QUADRATIC FUNCTORS AND METASTABLE HOMOTOPY II (HOMOTOPY GROUPS OF MOORE SPACES AND HOMOLOGY GROUPS OF EILENBERG–MAC LANE SPACES)

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For the sixtieth birthday of Keith Hardie

Abstract: Using a metastable approximation of certain fibers, resp. cofibers we obtain natural six term exact sequences for the homotopy groups of Moore spaces and for the (co) homology groups of Eilenberg–Mac Lane spaces respectively. As an application we describe a new homotopy invariant of an $(n-1)$–connected $(2n+1)$–dimensional closed manifold.
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We study quadratic functors which are given by homotopy groups and homology groups respectively, in particular we consider the homotopy groups of Moore spaces $\pi_m M(A,n)$, $m < 3n-2$, and the homology groups of Eilenberg–Mac Lane spaces $H_m K(A,n)$, $m < 3n$. The main results in this paper describe new natural six–term exact sequences for these groups, see (2.5), (3.5) and also (4.5). In particular we get the natural exact sequences

$$(1) \qquad \begin{aligned} 0 \longrightarrow\ & A*'\pi_m\{S^n\} \longrightarrow {}_\lambda\pi_{n+1}M(A,n) \longrightarrow A*''\pi_{m-1}\{S^n\} \longrightarrow \\ & A\otimes\pi_m\{S^n\} \longrightarrow \pi_m M(A,n) \longrightarrow {}_\lambda\pi_m M(A,n) \longrightarrow 0 \end{aligned}$$

$$(2) \qquad \begin{aligned} 0 \longleftarrow\ & A\otimes' H_m\{n\} \longleftarrow {}_\kappa H_{m-1}K(A,n-1) \longleftarrow A\otimes'' H_{m+1}\{n\} \longleftarrow \\ & A*H_m\{n\} \longleftarrow H_m K(A,n-1) \longleftarrow {}_\kappa H_m K(A,n-1) \longleftarrow 0 \end{aligned}$$

which are Eckmann-Hilton dual to each other. Here $\pi_m\{S^n\}$ is the quadratic $\mathbb{Z}$–module

(3) $\pi_m\{S^n\} = (\pi_m S^n \xrightarrow{H} \pi_m S^{2n-1} \xrightarrow{P} \pi_m S^n)$

given by the Hopf invariant H and by the Whitehead product map $P = [i_n, i_n]_*$. On the other hand $H_m\{n\}$ is the quadratic $\mathbb{Z}$–module

(4) $H_m\{n\} = (H_m K(\mathbb{Z},n) \xrightarrow{H} H_m K(\mathbb{Z},n) \wedge K(\mathbb{Z},n) \xrightarrow{P} H_m K(\mathbb{Z},n))$

given by the diagonal $H = \Delta_*$ and by the map P induced by the Hopf construction of the multiplication on $K(\mathbb{Z},n)$. The bifunctors $*', *'', \otimes$ and $\otimes', \otimes'', *$ are derived from the quadratic tensor product in [6], see (7.4)[6] where we also describe the properties of these functors. Using (1) and (2) we obtain for example natural isomorphisms

(5) $A *' \mathbb{Z}^\Gamma \cong R(A) \cong H_5 K(A,2)$ and

(6) $A \otimes' \mathbb{Z}^\Gamma \oplus A \otimes \mathbb{Z}/3 \cong \Omega(A) \oplus A \otimes \mathbb{Z}/3 \cong H_7 K(A,3)$

where $\mathbb{Z}^\Gamma \cong \pi_3\{S^2\} \cong H_4\{2\}$. Here R and Ω are the functors of Eilenberg-Mac Lane in [10]. Moreover we derive from (1) a new homotopy invariant

(7) $\tau(X) \in H_n(X) *' \pi_{2n-1}\{S^n\}$

for any (n–1)–connected (2n+1)–dimensional closed manifold X, $n \geq 2$. The exact sequence (1), resp. (2), is actually a very special case of an exact sequence obtained from the metastable approximation of a certain fiber, resp. cofiber. These approximations surprisingly are Eckmann-Hilton dual to each other, see §5 and §6.

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§1 *Quadratic homotopy functors*

We introduce additive categories of homotopy abelian co–H–groups and H–groups respectively and we describe quadratic functors on these categories. The functors are given by homotopy groups, homology groups, and cohomology groups respectively.

We first fix some notation. A bold face letter like $\underline{\mathbf{C}}$ denotes a category, we write $f \in \underline{\mathbf{C}}$ and $A \in \underline{\mathbf{C}}$ if f is a morphism and A an object in $\underline{\mathbf{C}}$. The set of morphisms

$A \longrightarrow B$ is $\underline{\underline{C}}(A,B)$. Maps which are injective, resp. surjective, are indicated by arrows $\rightarrowtail$, resp. $\twoheadrightarrow$; on the other hand such arrows as well are used for cofibrations, resp. fibrations, see [4]. Let $\underline{\underline{CW\text{-spaces}}}^*/\simeq$ be the homotopy category of CW–spaces with basepoint $*$; the set of morphisms $X \longrightarrow Y$ in this category is the set of homotopy classes $[X,Y]$. We write $\dim(Y) \le m$ if there is a homotopy equivalence $Y \simeq X$ where X is an m–dimensional CW–complex. Moreover we write $\mathrm{hodim}(Y) \le m$ if $\pi_i(Y) = 0$ for $i > m$. Let $\underline{\underline{A}}_n^k$, resp. $\underline{\underline{B}}_n^k$ be the full subcategories of $\underline{\underline{CW\text{-spaces}}}^*/\simeq$ consisting of $(n-1)$–connected spaces X with $\dim(X) \le n+k$, resp. $\mathrm{hodim}(X) \le n+k$. Let G be an abelian group. An Eilenberg–Mac Lane space $K(G,n)$ is a CW–space with $\pi_n(K(G,n)) = G$ and $\pi_j K(G,n) = 0$ for $j \ne n$. A Moore space $M(G,n)$ is a simply connected CW–space with homology groups $H_n M(G,n) = G$ and $H_j M(G,n) = 0$, $n \ne j \ge 1$. We clearly have $\mathrm{hodim}\, K(G,n) \le n$ and $\dim M(G,n) \le n+1$.

(1.1) Definition: Let $\underline{\underline{HA}}$ and $\underline{\underline{coHA}}$ be the following subcategories of $\underline{\underline{CW\text{-spaces}}}^*/\simeq$. Objects in $\underline{\underline{HA}}$ are homotopy abelian H–groups and morphism are H–maps. The objects in $\underline{\underline{coHA}}$ are homotopy abelian co–H–groups and morphisms are co–H–maps. Let $\underline{\underline{HA}}_n$, resp. $\underline{\underline{coHA}}_n$ be the full subcategories consisting of $(n-1)$–connected objects.

For example a double loop space $\Omega^2 Y$ and a double suspension $\Sigma^2 Y$ are objects in $\underline{\underline{HA}}$ and $\underline{\underline{coHA}}$ respectively. This shows that one has full inclusions

$$\underline{\underline{A}}_n^k \subset \underline{\underline{coHA}}_n \text{ and } \underline{\underline{B}}_n^k \subset \underline{\underline{HA}}_n \text{ for } k < n-1. \tag{1.2}$$

All categories in (1.2) are additive categories; the biproduct in $\underline{\underline{coHA}}$ is given by the one point union $X \vee Y$ of spaces and the biproduct in $\underline{\underline{HA}}$ is given by the product $X \times Y$ of spaces. For a CW–space K let π_m^K and π_K^m be the homotopy functors defined by

$$\pi_m^K(X) = [\Sigma^m K, X] \text{ and } \pi_K^m(X) = [X, \Omega^m K]. \tag{1.3}$$

As usual we have $\pi_m^K(X) = \pi_m(X)$ if $K = S^0$ is the 0–sphere and we have $\pi_K^m(X) = H^k(X,G)$ if $K = K(G,m+k)$. The sets in (1.3) are groups, resp. abelian groups, for $m = 1$, resp. $m \ge 2$. Let $\underline{\underline{Ab}}$ be the category of abelian groups. Using the homotopy functors (1.3) and the homology and cohomology functors we obtain the following four functors

(1) $\pi_m^K : \underline{\underline{coHA}}_n \longrightarrow \underline{\underline{Ab}}$ with $\dim(\Sigma^m K) < 3n-2$,

(2) $\pi_K^m : \underline{\underline{HA}}_n^{op} \longrightarrow \underline{\underline{Ab}}$ with $\mathrm{hodim}(\Omega^m K) < 3n$,

(3) $H^m(\_,G) : \underline{\underline{HA}}_n^{op} \longrightarrow \underline{\underline{Ab}}$ with $m < 3n$,

(4) $H_m(\_,G) : \underline{\underline{HA}}_n \longrightarrow \underline{\underline{Ab}}$ with $m < 3n$.

The functor (3) is a special case of (2) when we set $K = K(G,m+k)$. The conditions on the right hand side describe the <u>meta stable range</u> of these functors. It is well known that in this range the functors are quadratic. In the <u>stable range</u> (given by $\dim(\Sigma^m K) < 2n-1$, $\mathrm{hodim}(\Omega^m K) < 2n$, resp. $m < 2n$) the functors are additive. To this end we recall the following notation from [6].

(1.4) <u>Definition</u>: Let $\underline{\underline{A}}$ be an additive category and let $F : \underline{\underline{A}} \longrightarrow \underline{\underline{Ab}}$ be a quadratic functor, that is $\Delta(f,g) = F(f+g)-F(f)-F(g)$ is bilinear for $f,g \in \underline{\underline{A}}(X,Y)$. A biproduct $X \vee Y$ with inclusions i_τ and retractions r_τ $(\tau = 1,2)$ yields the quadratic cross effect

(1) $$F(X\,|\,Y) = \mathrm{im}\{\Delta(i_1r_1,i_2r_2) : F(X \vee Y) \longrightarrow F(X \vee Y)\}$$

such that $\Psi = (F(i_1),F(i_2),i_{12})$:

(2) $$F(X)\oplus F(Y)\oplus F(X\,|\,Y) \cong F(X\vee Y)$$

is an isomorphism where i_{12} is the inclusion given by (1). Let r_{12} be the retraction of i_{12} determined by Ψ. Then we obtain

(3) $$F\{X\} = (F(X) \xrightarrow{H} F(X\,|\,X) \xrightarrow{P} F(X))$$

by $H = r_{12}F(\mu)$ and $P = F(\nabla)i_{12}$ where $\mu = i_1 + i_2 : X \longrightarrow X \vee X$ and $\nabla = r_1+r_2 : X\vee X \longrightarrow X$. Next we obtain the isomorphism

(4) $$T : F(X\,|\,Y) \longrightarrow F(Y\,|\,X) \text{ with } TT = 1$$

by restriction of $F(i_2r_1+i_1r_2)$. In [6] we have seen that the tuple $F\{\underline{\underline{A}}\} = (F(\_),F(\_\,|\,\_),H,P,T)$ defined above satisfies the proposition of a "quadratic $\underline{\underline{A}}$–module". For a full subcategory $\underline{\underline{R}}$ of $\underline{\underline{A}}$ let $F\{\underline{\underline{R}}\}$ be given by restriction of $F\{\underline{\underline{A}}\}$ to $\underline{\underline{R}}$.

