}$ $(i=\pm 1)$. It is easy to see that $C_1C_0L_{s(z)}\neq 0$ $(z\in\mathbb{C}^*)$, so $L_{s(z)}\not\simeq L_{i,\varepsilon}$ $(z\in\mathbb{C}^*,\ i=\pm 1)$. The ($\blacktriangledown$) is proved for $c=c_1$.

In the above discussion we see that $(\phi_q)_{*,c}$ is an isomorphism when $q+1\neq 0$ and rank $\ker(\phi_q)_{*,c}=3$ when $q+1=0$.

(G). Since $(\phi_q)_*$ is an isomorphism if and only if $(\phi_q)_{*,c}$ is an isomorphism for any two-sided cell c of W, we see that 7.4(iv) is true according to (A-F).

Theorem 7.4 is proved.

7.5. In the proof of 7.4 we have determined the $\ker(\phi_q)_{*,c}$ explicitly for two-sided cell c in $W-c_0$. Now we determine $\ker(\phi_q)_{*,c_0}$, in a different way from that in chapter 5, which is 0 when $\sum_{w\in W_0} q^{l(w)}\neq 0$.

We denote w_0 the longest element of W_0. Let x_i $(i=1,2)$ be the i-th basic dominant weight in X, then $r_ix_j=x_jr_i\in c_0$ $(i\neq j\in\{1,2\})$, $x_1x_2\in c_0$. In $\mathbf{H}_q$ we have

(a) Case $\tilde{B}_2$: $C_{w_0}C_{x_1r_2}=[2](C_{w_0x_1}+([2]^2-1)C_{w_0})$,
$C_{w_0}c_{x_2r_1}=[2]C_{w_0x_2}$, $C_{w_0}C_{x_1x_2}=C_{w_0x_1x_2}+[2]^2C_{w_0x_2}$

(b) Case $\tilde{G}_2$: $C_{w_0}C_{x_1r_2}=[2](C_{w_0x_1}+[5]C_{w_0})$,

$$C_{w_0}C_{x_2r_1} = [2](C_{w_0x_2} + ([2]^2-1)C_{w_0x_1} + C_{w_0}),$$
$$C_{w_0}C_{x_1x_2} = C_{w_0}S_{x_1}S_{x_2} + ([2]^2-1)C_{w_0}S_{x_1}^2 + [5]C_{w_0}S_{x_2} + [2]^2C_{w_0}S_{x_1} + [5]C_{w_0}$$

where $[5] = q^2+q+1+q^{-1}+q^{-2}$, S_{x_1}, S_{x_2} are defined as in [X, II, 2.8].

Let $V(x_i)$ $(i=1,2)$, $V(x_1x_2)$ be the simple G-modules of highest weight x_i, x_1x_2, respectively. Then $s \to \varphi(s) = (\lambda_1, \lambda_2)$ defines a bijection between the set of semisimple conjugacy classes of G and $\mathbb{C}^2$, where $\lambda_i = tr(s, V(x_i))$ $(i=1,2)$. Let E_s be the simple $\mathbf{J}_{c_0}$-module corresponding to a semisimple conjugacy class s of G (see [X]). According to [X2, 3.9] and (a), (b) we have the following results.

(c) Case $\tilde{B}_2$: When $q^2+1=0$, $(\phi_q)_{*,c_0}(E_s)=0$ if and only if $\varphi(s)=(-1,0)$. When $q+1=0$, $(\phi_q)_{*,c_0}(E_s)=0$ if and only if $\lambda_1\lambda_2=\lambda_2$, i.e. $tr(s,V(x_1x_2))=0$. when $(q+1)(q^2+1)\neq 0$, we know that $(\phi_q)_*$ is an isomorphism.

(d) Case $\tilde{G}_2$: When $q^4+q^2+1=0$, $(\phi_q)_{*,c_0}(E_s)=0$ if and only if $\varphi(s)=(q^3,q^3)$, when $q+1=0$, $(\phi_q)_{*,c_0}(E_s)=0$ if and only if $\lambda_1\lambda_2-\lambda_1^2+\lambda_2+1=0$, i.e. $tr(s,V(x_1x_2))=0$. When $(q+1)(q^4+q^2+1)\neq 0$, we know that $(\phi_q)_*$ is an isomorphism.

7.6. Now we assume that W is an arbitrary irreducible affine Weyl group. Let e_n be the largest exponent of W_0. When q is a primitive (e_n+1)-th root of 1, then rank $\ker(\phi_q)_{*,c_0}=1$ (see [X2]). It is likely that rank $\ker(\phi_q)_*=1$ in this case, i.e. $(\phi_q)_{*,c}$ is an isomorphism when $c\neq c_0$.

7.7. Relations between various $\mathbf{H}_q$. Let G be a simple algebraic group over $\mathbb{C}$. Let W_0 be its Weyl group and W be its extended affine Weyl group. Let $\mathbf{H}_{q,W_0}$ be the Hecke algebra over $\mathbb{C}$ of W_0 with parameter $q\in\mathbb{C}^*$. When $\sum_{w\in W_0} q^{l(w)}\neq 0$, It is known that there are natural isomorphisms of $\mathbb{C}$-algebras

$$\mathbf{H}_{q,W_0}\simeq \mathbb{C}[W], \qquad \mathbf{H}_{q,W_0}\simeq \mathbf{J}_{W_0}, \quad \text{(see [L3, GU])},$$

where $\mathbf{J}_{W_0}\subset \mathbf{J}$ stands in an obvious sense.

Reall that $\mathbf{H}_q$ is the Hecke algebra over $\mathbb{C}$ of W with parameter $q\in\mathbb{C}^*$ and $\mathbf{J}$ is the asymototic algebra of $\mathbf{H}_q$, $q\in\mathbb{C}^*$ defined in 2.7. The homomorphism $\phi_q:\ \mathbf{H}_q\to\mathbf{J}$ is injective but never surjective. Actually it is impossible to find an isomorphism between

$\mathbf{H}_q$ and $\mathbf{J}$ for any $q \in \mathbb{C}^*$. Now we would like to look the relations between $\mathbf{H}_q$ and $\mathbf{H}_1 = \mathbb{C}[W]$.

When G is of type A_1, W is of type $\tilde{A}_1$. Let s, t be the simple reflections in W. When $q+1 \neq 0$, there is a unique isomorphism of $\mathbb{C}$-algebra between $\mathbf{H}_q$ and $\mathbb{C}[W]$ such that

$$T_s \to \frac{q+1}{2}s + \frac{q-1}{2}, \qquad T_t \to \frac{q+1}{2}t + \frac{q-1}{2}.$$

$\mathbf{H}_{-1}$ has two (resp. one) simple modules of dimension 1 when G is simply connected (resp. adjoint), $\mathbf{H}_1$ has four simple modules of dimension 1, so $\mathbf{H}_{-1} \not\simeq \mathbb{C}[W] = \mathbf{H}_1$.

Now we assume that $G = SL_3(\mathbb{C})$, the simply connected, simple algebraic group over $\mathbb{C}$ of type A_2. We shall show that

(a) $\mathbf{H}_q \not\simeq \mathbb{C}[w]$ *whenever* $q \neq 1$.

The extended affine Weyl group $W = \Omega \ltimes W'$ has three two-sided cells: $c_e = \Omega$, $c_1 = \{w \in W \mid a(w) = 1\}$, $c_0 = \{w \in W \mid a(w) = 3\}$. We have (see [X1-X2])

(b). $\mathbf{J}_{c_0} \simeq M_{6\times 6}(\mathbf{R}_G)$, $\quad \mathbf{J}_{c_1} \simeq M_{3\times 3}(\mathbf{A})$, $\quad \mathbf{J}_{c_e} \simeq \mathbb{C}[\Omega]$. Note that $\mathbf{A} = \mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$.

Each $\mathbf{J}_c$-module E gives rise to an $\mathbf{H}_q$-module E_q via the homomorphism $\phi_{q,c}\colon \mathbf{H}_q \to \mathbf{J} \to \mathbf{J}_c$, where c is a two-sided cell of W. Each $\mathbf{J}$-module E gives rise to an $\mathbf{H}_q$-module via the homomorphism $\phi_q : \mathbf{H}_q \to \mathbf{J}$. Note that $\mathbf{J} = \mathbf{J}_{c_e} \oplus \mathbf{J}_{c_1} \oplus \mathbf{J}_{c_0}$.

