The divisor class group of ordinary and symbolic blow-ups

by

A. Simis(*) and Ngô Viêt Trung

A. Simis(*)

Departamento de Matematica
Universidade Federal De
Pernambuco
Av. Luis Freire, s/n
50.000 - Recife, PE

Brasil

N.V. Trung

Institute of Mathematics
Box 631 Bò Hô
Hanoi

Vietnam

(*) Partially supported by a CNP_q grant.

MPI/87-29

ABSTRACT. To a filtration of a ring R by ideals is attached as usual a Rees algebra. If S is a Krull domain then the divisor class group $C\ell(S)$ is envisaged and it is shown that it is an extension of $C\ell(R)$ by a finitely generated free group. The as yet unsolved question of determining the rank of this free group is considered, as is the question of when the extension splits. Applications are given for ordinary Rees algebras ("blow-ups") and symbolic Rees algebras.

Acknowledgement

The second author would like to thank the Max-Planck-Institut for support and hospitality during the preparation of this paper.

1. Introduction

Recently, several authors considered the question of computing the divisor class group of a normal Rees algebra R[It] ("blow-up"). Roughly, the result should depend solely on the class group of the base ring R and on the blown-up ideal $I \subset R$. In [H-V] a special case of this problem was dealt with. Simultaneously, in [Si] a general exact sequence was obtained which emphasized the fact that the inlusion $R[It] \subset R[t]$ almost always satisfies condition "P D E" of Samuel [Sa], which, as is known allows to define a natural map $C\ell(R[It]) \to C\ell(R[t]) = C\ell(R)$ of the respectice class groups. Of course, in special cases, "ad hoc" methods allow to compute the divisor class group of $R[t]$ (e.g., [Br_1], [Br-Si]).

The aim of this paper is to give a unified method to deal with the above computation. Moreover, a suitable generalization to the set-up of Rees algebras associated with filtrations allows to draw consequences for the symbolic Rees algebras as well. We give a few samples of such results.

Theorem 1.1. *Let R be a noetherian domain and $I \subset R$ an ideal for which the (res. symbolic) Rees algebra S is a normal (res. Krull) domain. Then there is an exact sequence*

$$0 \longrightarrow \mathbb{Z}^r \longrightarrow C\ell(S) \longrightarrow C\ell(R) \longrightarrow 0 \ ,$$

where r is the number of height one prime ideals $P \supset IS$ whose contraction $P \cap R$ has height ≥ 2 .

Theorem 1.2. Let R be a normal (noetherian) domain and let $I \subset R$ be an ideal such that:

(i) I is radical and generically a complete intersection

(ii) $gr_I(R) := R[It]/IR[It]$ is R/I-torsion free.

Then $R[It]$ is normal and $C\ell(R[It]) = C\ell(R) \oplus \mathbb{Z}^r$, where $r :\#$ minimal primes of R/I of height ≥ 2 .

Theorem 1.3. Let R be a normal (noetherian) domain and let $I \subset R$ be an ideal which is radical and generically a complete intersection. Then the symbolic Rees algebra $\sum_{n \geq 0} I^{(n)}$ is a Krull domain and $C\ell(\sum_{n \geq 0} I^{(n)}) \cong C\ell(R) \oplus \mathbb{Z}^r$, where $r :\#$ minimal primes of R/I of height ≥ 2 .

Two basic questions touched upon, but not completely solved, in this paper are as follows:

Rank Question. To express the rank of free group $\mathbb{Z}^r$ in Th. 1.1 directly in terms of the ideal I .

The authors do not know the answer to this question even when R is a regular local ring, in which case r is the rank of $C\ell(R[It])$. If one further assumes that $gr_I(R)$ is reduced, then it has been proved [Hu-Si-V] that $gr_I(R)$ is actually R/I-torsion free, so in this case r is the number of minimal primes of R/I (cf. Lemma 2.3 in this work).

Splittig Conjecture. The exact sequence of Th. 1.1 is split.

Note Th. 1.2 gives some (weak) support to this conjecture. In Section 3 we give further evidence to the conjecture.

We now briefly explain the organization of the paper.

Section 2 is devoted to the proof of the fundamental exact sequence of the divisor class group of a Rees algebra and to its various applications.

Section 3 contains special cases of the splitting conjecture. A few applications are surveyed, however the emphasis as a whole is on a general framework that may prove useful in proving the conjecture.

Finally, in Section 4 we develop similar results for the extended Rees algebra associated with a filtration. There is however a surprising difference in that the freeness of the subgroup corresponding to the exceptional locus depends on a unimodular condition on the vector of the multiplicities of the components of the locus. As a particular instance of our procedure we obtain the result of Shimoda [Sh].

The authors wish to thank the hospitality of the Istituto di Matematica (Genova), where initial conversations on the subject were started. The first named author is also grateful to Craig Huneke and Wolmer Vasconcelos for valuable interchange around these ideas.