We now consider the cross effects and the structure maps H,P,T for the quadratic functors in (1.3)(1)...(4). For suspensions $X = \Sigma X'$, $Y = \Sigma Y'$ the Hilton–Milnor theorem shows

(1.5) $$\pi_m^K(\Sigma X' \wedge Y') \cong \pi_m^K(X\,|\,Y)\,.$$

Here the isomorphism is induced by the injection $\pi_m^K([i_1,i_2])$ where $[i_1,i_2] : \Sigma X' \wedge Y' \longrightarrow X \vee Y$ is the Whitehead product map. Using (1.5) as an identification the map T in (1.4) coincides with $-(\Sigma T_{21})_*$ where $T_{21} : X' \wedge Y' \longrightarrow Y' \wedge X'$ is the interchange map. Moreover the maps

$$\pi_m^K\{\Sigma X'\} = (\pi_m^K(\Sigma X') \xrightarrow{H} \pi_m^K(\Sigma X' \wedge X') \xrightarrow{P} \pi_m^K(\Sigma X')),$$

given by (1.4) and (1.5), coincide with the James–Hopf invariant $H = \gamma_2$ and the Whitehead product map $P = [1,1]_*$ where $1 = 1_X$ is the identity. These maps H and P are exactly the operators which appear in the classical EHP–sequence, see [12]. Next we obtain the cross effects of the functors (1.3)(2)(3)(4) by canonical isomorphisms

(1.6) $$\begin{aligned} &\pi_K^m(X \wedge Y) \cong \pi_K^m(X\,|\,Y),\\ &H^m(X \wedge Y,G) \cong H^m(X\,|\,Y,G),\\ &H_m(X \wedge Y,G) \cong H_m(X\,|\,Y,G) \end{aligned}$$

which are readily obtained by the cofiber sequence $X \vee Y \longrightarrow X \times Y \longrightarrow X \wedge Y$. For (1.3)(2)(3) the maps H,P,T correspond to $H = (H\mu)^*$, $P = \Delta^*$, $T = (T_{21})^*$ where $\Delta : X \longrightarrow X \wedge X$ is the reduced diagonal and where $H\mu : \Sigma X \wedge X \longrightarrow \Sigma X$ is the Hopf–construction of the H–space multiplication $\mu = r_1 + r_2 : X \times X \longrightarrow X$. In (1.3)(4) we get $H = \Delta_*$, $P = (H\mu)_*$ and $T = (T_{21})_*$. For the definition of $H\mu$ see for example (II 15.15) [4]. For $(H\mu)^*$ and $(H\mu)_*$ we use the canonical suspension isomorphisms $\pi_K^{m-1}(\Sigma X) = \pi_K^m(X)$ and $H_{m+1}(\Sigma X,G) = H_m(X,G)$.

§2 *Homotopy groups of Moore space*

We describe a six term exact sequence for the homotopy groups of Moore spaces which is useful for computation in the metastable range of these groups. As an application we obtain a new homotopy invariant $\tau(X)$ of an (n–1)–connected (2n+1)–dimensional closed manifold X.

Let $\underline{\underline{R}} \subset \underline{coHA}_n$ be a small subringoid consisting of suspensions $X = \Sigma X'$. A CW–space U gives us the $\underline{\underline{R}}^{op}$–module (= additive functor)

$$[\underline{\underline{R}},U] : \underline{\underline{R}}^{op} \longrightarrow \underline{Ab}$$

which carries $X \in \underline{\underline{R}}$ to the abelian group [X,U]. The quadratic $\underline{\underline{R}}$–module $\pi_m^K\{\underline{\underline{R}}\}$ associated to (1.3)(1), see (1.4), and the tensor product (3.1) [6] can be used for the natural homomorphism $(\dim \Sigma^m K < 3n-2)$

(2.1)
$$\lambda : [\underline{\underline{R}},U] \otimes_{\underline{\underline{R}}} \pi_m^K\{\underline{\underline{R}}\} \longrightarrow \pi_m^K(U)$$

which we call a tensor approximation of $\pi_m^Y(U)$. For $a \in [\Sigma X',U]$, $b \in [\Sigma Y',U]$, $(\Sigma X',\Sigma Y' \in \underline{\underline{R}})$, and for $\alpha \in [\Sigma^m Y,\Sigma X']$, $\beta \in [\Sigma^m Y,\Sigma X' \wedge Z']$ we define λ by $\lambda(a\otimes\alpha) = a\circ\alpha$ and $\lambda([a,b]\otimes\beta) = [a,b]\circ\beta$ where [a,b] is the Whitehead product. The image of λ is the subgroup generated by all compositions

$$\Sigma^m Y \xrightarrow{\alpha} X_1 \vee \ldots \vee X_k \xrightarrow{a} U$$

with $X_i \in \underline{\underline{R}}$, $k \geq 1$. The map α is in the metastable range, the composition $a\circ\alpha$, however, needs not to be in the metastable range.

(2.2) Lemma: λ in (2.1) is a well defined natural homomorphism. Moreover λ is an isomorphism if $U = X_1 \vee \ldots \vee X_k$ with $X_i \in \underline{\underline{R}}$ and if $[X,X_i] \subset \underline{\underline{R}}(X,X_i)$ for all $i = 1,\ldots,k$ and $X \in \underline{\underline{R}}$.

The lemma is a consequence of the distributivity laws [3] and of (4.4) [6].

(2.3) Remark: A natural description of the homotopy group $\pi_m^K M(A,n)$ of the Moore space M(A,n) can be obtained by the tensor approximation (2.1). For this we need to consider elementary More spaces $M(\mathbb{Z},n) = S^n$ or $M(\mathbb{Z}/p^i,n)$, p = prime. Let $\underline{\underline{R}}$ be the full homotopy category consisting of elementary Moore spaces. Then (2.1) yields the natural homomorphism, $n \geq 3$,

$$\lambda : [\underline{\underline{R}},M(A,n)] \otimes_{\underline{\underline{R}}} \pi_m^K\{\underline{\underline{R}}\} \longrightarrow \pi_m^K M(A,n)$$

which is an isomorphism if A is finitely generated. This follows from (2.2).

We now consider an example of λ in (2.1) where $\underline{\underline{R}} \cong \mathbb{Z}$ is the full subcategory consisting only of the sphere S^n and where $U = M(A,n)$. Then $\pi_m^K\{\underline{\underline{R}}\}$ is just the quadratic $\mathbb{Z}$–module $\pi_m^K\{S^n\} = (\pi_m^K(S^n) \xrightarrow{H} \pi_m^K(S^{2n-1}) \xrightarrow{P} \pi_m^K(S))$ which is defined by $F = \pi_m^K(\_)$ as in (1.4)(3); here H is the Hopf invariant and $P = [1,1]_*$ as in (1.5). Now (2.1) gives us the natural homomorphism

$$\text{(2.4)} \qquad \lambda : A \otimes_{\mathbb{Z}} \pi_m^K\{S^n\} \longrightarrow \pi_m^K M(A,n)$$

which is an isomorphism if A is a free abelian group (here A needs not to be finitely generated). It is an old result of Hopf that $\pi_3\{S^2\} \cong \mathbb{Z}^\Gamma = (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z})$. Therefore we derive from (2.3) the natural homomorphism $\lambda : \Gamma(A) = A \otimes \mathbb{Z}^\Gamma \cong \pi_3 M(A,2)$ which is actually an isomorphism for all abelian groups A, see [25] and (2.11), (4.9) in [6]. In general the map λ in (2.3) is not an isomorphism. Let $S \subset \pi_m^K M(A,n)$ be the subgroup generated by all compositions $\Sigma^m K \longrightarrow S^n \vee \ldots \vee S^n \longrightarrow M(A,n)$ and let

$$_\lambda\pi_m^K M(A,n) = \pi_m^K M(A,n)/S$$

be the quotient group. For $\dim \Sigma^m K < 3n-2$ this is the cokernel of λ in (2.4). Now λ is embedded in the following exact sequence which shows the relevance of the corresponding derived functors in (7.4) [6].

(2.5) Theorem: For $\dim(\Sigma^m K) < 3n-2$ there is a natural exact sequence

$$\begin{aligned} 0 \longrightarrow A *' \pi_m^K\{S^n\} &\longrightarrow {}_\lambda\pi_{m+1}^K M(A,n) \longrightarrow A *'' \pi_{m-1}^K\{S^n\} \xrightarrow{\partial} \\ &A \otimes \pi_m^K\{S^n\} \xrightarrow{\lambda} \pi_m^K M(A,n) \xrightarrow{q} {}_\lambda\pi_m^K M(A,n) \longrightarrow 0 \end{aligned}$$

where q is the quotient map.

We prove this theorem in (5.8). Computations based on (2.5) are described in the examples (2.11) below.

(2.6) Corollary: For $m \leq \mathrm{Min}\,(2n,3n-3)$ one has the natural short exact sequence

$$0 \longrightarrow A \otimes \pi_m\{S^n\} \xrightarrow{\lambda} \pi_m M(A,n) \longrightarrow A *' \pi_{m-1}\{S^n\} \longrightarrow 0$$

and the isomorphism $_\lambda\pi_{m+1} M(A,n) \cong A *' \pi_m\{S^n\}$.

Proof: Since $\pi_{2n-1}S^{2n-1} = \mathbb{Z}$ we see that $A*''\pi_{m-1}\{S^n\} = 0$ for $m \leq 2n$, compare (7.8) [6]. Whence (2.6) is a consequence of (2.5). In the stable range $m < 2n-1$ the sequence (2.6) is well known (see for example [1]); in this case we have $A \otimes \pi_m\{S^n\} = A \otimes \pi_m S^n$ and $A*'\pi_{m-1}\{S^n\} = A*\pi_{m-1}S^n$, see (7.5) [6].

Next we consider the cross effects of the exact sequence in (2.5). For this let

$$M(A\,|\,B,n) = M(A,n) \wedge M(B,n-1) \ \text{ and let } \ {}_\lambda\pi_m^K M(A\,|\,B,n) = \pi_m^K M(A\,|\,B,n)/S'$$

where S' is the subgroup generated by all compositions $\Sigma^m K \to S^{2n-1} \to M(A\,|\,B,n)$.

