Contractions on a manifold polarized by an ample vector bundle

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Introduction

An algebraic variety X of dimension n (over the complex field) together with an ample vector bundle E on it will be called a generalized polarized variety. The adjoint bundle of the pair (X, E) is the line bundle $K_X + det(E)$. Problems concerning adjoint bundles have drawn a lot of attention to algebraic geometer: the classical case is when E is a (direct sum of) line bundle (polarized variety), while the generalized case was motivated by the solution of Hartshorne-Frankel conjecture by Mori ([Mo]) and by consequent conjectures of Mukai ([Mu]).

A first point of view is to study the positivity (the nefness or ampleness) of the adjoint line bundle in the case $r = rank(E)$ is about $n = dimX$. This was done in a sequel of papers for $r \geq (n-1)$ and for smooth manifold X ([Ye-Zhang], [Fujita], [Andreatta-Ballico-Wisniewski]). In this paper we want to discuss the next case, namely when $rank(E) = (n-2)$, with X smooth; we obtain a complete answer which is described in the theorem (4.1). This is divided in three cases, namely when $K_X + det(E)$ is not nef, when it is nef and not big and finally when it is nef and big but not ample. If $n = 3$ a complete picture is already contained in the famous paper of Mori ([Mo1]), while the particular case in which $E = \oplus^{(n-2)}(L)$ with L a line bundle was also studied ([Fu1], [So]; in the singular case see [An]). The part 1 of the theorem was proved (in a slightly weaker form) by Zhang ([Zh]) and, in the case E is spanned by global sections, by Wisniewski ([Wi2]).

Another point of view can be the following: let (X, E) be a generalized polarized variety with X smooth and $rankE = r$. If $K_X + det(E)$ is nef, then by the Kawamata-Shokurov base point free theorem it supports a contraction (see (1.2)); i.e. there exists a map $\pi : X \to W$ from X onto a normal projective variety W with connected fiber and such that $K_X + det(E) = \pi^* H$ for some ample line bundle H on W. It is not difficult to see that, for every fiber F of π, we have $dimF \geq (r - 1)$, equality holds only if $dimX > dimW$. In the paper we study the "border" cases: we assume that $dimF = (r - 1)$ for every fibers and we prove that X has a $\mathbf{P}^r$-bundle structure given by π (theorem (3.2)). We consider also the case in which $dimF = r$ for every fibers and π is birational, proving that W is smooth and that π is a blow-up of a smooth subvariety (theorem (3.1)). This point of view was discussed in the case $E = \oplus^r L$ in the paper [A-W].

Finally in the section (4) we extend the theorem (3.2) to the singular case, namely for projective variety X with log-terminal singularities. In particular this gives the Mukai's conjecturel for singular varieties.

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1 Notations and generalities

(1.1) We use the standard notations from algebraic geometry. Our language is compatible with that of [K-M-M] to which we refer constantly. We just explain some special definitions and propositions used frequently.

In particular in this paper X will always stand for a smooth complex projective variety of dimension n. Let $Div(X)$ the group of Cartier divisors on X; denote by K_X the canonical divisor of X, an element of $Div(X)$ such that $\mathcal{O}_X(K_X) = \Omega^n_X$. Let $N_1(X) = \frac{\{1-cycles\}}{\equiv} \otimes \mathbf{R}$, $N^1(X) = \frac{\{divisors\}}{\equiv} \otimes \mathbf{R}$ and $\overline{< NE(X) >} = \overline{\{\text{effective 1-cycles}\}}$; the last is a closed cone in $N_1(X)$. Let also $\rho(X) = dim_{\mathbf{R}} N^1(X) < \infty$.

Suppose that K_X is not nef, that is there exists an effective curve C such that $K_X \cdot C < 0$.

Theorem 1.1 *[KMM] Let X as above and H a nef Cartier divisor such that $F := H^\perp \cap \overline{< NE(X) >} \setminus \{0\}$ is entirely contained in the set $\{Z \in N_1(X) : K_X \cdot Z < 0\}$, where $H^\perp = \{Z : H \cdot Z = 0\}$. Then there exists a projective morphism $\varphi : X \to W$ from X onto a normal variety W with the following properties:*

i) For an irreducible curve C in X, $\varphi(C)$ is a point if and only if $H.C = 0$, if and only if $cl(C) \in F$.

ii) φ has only connected fibers

iii) $H = \varphi^*(A)$ for some ample divisor A on W.

iv) The image $\varphi^* : Pic(W) \to Pic(X)$ coincides with $\{D \in Pic(X) : D.C = 0 \text{ for all } C \in F\}$.