2. The fundamental exact sequence of the divisor class group

Throughout R will stand for a (not necessarily noetherian) domain equipped with a (multiplicative) filtration $F : R = I_0 \supset I_1 \supset I_2 \supset \ldots, I_m I_n \subset I_{m+n}$, where I_n is an ideal for every $n \geq 0$. We will be concerned with the Rees algebra associated with F, namely, the graded R-algebra $R[F] := \sum_{n \geq 0} I_n t^n \subset R[t]$ (t an intermediate over R). The ideal $(F) := \sum_{n \geq 0} I_{n+1} t^n$ of $R[F]$ plays a distinguished role and will sometimes be referred to as the "exceptional locus" of $R[F]$.

For our purpose, the filtration F will henceforth be assumed to be "adic in codimension one", that is to say, it will satisfy the condition $I_n R_p = I_1^n R_p$ for every $n > 1$ and every height one prime $p \subset R$. This condition is automatically satisfied if $ht(I_1) \geq 2$.

Needless to emphasize, the main examples envisaged herein will be the filtration of the powers I^n of an ideal, the filtration of the symbolic powers $I^{(n)}$ of an ideal and the filtration $F_a : R_a \supset I_a \supset (I^2)_a \supset \ldots$, where $(I^n)_a$ denotes the integral closure of the ideal I^n. It is well known that $R_a[F_a]$ is the integral closure of $R[It]$ in $K(t)$ (or $K[t]$), where $K :=$ quotient field of R. Note that if R is noetherian, then $R_a[F_a]$ is a Krull domain, so the results will often be applicable to $R_a[F_a]$.

After these preliminaries, we can state the basic result of this section, where for a Krull domain A, $C\ell(A)$ stands for its divisor class group (cf. [Bou_2] or [Fo]).

Theorem 2.1 ("Fundamental exact sequence")

Let F be a filtration of a domain R, adic in codimension one. If $R[F]$ is a Krull domain then (so is R and) there exists an exact sequence of groups

$$0 \longrightarrow \mathbf{Z}^r \longrightarrow C\ell(R[F]) \longrightarrow C\ell(R) \longrightarrow 0$$

where r is the number of the height one prime ideals p containing the exceptional locus (F) and such that $ht(P \cap R) \geq 2$.

Supplement. If, moreover, $ht(I_1) \geq 2$, then the inclusion $R[F] \subset R[t]$ satisfies condition P D E of Samuel and the above map $C\ell(R[F]) \longrightarrow C\ell(R)$ can be taken to be the naturally induced map $C\ell(R[F]) \longrightarrow C\ell(R[t])$.

Proof. The proof rests on two elementary facts pertaining the nature of $R[F]$ which allow for the construction of a map $C\ell(R[F]) \longrightarrow C\ell(R)$.

Fact 1. Let $\mathrm{Div}^R(R[F]) := \langle [P] \in \mathrm{Div}(R[F]) \mid P \cap R \neq 0 \rangle$. Then the image of $\mathrm{Div}^R(R[F])$ in $C\ell(R[F])$ is the whole of $C\ell(R[F])$.

Fact 2. For every height one prime $p \subset R$, the extended ideal $pR_p[F_p]$ is prime (here $F_p : R_p \supset (I_1)_p \supset \ldots$).

The first of these facts is a standard argument using Nagata's theorem (cf. [Fo, proof of Th. 8.1]). The second fact is easy: if $I_1 \not\subset p$, $R_p[F_p] = R_p[t]$; otherwise, $R_p[F_p] = R_p[(I_1)_p t]$ since F is adic in codimension one. But if R_p is regular then $(I_1)_p$ is principal, hence $R_p[F_p]$ is again a polinomial ring over R_p . Thus, it remains to show that R_p is regular. But we have in fact that R is a Krull domain as $R = R[F] \cap K$.

Observe that Fact 2 means that for every height one prime of R there exists a unique height one prime $P \subset R[F]$ lying over p and such P is unramified.

We can now proceed to define a homomorphism $C\ell(R[F] \longrightarrow C\ell(R)$. First define it on the level of divisors by the assignment

$$\psi : \mathrm{Div}^R(R[F]) \longrightarrow \mathrm{Div}(R)$$

$$[P] \longmapsto \begin{cases} [P\cap R] & \text{if } \mathrm{ht}(P\cap R) = 1 \\ 0 & \text{if } \mathrm{ht}(P\cap R) \geq 2 \end{cases} .$$