We recall (see 3.9(e) and [X2])

(c) Any simple $\mathbf{H}_q$-module L is a quotient module of E_q for some simple $\mathbf{J}$-module E with $c_E = c_L$.

(d) Assume that E is a simple $\mathbf{J}$-module, then E_q has at most one simple constituent L such that $c_L = c_E$.

(e) Given two simple $\mathbf{J}$-modules E, E' such that E_q has a simple constituent L with $c_L = c_E$ and E'_q has a simple constituent L' with $c_{L'} = c_{E'}$. Then $L \simeq L'$ if and only if $E \simeq E'$.

For each semisimple element s of G, we have a simple $\mathbf{J}_{c_0}$-module E_s obtained through the simple representation ψ_s of $\mathbf{J}_{c_0}$:

$$\psi_s\colon \mathbf{J}_{c_0} \simeq M_{6\times 6}(\mathbf{R}_G) \to M_{6\times 6}(\mathbb{C}),$$

$$(m_{ij}) \to (tr(s, m_{ij})).$$

It is known that $E_s \simeq E_t$ if and only if s, t are conjugate in G.

For each element $a \in \mathbb{C}$, we have a simple $\mathbf{J}_{c_1}$-module E_a obtained through the simple representation ψ_s of $\mathbf{J}_{c_1}$:

$$\psi_a : \mathbf{J}_{c_1} \simeq M_{3\times 3}(\mathbf{A}) \to M_{3\times 3}(\mathbb{C}),$$
$$\text{specialize } \mathbf{q} \text{ to } a.$$

It is obvious that $E_a \simeq E_b$ if and only if $a = b$.

We have

(f) $E_{s,q}$ ($s \in G$ semisimple) has a simple constituent $L_{s,q}$ such that $c_{L_{s,q}} = c_0$ if and only if $\mathbf{g}_{s,q} = \mathcal{N}_{s,q}$ or $q = 1$. Moreover

$$\dim L_{s,q} = \begin{cases} 6, & \text{if } \mathcal{N}_{s,q} = \{0\}, \\ 3, & \text{if } \mathcal{N}_{s,q} \neq \{0\} \text{ and } \mathcal{N}_{s,q} \text{ doesnot contains} \\ & \quad \text{any regular nilpotent element of } \mathbf{g}, \\ 1, & \text{if } \mathcal{N}_{s,q} \text{ contains some regular nilpotent element of } \mathbf{g}. \end{cases}$$

(g) $E_{a,q}$ ($a \in \mathbb{C}$) has a unique simple constituent $L_{a,q}$ such that $c_{L_{a,q}} = c_1$ for any $a \in \mathbb{C}$. Moreover (see 8.3(A))

$$\dim L_{a,q} = \begin{cases} 3, & \text{if } (a + q^{\frac{1}{2}}\xi^i)(a + q^{-\frac{1}{2}}\xi^i) \neq 0, \\ 2, & \text{if } (a + q^{\frac{1}{2}}\xi^i)(a + q^{-\frac{1}{2}}\xi^i) = 0 \text{ and } q^2 + q + 1 \neq 0, \\ 1, & \text{if } (a + q^{\frac{1}{2}}\xi^i)(a + q^{-\frac{1}{2}}\xi^i) = 0 \text{ and } q = \xi^k \neq 1, \end{cases}$$

where ξ is a primitive 3rd root of 1.

(h) For any simple $\mathbf{J}_{c_e}$-module E, E_q is a simple $\mathbf{H}_q$-module. We always $\dim E_q = 1$.

By (f-h) we see that $\mathbb{C}[W]$ has three simple modules of dimensions 2, $\mathbf{H}_q$ has six simple modules of dimensions 2 when $q^3 - 1 \neq 0$, and $\mathbf{H}_q$ has no simple modules of dimensions 2 when $q^2 + q + 1 = 0$. Now the assertion (a) follows. I donot know whether $\mathbf{H}_q \simeq \mathbf{H}_p$ when $(q^3 - 1)(q^2 - 1) \neq 0$ and $(p^3 - 1)(p^2 - 1) \neq 0$.

Now assume W is arbitrary, it is likely that $\mathbf{H}_q \not\simeq \mathbf{H}_1$ whenever $q \neq 1$ and W is not of type $\tilde{A}_1 \times \cdots \tilde{A}_1$. It seems interesting to find relations among various $\mathbf{H}_q$ ($q \in \mathbb{C}^*$). Of course we have $\mathbf{H}_q \simeq \mathbf{H}_{q^{-1}}$ by 1.6(e).

8. Simple Modules Attached to c_1

Let G be a simply connected, simple complex algebraic group of rank n. Let W be its extended affine Weyl group (see 2.1), then $W = \Omega \ltimes W'$ for certain abelian group Ω and certain Affine Weyl group W'. The second highest two-sided cell c_1 of W is described in [L4]. We have

$$c_1 = \{w \in W \mid a(w) = 1\}.$$

In this chapter we prove the conjecture in [L14] for c_1, then classify the simple $\mathbf{H}_q$-modules to which the attached two-sided cell is c_1 and determine the dimensions of these simple $\mathbf{H}_q$-modules. From the results one can easily get the multiplicities of a simple $\mathbf{H}_q$-module in the standard modules $M_{s,N,q,\rho}$ when N is a subregular nilpotent element in the Lie algebra $\mathbf{g}$ of G. The multiplicities can be interpratated as the dimensions of certain cohomology groups (see [G3]). This chapter is based on preprint [X4].

8.1. For the two-sided cell c_1 of W, let $F_1 = F_{c_1}$ be the reductive complex algebraic group attached to c_1 as in 3.14, then

$$F_1 = \begin{cases} \mathbb{C}^*, & \text{type } \tilde{A}_n\ (n \geq 2) \\ \mathbf{Z}/2 \times \mathbf{Z}/2, & \text{type } \tilde{B}_n\ (n \geq 3) \\ \mathbf{Z}/2 \ltimes \mathbb{C}^*, & \text{type } \tilde{C}_n\ (n \geq 2) \\ \Omega, & \text{type } \tilde{D}_n\ (n \geq 4),\ \tilde{E}_n\ (n = 6, 7, 8) \\ \mathbf{Z}/2, & \text{type } \tilde{F}_4 \\ \mathfrak{S}_3, & \text{type } \tilde{G}_2. \end{cases}$$

We refer to chapter 3, especially 3.13 for notations. The main results of the chapter are the following.

Theorem 8.2. *We keep the set up in 8.1.*

(a) *There exists a finite F_1-set Y and a bijection*

$$\pi\colon c_1 \xrightarrow{\sim} \text{set of irreducible } F_1\text{-v.b. on } Y \times Y \text{ (up to isomorphism)}$$

such that

(i) *The* $\mathbb{C}$*-inear map* $\pi: \mathbf{J}_1 \to \mathbf{K}_{F_1}(Y \times Y),\ t_w \to \pi(w)$ *is an algebra isomorphism (preserving the unit element).*

(ii) $\pi(w^{-1}) = \widetilde{\pi(w)}$ $(w \in c_1)$, *where* $\mathbf{J}_1 = \mathbf{J}_{c_1}$.

(b) *Given a simple* $\mathbf{J}_1$*-module* E, E_q *(see 3.13 for the definition) has at most one simple constituent* L *such that* $c_L = c_1$. *Suppose that the simple* $\mathbf{J}_1$*-module* E *(resp.* E'*),* E_q *(resp.* E'_q*) has a simple constituent* L *(resp.* L'*) such that* $c_L = c_1$ *(resp.* $c_{L'} = c_1$*), then* $L \simeq L'$ *if and only if* $E \simeq E'$. *Thus The set* $\Lambda_1 = \{(\phi_q)_{*,c_1}(E) \mid E$ *a simple* $\mathbf{J}_1$*-module (up to isomorphism)*$\} - \{0\}$ *is the set of simple* $\mathbf{H}_q$*-modules (up to isomorphism) with attached two-sided cell* c_1 *(cf. 3.9(e)).*

The theorem supports the conjecture in [L14] and the idea in 3.13.