Definition 1.2 *The following terminology is mostly used ([KMM], definition 3-2-3). Referring to the above theorem,*

the map φ is called a contraction *(or an* extremal contraction*); the set F is an* extremal face, *while the Cartier divisor H is a* supporting divisor *for the map φ (or the face F). If $dim_{\mathbf{R}} F = 1$ the face F is called an* extremal ray, *while φ is called an* elementary contraction.

Remark We have also ([Mo1]) that if X has an extremal ray R then there exists a rational curve C on X such that $0 < -K_X \cdot C \leq n+1$ and $R = R[C] := \{D \in < NE(X) >: D \equiv \lambda C, \lambda \in \mathbf{R}^+\}$. Such a curve is called an extremal curve.
Remark Let $\pi : X \to V$ denote a contraction of an extremal face F, supported by $H = \pi^* A$([iii]1.1) . Let R be an extremal ray in F and $\rho : X \to W$ the contraction of R. Since $\pi^* A \cdot R = 0$, $\pi^* A$ comes from $Pic(W)$ ([iv]1.1). Thus π factors trough ρ.

Definition 1.3 *To an extremal ray R we can associate:*

i) its length *$l(R) := min\{-K_X \cdot C;$ for C rational curve and $C \in R\}$*

ii) the locus *$E(R) := \{$the locus of the curves whose numerical classes are in $R\} \subset X$.*

Definition 1.4 *It is usual to divide the elementary contractions associated to an extremal ray R in three types according to the dimension of $E(R)$: more precisely we say that φ is of* fiber type, *respectively* divisorial type, *resp.* flipping type, *if $dimE(R) = n$, resp. $n-1$, resp. $< n-1$. Moreover an extremal ray is said not nef if there exists an effective $D \in Div(X)$ such that $D \cdot C < 0$.*

The following very useful inequality was proved in [Io] and [Wi3].

Proposition 1.5 *Let φ the contraction of an extremal ray R, $E'(R)$ be any irreducible component of the exceptional locus and d the dimension of a fiber of the contraction restricted to $E'(R)$. Then*

$$dimE'(R) + d \geq n + l(R) - 1.$$

(1.2) Actually it is very useful to understand when a contraction is elementary or in other words when the locus of two distinct extremal rays are disjoint. For this we will use in this paper the following results.

Proposition 1.6 *[BS, Corollary 0.6.1] Let R_1 and R_2 two distinct not nef extremal rays such that $l(R_1)+l(R_2)>n$. Then $E(R_1)$ and $E(R_2)$ are disjoint.*

Something can be said also if $l(R_1)+l(R_2)=n$:

Proposition 1.7 *[Fu3, Theorem 2.4] Let $\pi : X \to V$ as above and suppose $n \geq 4$ and $l(R_i) \geq n-2$. Then the exceptional loci corresponding to different extremal rays, are disjoint with each other.*

Proposition 1.8 *[ABW1] Let $\pi : X \to W$ be a contraction of a face such that $dimX > dimW$. Suppose that for every rational curve C in a general fiber of π we have $-K_X \cdot C \geq (n+1)/2$ Then π is an elementary contraction except if*

a) $-K_X \cdot C = (n+2)/2$ for some rational curve C on X, W is a point, X is a Fano manifold of pseudoindex $(n+2)/2$ and $\rho(X)=2$

b) $-K_X \cdot C = (n+1)/2$ for some rational curve C, and $dim\, W \leq 1$

The following definition is used in the theorem:

Definition 1.9 *Let L be an an ample line bundle on X. The pair (X,L) is called a scroll (respectively a quadric fibration, respectively a del Pezzo fibration) over a normal variety Y of dimension m if there exists a surjective morphism with connected fibers $\phi : X \to Y$ such that*

$$K_X + (n-m+1)L \approx p^*\mathcal{L}$$

(respectively $K_X + (n-m)L \approx p^*\mathcal{L}$; respectively $K_X + (n-m-1)L \approx p^*\mathcal{L}$) for some ample line bundle $\mathcal{L}$ on Y. X is called a classical scroll (respectively quadric bundle) over a projective variety Y of dimension r if there exists a surjective morphism $\phi : X \to Y$ such that every fiber is isomorphic to $\mathbf{P}^{n-r}$ (respectively to a quadric in $\mathbf{P}^{(n-r+1)}$) and if there exists a vector bundle E of rank $(n-r+1)$ (respectively of rank $n-r+2$) on Y such that $X \simeq \mathbf{P}(E)$ (respectively exists an embedding of X as a subvariety of $\mathbf{P}(E)$).

2 A technical construction

Let E be a vector bundle of rank r on X and assume that E is ample, in the sense of Hartshorne.

Remark Let $f : \mathbf{P}^1 \to X$ be a non constant map, and $C = f(\mathbf{P}^1)$, then $detE \cdot C \geq r$.

In particular if there exists a curve C such that $(K_X + detE).C \leq 0$ (for instance if $(K_X + detE)$ is not nef) then there exists an extremal ray R such that $l(R) \geq r$.