Note one may restrict oneself to $\mathrm{Div}^R(R[F])$ due to Fact 1. Now consider an element $f \in K(t)$ - the quotient field of $(R[F])$ - such that the corresponding divisor $[f]$ belongs to $\mathrm{Div}^R(R[F])$. Say $[f] = \sum_{P\cap R \neq 0} v_p(f)[P]$. Then, $\psi[f] = \sum_{\mathrm{ht}(P\cap R)=1} v_p(f)[P\cap R]$. But since $K(t)$ is also the quotient field of $K[t]$ and $v_Q(f) = 0$ for every height one prime $Q \subset R[F]$ with $Q \cap R = 0$, it follows that f must be a unit of $K[t]$, hence $f \in K$. Finally, from Fact 2, $v_p(f) = v_{P\cap R}(f)$ if $\mathrm{ht}(P) = \mathrm{ht}(P\cap R) = 1$. We must conclude that $\psi[f]$ is the divisor of f in $\mathrm{Div}(R)$. This implies the induced homomorphism $C\ell(R[F]) \longrightarrow C\ell(R)$.

Again, by Fact 2, this homomorphism is surjective.

We now claim that

$$\ker(C\ell(R[F]) \longrightarrow C\ell(R)) = \langle [P] \varepsilon \mathrm{Div}(R[F]) \mid \mathrm{ht}(P\cap R) \geq 2 \rangle$$

(modulo principal divisors). For this, note that if $D \in \mathrm{Div}^R(R[F])$ is such that $\psi(D) = [f]_R$, with $f \in K$, then by Fact 2. $D-[f]_{R[F]}$ belongs to $\langle [P] \in \mathrm{Div}(R[F]) \mid \mathrm{ht}(P\cap R) \geq 2\rangle$.

Finally, we verify that this subgroup of $C\ell(R[F])$ is free on its generators $C\ell(P)$, $\mathrm{ht}(P\cap R) \geq 2$. Thus, let $f \in K(t)$ be such that the divisor $[f]_{R[F]}$ belongs to $\langle [P] \in \mathrm{Div}^R(R[F]) \mid \mathrm{ht}(P\cap R) \geq 2\rangle$. As before, $f \in K$. Again, by Fact 2, $v_p(f) = v_P(f) = 0$ for every height one prime $p \subset R$ and the unique $P \subset R[F]$ such that $P \cap R = p$. It then must be the case that f is a unit in R, so $[f]_{R[F]} = 0$ as required.

This finishes the proof of the theorem. The proof of the supplement goes along the following lines: Let $\mathfrak{p} \subset R[t]$ be a prime of height 1. Clearly, $\mathrm{ht}(p) \leq 1$, where $p = \mathfrak{p} \cap R$. Therefore, $I_1 \not\subset p$ and so

$$\begin{aligned} R[F]_{\mathfrak{p}\cap R[F]} &= R_p[(I_1)_p t]_{\mathfrak{p}R_p[(I_1)_p t]} = R_p[t]_{\mathfrak{p}R_p[t]} \\ &= R[t]_{\mathfrak{p}} . \end{aligned}$$

This implies that $\mathrm{ht}(\mathfrak{p}\cap R[F]) = \mathrm{ht}(\mathfrak{p}) = 1$, thus showing $R[F] \subset R[t]$ satisfies P D E condition.

The map $C\ell(R[F]) \longrightarrow C\ell(R)$ defined in Th. 2.1 will henceforth be referred to as the "canonical map".

As an immediate corollary we get our Theorem 1.1 in the first section. Special cases of Theorem 2.1 are Theorem 2.14 of [Vi], Theorem of [H-V] (the splitting part is given in our § 3) and Theorem 2.3 of [Br-Si]. We now indicate an interesting consequence of Th. 2.1 that contains Theorem, (a) of [H-V] as a particular case.

Corollary 2.2. Let $I \subset R$ be a divisorial ideal of the normal domain R such that $R[It]$ is normal. Then the following conditions are equivalent:

(i) $C\ell(R[It]) \cong C\ell(R)$ via the canonical map.

(ii) $gr_I(R)$ is R/I - torsion free.

Next we consider the rank question referred to in § 1. We will work in the general set-up fitrations. For this purpose, Let $Ass(R/I_n)$ denote the set of associated primes of I_n, i.e. primes p of R for which there exists an element $x \in R$ such that $p = I_n : x$.

Lemma 2.3. Let $R[F]$ be a Krull domain. Then

$$\{P \cap R \mid P \supseteq (F),\ ht P = 1\} = \bigcup_{n=1}^{\infty} Ass(A/I_n) .$$

Proof. We note that (F) as a module over $S := R[F]$ is isomorphic to S^+. Since S^+ is a height one prime ideal of S,

$$S^+ = S \cap S^+S_{S^+} = \bigcap_{ht P=1} S^+S_P .$$

Therefore, we also have

$$(F) = \bigcap_{htP=1} (F)S_P .$$

Hence (F) is a divisorial ideal of S, and there exist only a finite number of height one primes $P \supseteq (F)$.