8.3. The rest of the chapter will be concerned with the proof of 8.2. We do it case by case, also we determine the dimensions of c_1-modules. We make some conventions:

If $s_{i_1}s_{i_2}\dots s_{i_k}$ is a reduced express of an element $w \in W_a$, we often write $i_1i_2\dots i_k$, $C_{i_1i_2\dots i_k}$, $t_{i_1i_2\dots i_k}$ instead of w, C_w, t_w respectively. For any simple $\mathbf{J}_1$-module E, we also use $(\phi_q)_{*,c_1}(E)$ for the direct sum of simple consituents of E_q to which the attached two-sided cell c_1.

(A). Type $\tilde{A}_n$ $(n \geq 2)$. In this part we assume that W is of type $\tilde{A}_n$ $(n \geq 2)$ (we omit the case $\tilde{A}_1$, see [X2] for the case).

Let $\omega \in \Omega$ be such that $\omega s_n = s_0\omega$, $\omega s_i = s_{i+1}\omega$ $(0 \leq i \leq n-1)$. Then we have $c_1 = \{w(i,j,k),\ u(i,j,k) \mid 0 \leq i,j \leq n,\ k \geq 0\}$, where $w(i,j,k) = \omega^i s_1(\omega s_1)^k\omega^{-j}$, $u(i,j,k) = \omega^i s_1(\omega^{-1}s_1)^k\omega^{-j}$.

Let $Y = \{1, 2, \dots, n, n+1\}$ and let $F_1 = \mathbb{C}^*$ acts on Y trivially. We shall regard $\mathbf{q}^k$ $(k \in \mathbf{Z})$ as the simple representation $\mathbb{C}^* \to \mathbb{C},\ z \to z^k$. Then it is easy to check that the following bijection

$$\pi : c_1 \xrightarrow{\sim} \text{set of irreducible } F_1\text{-v.b. on } Y \times Y \text{ (up to isomorphism)}$$

$$w(i,j,k) \to \mathbf{q}^{k}_{(i+1,j+1)}, \qquad u(i,j,k) \to \mathbf{q}^{-k}_{(i+1,j+1)}$$

has the properties (i) and (ii) in 8.2(a).

The convolution algebra $\mathbf{K}_{F_1}(Y \times Y)$ is naturally isomorphic to $M_{n+1}(\mathbf{A})$, the $(n+1) \times (n+1)$ matrix ring over $\mathbf{A} = \mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$. For each $a \in \mathbb{C}^*$, specialize $\mathbf{q}$ to a, then we get a simple representation of $\mathbf{J}_1$:

$$\psi_a : \mathbf{J}_1 \to \mathbf{K}_{F_1}(Y \times Y) \to M_{n+1}(\mathbf{A}) \to M_{n+1}(\mathbb{C}).$$

Any simple representation of $\mathbf{J}_1$ is isomorphic to some ψ_a. Let E_a be a simple $\mathbf{J}_1$-module providing the representation ψ_a, then $\dim E_a = n+1$. E_a gives rise to an $\mathbf{H}_q$-module $E_{a,q}$ via ϕ_{q,c_1}. Let v_i $(1 \le i \le n+1)$ be the natural base of $E_{a,q}$, then we have

(a1) $C_\omega v_{n+1} = v_1,\ C_\omega v_i = v_{i+1}\ (0 \le i \le n-1.)$

(a2) $C_1 v_1 = [2] v_1,\ C_1 v_2 = a v_1,\ C_1 v_{n+1} = a^{-1} v_1,\ C_1 v_i = 0\ (2 \le i \le n).$

where $[2] = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$.

Since $C_w = h_1 C_1 h_2$ for some $h_1, h_2 \in \mathbf{H}_q$ when $w \in c_1$, by (a2) we see that $\dim C_1 E_{a,q} = 1$, so $E_{a,q}$ has a unique simple constituent to which the attached two-sided cell is c_1, we denote it by L_a. The eigenpolynomial of $C_{1\omega}$ on L_a is $(\lambda - a)\lambda^{b(a)}$, where $b(a) = \dim L_a - 1$. Therefore $L_a \simeq L_b$ if and only if $a = b$. Thus 8.2(b) is proved for type $\tilde{A}_n$ $(n \ge 2)$.

Now we determine the dimension of L_a. L_a in fact is the unique simple quotient module of $E_{a,q}$. Let N_a be the maximal submodule of $E_{a,q}$, then we have

$$N_a = \begin{cases} 0, & \text{if } (a + q^{\frac{1}{2}}\xi^i)(a + q^{-\frac{1}{2}}\xi^i) \ne 0, \\ < \kappa_i >, & \text{if } (a + q^{\frac{1}{2}}\xi^i)(a + q^{-\frac{1}{2}}\xi^i) = 0, \text{ and } \sum_{m=0}^{n} q^m \ne 0, \\ < \kappa_i,\ \kappa_{i+k} >, & \text{if } a + q^{\frac{1}{2}}\xi^i = 0 \text{ and } q = \kappa^k \ne 1, \\ < \kappa_i,\ \kappa_{i-k} >, & \text{if } a + q^{-\frac{1}{2}}\xi^i = 0 \text{ and } q = \kappa^k \ne 1, \end{cases}$$

where $\kappa_i = v_1 + \xi^{-i}v_2 + \cdots + \xi^{-ni}v_{n+1}$, ξ is a primitive $(n+1)$-th root of 1. By this we get the following fact.

$$\dim L_a = \begin{cases} n+1, & \text{if } (a+q^{\frac{1}{2}}\xi^i)(a+q^{-\frac{1}{2}}\xi^i) \neq 0, \\ n, & \text{if } (a+q^{\frac{1}{2}}\xi^i)(a+q^{-\frac{1}{2}}\xi^i) = 0 \text{ and } \sum_{m=0}^{n} q^m \neq 0, \\ n-1, & \text{if } (a+q^{\frac{1}{2}}\xi^i)(a+q^{-\frac{1}{2}}\xi^i) = 0 \text{ and } q = \xi^k \neq 1. \end{cases}$$

(B). Type $\tilde{B}_n$ $(n \geq 3)$. In this part we assume that W is of type $\tilde{B}_n$ $(n \geq 3)$.

Let $\omega \in \Omega$ be such that $\omega s_0 = s_1\omega$, $\omega s_1 = s_0\omega$, $\omega s_i = s_i\omega$ $(2 \leq i \leq n)$. Then we have

$$\begin{aligned} c_1 = \{&0\omega^p,\ 021\omega^p,\ 120\omega^p,\ 023\ldots n(n-1)\ldots 20\omega^p,\ n(n-1)n\omega^p \\ &023\ldots n(n-1)\ldots i\omega^p\ (1 \leq i \leq n-1), \quad 023\ldots i\omega^p\ (2 \leq i \leq n), \\ &i(i+1)\ldots j\omega^p\ (1 \leq i \leq j \leq n), \quad i(i-1)\ldots j\omega^p\ (1 \leq j \leq i \leq n), \\ &i(i+1)\ldots n(n-1)\ldots 20\omega^p\ (1 \leq i \leq n), \quad i(i-1)\ldots 20\omega^p\ (2 \leq i \leq n), \\ &i(i+1)\ldots n(n-1)\ldots j\omega^p\ (1 \leq i \leq n,\ 1 \leq j \leq n-1) \quad \mid p = 0,1\}. \end{aligned}$$

For each element $w \in c_1$, there exist unique i, j such that $l(s_iw) < l(w)$, $l(ws_j) < l(w)$. Assume that $w\omega^p \in W_a$, then w is completely determined by $i, j, l(w), p$, we then write $w(i,j,k,p)$ instead of w, where $k = l(w)$.

Let $Y = \{0, 1, \ldots, n, n'\}$. We define an action of $F_1 = \mathbf{Z}/2 \times \mathbf{Z}/2$ on Y by setting $ai = i$ $(0 \leq i \leq n-1)$, $a \in F_1$ and $a_1n = a_2n = n'$, where $a_1 = (\bar{1}, \bar{0})$, $a_2 = (\bar{0}, \bar{1}) \in F_1$. Let V_i $(1 \leq i \leq 4)$ be the simple F_1-module such that a_1, a_2 acts on V_1 by scalar $-1, 1$, on V_2 by scalar $1, -1$, on V_3 by scalar $-1, -1$, on V_4 by scalar $1, 1$. Let V_1', V_2' be simple $F_1' = \{e, a_1a_2\}$-module such that a_1a_2 acts on V_i' by scalar $(-1)^i$ $(i = 1, 2)$, where $e = (\bar{0}, \bar{0}) \in F_1$.