(2.1) Let $Y = \mathbf{P}(E)$ be the associated projective space bundle, $p : Y \to X$ the natural map onto X and ξ_E the tautological bundle of Y. Then we have the formula for the canonical bundle $K_Y = p^*(K_X + detE) - r\xi_E$. Note that p is an elementary contraction; let R be the associated extremal ray.

Assume that $K_X + detE$ is nef but not ample and that it is the supporting divisor of an elementary contraction $\pi : X \to W$. Then $\rho(Y/W) = 2$ and $-K_Y$ is $\pi \circ p$-ample. By the relative Mori theory over W we have that there exists a ray on $NE(Y/W)$, say R_1, of length $\geq r$, not contracted by p, and a relative elementary contraction $\varphi : Y \to V$. We have thus the following commutative diagram.

$$\begin{array}{ccc} \mathbf{P}(E) = Y & \xrightarrow{\varphi} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{\pi} & W \end{array} \qquad (1)$$

where φ and ψ are elementary contractions. Let $w \in W$ and let $F(\pi)_w$ be an irreducible component of $\pi^{-1}(w)$; choose also v in $\psi^{-1}(w)$ and let $F(\varphi)_v$ be an irreducible component of $\varphi^{-1}(v)$ such that $p(F(\varphi)_v) \cap F(\pi)_w \neq \emptyset$; then $p(F(\varphi)_v) \subset F(\pi)_w$. This is true by the commutativity of the diagram. Since p and φ are elementary contractions of different extremal rays we have that $dim(F(\varphi) \cap F(p)) = 0$, that is curve contracted by φ cannot be contracted by p.

In particular this implies that $dimp(F(\varphi)_v) = dimF(\varphi)_v$; therefore

$$dimF(\varphi)_v \leq dimF(\pi)_w.$$

Remark If $dimF(\varphi)_v = dimF(\pi)_w$, then $dimF(\psi)_w := dim(\psi^{-1}(w)) = r - 1$; if this holds for every $w \in W$ then ψ is equidimensional.

Proof. Let Y_w be an irreducible component of $p^{-1}\pi^{-1}(w)$ such that $\varphi(Y_w) = F(\psi)_w$. Then $dimF(\psi)_w = dimY_w - dimF(\varphi)_v = dimY_w - dimF(\pi)_w = dimF(p) = (r - 1)$.

□

(2.2)Slicing techniques

Let $H = \varphi^*(A)$ be a supporting divisor for φ such that the linear system $|H|$ is base point free. We assume as in (2) that $(K_X + detE)$ is nef and we refer to the diagram (1). The divisor $K_Y + r\xi_E = p^*(K_X + detE)$ is nef on Y and therefore $m(K_Y + r\xi_E + aH)$, for $m \gg 0$, $a \in \mathbf{N}$, is also a good supporting divisor for φ. Let Z be a smooth n-fold obtained by intersecting $r-1$ general divisor from the linear system H, i.e. $Z = H_1 \cap \ldots \cap H_{r-1}$ (this is what we call a slicing); let $H_i = \varphi^{-1}A_i$.

Note that the map $\varphi' = \varphi_{|Z}$ is supported by $m|(K_Y + r\xi_E + a\varphi^*A)_{|Z}|$, hence, by adjunction, it is supported by $K_Z + rL$, where $L = \xi_{E|Z}$. Let $p' = p_{|Z}$; by construction p' is finite.

If T is (the normalization of) $\varphi(Z)$ and $\psi' : T \to W$ is the map obtained restricting ψ then we have from (1) the following diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi'} & T \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle \psi'} \\ X & \xrightarrow{\pi} & W \end{array} \qquad (2)$$

In general one has a good comprehension of the map φ' (for instance in the case $r = (n-2)$ see the results in [Fu1] or in [An]). The goal is to "transfer" the information that we have on φ' to the map π. The following proposition is the major step in this program.

Proposition 2.1 *Assume that ψ is equidimensional (in particular this is the case if for every non trivial fiber we have $dimF(\varphi) = dimF(\pi)$). Then W has the same singularities of T.*

Proof. By hypothesis any irreducible reduced component F_i of a non trivial fiber $F(\psi)$ is of dimension $r-1$; this implies also that $F_i = \varphi(F(p))$ for some fiber of p.