Let $p \in Ass(A/I_n)$ for some $n \geqq 1$. If $p \neq P \cap R$ for all height one primes $P \supseteq (F)$, one can find an element $x \in p$ such that $x \notin P$ for all such primes P. Therefore

$$(F) : x = \bigcap_{htP=1} (F)S_P : x = \bigcap_{htP=1} (F)S_P = (F) .$$

That means $(I_{n+1} : x) \cap I_n = I_{n+1}$ or, equivalently, $I_{n+1} : x = I_{n+1}$, for all $n \geqq 1$. Hence $x \notin p$, a contradiction.

Conversely, let Q be an arbitrary height one prime containing (F) and $q = Q \cap R$. Then one can find a homogeneous element $f \in S$ such that $(F)S_Q : f = QS_Q$. Let $g \in S$ be a homogeneous element such that $g \notin Q$ and $g \in P$ for any height one prime $P \supseteq (F)$, $P \neq Q$. Then

$$(F) : fg^m = \bigcap_{htP=1} (F)S_P : fg^m = S \cap QS_Q = Q$$

for all large m. Write $fg^m = xt^{n-1}$, $x \in R$. Then $I_n : x = q$. Hence $q \in Ass(R(I_n)$, as required.

<u>Remark</u>. Since there are only a finite number of height one primes $P \supseteq (F)$, $\bigcup_{n=1}^{\infty} Ass(A/I_n)$ is a finite set. Thus,

$\bigcup_{n=1}^{\infty} \mathrm{Ass}(A/I_n) = \mathrm{Ass}(A/I_n)$ for n large (cf. [Bro] for a similar result in the noetherian case).

The following result answers the rank question for the ordinary blow-up in a special situation.

Proposition 2.4. Let R be a normal (noetherian) domain. Let $I \subset R$ be a nonzero ideal such that $gr_{I_p}(R_p)$ is a domain for every $p \in \mathrm{Ass}(R/I)$ (e.g., I is radical and generically a complete intersection). Then the following conditions are equivalent:

(i) $R[It]$ is normal and the kernel of the map $C\ell(R[It]) \to C\ell(R)$ has rank $= \#\{p \in \mathrm{Ass}(R/I) \mid \mathrm{ht}\, p \geq 2\}$.

(ii) $gr_I(R)$ is R/I-torsion free.

Proof. (i) $\Rightarrow$ (ii). From Theorem 2.1 and Lemma 2.3 we get $\{P \cap R \mid P \supseteq (F),\ \mathrm{ht}\, P = 1\} = \mathrm{Ass}(R/I)$. Note that $gr_I(R) = R[It]/IR[It]$. Then, since $gr_{I_p}(R_p)$ is a domain for every $p \in \mathrm{Ass}(R/I)$, using the pigeon-hole principle we can derive that $gr_I(R)$ is reduced. From this it follows that $gr_I(R)$ is R/I-torsion free.

(ii) $\Rightarrow$ (i). From the assumptions we can conclude that $gr_I(R)$ is reduced and that there is an one-to-one correspondence between the height one primes $P \supseteq (F)$ and the associated primes of I. Therefore, $R[It]$ is normal [Ba] and the kernel of the canonical map $C\ell(R[It]) \to C\ell(R)$ has rank $= \#\{p \in \mathrm{Ass}(R/I) \mid \mathrm{ht}\, p \geq 2\}$.

We now give results for the symbolic Rees algebra. These will often be more definite as we now imply.

First recall that for an ideal $I \subset R$ in a noetherian ring, we define its " $n^{\underline{th}}$ symbolic power" to be

$$I^{(n)} := I^n R_M \cap R, \; n \geq 0 ,$$

where $M = R \setminus \bigcup_p p, p \in \operatorname{Min} R/I$. In particular, $I^{(1)}$ stands for the non-embedded part of I .

The following observations explain why the symbolic filtrations are arithmetically simpler.

<u>Lemma 2.5</u>. Let R be a normal (noetherian) domain and $I \subset R$ an ideal. Then $\sum I^{(n)} t^{(n)}$ is a Krull domain if and only if $R_p[I_p\, t]$ is normal for every $p \in \operatorname{Min} R/I$.

<u>Proof</u>. We have:

$$\begin{aligned}\sum_n I^{(n)} t^n &= \sum_n (I^n R_M \cap R) t^n \\ &= (\sum_n (\bigcap_p I_p^n) t^n) \cap R[t] (p \in \operatorname{Min} R/I) \\ &= (\bigcap_p R_p[I_p\, t]) \cap R[t] \, .\end{aligned}$$

Since a finite intersection of normal domains is a Krull domain, (cf. [Bou] or [Fo]) we are through .

<u>Remark</u>. There is a complicated proof for Lemma 2.5 for the case I being a prime ideal by Katz and Ratliff [K-R] .

Using this lemma, along with Th. 2.1 and Lemma 2.3, we obtain the following results.