We define a bijection $\pi: c_1 \tilde{\rightarrow}$ set of irreducible F_1-v.b. on $Y \times Y$ (up to isomorphism) as follows.

If $i \neq n \neq j$, we set

$$\pi(w(i,j,k,p)) = \begin{cases} V_{1(i,j)}, & \text{if } p = 1 \text{ with } k \text{ maximal}, \\ V_{2(i,j)}, & \text{if } p = 1 \text{ with } k \text{ minimal}, \\ V_{3(i,j)}, & \text{if } p = 0 \text{ with } k \text{ maximal}, \\ V_{4(i,j)}, & \text{if } p = 0 \text{ with } k \text{ minimal}. \end{cases}$$

If $i \neq j$ and $i = n$ or $j = n$, we set

$$\pi(w(i,j,k,p)) = \begin{cases} V'_{1(i,j)}, & \text{if } p = 1 \ , \\ V'_{2(i,j)}, & \text{if } p = 0, \end{cases}$$

and we set

$$\pi(n) = V'_{2(n,n)},\ \pi(n\omega) = V'_{1(n,n)},\ \pi(n(n-1)n) = V'_{2(n,n')},\ \pi(n(n-1)n\omega) = V'_{1(n,n')}.$$

One may check that π induces an isomorphism π of $\mathbb{C}$-algebra between $\mathbf{J}_1$ and $\mathbf{K}_{F_1}(Y \times Y)$ and $\pi(w^{-1}) = \widetilde{\pi(w)}$ if $w \in c_1$. 8.2(a) is proved in this case.

Now we consider simple $\mathbf{J}_1$-modules. There are four semisimple conjugacy classes in F_1: e, a_1, a_2, a_1a_2. For any $a \in F_1$, we have $A(a) = F_1$. According to 3.4, we see that $\mathbf{J}_1$ has six simple modules (up to isomorphism): $E_1 = E_{a_1}$, $E_2 = E_{a_2}$, $E_3 = E_{a_1a_2}$, $E_4 = E_e$, $E_5 = E_{a_1a_2,V_3}$, $E_6 = E_{e,V_3}$. Via ϕ_{q,c_1}, E_p $(1 \leq p \leq 6)$ gives rise to an $\mathbf{H}_q$-module $E_{p,q}$.

By definition, $E_{p,q}$ $(p = 1,2)$ has a base v_i $(0 \leq i \leq n-1)$ defined by $v_i : Y^{a_p} \to \mathbb{C},\ j \to \delta_{ij},\ (0 \leq j \leq n-1)$, $p = 1,2$. We have

(b1) $C_0v_0 = [2]v_0,\ C_0v_2 = v_0,\ C_0v_i = 0\ (i \neq 0,2,\ 1 \leq i \leq n-1)$.

(b2) $C_\omega v_0 = (-1)^{p-1}v_1,\ C_\omega v_1 = (-1)^{p-1}v_0,\ C_\omega v_i = (-1)^{p-1}v_i\ (2 \leq i \leq n-1)$.

(b3) $C_2v_0 = C_2v_1 = C_2v_3 = v_2,\ C_2v_2 = [2]v_2,\ C_iv_i = 0\ (4 \leq i \leq n-1,\ n \geq 4)$ and $C_2v_0 = C_2v_1 = v_2,\ C_2v_2 = [2]v_2,$ when $n = 3$.

(b4) $C_iv_{i-1} = C_iv_{i+1} = v_i,\ C_iv_i = [2]v_i,\ C_iv_j = 0\ (0 \leq j \leq n-1,\ j \neq i, i-1, i+1,\ 3 \leq i \leq n-2,\ n \geq 4)$.

(b5) $C_{n-1}v_{n-2} = v_{n-1},\ C_{n-1}v_{n-1} = [2]v_{n-1},\ C_{n-1}v_j = 0\ (0 \leq j \leq n-2,\ n \geq 4)$.

We always have

(b6) $C_nE_{p,q} = 0\ (p = 1,2)$.

By (b1-b5) we see that $E_{p,q}$ $(p = 1,2)$ has a unique simple constituent to which the attached two-sided cell is c_1, we denote it by L_p. We have $L_1 \not\simeq L_2$ since the eigenpolynomial of $C_{20\omega}$ on L_p $(p = 1,2)$ is $(\lambda - (-1^p))\lambda^{\dim L_p - 1}$ $(p = 1,2)$.

Similarly we know that $E_{p,q}$ ($p = 3,4$) has a unique simple constituent to which the attached two-sided cell is c_1, we denote it by L_p. We claim that $L_3 \not\simeq L_4$. In fact, let $e_i \in E_{p,q}$ ($0 \le i \le n$) be defined by $e_i : Y \to \mathbb{C}, y \to \delta_{iy}$ ($y \in Y$, $1 \le i \le n-1$) and $e_n(n) = e_n(n') = 1$, $e_n(j) = 0$ ($0 \le j \le n-1$). Then we have

(b7) $C_0 e_0 = [2]e_0,\ C_0 e_2 = e_0,\ C_0 e_i = 0\ (i \ne 0,2,\ 1 \le i \le n)$.

(b8) $C_\omega e_0 = (-1)^p e_1,\ C_\omega e_1 = (-1)^p e_0,\ C_\omega v_i = (-1)^p v_i\ (2 \le i \le n)$.

(b9) $C_2 e_0 = C_2 e_1 = C_2 e_3 = e_2,\ C_2 e_2 = [2]e_2,\ C_2 v_i = 0\ (4 \le i \le n,\ n \ge 4)$ and $C_2 e_0 = C_2 e_1 = e_2,\ C_2 e_3 = 2e_2,\ C_2 e_2 = [2]e_2$ when $n = 3$.

(b10) $C_i e_i = [2]e_i,\ C_i e_{i-1} = C_i e_{i+1} = e_i,\ C_i e_j = 0\ (3 \le i \le n-2,\ \ 1 \le j \le n,\ n \ge 4)$.

(b11) $C_{n-1} e_{n-2} = e_{n-1},\ C_{n-1} e_n = 2e_{n-1},\ C_{n-1} e_{n-1} = [2]e_{n-1},\ C_{n-1} v_i = 0,\ (0 \le i \le n-3,\ n \ge 4)$.

(b12) $C_n e_{n-1} = e_n,\ C_n e_n = [2]e_n,\ C_n v_i = 0,\ (0 \le i \le n-2)$.

So the eigenpolynomial of $C_{20\omega}$ on L_p ($p = 3,4$) is $(\lambda + (-1)^p)\lambda^{\dim L_p - 1}$ and $L_3 \not\simeq L_4$.

We have $\dim E_{5,q} = \dim E_{6,q} = 1$ and C_i ($0 \le i \le n-1$) acts on $E_{p,q}$ ($p = 5,6$) by scalar zero and C_n acts on $E_{p,q}$ ($p = 5,6$) by scalar $[2]$, C_ω acts on $E_{p,q}$ ($p = 5,6$) by scalar $(-1)^p$. So $(\phi_q)_{*,c_1}(E_p) = 0$ ($p = 5,6$) if and only if $q+1 = 0$ and $(\phi_q)_{*,c_1}(E_p) = E_{p,q} = L_p$ ($p = 5,6$) is a c_1-module if $q+1 \ne 0$. Obviously we have $L_5 \not\simeq L_6$.

By (b7), (b11-b12) we see that $L_p \not\simeq L_{p'}$ ($p = 1,2$; $p' = 3,4,5,6$), we also have $L_p \not\simeq L_{p'}$ ($p = 3,4$; $p' = 5,6$) since $C_0 L_p \ne 0$, $C_0 L_{p'} = 0$. 8.2(b) is proved in this case.