(2.7) Corollary: For $\dim(\Sigma^m K) < 3n-2$ there is a natural exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Trp}(A,B,\pi_m^K S^{2n-1}) \longrightarrow {}_\lambda\pi_{m+1}^K M(A\,|\,B,n) \longrightarrow A*B*\pi_{n-1}^K S^{2n-1} \\ &\xrightarrow{\partial} A\otimes B\otimes \pi_m^K S^{2n-1} \longrightarrow \pi_m^K M(A\,|\,B,n) \longrightarrow {}_\lambda\pi_m^K M(A\,|\,B,n) \longrightarrow 0 \end{aligned}$$

Here Trp is the triple torsion product of Mac Lane [14], see also (7.7)(3) in [6]. Corollary (2.7), is the 'cross effect sequence' of (2.5) obtained by the formulas (7.7) [6]. It is an interesting problem to compute the boundary operators ∂ in (2.5) and (2.7) only in terms of 'some structure' of the homotopy groups $\pi_i^K(S^j)$ of spheres, in particular if $K = S^0$.

(2.8) Remark: There are many papers in the literature concerning the homotopy groups of Moore spaces $\pi_m M(A,n)$, see for example [22] and [8], [17]. We here are mainly interested in the functorial properties of $\pi_m M(A,n)$, $m < 3n-2$, which are not so well understood; an early approach in this direction is due to Barratt [1] for $m < 2n-1$.

The functorial properties of the groups $\pi_m M(A,n)$ are of special interest for the homotopy classification of manifolds and Poincaré–complexes respectively. Let P_n^k be the class of $(n-1)$–connected $(2n+k)$–dimensional Poincaré–complexes.

(2.9) Examples: Let $n \geq 2$. For $X \in P_n^0$ there is a homotopy invariant

$$\epsilon(X) \in H_n(X) \otimes \pi_{2n-1}\{S^n\}$$

where H_nX is a finitely generated free abelian group. In fact X is the mapping cone $X \simeq C_f$ of a map $f: S^{2n-1} \longrightarrow M(H_nX,n)$ and $\epsilon(X) = \lambda^{-1}(f)$ is given by the isomorphism λ in (2.4). Whence $\epsilon(X)$ is a complete homotopy invariant of X, that is for $X,Y \in P_n^0$ there is an orientation preserving homotopy equivalence $X \simeq Y$ iff there is an isomorphism $\varphi: H_nX \cong H_nY$ with $(\varphi \otimes 1)\epsilon(X) = \epsilon(Y)$. We can write the invariant $\epsilon(X)$ in terms of the cohomology $H^n(X)$ as follows. Since $H_n(X) = \mathrm{Hom}(H^n(X);\mathbb{Z})$ we have by (5.5) [6] the isomorphism

$$\chi: H_n(X) \otimes \pi_{2n-1}\{S^n\} \cong \mathrm{Hom}(H^n(X), \pi_{2n-1}\{S^n\})$$

Therefore $\chi\epsilon(X) = (\alpha_e, \alpha_{ee})$ is a quadratic form with $\alpha_e : H^n(X) \longrightarrow \pi_{2n-1}S^n$ and with $\alpha_{ee} : H^n(X) \times H^n(X) \longrightarrow \pi_{2n-1}S^{2n-1} \cong \mathbb{Z}$. Here α_{ee} is just the cup product pairing in X. Moreover $\alpha_e = \Psi$ is exactly the cohomology operation considered by Kervaire–Milnor in 8.2 [13]; (the formula there is equivalent to the fact that (α_e, α_{ee}) is a quadratic form, compare the first equation in (5.1)(2) [6]).

(2.10) <u>Example</u>: For $X \in P_n^1$ $(n \geq 2)$ we define a new homotopy invariant

$$\tau(X) \in H_n(X) *' \pi_{2n-1}\{S^n\}$$

which we call the <u>torsion–invariant</u> of X. We obtain $\tau(X)$ by a homotopy equivalence $X \simeq C_f$ where $f: S^{2n} \longrightarrow M(H_{n+1}X,n+1) \vee M(H_nX,n)$. Let $r_2 f \in \pi_{2n}M(H_nX,n)$ be given by the retraction r_2 and let $\tau(X)$ be the image of $r_2 f$ under the homomorphism

$$\pi_{2n}M(H_nX,n) \longrightarrow\!\!\!\!\rightarrow {}_\lambda\pi_{2n}M(H_nX,n) \cong H_n(X) *' \pi_{2n-1}\{S^n\}$$

given by (2.6). One can check that an orientation preserving map $v: X \longrightarrow Y$ with $X,Y \in P_n^1$ satisfies

$$(H_n(v) *' 1)(\tau(X)) = \tau(Y)$$

so that $\tau(X)$ is a well defined homotopy invariant. For $n \geq 3$ the exact sequence (2.6) can be used for the computation of all possible f which yield the same torsion invariant. This yields a kind of homotopy classification of objects in P_n^1, (using different invariants such a classification is intensively studied in [19], [20], [18], [11], [24]).

(2.11) <u>Examples of computations</u>: The following list shows some examples of the quadratic $\mathbb{Z}$–modules $\pi_m\{S^n\}$ where we use the notation for indecomposable quadratic $\mathbb{Z}$–modules in (2.4), (2.11) [6]. These examples can be deduced from Toda's computations [23]. In the list we denote a cyclic group $\mathbb{Z}/n$ simply by n and we denote a direct sum $\mathbb{Z}/n\oplus\mathbb{Z}/m$ by $n\oplus m$. Moreover (n,m) and (n,m,r) are the greatest common divisors.

n,m	$\pi_m\{S^n\}$	$\mathbb{Z}/k\otimes\pi_m\{S^n\}$	$\mathbb{Z}/k*'\pi_m\{S^n\}$	$\mathbb{Z}/k*''\pi_m\{S^n\}$	
2,3	$\mathbb{Z}^\Gamma$	$(k^2,2k)$	$(k,2)$	0	
3,5	$\mathbb{Z}^\Lambda\oplus 2$	$(k,2)$	$(k,2)\oplus k$	0	
3,6	$H(2,1)\oplus 3$	$(k,12)$	$(k,12)\oplus(k,2)$	$(k,2)$	
4,7	$E^\Gamma\oplus 3$	$(k^2,2k)\oplus\varepsilon_k$ $\oplus(k,3)$	$(k,24)$	0	
4,8	$(\mathbb{Z}/2)^{\tilde\Gamma}$	0	0	0	$k\equiv 1(2)$
		2	2	0	$k\equiv 2(4)$
4,9		$2\oplus 2$	$2\oplus 2$	0	$k\equiv 0(4)$
5,9	$P(1)$	$(k,2)$	$(k^2,2k)$	0	
5,10	$(\mathbb{Z}/2)^S$	$(k,2)$	$(k,2)$	0	
5,11	$(\mathbb{Z}/2)^\Lambda\oplus 2$	$(k,2)$	$(k,2)\oplus(k,2)$	$(k,2)$	
5,12	$H(1,3)\oplus 15$ $\oplus(\mathbb{Z}/3)^\Lambda$	$(k,2)\oplus(k,15)$	$(k,2)\oplus(k,3)$ $\oplus(k,8)\oplus(k,15)$	$(k,3)\oplus$ $(k,8)$	
6,11	$\mathbb{Z}^S$	k	0	0	
6,12	$(\mathbb{Z}/2)^\Lambda\oplus 2$	$(k,2)$	$(k,2)\oplus(k,2)$	$(k,2)$	
6,13	$H(2,1)\oplus 15$	$(k,60)$	$(k,60)\oplus(k,2)$	$(k,2)$	
6,14	$(\mathbb{Z}/8)^\Gamma\oplus 2$ $\oplus(\mathbb{Z}/3)^S$	$(k,2)$ $\oplus(k^2,2k,24)$	$(k,2)\oplus(k,2)$ $\oplus(k^2,2k,24)$	$(k,2)$	
6,15	$2\oplus 2\oplus 2$	$(k,2)\oplus(k,2)$ $\oplus(k,2)$	$(k,2)\oplus(k,2)$ $\oplus(k,2)$	0	

The quadratic $\mathbb{Z}$–module E^{Γ} (see $(n,m) = (4,7)$) is given by

$$E^{\Gamma} = (\mathbb{Z}\oplus\mathbb{Z}/4 \xrightarrow{(1,0)} \mathbb{Z} \xrightarrow{(2,-1)} \mathbb{Z}\oplus\mathbb{Z}/4)$$

and ϵ_k in this line is

$$\epsilon_k = \begin{cases} 2 & k \equiv 0\ (4),\ k \not\equiv 0\ (8), \\ 4 & k \equiv 0\ (8), \\ 0 & \text{otherwise}. \end{cases}$$

Moreover for $(n,m) = (4,8), (4,9)$ we use $(\mathbb{Z}/2)^{\tilde{\Gamma}} = [\mathbb{Z}^{\tilde{\Gamma}}]\otimes\mathbb{Z}/2$ as defined in (2.1) [6].

The computation of the groups in this list is readily obtained by (7.9) [6]. Combining the groups in the list with the exact sequence (2.5), (2.6) we immediately get the following short exact sequences.

(1) $\mathbb{Z}/(k,12) \rightarrowtail \pi_6 M(\mathbb{Z}/k,3) \twoheadrightarrow \mathbb{Z}(k,2) \oplus \mathbb{Z}/k$

(2) $\left.\begin{array}{ll} k \equiv 1(2) & 0 \\ k \equiv 2(4) & \mathbb{Z}/2 \\ k \equiv 0(4) & \mathbb{Z}/2\oplus\mathbb{Z}/2 \end{array}\right\} \rightarrowtail \pi_8 M(\mathbb{Z}/k,4) \twoheadrightarrow \mathbb{Z}/(k,24)$

(3) $\left.\begin{array}{ll} k \equiv 1(2) & 0 \\ k \equiv 2(4) & \mathbb{Z}/2 \\ k \equiv 0(4) & \mathbb{Z}/2\oplus\mathbb{Z}/2 \end{array}\right\} \rightarrowtail \pi_9 M(\mathbb{Z}/k,4) \twoheadrightarrow \begin{cases} 0 \\ \mathbb{Z}/2 \\ \mathbb{Z}/2\oplus\mathbb{Z}/2 \end{cases}$

(4) $\mathbb{Z}/(k,2) \rightarrowtail \pi_{10} M(\mathbb{Z}/k,5) \twoheadrightarrow \mathbb{Z}/(2k,k^2)$

(5) $\mathbb{Z}/(k,2) \rightarrowtail \pi_{11} M(\mathbb{Z}/k;5) \twoheadrightarrow \mathbb{Z}/(k,2)$

By a result of Sasao [17] the sequence (1) is non split only for $k \equiv 0(2)$ and $k/(k,12) \equiv 1(2)$; in this case one has $\pi_6 M(\mathbb{Z}/k,3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2k \oplus \mathbb{Z}/(k,12)/2$. Moreover Tipple [22] showed that (3) is split and that (4) is non split only for $k \equiv 2(4)$. Finally we deduce $\pi_{12} M(\mathbb{Z}/k,6) = \mathbb{Z}/(k,12)$ from the list above. We leave it to the reader to describe further examples for the exact sequences (2.5).