Proposition 2.6. Let R be a normal (noetherian) domain and $I \subset R$ a divisorial ideal: Then $\sum I^{(n)}t^n$ is a Krull domain and $C\ell(\sum I^{(n)}t^n) = C\ell(R)$.

Proposition 2.7. Let R be a normal (noetherian) domain and $I \subset R$ an ideal for which $gr_{I_p}(R_p)$ is a domain for every $p \in \mathrm{Min}\ R/I$ (e.g. I is radical and generically a complete intersection). Then $\sum I^{(n)}t^n$ is a Krull domain and there is an exact sequence

$$0 \longrightarrow \mathbb{Z}^r \longrightarrow C\ell(\textstyle\sum I^{(n)}t^n) \longrightarrow C\ell(R) \longrightarrow 0$$

where $r = \#\{p \in \mathrm{Min}\ R/I \mid \mathrm{height}(p) \geq 2\}$.

To close this section, here are some critical examples to bear in mind.

Example A. Let $F + G \in k[X_1,\ldots,X_m]$, where F and G are forms of degree s and $s+1$, respectively, with no common proper factors. Set $R := k[X_1,\ldots,X_n]/(F+G)$ and assume R is normal (e.g., $F = X_1 \ldots X_s$, $s \leq m-1$, and G a form in the remaining variables $X_{s+1},\ldots,X_m$). Set $I := (X_1,\ldots,X_m)R$, $\underline{T} = T_1,\ldots,T_m$. Then $gr_I(R) \cong k[\underline{T}]/(F(\underline{T}))$. If, moreover, F is square-free - say, $F = F_1 \ldots F_r$, F_i distinct and irreducible - then $R[It]$ is normal and the height one primes containing $IR[It]$ correspond to $(I,F_i(\underline{T}))$, $1 \leq i \leq r$. Clearly, $(I,F_i(\underline{T})) \cap R = I$ for every $1 \leq i \leq r$. This shows:

(A.1) $gr_I(R)$ is R/I-torsion free

(A.2) The rank of $\ker(C\ell(R[It]) \to C\ell(R))$ is r.

However, $\#\, \mathrm{Ass}\, R/I = 1$.

<u>Example B</u>. Let $R := k[X_1,X_2,X_3,X_4]/(X_2^2 - X_3X_4)$ and $I := (X_1,X_2,X_3)R$. Clearly, $gr_I(R) \cong k[X_4,T_1,T_2,T_3]/(X_4T_3)$ and R is normal. Therefore, $R[It]$ is normal with exceptional primes $(X_1,X_2,X_3,X_3t)R[It]$ and $(X_1,X_2,X_3,X_4)R[It]$. Thus:

(B.1) $gr_I(R)$ is not R/I-torsion free, as $(X_1,X_2,X_3,X_4)R$ is an associated prime of R/I^2 but not of R/I.

(B.2) The rank of $\ker(C\ell(R[It]) \to C\ell(R))$ is 2.

(B.3) I is generically a complete intersection.

However, $\#\, \mathrm{Ass}\, R/I = 1$.

<u>Example C</u>. Let $R := k[x,y,z] = k[X,Y,Z]/(Y^2-XZ)$ and $I := (x,y)$. Then $R[It]$ is normal. By Prop. 2.6, $C\ell(\sum I^{(n)}t^n) \cong C\ell(R) \cong \mathbb{Z}/2\mathbb{Z}$. As a matter of suggesting that Prop. 2.6 is a non-trivial result, we give an explicit presentation of $\sum I^{(n)}t^n$. It is easy to check that $\sum I^{(n)}t^n = R[xt,yt,xt^2] \subset R[t]$. Define a map

$$R[T,U,V] \xrightarrow{\psi} \sum I^{(n)}t^n$$

with $\deg(T) = \deg(U) = 1$, $\deg(V) = 2$ and $\psi(T) = xt$, $\psi(U) = yt$, $\psi(V) = xt^2$. A little computation yields that $\ker\psi$ lifted to

$k[X,Y,Z,T,U,V]$ contains $(Y^2-XZ, XU-YT, YU-ZT, U^2-ZV, YV-UT, T^2-XV)$. Regrading X,Y,Z to have degree 1 each and V to have degree 1, we see that the latter ideal is the ideal of 2x2 minors of the 3x3 symmetric generic matrix

$$\begin{pmatrix} Z & Y & U \\ Y & X & T \\ U & T & V \end{pmatrix}$$

hence must be the whole of $\ker \psi$.

We have thus obtained that for the ideal J of 2x2 minors of a 3x3 symmetric generic, the factor ring is normal and its class group is $\mathbf{Z}/2\mathbf{Z}[Go]$.

3. Splitting principles (with an application to computing the canonical class).

We ask for conditions under which the fundamental sequence of Th. 2.1 splits. One expects this to be always the case. In the lack of such a result in all generality, we indicate a technique that ought to work under fairly broad hypotheses.