One may check that L_p is the unique simple quotient module of $E_{p,q}$ ($1 \le p \le 6$). Let N_p be the maximal submodule of $E_{p,q}$, then

(b13) When $p = 1,2$, we have

$$N_p = \begin{cases} 0, & \text{if } (q+1)(q^{n-1}+1) \ne 0, \\ < v_0 - v_1 >, & \text{if } q+1 = 0 \text{ and } n \text{ odd}, \\ < v_0 - v_1, \sum_{i=1}^{n/2} (-1)^i v_{2i-1} >, & \text{if } q+1 = 0 \text{ and } n \text{ even}, \\ < \sum_{i=0}^{n-1} \alpha_i v_i >, & \text{if } q+1 \ne 0 \text{ and } q^{n-1}+1 = 0, \end{cases}$$

where $\alpha_0 = \alpha_1 = 1,\ \alpha_i = (-q)^{\frac{i-1}{2}} + (-q)^{\frac{1-i}{2}},\ 2 \le i \le n-1$. Thus

$$\dim L_p = \begin{cases} n, & \text{if } (q+1)(q^{n-1}+1) \neq 0, \\ n-1, & \text{if } q+1=0,\ n \text{ odd or if } q+1 \neq 0 \text{ but } q^{n-1}+1=0, \\ n-2, & \text{if } q+1=0 \text{ and } n \text{ even.} \end{cases}$$

(b14) When $p = 3, 4$, we have

$$N_p = \begin{cases} 0, & \text{if } (q+1)(q^{n-1}-1) \neq 0, \\ < e_0 - e_1 >, & \text{if } q+1=0 \text{ and } n \text{ even}, \\ < e_0 - e_1, \sum_{i=1}^{\frac{n+1}{2}} (-1)^i (2 - \delta_{n,2i-1}) e_{2i-1} > & \text{if } q+1=0 \text{ and } n \text{ odd}, \\ < \sum_{i=0}^{n} \alpha_i e_i >, & \text{if } q+1 \neq 0,\ q^{n-1}-1=0, \end{cases}$$

where $\alpha_0 = \alpha_1 = 1,\ \alpha_i = (-q)^{\frac{i-1}{2}} + (-q)^{\frac{1-i}{2}},\ 2 \le i \le n-1,\ \alpha_n = (-q)^{\frac{n-1}{2}}$. Thus

$$\dim L_p = \begin{cases} n+1, & \text{if } (q+1)(q^{n-1}+1) \neq 0, \\ n, & \text{if } q+1=0,\ n \text{ even or if } q+1 \neq 0 \text{ but } q^{n-1}-1=0, \\ n-1, & \text{if } q+1=0 \text{ and } n \text{ odd.} \end{cases}$$

Finally we have $\dim L_p = 1$ $(p = 5, 6)$ when $q + 1 \neq 0$.

(C). Type $\tilde{C}_n$ $(n \ge 2)$: In this part we assume that W is of type $\tilde{C}_n$ $(n \ge 2)$. Then $F_1 = \mathbf{Z}/2 \ltimes \mathbb{C}^*$, where $\mathbf{Z}/2$ acts on $\mathbb{C}^*$ by $z \to z^{-1}$.

Let $\omega \in \Omega$ be such that $\omega s_i = s_{n-i}\omega$ $(0 \le i \le n)$. For $k \ge 0$, let

$$w(0,0,k) = (012\ldots(n-1)\omega)^k 0, \qquad w(n,n,k) = \omega w(0,0,k)\omega,$$
$$w(0,n,k) = w(0,0,k)\omega, \qquad w(n,0,k) = \omega w(0,0,k).$$

For $0 < i,j < n,\ k \geq 1,\ k' \geq 0$, let

$$\begin{aligned}
w(0,i,k) &= w(0,0,k-1)12\dots(n-i)\omega, & u(i,0,k) &= w(0,i,k)^{-1},\\
u(0,i,k') &= w(0,0,k')12\dots i, & w(i,0,k') &= u(0,i,k')^{-1},\\
w(n,i,k) &= w(n,n,k-1)(n-1)\dots i, & u(i,n,k) &= w(n,i,k)^{-1},\\
u(n,i,k') &= w(n,n,k')(n-1)\dots(n-i)\omega, & w(i,n,k') &= u(n,i,k')^{-1},\\
w(i,j,k) &= i\dots 1w(0,0,k-1)1\dots(n-j)\omega, & w(i,j,0) &= i\dots j,\\
u(i,j,k) &= i\dots(n-1)w(n,n,k-1)(n-1)\dots(n-j)\omega, & u(i,j,1) &= i\dots(n-j)\omega,\\
w(i,j,k'+2) &= i\dots(n-1)w(n,n,k')(n-1)\dots j, & &\\
w(i,j,k)' &= i\dots 1w(0,0,k)1\dots j. & &
\end{aligned}$$

We set $w(i,j,0) = u(i,j,0)$, $w(i,j,0)' = u(i,j,0)'$, $(0 \leq i,j \leq n)$. Then

$$c_1 = \{w(i,j,k), u(i,j,k), w(i,j,k)', u(i,j,k)', 010, 010\omega, \omega010, \omega010\omega \,|\, 0 \leq i,j \leq,\ k \geq 0\}.$$

Let $Y = \{0,n,i,i' \,|\, 0 \leq i \leq n\}$ be a F_1-set such that as F_1 sets we have $\{0\} \simeq \{n\} \simeq F_1/F_1,\ \{i,i'\} \simeq F_1/F_1^0$ $(0 < i < n)$. Then the bijection

$$\pi \colon c_1 \xrightarrow{\sim} \text{set of irreducible } F_1\text{-v.b. on } Y \times Y \text{ (up to isomorphism)}$$

defined by

$$\begin{aligned}
&w(i,j,k) \to \sigma(k)_{(i,j)}, && i,j = 0,n, && &&\\
&010 \to \varepsilon_{(0,0)}, && 010\omega \to \varepsilon_{(0,n)}, && \omega010 \to \varepsilon_{(n,0)}, && \omega010\omega \to \varepsilon_{(n,n)},\\
&w(i,j,k) \to \mathbf{q}^k_{(i,j)}, && u(i,j,k) \to \mathbf{q}^{-k}_{(i,j)}, && 0 \leq i,j \leq n, && i \neq 0,n, \text{ or } j \neq 0,n,\\
&w(i,j,k)' \to \mathbf{q}^k_{(i,j')}, && u(i,j,k)' \to \mathbf{q}^{-k}_{(i,j')}, && 0 < i,j < n, &&
\end{aligned}$$

satisfying (i) and (ii) in 8.2(a), where $\sigma(k)$ $(k > 0)$ is the simple representation of F_1 such that its restriction to $\mathbb{C}^*$ is the direct sum of $\mathbf{q}^k$ and $\mathbf{q}^{-k}$, $\sigma(0) = 1$ is the unit representation of F_1, ε is the one dimensional representation of F_1 which is not isomorphic to the unit representation of F_1. The part (ii) is obvious. To see that part (i) is true we need to check that (c1) $t_w t_u = \pi(w)\pi(u)$ for $w, u \in c_1$.

The proof is similar to part (C) in the proof of 7.2. First we have

(c2) $t_i t_w = t_w t_j = t_w$, if $l(s_i w) = l(w s_j) < l(w)$ and $t_u t_w = t_w t_u = 0$, if $l(us) > l(u)$ but $l(sw) < l(w)$ for some $s \in S$, see [L12].

(c3) It is easy to check that $(*)$ for $w = \omega^p 010 \omega^{p'}$ $(p, p' = 0, 1)$.

Now suppose that $w = w(i,j,k)$ or $u(i,j,k)$ or $w(i,j,k)'$ or $u(i,j,k)'$, when $k = 1$ it is not difficult to verify $(*)$. Using this fact and (c2-c3) we see that (c1) is true. 8.2(a) is proved for type $\tilde{C}_n$ $(n \geq 2)$.

Now we prove 8.2(b) . Let $\alpha \in \mathbf{Z}/2$ be such that $\alpha z = z^{-1}\alpha$ for any $z \in \mathbb{C}^*$. Then the conjugacy class containing α is $s(\alpha) = \{\alpha z \mid z \in \mathbb{C}^*\}$. For any $z \in \mathbb{C}^*$, let $s(z)$ be the conjugacy class containing z, then $s(z) = \{z, z^{-1}\}$, $s(1) = \{1\}$ $s(-1) = \{-1\}$. According to 3.4 and (i) we see that $\{E_{s(\alpha)}, E_{s(z)}\ (z \in \mathbb{C}^*), E_{s(1),\varepsilon}, E_{s(-1),\varepsilon}\}$ is a complete set of simple $\mathbf{J}_1$-modules, where ε is the restriction to $\mathbf{Z}/2$ of the one dimensional representation ε of F_1.