§3 *Homology of Eilenberg-Mac Lane complexes*

We describe a six term exact sequence for the metastable homology groups of Eilenberg-Mac Lane complexes. This sequence is a kind of Eckmann–Hilton dual of the corresponding exact sequence for metastable homotopy groups of Moore spaces in §2.. Moreover we use the operators in Whitehead's certain exact sequence for a map which carries the homotopy groups of Moore spaces to the homology groups of Eilenberg-Mac Lane spaces.

Let $\underline{\underline{R}} \subset \underline{\underline{HA}}_n$ be a small subringoid, see (1.1). A homotopy abelian H–space U, $U \in \underline{\underline{HA}}$, gives us the $\underline{\underline{R}}^{op}$–module

$$[\underline{\underline{R}},U]' : \underline{\underline{R}}^{op} \longrightarrow \underline{Ab}$$

which carries $X \in \underline{\underline{R}}$ to the abelian group of H–maps $[X,U]' = \underline{\underline{HA}}(X,U)$ which is a subgroup of $[X,U]$. The quadratic $\underline{\underline{R}}$–module $H_m\{\underline{\underline{R}},G\}$ associated to (1.3)(4), see (1.4), and the tensor product (3.1) [6] yield the natural homomorphism ($m < 3/n$)

$$\lambda : [\underline{\underline{R}},U]' \otimes_{\underline{\underline{R}}} H_m\{\underline{\underline{R}},G\} \longrightarrow H_m(U,G) \tag{3.1}$$

as follows. For $a \in [X,U]'$, $b \in [Y,U]'$, $\alpha \in H_m(X,G)$, $\beta \in H_m(X \wedge Y,G)$ let $\lambda(a \otimes \alpha) = a_*(\alpha)$ and $\lambda([a,b] \otimes \beta) = H(\mu)_*(a \wedge b)_*(\beta)$, compare (1.6). The image of λ is the subgroup of $H_m(U,G)$ generated by all elements $\alpha_*(a)$ where $\alpha : X_1 \times \ldots \times X_k \longrightarrow U$ is an H–map, $X_i \in \underline{\underline{R}}$, $k \geq 1$, and where $a \in H_m(X_1 \times \ldots \times X_k,G)$.

(3.2) <u>Lemma</u>: λ in (3.1) is a well defined natural homomorphism. Moreover λ is an isomorphism if $U = X_1 \times \ldots \times X_k$, $X_i \in \underline{\underline{R}}$ for $i = 1,\ldots,k$ and if $\underline{\underline{R}}$ is a full subringoid of $\underline{\underline{HA}}_n$.

Similarly as in (2.2) the lemma is a consequence of (4.4) [6].

(3.3) <u>Remark</u>: A natural description of $H_m(K(A,n),G)$, $m < 3n$, can be obtained by (3.1). For this let $\underline{\underline{R}}$ be the full homotopy category consisting of elementary Eilenberg–Mac Lane spaces $K(\mathbb{Z},n)$ or $K(\mathbb{Z}/p^i,n)$, p = prime. Then (3.1) yields the natural homomorphism ($n \geq 2$)

$$\lambda : [\underline{\underline{R}},K(A,n)] \otimes_{\underline{\underline{R}}} H_m\{\underline{\underline{R}},G\} \xrightarrow{\cong} H_m(K(A,n),G)$$

which is an isomorphism for all $A \in \underline{Ab}$. This follows essentially from (3.2), compare (4.6) [6]. We clearly have $[\underline{\underline{R}},K(A,n)] = [\underline{\underline{R}},K(A,n)]'$.

We now consider a special case of λ in (3.1). For this let $\underline{\underline{R}} \cong \mathbb{Z}$ be the full subcategory consisting only of $K(\mathbb{Z},n)$ and let $U = K(A,n)$. Then $H_m\{\underline{\underline{R}},G\}$ is the quadratic $\mathbb{Z}$–module (see (1.6))

$$H_m^G\{n\} = (H_m(K(\mathbb{Z},n),G) \xrightarrow{H} H_m(K(\mathbb{Z},n) \wedge K(\mathbb{Z},n),G) \xrightarrow{P} H_m(K(\mathbb{Z},n),G))$$

and we get by (3.1) the natural homomorphism

$$\lambda : A \otimes_{\mathbb{Z}} H_m^G\{n\} \longrightarrow H_m(K(A,n),G) \tag{3.4}$$

which is an isomorphism if A is free abelian; here A needs not to be finitely generated. In fact λ is the tensor approximation of the functor $\underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$ which carries A to $H_m(K(A,n),G)$, compare (4.8) [6]. For $G = \mathbb{Z}$ we set $H_m\{n\} = H_m^{\mathbb{Z}}\{n\}$. Since $K(\mathbb{Z},2) = \mathbb{C}P_\infty$ we readily see that $H_4\{2\} \cong \mathbb{Z}^\Gamma$. Therefore we derive form (3.4) the natural homomorphism $\lambda : \Gamma(A) = A \otimes \mathbb{Z}^\Gamma \cong H_4K(A,2)$ which is actually an isomorphism for all A, compare [10]. The following list shows some examples of quadratic $\mathbb{Z}$–modules $H_m\{n\}$. We use in this list the notation for indecomposable quadratic $\mathbb{Z}$–modules in (2.4), (2.11) [6]; the examples can be deduced from the computations in [10].

m	n	$H_m\{n\}$	$H_m(K(A,n))$
3	2	0	0
4	2	$\mathbb{Z}^\Gamma$	$\Gamma(A)$
5	2	0	$R(A)$
5	3	$\mathbb{Z}/2$	$\mathbb{Z}/2 \otimes A$
6	3	$\mathbb{Z}^\Lambda$	$\mathbb{Z}/2 * A \oplus \Lambda^2(A)$
7	3	$\mathbb{Z}/3$	$\mathbb{Z}/3 * A \oplus \Omega(A)$
8	3	$(\mathbb{Z}/2)^\otimes$	$\mathbb{Z}/3 * A \oplus A \otimes A \otimes \mathbb{Z}/2$
7	4	0	$\mathbb{Z}/2 * A$
8	4	$\mathbb{Z}^\Gamma \oplus \mathbb{Z}/3$	$\mathbb{Z}/3 \otimes A \oplus \Gamma(A)$
9	4	0	$\mathbb{Z}/3 * A \oplus R(A)$
9	5	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2 \oplus \mathbb{Z}/3) \otimes A$
10	5	$\mathbb{Z}^\Lambda$	$(\mathbb{Z}/2 \oplus \mathbb{Z}/3) * A \oplus \Lambda^2(A)$

In general the map λ in (3.4) is not an isomorphism. As an analogue of theorem (2.5) we proof in (6.13) below the following result, in which, however λ above does not appear. Again we use the derived functors in (7.4) [6].

(3.5) <u>Theorem</u>: Let $m \leq 3n-3$. Then there is a natural map $\kappa : H_m(K(A,n-1),G) \longrightarrow A*H_m^G\{n\}$ such that ${}_\kappa H_m(K(A,n-1),G) = \mathrm{kernel}(\kappa)$ is embedded in the natural exact sequence

$$\begin{aligned} 0 \longleftarrow\ & A\otimes' H_m^G\{n\} \longleftarrow {}_\kappa H_{m-1}(K(A,n-1),G) \longleftarrow A\otimes'' H_{m+1}^G\{n\} \longleftarrow \\ & A*H_m^G\{n\} \xleftarrow{\ \kappa\ } H_m(K(A,n-1),G) \xleftarrow{\ i\ } {}_\kappa H_m(K(A,n-1),G) \longleftarrow 0 \end{aligned}$$

where i is the inclusion.

In the stable range $m < 2n-2$ this yields just the short exact sequence

$$A*H_m(K(\mathbb{Z},n),G) \twoheadleftarrow\!\!\xleftarrow{\ \kappa\ } H_m(K(A,n-1),G) \hookleftarrow A\otimes H_{m+1}(K(\mathbb{Z},n),G) \tag{3.6}$$

which is a kind of Eckmann–Hilton dual of (2.6). Using the formulas in (7.7) [6] it is easy to obtain the exact "cross effect sequence" of (3.5), this is a sequence of a similar nature as in (2.7).

(3.7) <u>Examples</u>: We describe some applications of (3.5) where we use the list in (3.4). Since $H_7\{4\} = 0$ and since $H_9\{4\} = 0$ we obtain the isomorphism

$$\begin{aligned} A\otimes'\mathbb{Z}^\Gamma \oplus A\otimes\mathbb{Z}/3 &= A\otimes' H_8(4) \\ &\cong {}_\kappa H_7 K(A,3) \\ &= H_7K(A,3) \cong \Omega A\oplus A\otimes\mathbb{Z}/3 \end{aligned}$$

which corresponds to the isomorphism $A\otimes'\mathbb{Z}^\Gamma \cong \Omega A$ in (7.12)[6]. Since $H_7\{3\} = \mathbb{Z}/3$ we have $A\otimes'' H_7\{3\} = 0$ so that ${}_\kappa H_5K(A,2) \cong A\otimes'\mathbb{Z}^\Lambda$ where $\mathbb{Z}^\Lambda = H_6\{3\}$. Moreover we have $H_4\{3\} = 0$ so that ${}_\kappa H_4K(A,2) = H_4K(A,2) \cong \Gamma(A)$. Therefore we derive from (3.5) the exact sequence

$$A\otimes\mathbb{Z}/2 \twoheadleftarrow \Gamma(A) \hookleftarrow A\otimes''\mathbb{Z}^\Lambda \xleftarrow{\ 0\ } A*\mathbb{Z}/2 \twoheadleftarrow\!\!\xleftarrow{\ \kappa\ } R(A) \hookleftarrow A\otimes'\mathbb{Z}^\Lambda$$

which is the union of two natural short exact sequences. By (2.10) [6] this shows that there are natural isomorphisms $A\otimes''\mathbb{Z}^\Lambda \cong SP^2(A) \cong A\otimes\mathbb{Z}^S$.

(3.8) Remark: J. Decker got a formula for $H_mK(A,n)$, $m < 3n$, in terms of a list of homology operations α, see III (4.3) [9]. This list of homology operations (based on results of Cartan [7]) allows in principle the computation of $H_mK(A,n)$ as a functor and whence we can derive the quadratic $\mathbb{Z}$–module $H_m\{n\}$. The exact sequence (3.5) still is helpful for understanding the somewhat intricate functors Ω_q and R_q which appear in Decker's formula. They generalize the functors Ω and R of Eilenberg–Mac Lane [10], that is $\Omega_0 = \Omega$, $R_0 = R$.