Let $M \subset R$ be a multiplicative set and let $F : R \supset I_1 \supset \ldots$ be a filtration, adic in codimension one. Assume $R[F]$ is a Krull domain. By Th. 2.1, we have an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & C\ell(R[F]) & \longrightarrow & C\ell(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^M & \longrightarrow & C\ell(R_M[F_M]) & \longrightarrow & C\ell(R_M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where G is freely generated by the classes of height one primes $P \supset (F)$ such that $ht(P \cap R) \geq 2$ and G^M is similarly obtained. Clearly, there is a (natural) splitting $G \rightleftarrows G^M$.

Lemma 3.1. Assume the following conditions hold for M :

(a) $G^M \neq (0)$

(b) The sequence $0 \longrightarrow G^M \longrightarrow C\ell(R_M[F_M]) \longrightarrow C\ell(R_M) \longrightarrow 0$ splits (e.g., $C\ell(R_M) = (0)$) .

Then the sequence $0 \longrightarrow G \longrightarrow C\ell(R[F]) \longrightarrow C\ell(R) \longrightarrow 0$ splits.

Proof. This is clear by composing along the various sections

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d, shift left] & C\ell(R[F]) \\ 0 \arrow[r] & G^M \arrow[r, shift left] \arrow[u, shift left] & C\ell(R_M[F_M]) \arrow[l, bend left] \end{tikzcd} .$$

We emphasize that a (free) generator of G^M is the class of a height one prime $P \supset (F)$ such that $ht(P \cap R) \geqq 2$ and $P \cap M = 0$.

Theorem 3.2. Let $F : R \supset I_1 \supset \dots$ be a filtration, adic in codimension one, for which $R[F]$ is a Krull domain. Assume that R_p is factorial for every $p \in \bigcup_{n\geq 1} \operatorname{Ass} R/I_n$. Then $C\ell(R[F]) \cong \mathbb{Z}^r \oplus C\ell(R)$, for some $r \geq 0$.

Proof. We apply Lemma 3.1 with $M = R \setminus \bigcup_{n\geq 1} \operatorname{Ass} R/I_n$. By Lemma 2.3, $G \cong G^M = \mathbb{Z}^r$ for some $r \geq 0$. Since $\bigcup_{n\geq 1} \operatorname{Ass} R/I_n$ is finite, R_M is a semi local domain whose localizations at the maximal ideals are factorial rings. In this case, every height one prime ideal of R_M is principal [Na, (28.9)]. Hence R_M is factorial and $C\ell(R_M) = 0$ [Bou 2, Theorem 1, p. 32].

Collecting Th. 3.2 and Prop. 2.4 (resp. Prop. 2.7) one easily obtains Th. 1.2 (resp. Th. 1.3) mentioned in § 1 (In fact, Th. 1.2 is a consequence of Th. 1.3).

We now proceed to the main application of this section, namely, to an explicit computation of the canonical class of $R[F]$. We shall derive our results only for the symbolic Rees algebras, leaving to the reader the suitable modifications needed to produce the general case. Our computations of the canonical class generalizes the one in [H-V] and is based on the ideas of loc. cit. The result, however, will have independent interest as we point out afterwards.

For a ring A, we denote by w_A its canonical module (provided it exists). In order to avoid tedious repetitions, we shall make the general proviso that all relevant rings in sight admit a canonical module. For an ideal $\mathfrak{a}$ in a Krull domain A, $c\ell(\mathfrak{a})$ will denote the class in $C\ell(A)$ of its divisorial closure.

<u>Theorem 3.3</u>. Let R be a normal (noetherian) domain and let $I \subset R$ be an ideal of height ≥ 2 such that:

(a) I is radical

(b) I is generically a complete intersection.

Then $S := \sum_n I^{(n)}$ is a Krull domain and

$$c\ell(w_S) = c\ell(w_R) - \sum_P (\mathrm{ht}(P \cap R) - 2)c\ell(P) \quad ,$$

where P runs through the height one primes of the exceptional locus $I^{(1)} + I^{(2)}t + \ldots$ and $c\ell(w_R)$ is viewed as an element of $C\ell(S)$ via the splitting of Th. 3.2.

<u>Proof</u>. From (b), using Prop. 2.7, we know that S is a Krull domain. Since (a) implies, moreover, that R_p is regular for every $p \in \mathrm{Min}\, R/I$, we have a splitting of $C\ell(S)$ as in

Th. 3.2. Reading that splitting more closely, we can write

$$c\ell(w_S) = C - \sum_P n_P c\ell(P) \ ,$$

for suitable $C \in C\ell(R)$ and integers n_P, with P running through the height one primes of the exceptional locus.