It is easy to see that $E_{s(\alpha),q}$ is a simple $\mathbf{H}_q$-module of dimension 2 and $C_0 E_{s(\alpha),q} \neq 0$ when $q + 1 \neq 0$, and $(\phi_q)_{*,c_1}(E_{s(\alpha)}) = 0$ when $q + 1 = 0$. We always have $C_i E_{s(\alpha),q} = 0$ $(1 \leq i \leq n-1)$.

One verifies that $E_{s(z),q}$ $(z \in \mathbb{C}^*)$ has exactly one simple constituent to which the attached two-sided cell is c_1, we denote it by $L_{s(z)}$. The eigenpolynomial of $C_{10\omega}$ on $L_{s(z)}$ is $(\lambda - z - z^{-1})\lambda^{b(z)}$, where $b(z) = \dim L_{s(z)} - 1$. So $L_{s(z)} \simeq L_{s(z')}$ if and only if $s(z) = s(z')$ when $z, z' \in \mathbb{C}^*$. One checks that for any $z \in \mathbb{C}^*$, $C_i L_{s(z)} \neq 0$ $(1 \leq i \leq n-1)$ if $q + 1 \neq 0$, thus $L_{s(z)} \not\simeq E_{s(\alpha),q}$ when $q + 1 \neq 0$ for any $z \in \mathbb{C}^*$.

We have $(\phi_q)_{*,c_1}(E_{s(p),\varepsilon}) = 0$ $(p = \pm 1)$ if and only if $n = 2$ and $q + 1 = 0$. If $(\phi_q)_{*,c_1}(E_{s(p),\varepsilon}) \neq 0$, it is easy to verify that $(\phi_q)_{*,c_1}(E_{s(p),\varepsilon}) = L_{p,\varepsilon}$ is attached to c_1. C_0 acts on $L_{p,\varepsilon}$ by scalar 0. The eigenpolynomial of $C_{12\ldots(n-2)\omega}$ $(n \geq 3)$ or C_ω $(n = 2)$ on $L_{p,\varepsilon}$ is $(\lambda - p)\lambda^{\dim L_{p,\varepsilon} - 1}$. So we have $L_{1,\varepsilon} \not\simeq L_{-1,\varepsilon}$, $L_{p,\varepsilon} \not\simeq E_{s(\alpha),q}$ $(p = \pm 1)$, and $L_{p,\varepsilon} \not\simeq L_{s(z)}$ $(p = \pm 1,\ z \in \mathbb{C}^*)$.

The 8.2(b) is proved in this case.

For a simple $\mathbf{J}_1$-module E, One verifies that $(\phi_q)_{*,c_1}(E)$ is the unique simple quotient module of E_q when $(\phi_q)_{*,c_1}(E) \neq 0$. Wh shall determine the maximal submodule of E_q when $(\phi_q)_{*,c_1}(E) \neq 0$.

When $q+1 \neq 0$, we have $\dim L_{s(\alpha)} = 2$. When $q+1 \neq 0$, $n=2$, we have $\dim L_{p,\varepsilon} = 1$ $(p = \pm 1)$. Now assume that $n \geq 3$, By definition, $L_{p,\varepsilon}$ $(p = \pm 1)$ has a base v_i $(1 \leq i \leq n-1)$ defined by $v_i : Y \to \mathbb{C}$, $i \to 1$, $i' \to -1$, $y \to 0$ if $y \neq i, i'$, $y \in Y$. Let $N_{p,\varepsilon}$ be the maximal submodule of $E_{s(p),\varepsilon,q}$, then we have

$$N_{p,\varepsilon} = \begin{cases} < \sum_{i=1}^{\frac{n}{2}} (-1)^i v_{2i-1} >, & \text{if } q+1=0 \text{ and } n \text{ even,} \\ < \sum_{i=1}^{n-1} \alpha_i v_1 >, & \text{if } q+1 \neq 0 \text{ and } q^n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha_i = (-q)^{\frac{i-1}{2}} + (-q)^{\frac{i-3}{2}} + \cdots + (-q)^{\frac{3-i}{2}} + (-q)^{\frac{1-i}{2}}$, $1 \leq i \leq n-1$.

So if $n \geq 3$, we have

$$\dim L_{p,\varepsilon} = \begin{cases} n-2, & \text{if } q+1=0 \text{ and } n \text{ even; or } q+1 \neq 0 \text{ and } q^n = 1, \\ n-1, & \text{otherwise.} \end{cases}$$

Now we consider $L_{s(z)}$ $(z = \pm 1)$, By definition, $E_{s(z),q}$ $(z = \pm 1)$ has a base v_i $(0 \leq i \leq n)$ defined by $v_i : Y \to \mathbb{C}$, $i \to 1$, $i' \to 1$, $y \to 0$ if $y \neq i, i'$, $y \in Y$ (we set $0' = 0, n' = n$). Let $N_{s(z)}$ be the maximal submodule of $E_{s(z),q}$, then we have

$$N_{s(z)} = \begin{cases} < \sum_{i=0}^{\frac{n}{2}} (-1)^i v_{2i-1} >, & \text{if } q+1=0 \text{ and } n \text{ even,} \\ < \sum_{i=0}^{n} \alpha_i v_1 >, & \text{if } q+1 \neq 0 \text{ and } q^{n+1}+1 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha_i = (-q)^{\frac{i}{2}} + (-q)^{\frac{-i}{2}}$ $(0 \leq i \leq n-1)$, $\alpha_n = z(-q)^{\frac{n}{2}} + z(-q)^{\frac{-n}{2}}$.

Thus we have $(z = \pm 1)$

$$\dim L_{s(z)} = \begin{cases} n, & \text{if } q+1=0 \text{ and } n \text{ even; or } q+1 \neq 0 \text{ and } q^{n+1}+1 = 0, \\ n+1, & \text{otherwise.} \end{cases}$$

For $z \in \mathbb{C}^*$, $z^2 \neq 1$, $L_{s(z),q}$ has a base v_a $(a \in Y)$ defined by $v_a : Y \to \mathbb{C}$, $a \to 1$, $y \to 0$ if $y \neq a$, $y \in Y$. Let $N_{s(z)}$ be the maximal submodule of $E_{s(z),q}$, then we have

$$N_{s(z)} = \begin{cases} <v', v''>, & \text{if } q+1=0,\ n \text{ odd and } z^2+1=0, \\ <e'>, & \text{if } q+1 \neq 0 \text{ and } q^n z^2 - 1 = 0 \text{ but } q^n z^{-2} - 1 \neq 0, \\ <e''>, & \text{if } q+1 \neq 0 \text{ and } q^n z^2 - 1 \neq 0 \text{ but } q^n z^{-2} - 1 = 0, \\ <e', e''>, & \text{if } q+1 \neq 0 \text{ and } q^n z^2 - 1 = 0,\ q^n z^{-2} - 1 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$v' = \sum_{i=0}^{\frac{n-1}{2}} (-1)^i (v_{2i-1} - v_{(2i-1)'}) - (-1)^{\frac{n+1}{2}} z v_n,$$

$$v'' = v_0 + \sum_{i=0}^{\frac{n-1}{2}} (-1)^i (v_{2i} - v_{(2i)'}),$$

$$e' = \sum_{i=0}^{n-1} (-q)^{\frac{i}{2}} v_i + \sum_{i=1}^{n-1} (-q)^{-\frac{i}{2}} v_{i'} + z(-q)^{\frac{n}{2}} v_n,$$

$$e'' = \sum_{i=0}^{n-1} (-q)^{-\frac{i}{2}} v_i + \sum_{i=1}^{n-1} (-q)^{\frac{i}{2}} v_{i'} + z(-q)^{-\frac{n}{2}} v_n.$$

Thus for $z \in \mathbb{C}^*$ $(z \neq \pm 1)$ we have

$$\dim L_{s(z)} = \begin{cases} 2n-2, & \text{if } q=-1,\ n \text{ odd},\ z^2=-1; \text{ or } q \neq -1 \text{ and } q^n = z^2 = -1, \\ 2n-1, & \text{if } q+1 \neq 0,\ z^2 \neq -1, \text{ and } q^n z^2 = 1 \text{ or } q^n z^{-2} = 1, \\ 2n, & \text{otherwise.} \end{cases}$$

(D-E). Type $\tilde{D}_n$ $(n \geq 4)$, $\tilde{E}_n$ $(n = 6,7,8)$ In this part we assume that W is of type $\tilde{D}_n$ $(n \geq 4)$, $\tilde{E}_n$ $(n = 6,7,8)$. Then $F_1 = \Omega$. We write s_{n+1} instead of s_0.