We now describe a connection between homotopy groups of Moore spaces and homology groups of Eilenberg-Mac Lane spaces. To this end recall that the Hurewicz homomorphism h is embedded in a long exact sequence [25]

$$\longrightarrow H_{n+1}X \xrightarrow{b} \Gamma_nX \xrightarrow{i} \pi_nX \xrightarrow{h} H_nX \xrightarrow{b} \Gamma_{n-1}X \longrightarrow$$

which is natural for simply connected spaces X. For an abelian group A we have the canonical map $(n \geq 2)$

$$k : M(A,n) \longrightarrow K(A,n)$$

which induces the identity $H_n(k) = 1_A$ of A. This map induces the natural homomorphism

$$(3.9) \qquad Q_1 = b^{-1}\Gamma_m(k)i^{-1} : \pi_mM(A,n) \longrightarrow H_{m+1}K(A,n)$$

where we use i and b in the exact sequence above. J.H.C. Whitehead [25] showed that Q_1 is an isomorphism for $m = n+1$. In the meta stable range Q_1 is part of the following commutative diagram where we use $\Sigma M(A,n-1) = M(A,n)$, $m < 3n-2$.

$$(3.10) \qquad \begin{array}{ccccc} \pi_mM(A,n) & \xrightarrow{H} & \pi_mM(A,n)\wedge M(A,n-1) & \xrightarrow{P} & \pi_mM(A,n) \\ \downarrow Q_1 & & \downarrow Q_2 & & \downarrow Q_1 \\ H_{m+1}K(A,n) & \xrightarrow{H} & H_{m+1}K(A,n)\wedge K(A,n) & \xrightarrow{P} & H_{m+1}K(A,n) \end{array}$$

The maps H and P are defined as in (1.5) and (1.6) respectively. The map Q_2 is defined by $Q_2 = h\pi_{m+1}(k\wedge k)\Sigma$ where Σ is the suspension operator and where h is the Hurewicz map. Whence Q_2 is an isomorphism for $m = 2n-1$. The commutativity of the diagram shows that $Q = (Q_1,Q_2)$ is a map between quadratic $\mathbb{Z}$–modules. We obtain the commutativity of (3.10) by the homotopy commutativity of

$$(3.11) \qquad \begin{array}{ccccc} M(A,n) & \xrightarrow{\mu'} & M(A,u)\vee M(A,n) & \xrightarrow{\nabla} & M(A,n) \\ \downarrow k & & \downarrow k' & & \downarrow k \\ K(A,n) & \xrightarrow{\Delta} & K(A,n)\times K(A,n) & \xrightarrow{\mu} & K(A,n) \end{array}$$

Here μ' and μ are the comultiplication and multiplication respectively and k' is given by $k \vee k$ and the inclusion. By applying the functor Γ_m to (3.11) we essentially get (3.10).

For any (n–1)–connected space X with $H_nX \cong A$ we have maps

$$k : M(A,n) \xrightarrow{k'} X \xrightarrow{k''} K(A,n) \tag{3.12}$$

which induce isomorphisms in homology H_n. Here the homotopy class of k'' is unique, the homotopy class of k', however, is not unique. From (3.10) we derive for $m < 3n-2$ the commutative diagram

(3.13)
$$\begin{tikzcd}
A\otimes\pi_m\{S^n\} \arrow[rr] \arrow[d, "\lambda"] & & A\otimes H_{m+1}\{n\} \arrow[d, "\lambda"] \\
\pi_m M(A,n) \arrow[r, "k'_*i^{-1}"] \arrow[rr, bend right, "Q_1"'] & \Gamma_m X \arrow[r, "b^{-1}k''_*"] & H_{m+1}K(A,n)
\end{tikzcd}$$

which shows that $\Gamma_m X$ is non trivial if Q_1 is non trivial. The following lemma gives information on part of the kernel of Q_1.

(3.14) <u>Lemma</u>: Let $\alpha \in \pi_m(M(A,n))$ be a map which admits a factorization $\alpha : S^m \longrightarrow Y \longrightarrow M(A,n)$ where Y is n–connected and $\dim(Y) \leq m-1$. Then we have $Q_1(\alpha) = 0$. In particular we have $Q_1([\xi,\eta]) = 0$ for all Whitehead products $[\xi,\eta]$ with $\xi \in \pi_t M(A,n)$, $t > n$.

(3.15) <u>Example</u>: All arrows in (3.13) are isomorphisms for $n = 2$, $m = 3$. Moreover the map

$$Q_1 : \pi_4 M(A,2) \longrightarrow H_5K(A,2) \cong R(A)$$

is surjective and its kernel is the subgroup S in (2.5). Whence we have the natural isomorphisms

$$A*'\mathbb{Z}^\Gamma \cong {}_\lambda\pi_4 M(A,2) \cong H_5K(A,2) \cong R(A),$$

compare (2.6) and (7.12) [6].

§4 *Cohomology of Eilenberg-Mac Lane spaces*

Here we obtain a six term exact sequence for the cohomology groups of Eilenberg-Mac Lane spaces in the metastable range.

Let $\underline{\underline{R}} \subset \underline{\underline{HA}}_n$ be a small subringoid, see (1.1). A homotopy abelian H–space gives us the $\underline{\underline{R}}^{op}$–module $[\underline{\underline{R}},U]'$ as in (3.1). Now the quadratic $\underline{\underline{R}}^{op}$–modules $H^m\{\underline{\underline{R}},G\}$ and $\pi_K^m\{\underline{\underline{R}}\}$ associated to the functors (1.3)(3) and (1.3)(2) respectively yield the natural homomorphisms $(m < 3n$, resp. $\operatorname{hodim}(\Omega^m K) < 3n)$

$$(4.1) \qquad \lambda : H^m(U,G) \longrightarrow \operatorname{Hom}_{\underline{\underline{R}}^{op}}([\underline{\underline{R}},U]', H^m\{\underline{\underline{R}},G\}),$$

$$\lambda : \pi_K^m(U) \longrightarrow \operatorname{Hom}_{\underline{\underline{R}}^{op}}([\underline{\underline{R}},U]', \pi_K^m\{\underline{\underline{R}}\})$$

Compare (5.7) [6]. By (5.8) [6] we know

(4.2) Proposition: The homomorphisms λ in (4.1) are isomorphisms if $U = X_1 \times \ldots \times X_r$ is a finite product with $X_i \in \underline{\underline{R}}$ for $i = 1,\ldots,r$ and if $\underline{\underline{R}}$ is a full subringoid of $\underline{\underline{HA}}_n$.

(4.3) Remark: Let $\underline{\underline{R}}$ be the ringoid of elementary Eilenberg-Mac Lane spaces as in (3.3). Then (4.1) yields the natural homomorphism

$$\lambda : \pi_K^m(K(A,n),G) \longrightarrow \operatorname{Hom}_{\underline{\underline{R}}^{op}}([\underline{\underline{R}},K(A,n)], \pi_K^m\{\underline{\underline{R}}\})$$

which is an isomorphism if A is finitely generated.

We now consider a special case of λ in (4.1). For this let $\underline{\underline{R}} \cong \mathbb{Z}$ be the full subcategory consisting only of $K(\mathbb{Z},n)$ and let $U = K(A,n)$. Then $H^m\{\underline{\underline{R}},G\}$ and $\pi_K^m\{\underline{\underline{R}}\}$ are the quadratic $\mathbb{Z}$–modules

$$H_G^m\{n\} = (H^m(K(\mathbb{Z},n),G) \xrightarrow{H} H^m(K(\mathbb{Z},n) \wedge K(\mathbb{Z},n),G) \xrightarrow{P} H^m(K(\mathbb{Z},n),G)),$$

$$\pi_K^m\{n\} = (\pi_K^m K(\mathbb{Z},n) \xrightarrow{H} \pi_K^m K(\mathbb{Z},n) \wedge K(\mathbb{Z},n) \xrightarrow{P} \pi_K^m K(\mathbb{Z},n))$$

respectively defined as in (1.6). Now (4.1) yields the natural homomorphisms

$$(4.4) \qquad \begin{aligned} \lambda &: H^m(K(A,n),G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A,H_G^m\{n\}) \\ \lambda &: \pi_K^m(K(A,n)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A,\pi_K^m\{n\}) \end{aligned}$$

which are isomorphisms if A is a free abelian group (here A needs not to be finitely generated). In the next result we use the derived functors in (7.4) [6].

(4.5) <u>Theorem</u>: Let $m \leq 3n-2$. Then there is a natural map $\kappa : \mathrm{Ext}(A,H_G^m\{n\}) \longrightarrow H^m(K(A,n-1),G)$ such that ${}_\kappa H^m(K(A,n-1),G) = \mathrm{cokernel}(\kappa)$ is embedded in the natural exact sequence

$$\begin{aligned} 0 \longrightarrow \ & \mathrm{Hom}'(A,H_G^m\{n\}) \longrightarrow {}_\kappa H^{m-1}(K(A,n-1),G) \longrightarrow \mathrm{Hom}''(A,H_G^{m+1}\{n\}) \longrightarrow \\ & \mathrm{Ext}(A,H_G^m\{n\}) \xrightarrow{\kappa} H^m(K(A,n-1),G) \xrightarrow{q} {}_\kappa H^m(K(A,n-1),G) \longrightarrow 0 \end{aligned}$$

where q is the quotient map.

In the stable range $m < 2n-2$ this sequence is equivalent to the short exact sequence

$$(4.6) \quad 0 \longrightarrow \mathrm{Ext}(A,H_G^m\{n\}) \longrightarrow H^m(K(A,n-1),G) \longrightarrow \mathrm{Hom}(A,H_G^{m+1}\{n\}) \longrightarrow 0$$

where $H_G^m\{n\} = H^m(K(\mathbb{Z},n),G)$ is an abelian group. Theorem (4.5) is a special case of the next result.

(4.7) <u>Theorem</u>: Let $\mathrm{hodim}\ (\Omega^m K) \leq 3n-2$. Then there is a natural map $\kappa : \mathrm{Ext}(A,\pi_K^m\{n\}) \longrightarrow \pi_K^m K(A,n-1)$ such that ${}_\kappa\pi_K^m K(A,n-1) = \mathrm{cokernel}\ (\kappa)$ is embedded in the natural exact sequence

$$\begin{aligned} 0 \longrightarrow \ & \mathrm{Hom}'(A,\pi_K^m\{n\}) \longrightarrow {}_\kappa\pi_K^{m+1} K(A,n-1) \longrightarrow \mathrm{Hom}''(A,\pi_K^{m-1}\{n\}) \longrightarrow \\ & \mathrm{Ext}(A,\pi_K^m\{n\}) \xrightarrow{\kappa} \pi_K^m K(A,n-1) \xrightarrow{q} {}_\kappa\pi_K^m K(A,n-1) \longrightarrow 0 \end{aligned}$$

where q is the quotient map.