By changing to a polynomial ring $R[X]$, if needed, we may assume the existence of a prime element $x \in I$([H-V], [Tr, Korollar 2.3]). Then $C\ell(R) \cong C\ell(R_x)$ by Nagata's trick and, clearly, $S_x = R_x[t]$. Hence the preceding formula yields $C = c\ell(w_{S_x}) = c\ell(w_{R_x}) = c\ell(w_R)$ with the proper identifications.

To determine n_P we localize at $p := P \cap R$. By assumption, R_p is regular and, therefore, $C\ell(R_p) = 0$ and $PS_p = pS_p$, thus yielding $c\ell(w_{S_p}) = n_P c\ell(pS_p)$. But, S_p is a determinantal ring, hence $n_P = ht(p) - 2$ [Br_2], as required.

<u>Corollary 3.4</u>. Let R be a normal domain which is a factor of a Gorenstein ring. Let $I \subset R$ be an ideal of height ≥ 2 such that:

(a) I is radical and generically a complete intersection

(b) $\sum I^{(n)}t^n$ is noetherian.

Then ($\sum I^{(n)}t^n$ is normal and) the following conditions are equivalent:

(i) R is quasi-Gorenstein and $ht(I) = 2$

(ii) $\sum I^{(n)}t^n$ is quasi-Gorenstein.

If, moreover, R and $\sum I^{(n)}t^n$ are Cohen-Macaulay rings, then "quasi-Gorenstein" can be everywhere replaced by "Gorenstein". An occurence of application of Cor. 3.4 are the noetherian cases of symbolic algebras of monomial curves in A^3[Hu] .

Remark. If $I^{(n)} = I^n$ for all n (e.g. if $gr_I(R)$ is R/I-torsion free), one can apply Theorem 3.3 and Corollary 3.4 to obtain similar results on the ordinary Rees algebra $R[It]$ and one can recover results of [H-V] and [Ro].

Here's an improvement of Th. 3.3, in the spirit of [H-S-V, Th. 2.7]. The proof is similar and is left to the reader.

Proposition 3.5. Let R be a normal domain, factor of a Gorenstein ring. Let $I \supset R$ be an ideal of height ≥ 2 such that R_p is factorial and $gr_{I_p}(R_p)$ is a Cohen-Macaulay domain for every $p \in \mathrm{Min}\, R/I$. Then

$$c\ell(w_S) = c\ell(w_R) - \sum_P (ht(P \cap R) - \rho(I_{P \cap R}) - 1)c\ell(P)$$

where $S := \sum I^{(n)}t^n$, P runs through the height one primes of the exceptional locus and $\rho(I_{P \cap R})$ is the reduction number of $I_{P \cap R}$ in $R_{P \cap R}$.

4. Extended Rees Algebras

Let $F : R \supset I_1 \supset \ldots$ be a filtration as in § 2. We now deal with the extended Rees algebra

$$S := R[F][t^{-1}] \subset R[t,t^{-1}] .$$

Firstly, we remark that there is an one-to-one correspondence between the prime ideals of $R[F]$ containing (F) and the prime ideals of S containing t^{-1}. For, if we denote by $gr_F(R)$ the associated graded ring $\bigoplus_{n=0}^{\infty} I_n/I_{n+1}$, then

$$gr_F(R) = R[F]/(F) = S/t^{-1}S .$$

Proposition 4.1. S is a Krull domain if and only $R[F]$ is a Krull domain.

Proof. If S is a Krull domain, then $R = S \cap K$ and therefore $R[F] = S \cap R[t]$ is a Krull domain. Conversely, let p be a height one prime ideal of S. If $t^{-1} \in p$, then p is a minimal prime over $t^{-1}S$. Therefore, the corresponding prime ideal $P := p \cap R[F]$ of p in $R[F]$ is minimal over (F). Since (F) is a divisorial ideal, ht $P = 1$. Since $P \nsupseteq R[F]^+$, $R[F]_P \cong S_p$. If $t^{-1} \notin g$, S_p is a localization of $R[t,t^{-1}]$. In any case, since $R = R[F] \cap K$ is a Krull domain, S_p is a discrete valuation ring. Moreover, we have

$$\bigcap_{\text{all } p} S_p = \bigcap_{t^{-1} \in p} S_p \cap R[t,t^{-1}] = \bigcap_{(F) \subseteq p} R[F]_p \cap R[t,t^{-1}]$$

$$\bigcap_{(F) \subseteq P} R[F]_p \cap R[t] = \bigcap_{\text{all } p} R[F]_p = R[F] .$$

Hence $\bigcap_{\text{all } p} S_p = R[F][t^{-1}] = S$, and we can conclude that S is a Krull domain.

We now discuss the main result of this section.

In order to avoid tedious technical circumventions we assume, once for all that the leading ideal I_1 of the filtration F is not contained in any principal prime.