Now, let us follow an argument as in [Fu1, Lemma 2.12]. We can assume that the divisor A is very ample; we will choose $r-1$ divisors $A_i \in |A|$ as above such that, if $T = \bigcap_i A_i$, then $T \cap \psi^{-1}(w)_{red} = N$ is a reduced 0-cycle and $Z = H_1 \cap \ldots \cap H_{r-1}$ is a smooth n-fold, where $H_i = \varphi^{-1}A_i$. This can be done by Bertini theorem. Moreover the number of points in N is given by $A^{r-1} \cdot \psi^{-1}(w)_{red} = \sum_i A^{r-1} \cdot F_i = \sum_i d_i$. Note that, by projection formula, we have $A^{r-1} \cdot F_i = \varphi^* A^{r-1} \cdot F(p)$; moreover, since p is a projective bundle, the last number is constant i.e. $\varphi^* A^{r-1} \cdot F(p) = d$ for all fiber $F(p)$, that is the d_i's are constant.

Now take a small enough neighborhood U of w, in the metric topology, such that any connected component U_λ of $\psi^{-1}(U) \cap T$ meets $\psi^{-1}(w)$ in a single point. This is possible because $\psi' := \psi_{|T} : T \to W$ is proper and finite over w. Let ψ_λ the restriction of ψ at U_λ and m_λ its degree. Then $deg\psi' = \sum m_\lambda \geq \sum_i d_i = \sum_i d$ and equality holds if and only if ψ is not ramified at w (remember that $\sum_i d_i$ is the number of U_λ).

The generic $F(\psi)_w$ is irreducible and generically reduced. Note that we can choose $\tilde{w} \in W$ such that $\psi^{-1}(\tilde{w}) = \varphi(F(p))$ and $deg\psi' = A^{r-1} \cdot \psi^{-1}(\tilde{w})$, the latter is possible by the choice of generic sections of $|A|$. Hence, by projection formula $deg\psi' = A^{r-1} \cdot \psi^{-1}(\tilde{w}) = \varphi^* A^{r-1} \cdot F(p) = d$, that is $m_\lambda = 1$ and the fibers are irreducible. Since W is normal we can conclude, by Zarisky's Main theorem, that W has the same singularity as T.

□

3 Some general applications

As an application of the above construction we will prove the following proposition; the case $r = (n-1)$ was proved in [ABW2].

Proposition 3.1 *Let X be a smooth projective complex variety and E be an ample vector bundle of rank r on X. Assume that $K_X + detE$ is nef and big but not ample and let $\pi : X \to W$ be the contraction supported by $K_X + detE$. Assume also that π is a divisorial elementary contraction, with exceptional divisor D, and that $dimF \leq r$ for all fibers F. Then W is smooth, π is the blow up of a smooth subvariety $B := \pi(D)$ and $E = \pi^* E' \otimes [-D]$, for some ample E' on W.*

Proof. Let R be the extremal ray contracted by π and $F := F(\pi)$ a fiber. Then $l(R) \geq r$ and thus $dimF \geq r$ by proposition (1.5). Hence all the fibers of π have dimension r. Consider the commutative diagram (1); let R_1 be the ray contracted by φ. Since $l(R_1) \geq r$, again by proposition (1.5), we have that $dimF(\varphi) \geq r$ (note that R_1 is not nef). Therefore, since $dimF(\varphi) \leq dimF$, we have that $dimF(\varphi) = dimF = r$, $l(R) = l(R_1) = r$ and $\xi_E \cdot C_1 = 1$, where C_1 is a (minimal) curve in the ray R_1. Via slicing we obtain the map $\varphi' : Z \to T$ which is supported by $K_Z + r\xi_{E|Z}$. This last map is very well understood: namely by [AW, Th 4.1 (iii)] it follows that T is smooth and φ' is a blow up along a smooth subvariety. By proposition (2.1) also W is smooth. Therefore π is a birational morphism between smooth varieties with exceptional locus a prime divisor and with equidimensional non trivial fibers; by [AW, Corollary 4.11] this implies that π is a blow up of a smooth subvariety in W.

We want to show that $E = \pi^* E' \otimes [-D]$. Let D_1 be the exceptional divisor of φ; first we claim that $\xi_E + D_1$ is a good supporting divisor for φ. To see this observe that $(\xi_E + D_1) \cdot C_1 = 0$, while $(\xi_E + D_1) \cdot C > 0$ for any curve C with $\varphi(C) \neq pt$ (in fact ξ_E is ample and $D_1 \cdot C \geq 0$ for such a curve). Thus $\xi_E + D_1 = \varphi^* A$ for some ample $A \in Pic(V)$; moreover by projection formula $A \cdot l = 1$, for any line l in the fiber of ψ. Hence by Grauert theorem $V = \mathbf{P}(E')$ for some ample vector bundle E' on W. This yields, by the commutativity of diagram (1), to $E \otimes D = p_*(\xi_E + D_1) = p_* \varphi^* A = \pi^* \psi_* A = \pi^* E'$.

□

We now want to give a similar proposition for the fiber type case.