We prove (4.7) and (4.5) in (6.8) below. Again it is obvious how to describe the "cross effect sequence" of (4.7) by the formulas in (7.7) [6].

§5 *An exact sequence for homotopy groups of a cofiber*

The main result in this section describes an approximation of a certain fiber from which we deduce various exact sequences, this way we obtain a proof of the exact sequence for homotopy groups of Moore spaces in (2.5).

Let $g : A \longrightarrow B$ be a map in $\underline{\mathrm{Top}}^*$ and consider the following commutative diagram.

$$\begin{tikzcd}
C_g & CA \vee B \arrow[l, "{(\pi_g,i)}"'] & \\
B \arrow[u, "i"] \arrow[ur, phantom, "\text{push}"] & A \vee B \arrow[l, "{(g,1)}"] \arrow[u, "i \vee 1"'] & \\
P_i \arrow[u] & P_{i \vee 1} \arrow[l, "r(g)"] \arrow[u] & P_{r(g)} \arrow[l] \arrow[ul, "W_g"']
\end{tikzcd}$$

Here CB is the cone on B and $i : B \rightarrowtail CB$ is the inclusion. The <u>mapping cone</u> C_g is defined by the subdiagram 'push' which is a push out diagram. Let $r(g)$ be the induced map between homotopy theoretic fibers and let $P_{r(g)}$ be the homotopy theoretic fiber of $r(g)$. Let W_g be the composition of fibrations $\twoheadrightarrow$ as indicated in the diagram.

(5.2) <u>Theorem</u>: Let A be a suspension, $A = \Sigma A'$, which is (a–1)–connected and assume B is (b–1)–connected, $b \geq 2$. Then there is a map

$$\delta : \Sigma A' \wedge A' \longrightarrow P_{r(g)}$$

which is $\mathrm{Min}(a,b) + 2a - 3$ connected. Moreover the composition $W_g \delta$ satisfies the equation

$$\begin{cases} W_g \delta : \Sigma A' \wedge A' \longrightarrow A \vee B \\ W_g \delta = [i_1, i_1 - i_2 g] \end{cases}$$

Here i_1 (resp. i_2) is the inclusion of A (resp. B) into $A \vee B$ and $[\ ,\ ]$ is the Whitehead product.

<u>Proof</u>: The theorem is a partial reformulation of (6.9) in [5].

The next corollary of (5.2) yields an exact sequence which is dual to the sequence of E. Thomas in (6.4) below.

(5.3) Corollary: Let A be an $(a-1)$–connected suspension, $a \geq 1$, and let $g : A \longrightarrow B$ be a map where B is 1–connected. Then we have for $\dim(\Sigma^m K) < 2a-2$ the exact sequence

$$\pi_{m+1}^K(B) \xrightarrow{i_*} \pi_{m+1}^K(C_g) \longrightarrow \pi_m^K(A \vee B)_2 \xrightarrow{\partial} \pi_m^K(B) \xrightarrow{i_*} \pi_m^K(C_g)$$

Here $\pi_m^K(A \vee B)_2$ denotes the kernel of the map $(1,0)_* : \pi_m^K(A \vee B) \longrightarrow \pi_m^K(B)$ and $\partial = (g,1)_*$ is induced by $(g,1) : A \vee B \longrightarrow B$.

Compare also (6.9) in [5].

Proof of (5.3): By (5.2) we see that $P_{r(g)}$ is $(2a-2)$–connected. Therefore $r(g)$ induces an isomorphism $\pi_m^K(P_i) \cong \pi_m^K(P_{i \vee 1}) \cong \pi_m^K(A \vee B)_2$ since $\dim(\Sigma^m K) < 2a-2$. Now the fiber sequence for P_i yields the exact sequence in (5.3).

(5.4) Corollary: Let $g : \Sigma X_1 \longrightarrow \Sigma X_0$ be a map between $(n-1)$–connected suspensions in $\underline{\underline{coHA}}_n$ (see (1.1)) and let $\dim(\Sigma^m K) < 3n-2$. Then we have the following commutative diagram in which the column and the row are exact sequences.

$$\begin{tikzcd}
 & \pi_m^K(\Sigma X_1 \wedge X_1) \arrow[d, "D_2"] & & \\
 & \pi_m^K(\Sigma X_1) \oplus \pi_m^Y(\Sigma X_1 \wedge X_0) \arrow[d, "\overline{E}"] \arrow[dr, "D_1"] & & \\
\pi_{m+1}^K(C_g) \arrow[r, "j"] & \pi_{m+1}^K(C_g, \Sigma X_0) \arrow[r, "\partial"] \arrow[d, "\overline{H}"] & \pi_m^K(\Sigma X_0) \arrow[r, "i"] & \pi_m^K(C_g) \\
 & \pi_{m-1}^K(\Sigma X_1 \wedge X_1) \arrow[d, "D_2"] & & \\
 & Y_{m-1}^K(\Sigma X_1) \oplus \pi_{m-1}^K(\Sigma X_1 \wedge X_0) & &
\end{tikzcd}$$

The row is the exact homotopy sequence of the pair $(C_g, \Sigma X_0)$. The operators D_1 and D_2 are given by

$$D_2 = (P, -(1|g)_*),\ D_1 = (g_*, P(g|1)_*)$$

where P, $(g|1)_*$ and $(1|g)_*$ are defined as in (1.5). In fact $(1|g)_*$ is $(1 \wedge g)_*$ up to the switch of suspension coordinates and $(g|1)_* = (g \wedge 1)_*$.

Proof of (5.4): We apply the operator π_m^Y to the fiber sequence in the bottom row of (5.1). This yields the exact column of (5.4) since δ in (5.2) is highly connected. Here we use the isomorphisms

$$\begin{aligned} \pi_m^K P_{i \vee 1} &\cong \pi_{m+1}^K (CA \vee B, A \vee B), \\ &\cong \pi_m^K (A \vee B)_2, \text{ see (5.3)}, \\ &\cong \pi_m^K (\Sigma X_1) \oplus \pi_m^K (\Sigma X_1 \wedge X_0) \end{aligned}$$

with $A = \Sigma X_1$, $B = \Sigma X_0$. The map D_1 is induced by $(g,1) : A \vee B \longrightarrow B$. Moreover D_2 is induced by $W_g \delta$ in (5.2).

The operations D_1, D_2 form a chain complex, that is $D_1 D_2 = 0$. This chain complex corresponds exactly to the chain complex $M_*(d)$ in (6.3) [6] where $d = g$ and where $M = \{\pi_m^K\}$ is the quadratic $\underline{\underline{\text{coHA}}}_n$–module associated to the quadratic functor π_m^K in (1.3)(1), see (1.4) or (3.5) [6]. We therefore have the homology groups

$$(5.5) \qquad \begin{cases} H_0\{\pi_m^K\}_*(g) = \text{cokernel } (D_1) \\ H_1\{\pi_m^K\}_*(g) = \text{kernel } (D_1)/\text{image } (D_2) \\ H_2\{\pi_m^K\}_*(g) = \text{kernel } (D_2) \end{cases}$$

These homology groups appear in the exact sequence of the next corollary where

$$(5.6) \qquad j\pi_{m+1}^K (C_g) = \text{kernel } \partial$$

is the image of the operator j in (5.4).

(5.7) Corollary: With the assumptions in (5.4) there is the exact sequence

$$\begin{aligned} 0 \longrightarrow\ & H_1\{\pi_m^K\}_*(g) \xrightarrow{e} j\pi_{m+1}^K (C_g) \xrightarrow{h} H_2\{\pi_{m-1}^K\}_*(g) \xrightarrow{\partial} \\ & H_0\{\pi_m^K\}_*(g) \xrightarrow{i} \pi_m^K (C_g) \xrightarrow{j} j\pi_m^K (C_g) \longrightarrow 0 \end{aligned}$$

Proof: The maps are given as follows. The inclusion e is induced by $\overline{E}$ and the map h is the restriction of $\overline{H}$. The map ∂ is induced by $\partial(\overline{H})^{-1}$ and i and j are derived from the corresponding maps i and j in (5.4).

The exact sequences (5.4) and (5.6) are natural with respect to principal maps $F : C_g \longrightarrow C_{g'}$ between mapping cones, compare [4] for the definition of principal maps.

(5.8) Proof of (2.5): Theorem (2.3) is a special case of (5.7). For this let $X_1 \xrightarrow{d} X_0 \twoheadrightarrow A$ be a short free resolution of the abelian group A and let $g : M(X_1,n) \longrightarrow M(X_0,n)$ be a map which induces d. The mapping cone of g is the Moore space $M(A,n) = C_g$. Using the isomorphism λ in (2.4) (where we replace A by X_1 and X_0 respectively) we obtain isomorphisms

$$H_0\{\pi_m^K\}_*(g) = A\otimes\pi_m^K\{S^n\},$$
$$H_1\{\pi_m^K\}_*(g) = A*'\pi_m^K\{S^n\},$$
$$H_2\{\pi_m^K\}_*(g) = A*''\pi_m^K\{S^n\}.$$

Compare the definition in (7.4) [6]. Now it is easy to see that i in (5.7) corresponds to λ in (2.5). Therefore (2.5) is just a special case of (5.7).

§6 *An exact sequence for (co)homology groups of a fiber*

The main result in this section describes an approximation of a certain cofiber. This result is the Eckmann–Hilton dual of the result in §5, the proofs however, are not dual. We derive exact sequences which, in particular, yield the exact sequences for homology and cohomology groups of Eilenberg–Mac Lane spaces in §3 and §4 above.

We first describe the Eckammn–Hilton dual of diagram (5.1). Let $g : B \longrightarrow A$ be a map in $\underline{\underline{Top}}^*$ and consider the following commutative diagram

(6.1)
$$\begin{tikzcd}
P_g \arrow[r, "(\pi_g{,}q)"] \arrow[d, "q"] \arrow[dr, phantom, "\text{pull}"] & WA\times B \arrow[d, "q\times 1"] & \\
B \arrow[r, "(g{,}1)"'] \arrow[d, tail] & A\times B \arrow[d, tail] \arrow[dr, tail, bend left, "W_g"] & \\
C_q \arrow[r, "r(g)"'] & C_{q\times 1} \arrow[r, tail] & C_{r(g)}
\end{tikzcd}$$

Here WB is the contractible path object and P_g is the homotopy theoretic fiber of g which is defined by the subdiagram 'pull' which is a pull back diagram. Let $r(g)$ be the induced map between mapping cones and let $C_{r(g)}$ be the mapping cone of $r(g)$. Let W_g be the composition of cofibrations as indicated in the diagram.