For each $w \in c_1$, there exist unique $i,j \in [1, n+1]$, $\omega \in \Omega$ such that $l(s_i w) = l(w s_j) = l(w) - 1$, $w\omega^{-1} \in W_a$, we then write $w(i,j,\omega)$ instead of w. Let $\pi(w(i,j,\omega)) \in L_{n+1}(\mathbb{C}[\Omega])$ (the $(n+1) \times (n+1)$ matrix ring over the group ring $\mathbb{C}[\Omega]$) be such that its (i,j)- entry is ω and other entries are 0. Then the map $w(i,j,\omega) \to \pi(w(i,j,\omega)) \in L_{n+1}(\mathbb{C}[\Omega])$ defines an isomorphism of $\mathbb{C}$-algebra between $\mathbf{J}_1$ and $L_{n+1}(\mathbb{C}[\Omega])$. Since Ω is a finite abelian group, according to 3.4 we see that 8.2(a) is true.

Each $\mathbb{C}$-algebra homomorphism $f : \mathbb{C}[\Omega] \to \mathbb{C}$ gives rise to a simple representation ψ_f of $\mathbf{J}_1$:

$$\psi_f : \mathbf{J}_1 \to M_{n+1}(\mathbb{C}[\Omega]) \to M_{n+1}(C).$$

Let E_f be a simple $\mathbf{J}_1$-module providing ψ_f. As the same way as in part A we know that $E_{f,q}$ has exactly one simple constituent to which the attached two-sided cell is c_1, we denote it by L_f. For different $\mathbb{C}$-algebra homomorphism $f, f' : \mathbb{C}[\Omega]) \to \mathbb{C}$, we have $L_f \not\simeq L_{f'}$. 8.2(b) is proved.

It is easy to see that L_f is the unique simple quotient module of $E_{f,q}$. Let v_i $(1 \le i \le n+1)$ be the natural base of $E_{f,q}$ and let N_f be the maximal submodule of $E_{f,q}$. then we have

Type $\tilde{E}_8$:

$$N_f = \begin{cases} < v_2 - v_3, v_3 - v_5 + v_7 - v_9 >, & \text{if } q+1=0, \\ < \sum\limits_{i=1}^{9} \alpha_i v_i >, & \text{if } (q^3-1)(q^5-1)=0, \\ 0, & \text{otherwise,} \end{cases}$$

Where $\alpha_1 = q^{\frac{7}{2}} + q^{-\frac{7}{2}}$, $\alpha_2 = -q^2 - q^{-2} - 1$, $\alpha_3 = -q^3 - q - q^{-1} - q^{-3}$, $\alpha_i = (-1)^{\frac{10-i}{2}}(q^{\frac{10-i-1}{2}} + q^{\frac{10-i-3}{2}} + \cdots + q^{\frac{i+3-10}{2}} + q^{\frac{i+1-10}{2}})$ $(4 \le i \le 9)$. So

$$\dim L_f = \begin{cases} 7, & \text{if } q+1=0, \\ 8, & \text{if } (q^3-1)(q^5-1)=0, \\ 9, & \text{otherwise.} \end{cases}$$

Type $\tilde{E}_7$:

$$N_f = \begin{cases} < v_2 - v_5 + v_7, v_3 - v_5 + v_7 - v_8 >, & \text{if } q+1=0, \\ < \sum\limits_{i=1}^{8} \alpha_i v_i >, & \text{if } (q^3-1)(q^2+1)=0, \\ 0, & \text{otherwise,} \end{cases}$$

Where $\alpha_1 = q^{\frac{5}{2}} + q^{-\frac{5}{2}}$, $\alpha_2 = -q - q^{-1}$, $\alpha_3 = -q^2 - 1 - q^{-2}$, $\alpha_8 = -q^3 + 1 - q^{-3}$, $\alpha_i = (-1)^{\frac{8-i}{2}}(q^{\frac{8-i-1}{2}} + q^{\frac{8-i-3}{2}} + \cdots + q^{\frac{i+3-8}{2}} + q^{\frac{i+1-8}{2}})$ $(4 \le i \le 7)$. So

$$\dim L_f = \begin{cases} 6, & \text{if } q+1=0, \\ 7, & \text{if } (q^3-1)(q^2+1)=0, \\ 8, & \text{otherwise.} \end{cases}$$

Type $\tilde{E}_6$:

$$N_f = \begin{cases} < v_1 - v_4 + v_6 + v_7 >, & \text{if } q+1=0, \\ < v_1 \pm v_2 \mp v_3 - v_7, v_1 \mp v_3 \pm v_5 - v_6 >, & \text{if } (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) = \pm 1, \\ < v_1 \mp 2v_2 \mp 2v_3 + 3v_4 \mp v_5 + v_6 + v_7 >, & \text{if } (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) = \pm 2, \\ 0, & \text{otherwise.} \end{cases}$$

So

$$\dim L_f = \begin{cases} 5, & \text{if } (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) = \pm 1, \\ 6, & \text{if } (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) = 0 \text{ or } \pm 2, \\ 7, & \text{otherwise.} \end{cases}$$

Type $\tilde{D}_n$ $(n \geq 4)$:

$$N_f = \begin{cases} < v_1 - v_{n+1}, v_{n-1} - v_n, \sum_{i=1}^{\frac{n}{2}} (-1)^i v_{2i-1} >, & \text{if } q+1=0,\ n \text{ even}, \\ < v_1 - v_{n+1}, v_{n-1} - v_n, >, & \text{if } q+1=0,\ n \text{ odd}, \\ < \sum_{i=1}^{n+1} \alpha_i v_i >, & \text{if } q^{n-2} - 1 = 0, \text{ but } q+1 \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha_1 = \alpha_{n+1} = 1$, $\alpha_{n-1} = \alpha_n = -\frac{(-q)^{\frac{n-3}{2}} + (-q)^{\frac{3-n}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}$, $\alpha_i = q^{\frac{i-1}{2}} + q^{\frac{1-i}{2}}$ $(2 \leq i \leq n-2)$.
So

$$\dim L_f = \begin{cases} n-2, & \text{if } q+1=0,\ n \text{ even}, \\ n-1, & \text{if } q+1=0,\ n \text{ odd}, \\ n, & \text{if } q^{n-2} - 1 = 0, \text{ but } q+1 \neq 0, \\ n+1, & \text{otherwise,} \end{cases}$$

(F). Type $\tilde{F}_4$. In this part we assume that W is of type $\tilde{F}_4$. Then $F_1 = \mathbf{Z}/2$. We have

$c_1 = \{0, 01, 012, 0123, 01234, 01232, 012321, 0123210, 1, 10, 12, 123, 1234,$
$1232, 12321, 123210, 2, 21, 210, 23, 234, 232, 2321, 23210, 3, 32, 321, 3210, 34, 323, 3234, 4, 43,$
$432, 4321, 43210, 4323, 43234\}$.