Theorem 3.2 *Let X be a smooth projective complex variety and E be an ample vector bundle of rank r on X. Assume that $K_X + detE$ is nef and let $\pi : X \to W$ be the contraction supported by $K_X + detE$. Assume that $r \geq (n+1)/2$ and $dimF \leq r-1$ for any fiber F of π. Then W is smooth, for any fiber $F \simeq \mathbf{P}^{r-1}$ and $E_{|F} = \oplus^r \mathcal{O}(1)$.*

Proof. Note that by proposition (1.5) π is a contraction of fiber type and all the fibers have dimension $r-1$. Moreover the contraction is elementar, as it follows from proposition (1.8).

We want to use an inductive argument to prove the thesis. If $dimW = 0$ then this is Mukai's conjecture1; it was proved by Peternell, Kollár, Ye-Zhang (see for instance [YZ]). Let the claim be true for dimension $m-1$. Note that the locus over which the fiber is not $\mathbf{P}^{r-1}$ is discrete and W has isolated singularities. In fact take a general hyperplane section A of W, and $X' = \pi^{-1}(A)$ then $\pi_{|X'} : X' \to A$ is again a contraction supported by $K_{X'} + detE_{|X'}$, such that $r \geq ((n-1)+1)/2$. Thus by induction A is smooth, hence W has isolated singularities.

Let U be an open disk in the complex topology, such that $U \cap SingW = \{0\}$. Then by lemma below 3.3 we have locally, in the complex topology, a π-ample line bundle L such that restricted to the general fiber is $\mathcal{O}(1)$. As in [Ful, Prop. 2.12] we can prove that U is smooth and all the fibers are $\mathbf{P}^{r-1}$.

□

Lemma 3.3 *Let X be a complex manifold and $(W,0)$ an analityc germ such that $W \setminus \{0\} \simeq \Delta^m \setminus \{0\}$. Assume we have an holomorphic map $\pi : X \to W$ with $-K_X$ π-ample; assume also that $F \simeq \mathbf{P}^r$ for all fibers of π, $F \neq F_0 = \pi^{-1}(0)$, and that $codimF_0 \geq 2$. Then there exists a line bundle L on X such that L is π-ample and $L_{|F} = \mathcal{O}(1)$.*

Proof. (see also [ABW2, pag 338, 339]) Let $W^* = W \setminus \{0\}$ and $X^* = X \setminus F_0$. By abuse of notation call $\pi = \pi_{|X^*} : X^* \to W^*$; it follows immediately that $R^1\pi_*\mathbf{Z}_{X^*} = 0$ and $R^2\pi_*\mathbf{Z}_{X^*} = \mathbf{Z}$.

If we look at Leray spectral sequence, we have that:

$$E_2^{0,2} = \mathbf{Z} \text{ and } E_2^{p,1} = 0 \text{ for any p.}$$

Therefore $d_2 : E_2^{0,2} \to E_2^{2,1}$ is the zero map and moreover we have the following exact sequence

$$0 \to E_\infty^{0,2} \to E_2^{0,2} \xrightarrow{d_3} E_2^{3,0},$$

since the only non zero map from $E_2^{0,2}$ is d_3 and hence $E_\infty^{0,2} = kerd_3$. On the other hand we have also, in a natural way, a surjective map $H^2(X^*, \mathbf{Z}) \to$

$E_\infty^{0,2} \to 0$. Thus we get the following exact sequence

$$H^2(X^*, \mathbf{Z}) \xrightarrow{\alpha} E_2^{0,2} \to E_2^{3,0} = H^3(W^*, \mathbf{Z}).$$

We want to show that α is surjective. If $dimW := w \geq 3$ then $H^3(W^*, \mathbf{Z}) = 0$ and we have done. Suppose $w = 2$ then $H^3(W^*, \mathbf{Z}) = \mathbf{Z}$; note that the restriction of $-K_X$ gives a non zero class (in fact it is $r+1$ times the generator) in $E_2^{0,2}$ and is mapped to zero in $E_2^{0,3}$ thus the mapping $E_2^{0,2} \to E_2^{3,0}$ is the zero map and α is surjective. Since F_0 is of codimension at least 2 in X the restriction map $H^2(X, \mathbf{Z}) \to H^2(X^*, \mathbf{Z})$ is a bijection. By the vanishing of $R_i\pi_*\mathcal{O}_X$ we get $H^2(X, \mathcal{O}_X) = H^2(W, \mathcal{O}_W) = 0$ hence also $Pic(X) \to H^2(X, \mathbf{Z})$ is surjective. Let $L \in Pic(X)$ be a preimage of a generator of $E_2^{0,2}$. By construction L_t is $\mathcal{O}(1)$, for $t \in W^*$. Moreover $(r+1)L = -K_X$ on X^* thus, again by the codimension of X^*, this is true on X and L is π-ample.