(6.2) <u>Theorem</u>: Let A be an H–group which is (a–1)–connected and assume B is (b–1)–connected. Then there is a map
$$\delta : C_{r(g)} \longrightarrow A \wedge A$$
which is $\mathrm{Min}(a,b)+2a-1$ connected. Moreover the suspension $\Sigma(\delta W_g)$ satisfies the equation:
$$\Sigma(\delta W_g) : \Sigma(A\times B) \longrightarrow \Sigma(A\wedge A)$$
$$\Sigma(\delta W_g) = \Sigma(\Delta_A p_A) - \Sigma((1\wedge g)\hat{q}) + \lambda$$
Here $p_A : A\times B \longrightarrow A$ is the projection, $\hat{q} : A\times B \longrightarrow A\wedge B$ is the quotient map, and $\Delta_A : A \longrightarrow A\wedge A$ is the reduced diagonal. Finally λ is a map which admits a factorization $\lambda : \Sigma A\wedge B \longrightarrow Y \longrightarrow \Sigma A\wedge A$ where Y is (3a)–connected.

An explicit description of δW_g is given in the proof of the theorem, see (6) in §7 below. In the stable range we obtain by (6.2) the following exact sequence due to E. Thomas [21], see also [16]; a generalization of this sequence can be derived from the 'general loop theorem' (V. 10.6) in [4], see also (6.4.7) [2].

(6.3) <u>Corollary</u>: Let A be an H–group which is (a–1)–connected and let $g : B \longrightarrow A$ be a map. For $\mathrm{hodim}(\Omega^m Y) < 2a-1$ we have the exact sequence, $m \geq 1$,
$$\pi_Y^{m+1}(B) \xrightarrow{q^*} \pi_Y^{m+1}(P_g) \longrightarrow \pi_Y^m(A\rtimes B) \xrightarrow{\partial} \pi_Y^m(B) \xrightarrow{q^*} \pi_Y^m(P_g)$$
where $A\rtimes B$ is the cofiber of $i_2 : B \longrightarrow A\times B$. The boundary $\partial = (p(g,1))^*$ is induced by the composition $p(g,1) : B \longrightarrow A\times B \longrightarrow A\rtimes B$.

<u>Proof</u>: By (6.2) we see that $C_{r(g)}$ is (2a–1)–connected. Therefore $r(g)$ induces the isomorphism

$$\pi_m^Y(C_q) \cong \pi_m^Y(C_{q\times 1}) \cong \pi_m^Y(A \rtimes B)$$

since we assume $\operatorname{hodim}(\Omega^m Y) < 2a-1$. Now the Puppe sequence for C_q yields the exact sequence (6.3). The next corollary of (6.2) is dual to (5.4).

(6.4) <u>Corollary</u>: Let $g : X_0 \longrightarrow X_1$ be a map between (n–1)–connected homotopy abelian H–groups in $\underline{\underline{HA}}_n$ (see (1.1)) and let $\operatorname{hodim}(\Omega^m Y) \leq 3n-2$. Then we have the following commutative diagram in which the column and the row are exact sequences.

$$\begin{tikzcd}
 & \pi_Y^m(X_1 \wedge X_1) \arrow[d, "D_2"] & & \\
 & \pi_Y^m(X_1) \oplus \pi_Y^m(X_1 \wedge X_0) \arrow[d, "\overline{E}"] \arrow[dr, "D_1"] & & \\
\pi_Y^{m+1}(P_g) \arrow[r, "j"'] & \pi_Y^{m+1}(P_g \mid B) \arrow[r, "\partial"'] \arrow[d, "H"] & \pi_Y^m(X_0) \arrow[r, "q^*"'] & \pi_Y^m(P_g) \\
 & \pi_Y^{m-1}(X_1 \wedge X_1) \arrow[d, "D_2"] & & \\
 & \pi_Y^{m-1}(X_1) \oplus \pi_Y^{m-1}(X_1 \wedge X_0) & &
\end{tikzcd}$$

The row is the exact cofiber sequence (Puppe sequence) of the mapping cone C_q of $q : P_g \longrightarrow X_0$, here we write $\pi_Y^{m+1}(P_g \mid B) = \pi_Y^m(C_q)$ since this group is dual to the corresponding relative homotopy group $\pi_{m+1}^Y(C_g, \Sigma X_0)$ in (5.3), compare also the notation in (II.14.5) [4]. The operators D_1 and D_2 are given similarly as in (5.4), namely

$$D_2 = (D, -(1 \mid g)^*), \quad D_1 = (g^*, P(g \mid 1)^*)$$

Here $P = \Delta^*$, $(1 \mid g)^*$ and $(g \mid 1)^*$ are defined by (1.6). In fact we have $(1 \mid g)^* = (1 \wedge g)^*$ and $(g \mid 1)^* = (g \wedge 1)^*$.

Proof of (6.4): We apply the operator π_Y^m to the cofiber sequence in the bottom row of (6.1). This yields the exact column of (6.4) since δ in (6.2) is (3n–1)–connected. Here we also use the isomorphism

$$\begin{aligned}\pi_Y^m(C_{q\times 1}) &= \pi_Y^{m+1}(WA\times B \,|\, A\times B)\\ &= \pi_Y^m(A\times B/B)\\ &= \pi_Y^m(A)\oplus\pi_Y^m(A\wedge B)\end{aligned}$$

with $A = X_1$, $B = X_0$. The map D_1 is induced by $(g,1) : B \longrightarrow A\times B$. Moreover D_2 is induced by $(\delta W_g)^*$. The formula in (6.2) for $\Sigma(\delta W_g)$ yields the formula for D_2 above.

The operators D_1,D_2 form a chain complex, that is $D_1D_2 = 0$. This chain complex corresponds exactly to the chain complex $M_*(d)$ in (6.3) [6] where $d = g^{op}$ and where $M = \{\pi_K^m\}$ is the quadratic $\underline{\underline{HA}}_n^{op}$–module associated to the quadratic functor π_K^m in (1.3)(2), see (1.4) above or (3.5) [6]. The homology groups of this chain complex are the groups

$$(6.5)\qquad \begin{cases} H_0\{\pi_K^m\}_*(g^{op}) = \text{cokernel } (D_1),\\ H_1\{\pi_K^m\}_*(g^{op}) = \text{kernel } (D_1)/\text{image } D_2,\\ H_2\{\pi_K^m\}_*(g^{op}) = \text{kernel } (D_2)\,. \end{cases}$$

They appear in the exact sequence of the next corollary where

$$(6.6)\qquad j\pi_K^{m+1}(P_g) \cong \text{kernel}(\partial)$$

is the image of the operator j in (6.4).

(6.7) Corollary: With the assumptions in (6.4) there is the exact sequence

$$\begin{aligned}0 \longrightarrow\ & H_1\{\pi_K^m\}_*(g^{op}) \xrightarrow{e} j\pi_K^{m+1}(P_g) \xrightarrow{h} H_2\{\pi_K^{m-1}\}_*(g^{op}) \xrightarrow{\partial}\\ & H_0\{\pi_K^m\}_*(g^{op}) \xrightarrow{i} \pi_K^m(P_g) \xrightarrow{j} j\pi_K^m(P_g) \longrightarrow 0\end{aligned}$$

The proof of (6.7) is the same as the one of (5.7). The exact sequence (6.7) is natural with respect to principal maps $F : P_g \longrightarrow P_{g'}$ between fiber spaces, compare [4] for the definition of principal maps.

(6.8) Proof of (4.7): In fact theorem (4.7) is a special case of (6.7). For this let $g : K(X_1,n) \longrightarrow K(X_0,n)$ be a map which induces d in (5.8). Then the fiber of g is the Eilenberg–Mac Lane space $K(A,n-1) = P_g$. Therefore we can apply (6.7). Using the isomorphism λ in (4.4) (where we replace A by X_0 and X_1 respectively) we get the isomorphisms

$$H_0\{\pi_K^m\}(g^{op}) = \mathrm{Hom}(A,\pi_K^m\{n\}),$$
$$H_1\{\pi_K^m\}(g^{op}) = \mathrm{Ext}'(A,\pi_K^m\{n\}),$$
$$H_2\{\pi_K^m\}(g^{op}) = \mathrm{Ext}''(A,\pi_K^m\{n\}).$$

Compare the definition in (7.4) [6]. Now i in (6.7) yields the homomorphism κ in (4.7). Therefore (4.7) and also (4.5) is just a special case of (6.7).

Next we derive a further corollary from (6.2) concerning homology groups of the fiber P_g.

(6.9) Corollary: Let $g : X_0 \longrightarrow X_1$ be a map in $\underline{\underline{HA}}_n$ as in (6.4). Then we have for $m \leq 3n-3$ the following commutative diagram in which the column and the row are exact sequences.

$$\begin{tikzcd}
& H_m(X_1 \wedge X_1,G) & & \\
& H_m(X_1,G) \oplus H_m(X_1 \wedge X_0,G) \arrow[u, "D^1"] & & \\
H_{m-1}(P_g,G) & H_m(X_0,P_g;G) \arrow[l, "\partial"] \arrow[u, "\overline{E}"] & H_m(X_0,G) \arrow[l, "j"] \arrow[ul, "D^0"'] & H_m(P_g,G) \arrow[l, "q_*"] \\
& H_{m+1}(X_1 \wedge X_1,G) \arrow[u, "\overline{H}"] & & \\
& H_{m+1}(X_1,G) \oplus H_m(X_1 \wedge X_0,G) \arrow[u, "D^1"] & &
\end{tikzcd}$$

The row is the exact homology sequence of the pair (X_0,P_g) where we use the mapping cylinder of $q : P_g \longrightarrow X_0$. The operators D^0 and D^1 are given by

$$D^1 = (H,-(1\,|\,g)_*) \quad \text{and} \quad D^0 = (g_*,(1\,|\,g)_*H)$$

where H, $(1\,|\,g)_*$ and $(g\,|\,1)_*$ are defined as in (1.6). In fact we have $(1\,|\,g)_* = (1 \wedge g)_*$ and $(g\,|\,1)_* = (g \wedge 1)_*$.

Proof of (6.9): We apply the functor $H_m(\_,G)$ to the cofiber sequence in the bottom row of (6.1). This yields the exact column of (6.9) since δ in (6.2) is (3n–1)–connected. Here we also use the isomorphism

$$\begin{aligned} H_m(C_{q\times 1},G) &\cong H_m(A\times B/B;G) \\ &\cong H_m(A,G)\oplus H_m(A\wedge B;G) \end{aligned}$$

where $A = X_1$, $B = X_0$. The map D^0 is induced by $(g,1) : B \longrightarrow A\times B$. Moreover D^1 is induced by $(\delta W_g)_*$. The formulas in (6.2) for $\Sigma(\delta W_g)$ yields the formula for D^1 above.