Theorem 4.2. Assume S is a Krull domain and let $(F) = \bigcap_{i=1}^{r} P_i^{(\ell_i)}$ be the primary decomposition in $R[F]$ of the exceptional locus. Then there is an exact sequence

$$0 \longrightarrow \mathbf{Z}^r/\mathbf{Z}\cdot(\ell_1,\ldots,\ell_r) \longrightarrow C\ell(S) \longrightarrow C\ell(R) \longrightarrow 0 .$$

Proof. We have $S \cap K = R$, hence R is a Krull domain. Moreover, $S_{t^{-1}} = R[t,t^{-1}]$. Therefore, an easy application of Nagata's lemma yields an exact sequence

$$0 \longrightarrow \sum_p \mathbf{Z} c\ell(p) \longrightarrow C\ell(S) \longrightarrow C\ell(R) \longrightarrow 0 ,$$

where p runs through the primes (t^{-1},P_i), $1 \leq i \leq r$.

Consider the group isomorphism

$$\mathbf{Z}^r \xrightarrow[\sim]{\psi} \sum_{i=1}^{r} \mathbf{Z}[p_i], \ p_i = (t^{-1}, P_i)$$

that sends the basis vector $(0,\dots,0,\overset{i}{1},0,\dots,0)$ to $[p_i]$, $1 \leq i \leq r$. Since $t^{-1}S = \bigcap_{i=1}^{r} p_i^{(\ell_i)}$, we see that $\psi((\ell_1,\dots,\ell_r)) = \sum_i \ell_i[p_i]$ is a principal divisor. Therefore, we obtain an induced map

$$\mathbf{Z}^r/\mathbf{Z}(\ell_1,\dots,\ell_r) \longrightarrow \left(\sum_{i=1}^{r} \mathbf{Z}[p_i] + \rho\right)/\rho ,$$

where ρ is the group of principal divisors of S. Note that $[p_i] \notin \rho$, $1 \leq i \leq r$. Indeed, if some p_i is principal, then is P_i, hence also $P_i \cap R$. But the latter contains I_1 and this would contradict our proviso.

Moreover, if $\psi((n_1,\dots,n_r))$ is a principal divisor then, by the "unit trick" of [Br], we easily see that it must be $\operatorname{div}(\alpha t^{-m})$, for some unit $u \in R$ and some $m \in \mathbf{Z}$. That is, $(n_1,\dots,n_r) = m(\ell_1,\dots,\ell_r) \in \mathbf{Z}(\ell_1,\dots,\ell_r)$. This proves the result.

<u>Remark</u>. Clearly, $\mathbf{Z}^r/\mathbf{Z}(\ell_1,\dots,\ell_r)$ has rank $r-1$ and is free if and only if $\gcd(\ell_1,\dots,\ell_r) = 1$.

<u>Corollary 4.3</u>. If R is a normal domain and $gr_F(R)$ reduced, then S is a Krull domain and there is an exact sequence

$$0 \longrightarrow \mathbf{Z}^{r-1} \longrightarrow C\ell(S) \longrightarrow C\ell(R) \longrightarrow 0 ,$$

where $r = \#$ minimal primes of $gr_F(R)$.

Proof. Using the relations $gr_F(R) = S/t^{-1}S$ and $S[t] = R[t,t^{-1}]$, it is not hard to see that S is a Krull domain.

Corollary 4.4. If R is a normal domain, the following are equivalent:

(i) $gr_F(R)$ is a domain

(ii) S is normal and the canonical map

$$C\ell(S) \longrightarrow C\ell(R) \text{ is an isomorphism.}$$

Proof. (i) $\to$ (ii) Apply Cor. 4.4.
(ii) $\to$ (i) Apply Th. 4.2 to get $\mathbb{Z}^r/\mathbb{Z}(\ell_1,\ldots,\ell_r) = (0)$. It must be the case that $r = 1$ and $\ell_1 = 1$, i.e., $gr_F(R) = R[F]/(F)$ is a domain.

The localization procedure of § 3 allows us to derive entirely analogous splitting results for the extended Rees algebra. Thus, Th. 3.2 goes through in the case of S by replacing $\mathbb{Z}^r$ by $\mathbb{Z}^r/\mathbb{Z}(\ell_1,\ldots,\ell_r)$, etc. Since the Veronese subrings $R[I^{\ell}t,t^{-1}]$ of a normal extended Rees algebra $R[It,t^{-1}]$ are normal again, from the analogous splitting condition for the exact sequence of Theorem 4.2 we immediately obtain the following application.

Corollary 4.5. [Sh] Let R be a normal domain, let $X_1,\ldots,X_n$ be indeterminates over R and let $I := (X_1,\ldots,X_n)^{\ell} \subset R[X] = R[X_1,\ldots,X_n]$. Then $S := R[X][It,t^{-1}]$ is a normal domain and

$$C\ell(S) \cong (\mathbb{Z}/\ell\mathbb{Z}) \oplus C\ell(R) .$$

R E F E R E N C E S

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