□

4 An approach to the singular case

The following theorem arose during a discussion between us and J.A. Wisniewski; we would like to thank him. The idea to investigate this argument came from a preprint of Zhang [Zh2] where he proves the following result under the assumption that E is spanned by global sections. For the definition of log-terminal singularity we refer to [KMM].

Theorem 4.1 *Let X be an n-dimensional log-terminal projective variety and E an ample vector bundle of rank $n+1$, such that $c_1(E) = c_1(X)$. Then $(X, E) = (\mathbf{P}^n, \oplus^{n+1}\mathcal{O}_{\mathbf{P}^n}(1))$.*

Proof. We will prove that X is smooth, then we can apply proposition (3.2). We consider also in this case the associated projective space bundle Y and the commutative diagram

$$\begin{array}{ccc} \mathbf{P}(E) = Y & \xrightarrow{\varphi} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{\pi} & pt \end{array} \qquad (3)$$

as in (1); it is immediate that Y is a weak Fano variety (i.e. Y is Gorenstein, log-terminal and $-K_Y$ is ample; in particular it has Cohen-Macaulay singularities); moreover, as in (3.1), $dimF(\varphi) \leq dimF(\pi) = n$ and the map φ is supported by $K_Y + (n+1)H$, where $H = \xi_E + A$, with ξ_E the tautological line bundle and A a pull back of a ample line bundle from V. It is known that a contraction supported by $K_Y + rH$ on a log terminal variety has to have fibers of dimension

$\geq (r-1)$ and of dimension $\geq r$ in the birational case ([AW, remark 3.1.2]). Therefore in our case φ can not be birational and all fibers have dimension n; moreover, by the Kobayashi-Ochiai criterion the general fiber is $F \simeq \mathbf{P}^n$. We want to adapt the proof of [BS, Prop 1.4]; to this end we have only to show that there are no fibers of φ entirely contained in $Sing(Y)$. Note that, by construction, $Sing(Y) \subset p^{-1}(SingX)$ hence no fibers F of φ can be contained in $Sing(Y)$. Hence the same proof of [BS, Prop 1.4] applies and we can prove that V is nonsingular and $\varphi : Y \longrightarrow V$ is a classical scroll. In particular Y is nonsingular and therefore also X is nonsingular.

□

More generally we can prove the following.

Theorem 4.2 *Let X be an n-dimensional log-terminal projective variety and E be an ample vector bundle of rank r. Assume that $K_X + detE$ is nef and let $\pi : X \longrightarrow W$ be the contraction supported by $K_X + detE$. Assume also that for any fiber F of π $dimF \leq r-1$, that $r \geq (n+1)/2$ and $codimSing(X) > dimW$. Then X is smooth and for any fiber $F \simeq \mathbf{P}^{r-1}$.*

Proof. The proof that X is smooth is as in the theorem above and then we use proposition (3.2)

□

5 Main theorem

This section is devoted to the proof of the following theorem.

Theorem 5.1 *Let X be a smooth projective variety over the complex field of dimension $n \geq 3$ and E an ample vector bundle on X of rank $r = (n-2)$. Then we have*

1) $K_X + det(E)$ is nef unless (X, E) is one of the following:

i) there exist a smooth n-fold, W, and a morphism $\phi : X \longrightarrow W$ expressing X as a blow up of a finite set B of points and an ample vector bundle E' on W such that $E = \phi^* E' \otimes [-\phi^{-1}(B)]$.

Assume from now on that (X, E) is not as in (i) above (that is eventually consider the new pair (W, E') coming from (i)).

ii) $X = \mathbf{P}^n$ and $E = \oplus^{(n-2)}\mathcal{O}(1)$ or $\oplus^2\mathcal{O}(2) \oplus^{(n-4)} \mathcal{O}(1)$ or $\mathcal{O}(2) \oplus^{(n-3)} \mathcal{O}(1)$ or $\mathcal{O}(3) \oplus^{(n-3)} \mathcal{O}(1)$.

iii) $X = \mathbf{Q}^n$ and $E = \oplus^{(n-2)}\mathcal{O}(1)$ or $\mathcal{O}(2) \oplus^{(n-3)} \mathcal{O}(1)$ or $\mathbf{E}(2)$ with $\mathbf{E}$ a spinor bundle on $\mathbf{Q}^n$.

iv) $X = \mathbf{P}^2 \times \mathbf{P}^2$ and $E = \oplus^2\mathcal{O}(1,1)$

v) X is a del Pezzo manifold with $b_2 = 1$, i.e. $Pic(X)$ is generated by an ample line bundle $\mathcal{O}(1)$ such that $\mathcal{O}(n-1) = \mathcal{O}(-K_X)$ and $E = \oplus^{(n-1)}\mathcal{O}(1)$.

vi) X is a classical scroll or a quadric bundle over a smooth curve Y.

vii) X is a fibration over a smooth surface Y with all fibers isomorphic to $\mathbf{P}^{(n-2)}$.