Let $Y = \{0, 1, 2, 3, 3', 4, 4'\}$ be a F_1-set such that as F_1-sets we have $\{0\} \simeq \{1\} \simeq \{2\} \simeq F_1/F_1$, $\{3, 3'\} \simeq \{4, 4'\} \simeq F_1$. We define a bijection

$$\pi : c_1 \xrightarrow{\sim} \text{set of irreducible } F_1\text{-v.b. on } Y \times Y \text{ (up to isomorphism)}$$

as follows:

$$
\begin{array}{rrrr}
0 \to 1_{(0,0)}, & 0123210 \to \varepsilon_{(0,0)}, & 01 \to 1_{(0,1)}, & 012321 \to \varepsilon_{(0,1)}, \\
012 \to 1_{(0,2)}, & 01232 \to \varepsilon_{(0,2)}, & 0123 \to 1_{(0,3)}, & 01234 \to 1_{(0,4)}, \\
1 \to 1_{(1,1)}, & 12321 \to \varepsilon_{(1,1)}, & 10 \to 1_{(1,0)}, & 123210 \to \varepsilon_{(1,0)}, \\
12 \to 1_{(1,2)}, & 1232 \to \varepsilon_{(1,2)}, & 123 \to 1_{(1,3)}, & 1234 \to 1_{(1,4)}, \\
2 \to 1_{(2,2)}, & 232 \to \varepsilon_{(2,2)}, & 21 \to 1_{(2,1)}, & 2321 \to 1_{(2,1)}, \\
210 \to 1_{(2,0)}, & 23210 \to 1_{(2,0)}, & 23 \to 1_{(2,3)}, & 1234 \to 1_{(2,4)}, \\
3 \to 1_{(3,3)}, & 323 \to 1_{(3,3')}, & 34 \to 1_{(3,4)}, & 3234 \to 1_{(3,4')}, \\
4 \to 1_{(4,4)}, & 43234 \to 1_{(4,4')}, & 43 \to 1_{(4,3)}, & 4323 \to 1_{(4,3')}, \\
32 \to 1_{(3,2)}, & 321 \to 1_{(3,1)}, & 3210 \to 1_{(3,0)}, & \\
432 \to 1_{(4,2)}, & 4321 \to 1_{(4,1)}, & 43210 \to 1_{(4,0)}. &
\end{array}
$$

Where ε is the sign representation of F_1. A direct computation shows that the bijection π satisfies (i) and (ii). 8.2(a) is proved.

Let $\alpha = \bar{1}, e = \bar{0} \in \mathbf{Z}/2$. By 3.4 we see that $\mathbf{J}_1$ has three simple modules (up to isomorphism): $E_1 = E_\alpha,\ E_2 = E_e,\ E_3 = E_{e,\varepsilon},$ where ε is the sign representation of $\mathbf{Z}/2$. Via $\phi_{q,c_1}: \mathbf{H}_q \to \mathbf{J}_1$, they give rise to three $\mathbf{H}_q$-modules: $E_{1,q},\ E_{2,q},\ E_{3,q},$. $E_{p,q}$ $(p = 1,2,3)$ has a unique simple quotient, which is just $(\phi_q)_{*,c_1}(E_p)$. For simplicity, we denote it by L_p, then $C_iL_1 \neq 0,\ C_iL_3 = 0,\ (i = 0,1,2)\ C_jL_1 = 0,\ C_jL_3 \neq 0,\ (j = 3,4)\ C_kL_2 \neq 0,\ (k = 0,1,2,3,4)$. Thus we have $L_1 \not\simeq L_2 \not\simeq L_3 \not\simeq L_1$. 8.2(b) is proved.

Let N_p $(p = 1,2,3)$ be the maximal submodule of $E_{p,q}$, then we have

$$
N_1 = \begin{cases} < v_0 - v_2 >, & \text{if } q+1 = 0, \\ < v_0 - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})v_1 + v_2 >, & \text{if } q^2+1 = 0, \\ 0, & \text{otherwise,} \end{cases}
$$

where $v_i : \{0,1,2\} \to \mathbf{C}$ defined by $v_i(j) = \delta_{ij}$ $(i,j = 0,1,2)$. Therefore

$$
\dim L_1 = \begin{cases} 2, & \text{if } (q+1)(q^2+1) = 0, \\ 3, & \text{otherwise.} \end{cases}
$$

$$N_2 = \begin{cases} < v_0 - v_2 + v_4 >, & \text{if } q+1=0, \\ < \sum_0^4 \alpha_i v_i >, & \text{if } q^3-1=0, \\ 0, & \text{otherwise,} \end{cases}$$

where $v_i : Y \to \mathbb{C}$ defined by $v_i(j) = v_i(j') = \delta_{ij}$ $(i,j = 0,1,2,3,4)$ (we set $0 = 0', 1 = 1', 2 = 2'$), $\alpha_0 = 1$, $\alpha_1 = -q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, $\alpha_2 = q + 1 + q^{-1}$, $\alpha_3 = -\frac{1}{2}(q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}})$, $\alpha_4 = \frac{1}{2}(q + 1 + q^{-1})$. Therefore

$$\dim L_2 = \begin{cases} 4, & \text{if } (q+1)(q^3-1) = 0, \\ 5, & \text{otherwise.} \end{cases}$$

$$N_3 = \begin{cases} < v_3 - (q + q^{-\frac{1}{2}})v_4 >, & \text{if } q^2+q+1=0, \\ 0, & \text{otherwise,} \end{cases}$$

where $v_i : Y \to \mathbb{C}$ defined by $v_i(j) = -v_i(j') = \delta_{ij}$ $(i = 3,4;\ j = 0,1,2,3,4)$ (we set $0 = 0', 1 = 1', 2 = 2'$). Therefore

$$\dim L_3 = \begin{cases} 1, & \text{if } q^2+q+1=0, \\ 2, & \text{otherwise.} \end{cases}$$

Type G_2: For type $\tilde{G}_2$, 8.2 has be proved in chapter 7. We only determine the dimensions of the simple $\mathbf{H}_q$-modules to which the attached two-sided cell is c_1.

By 3.4, 8.2(a) we see that $\mathbf{J}_1$ has four simple modules $E_1 = E_{\overline{e}}$, $E_2 = E_{\overline{s_0}}$, $E_3 = E_{\overline{s_0 s_1}}$, $E_4 = E_{\overline{e},\sigma}$, where $\overline{e} = \{e\}$, (e the unit of $\mathfrak{S}_3$), $\overline{s_0} = \{s_0, s_1, s_0 s_1 s_0\}$, $\overline{s_0 s_1} = \{s_0 s_1, s_1 s_0\}$. $\dim E_1 = \dim E_2 = 3$, $\dim E_3 = 2$, $\dim E_4 = 1$. When $q + 1 \neq 0$, one verifies that $E_{i,q}$ $(1 \leq i \leq 4)$ has a unique simple quotient module L_i and $L_i = (\phi_q)_{*,c_1}(E_i)$. Moreover $L_i \not\simeq L_j$ if $i \neq j$. When $q + 1 = 0$, one can check that $E_{i,q}$ $(1 \leq i \leq 3)$ has a unique simple quotient L_i and $L_i = (\phi_q)_{*,c_1}(E_i)$. We also have $(\phi_q)_{*,c}(E_4) = 0$.

Let N_i $(i = 1,2,3)$ be the maximal submodule of $E_{i,q}$, then we have

$$N_1 = \begin{cases} < 3v_1 - v_3 >, & \text{if } q+1=0, \\ < v_1 \mp 2v_2 + v_3 >, & \text{if } q=1, \\ 0, & \text{otherwise,} \end{cases}$$

where $v_i : Y \to \mathbb{C}$ is defined by $v_i(j) = \delta_{ij}$ $(i = 1, 2, 1 \leq j \leq 5)$ and $v_3(p) = 1,\ v_3(i) = 0$ $(p = 3, 4, 5;\ i = 1, 2)$. So

$$\dim L_1 = \begin{cases} 2, & \text{if } q^2 - 1 = 0, \\ 3, & \text{otherwise.} \end{cases}$$

$$N_2 = \begin{cases} < v_1 - v_3 >, & \text{if } q + 1 = 0, \\ < v_1 \mp v_2 + v_3 >, & \text{if } q^2 + 1 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $v_i : \{1, 2, 3\} \to \mathbb{C}$ defined by $v_i(j) = \delta_{ij}$ $(i, j = 1, 2, 3)$. So

$$\dim L_2 = \begin{cases} 2, & \text{if } (q^2 + 1)(q + 1) = 0, \\ 3, & \text{otherwise.} \end{cases}$$

$$N_3 = \begin{cases} < v_1 - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})v_2 >, & \text{if } q^2 + q + 1 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $v_i : \{1, 2\} \to \mathbb{C}$ defined by $v_i(j) = \delta_{ij}$ $(i, j = 1, 2)$. So

$$\dim L_3 = \begin{cases} 1, & \text{if } (q^2 + q + 1) = 0, \\ 2, & \text{otherwise.} \end{cases}$$

Note that $\dim E_4$=1 and $E_{4,q}$ is always a simple $\mathbf{H}_q$-module (cf. 3.10).

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