The operators D^0, D^1 form a cochain complex, that is $D^1D^0 = 0$. This cochain complex corresponds to the cochain complex $M^*(d)$ in (6.3) [6] where $d = g$ and where $M = \{H_m^G\}$ is the quadratic $\underline{\underline{HA}}_n$–module associated to the quadratic functor $H_m(\_,G)$ in (1.3)(4), see (1.4) above or (3.5) [6]. The cohomology groups of this cochain complex are the groups

$$(6.10)\qquad \begin{cases} H^0\{H_m^G\}_*(g) = \text{kernel } (D^0), \\ H^1\{H_m^G\}_*(g) = \text{kernel } (D^1)/\text{image } (D^0), \\ H^2\{H_m^G\}_*(g) = \text{cokernel } (D^1) \end{cases}$$

They appear in the exact sequence of the next corollary where

$$(6.11)\qquad H_{m-1}(P_g,G)_\partial = \text{image } \partial$$

is the image of the operator ∂ in (6.9).

(6.12) Corollary: With the assumptions in (6.9) there is the exact sequence

$$\begin{aligned} 0 \longleftarrow\ & H^1\{H_m^G\}_*(g) \xleftarrow{e} H_{m-1}(P_g,G)_\partial \xleftarrow{h} H^2\{H_{m+1}^G\}_*(g) \xleftarrow{\delta} \\ & H^0\{H_m^G\}_*(g) \xleftarrow{q} H_m(P_g,G) \xleftarrow{i} H_m(P_g,G)_\partial \longleftarrow 0 \end{aligned}$$

The proof of (6.12) is similar to the proof of (5.7). The map e is induced by $\overline{E}$ and the map h is induced by $\partial\overline{H}$, moreover δ is essentially the restriction of j in (6.9), the map q is given by q_* and i is the inclusion. The exact sequence (6.12) is natural with respect to principal maps $F : P_g \longrightarrow P_{g'}$ between fiber spaces.

(6.13) Proof of (3.5): Theorem (3.5) is a special case of (6.12). For this we consider $K(A,n-1) = P_g$ as in (6.8). Using the isomorphism λ in (3.4) (where we replace A by X_0 and X_1 respectively) we get the isomorphisms

$$H_0\{H_m^G\}_*(g) \cong A * H_m^G\{n\}\ ,$$
$$H_1\{H_m^G\}_*(g) \cong A \otimes' H_m^G\{n\}\ ,$$
$$H_2\{H_m^G\}_*(g) \cong A \otimes'' H_m^G\{n\}\ .$$

Compare the definition in (7.4) [6]. Whence (3.5) is just a special case of (6.12).

§7 *Proof of theorem (6.2)*

We here prove theorem (6.2); the proof relies on a result of M. Mather [15]. We first consider the commutative diagram

(1)
$$\begin{tikzcd}
P_g \arrow[rrr] \arrow[dd, "q"'] \arrow[dr, "q''"] \arrow[drr, phantom, "\text{push}"] & & & WA\times B \arrow[dl, "q'"'] \arrow[dd] & \\
& R \arrow[r] \arrow[dl, "\simeq"'] \arrow[dd] & Q \arrow[dr, "j"] \arrow[dd] & & \\
B \arrow[rrr] \arrow[dd, tail] & & & A\times B \arrow[r, tail] \arrow[dd, tail] & C_j \arrow[dd, "\simeq"] \\
& C_{q''} \arrow[r, "\simeq"] \arrow[dl, "\simeq"'] & C_{q'} \arrow[dr] & & \\
C_q \arrow[rrr, "r(g)"'] & & & C_{q\times 1} \arrow[r] & C_{r(g)}
\end{tikzcd}$$

which is an extension of diagram (6.1). The subdiagram 'push' is a push out obtained by the mapping cylinder R of q, and j is the induced map. It is clear that the diagram yields a homotopy equivalence $C_j \simeq C_{r(g)}$. Next we consider the following diagram which is obtained from the top part of (1) by setting $B = A$ and $g = 1$.

$$\begin{tikzcd}
* \simeq WA \arrow[rrr] \arrow[dd] \arrow[dr] & & & WA\times A \arrow[dl] \arrow[dd] & \\
& R' \arrow[r] \arrow[dl] & Q' \arrow[dr, "j'"] \arrow[dd, "\simeq"', "h'"] & & \\
A \arrow[rrr, "{(1,1)}"'] & & & A\times A \arrow[r, tail] \arrow[d, "\simeq"', "h"] & C_{j'} \arrow[d, "\simeq"] \\
& & A\vee A \arrow[r, "j''"'] & A\times A \arrow[r, "\hat{q}"] & A\wedge A
\end{tikzcd}$$

The map $(1,1)$ is the diagonal in this case. For the definition of the homotopy equivalence h we use the assumption that A is a H–group with multiplication μ and homotopy invers v. We set

(3) $$h : A\times A \longrightarrow A\times A\times A \longrightarrow A\times A\times A \longrightarrow A\times A .$$

Here the first map carries (x,y) to (x,x,y), the second map is $1\times 1\times \nu$, and the third map is $1\times\mu$. We also write $h(x,y) = (x,x-y)$. The map j'' is the inclusion and $\hat{q}$ is the quotient map. Moreover the map h' corresponds to $1 \vee v$ since $hi_2 \simeq i_2 v$ and $hi_1 \simeq i_1$. Now it is clear that h induces a homotopy equivalence $C_{j'} \xrightarrow{\simeq} A\wedge A$.

The map g induces a 'map' from the top part of (1) to the top part of (2) given by

(4) $$\begin{cases} \pi_g : P_g \longrightarrow WA \\ 1\times g : WA\times B \longrightarrow WA\times A \\ g : B \longrightarrow A \\ 1\times g : A\times B \longrightarrow A\times A \end{cases}$$

Now (4) yields a map $g_Q : Q \longrightarrow Q'$ for which the following diagram homotopy commutes

(5) $$\begin{tikzcd} F \arrow[r, dashed, "i"] \arrow[d, "1"] & Q \arrow[r, "j"] \arrow[d, "g_Q"] & A\times B \arrow[r, tail] \arrow[d, "1\times g"] & C_j \arrow[d] & \Sigma F \arrow[l, dashed, "k"'] \arrow[d, "1"] \\ F \arrow[r, "i'"] \arrow[d, "\simeq"', "h''"] & Q' \arrow[r, "j'"] \arrow[d, "\simeq"', "h'"] & A\times A \arrow[r, tail] \arrow[d, "\simeq"', "h"] & C_{j'} \arrow[d, "\simeq"] & \Sigma F\Sigma \arrow[l, dashed, "k'"'] \arrow[d, "\Sigma v_F"] \\ F \arrow[r, "i''"] & A\vee A \arrow[r, "j''"] & A\times A \arrow[r, "\hat{q}"] & A\wedge A & \Sigma F \arrow[l, "k''"'] \end{tikzcd}$$

Here $F = \Sigma\Omega A \wedge \Omega A$ is the homotopy theoretic fiber of j, j' and j'' respectively as follows from a theorem of M. Mather (theorem 47 of [15]). Moreover the map i'' is the Whitehead product $[i_1R_A, i_2R_A]$ where $R_A : \Sigma\Omega A \longrightarrow A$ is the evaluation map. By naturality the map g_Q induces the identity on fibers and h' induces a homotopy equivalence h''. The maps k, k', and k'' are the canonical maps from the suspension of the fiber to the cofiber. The map k is $(\mathrm{Min}(a,b) + 2a-1)$–connected and the maps k', k'' are $(3a-1)$ connected, this follows from lemma (11) below. This shows that the composition

(6) $$\delta : C_{r(g)} \simeq C_j \longrightarrow C_{j'} \simeq A \wedge A$$

is $(\mathrm{Min}(a,b)+2a-1)$–connected. Hence the first part of the theorem is proved. For the second part we have to consider the maps

(7) $$\delta Wg = \hat{q}h(1\times g) : A\times B \longrightarrow A\wedge A$$

given by (5). By definition of h in (3) we get

(8) $$\begin{cases} \Sigma(qh) = \Sigma(\Delta_A p_1) + \Sigma((1\times v)\hat{q}) + \lambda_1 \\ \lambda_1 = (A\wedge H\mu)(\Sigma(\Delta_A \wedge v)) \end{cases}$$

Here H_μ is the Hopf construction of μ. The map λ_1 factors through the (3a)–connected space $\Sigma A\wedge A\wedge A$. Moreover we have

(9) $$\Sigma(1\wedge v) = -\Sigma(1\wedge 1) + \lambda_2$$

where λ_2 factors through the 3a–connected space $\Sigma A\wedge(A/A^{2a-1})$ where A^{2a-1} is the (2a–1)–skeleton of A. We see this by the following argument. There is a homotopy equivalence $\Sigma A' \simeq A^{2a-1}$ since A is (a–1)–connected. Moreover we have the homotopy commutative diagram

$$\begin{tikzcd}
 & & & A^{2a-1} \arrow[d, phantom, "\cap" rotate=90] \\
A^{2a-1}\vee A^{2a-1} \arrow[urrr, bend left=10, "\nabla"] \arrow[r, phantom, "="] & (A\times A)^{2a-1} \arrow[r, hook] & A\times A \arrow[r, "\mu"] & A \\
 & A^{2a-1} \arrow[ul, dashed, "i_1+i_2v'"] \arrow[u] \arrow[r, hook] & A \arrow[u, "(1,v)"] \arrow[ur, bend right=40, "0"'] &
\end{tikzcd}$$

Here v' is the restriction of the cellular map v and ∇ is the folding map. The diagram shows that the composition

$$\Sigma A^{2a-1} \subset \Sigma A \xrightarrow{1+\Sigma v} \Sigma A$$

is null homotopic. Whence $1+\Sigma v$ admits a factorization $\lambda_3 : \Sigma A \longrightarrow \Sigma(A/A^{2a-1}) \longrightarrow \Sigma A$, which proves (9) where we set $\lambda_2 = 1\wedge\lambda_3$. This completes the proof of (6.2). It remains to consider the following lemma which we used in (6).

Let $p : E \longrightarrow B$ be a fibration with fiber F and consider the following diagram in which all columns and rows are cofiber sequences.

(10) $$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "p"] & E/F \arrow[d, "q"] \\
* \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B \arrow[d] \\
\Sigma F \arrow[r, "k"] & C_p \arrow[r] & D \simeq C_q \simeq C_k
\end{tikzcd}$$

By the result of Mather we see that $\Sigma F \wedge \Omega B$ is the fiber of q. This shows:

(11) Lemma: Let F be (f–1)–connected, $f \geq 1$, and let B be (b–1)–connected. Then q and k are (f+b)–connected.

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