2) If $K_X + det(E)$ is nef then it is big unless there exists a morphism $\phi : X \to W$ onto a normal variety W supported by (a large multiple of) $K_X + det(E)$ and $dim(W) \leq 3$; let F be a general fiber of ϕ and $E' = E_{|F}$. We have the following according to $s = dimW$:

i) If $s = 0$ then X is a Fano manifold and $K_X + det(E) = 0$. If $n \geq 6$ then $b_2(X) = 1$ except if $X = \mathbf{P}^3 \times \mathbf{P}^3$ and $E = \oplus^4\mathcal{O}(1,1)$.

ii) If $s = 1$ then W is a smooth curve and ϕ is a flat (equidimensional) map. Then (F, E') is one of the pair described in [PSW], in particular F is either $\mathbf{P}^n$ or a quadric or a del Pezzo variety. If $n \geq 6$ then π is an elementary contraction. If the general fiber is P^{n-1} then X is a classical scroll while if the general fiber is $\mathbf{Q}^{n-1}$ then X is a quadric bundle.

iii) If $s = 2$ and $n \geq 5$ then W is a smooth surface, ϕ is a flat map and (F, E') is one of the pair described in the Main Theorem of [Fu2]. If the general fiber is $\mathbf{P}^{n-2}$ all the fibers are $\mathbf{P}^{n-2}$.

iv) If $s = 3$ and $n \geq 5$ then W is a smooth 3-fold and all fibers are isomorphic to $\mathbf{P}^{n-3}$.

3) Assume finally that $K_X + det(E)$ is nef and big but not ample. Then a high multiple of $K_X + det(E)$ defines a birational map, $\varphi : X \to X'$, which contracts an "extremal face" (see section 2). Let R_i, for i in a finite set of index, the extremal rays spanning this face; call $\rho_i : X \to W$ the contraction associated to one of the R_i. Then we have that each ρ_i is birational and divisorial; if D is one of the exceptional divisors (we drop the index) and $Z = \rho(D)$ we have that $dim(Z) \leq 1$ and the following possibilities occur:

i) $dimZ = 0$, $D = \mathbf{P}^{(n-1)}$ and $D_{|D} = \mathcal{O}(-2)$ or $\mathcal{O}(-1)$; moreover, respectively, $E_{|D} = \oplus^{n-2}\mathcal{O}(1)$ or $E_{|D} = \oplus^{n-1}\mathcal{O}(1) \oplus \mathcal{O}(2)$.

ii) $dimZ = 0$, D is a (possible singular) quadric, $\mathbf{Q}^{(n-1)}$, and $D_{|D} = \mathcal{O}(-1)$; moreover $E_{|D} = \oplus^{n-2}\mathcal{O}(1)$.

iii) $dimZ = 1$, W and Z are smooth projective varieties and ρ is the blow-up of W along Z. Moreover $E_{|F} = \oplus^{n-2}\mathcal{O}(1)$.

If $n > 3$ then φ is a composition of "disjoint" extremal contractions as in i), ii) or iii).

Proof. Proof of part 1) of the theorem

Let (X, E) be a generalized polarized variety and assume that $K_X + det(E)$ is not nef. Then there exist on X a finite number of extremal rays, $R_1, \ldots, R_s$, such that $(K_X + det(E)) \cdot R_i < 0$ and therefore, by the remark in section (2), $l(R_i) \geq (n-1)$.

Consider one of this extremal rays, $R = R_i$, and let $\rho : X \to Y$ be its associated elementary contraction. Then $L := -(K_X + det(E))$ is ρ-ample and also the vector bundle $E_1 := E \oplus L$ is ρ-ample; moreover $K_X + det(E_1) = \mathcal{O}_X$ relative to ρ. We can apply the theorem in [ABW2] which study the positivity of the adjoint bundle in the case of $rankE_1 = (n-1)$. More precisely we need a relative version of this theorem, i.e. we do not assume that E_1 is ample but that it is ρ-ample (or equivalently a local statement in a neighborhood of the exceptional locus of the extremal ray R). We just notice that the theorem in [ABW2] is true also in the relative case and can be proved exactly with the same proof using the relative minimal model theory (see [K-M-M]; see also the section 2 of the paper [AW] for a discussion of the local set up).

Assume first that ρ is birational, then $K_X + det(E_1)$ is ρ-nef and ρ-big; note also that, since $l(R_i) \geq (n-1)$, ρ is divisorial. Therefore we are in the (relative) case C of the theorem in [ABW2] (see also the proposition 3.1 with $r = (n-1)$); this implies that Y is smooth and ρ is the blow up of a point in Y. Since $l(R_i) \geq (n-1)$, the exceptional loci of the birational rays are pairwise disjoint by proposition (1.6). This part give the point (i) of the theorem 5.1; i.e. the birational extremal rays have disjoint exceptional loci which are divisors isomorphic to $\mathbf{P}^{(n-1)}$ and which contract simultaneously to smooth distinct points on a n-fold W. The description of E follows trivially (see also [ABW2]).

If ρ is not birational then we are in the case B of the theorem in [ABW2]; from this we obtain similarly as above the other cases of the theorem 5.1, with some trivial computations needed to recover E from E_1.

□

Proof of the part 2) of the theorem

Let $K_X + detE$ be nef but not big; then it is the supporting divisor of a face $F = (K_X + detE)^{\perp}$. In particular we can apply the theorems of section (2): therefore there exist a map $\pi : X \to W$ which is given by a high multiple of $K_X + detE$ and which contracts the curves in the face. Since $K_X + detE$ is not big we have that $dimW < dimX$. Moreover for every rational curve C in a general fiber of π we have $-K_X \cdot C \geq (n-2)$ by the remark in section (2). We apply proposition (1.8), which, together with the above inequality on $-K_X \cdot C$, says that π is an elementary contraction if $n \geq 5$ unless either $n = 6$, W is a point and X is a Fano manifold of pseudoindex 4 and $\rho(X) = 2$ or $n = 5$ and $dimW \leq 1$.

By proposition (1.5) we have the inequality

$$n + dimF \geq n + n - 2 - 1;$$

in particular it follows that $dimW \leq 3$.

(5.1) Let $dimW = 0$, that is $K_X + detE = 0$ and therefore X is a Fano manifold. By what just said above we have that $b_2(X) = 1$ if $n \geq 6$ with an exception which will be treated in the following lemma.

Lemma 5.2 *Let X be a 6 dimensional projective manifold, E is an ample vector bundle on X of rank 4 such that $K_X + detE = 0$. Assume moreover that $b_2 \geq 2$. Then $X = \mathbf{P}^3 \times \mathbf{P}^3$ and $E = \oplus^4 \mathcal{O}(1,1)$.*

Proof. The lemma is a slight generalizzation of [Wil, Prop B] for dimension 6; the poof is similar and we refer to this paper. In particular as in [Wil] we can see that X has two extremal rays whose contractions, $\pi_i, i = 1, 2$, are of fiber type with equidimensional fibers onto 3-folds W_i and with general fiber $F_i \simeq \mathbf{P}^3$. We claim that the W_i are smooth and thus $W_i \simeq \mathbf{P}^3$. First of all note that W_i can have only isolated singularity and only isolated points over which the fiber is not $\mathbf{P}^{n-3}$; in fact let S be a general hyperplane section of W_i and $T_i = \pi_i^*(S)$, then $(\pi_i)_{|T_i}$ is an extremal contraction, by proposition 1.8; hence by [ABW2, Prop 1.4.1] S is smooth; moreover the contraction is supported by $K_{T_i} + detE_{T_i}$ hence all fibers are $\mathbf{P}^3$ by the main theorem of [ABW2]. Now we are (locally) in the hypothesis of lemma 3.3 so we get, locally in the complex topology, a tautological bundle and we can conclude, by [Ful, Prop 2.12], that W_i is smooth. Let $T = H_1 \cap H_2$, where H_i are two general elements of $\pi_1^*(\mathcal{O}(1)$. T is smooth, we claim that $T \simeq \mathbf{P}^1 \times \mathbf{P}^3$. In fact $\pi_{1_{|T}}$ makes T a projective bundle over a line (since $H^2(\mathbf{P}^1, \mathcal{O}^*) = 0$), that is $T = \mathbf{P}(\mathcal{F})$. Moreover $\pi_{2_{|T}}$ is onto $\mathbf{P}^3$, therefore the claim follows. Therefore we conclude that $\pi_i^* \mathcal{O}_{\mathbf{P}^3}(1)_{|F_i} \simeq \mathcal{O}_{\mathbf{P}^3}(1)$ for $i = 1, 2$. This implies by Grauert Theorem that the two fibrations are classical scroll, that is $X = \mathbf{P}(\mathcal{F}_i)$, for $i = 1, 2$; moreover computing the canonical class of X the $\mathcal{F}_i$ are ample and the lemma easily follows.

□

(5.2) Let $dimW = 1$. Then W is a smooth curve and π is a flat map. Let F be a general fiber, then F is a smooth Fano manifold and $E_{|F}$ is an ample vector bundle on F of rank $(n-2) = dimF - 1$ such that $-K_F = det(E_{|F})$. These pairs $(F, E_{|F})$ are classified in the Main Theorem of [PSW]; in particular if $dimF \geq 5$ F is either $\mathbf{P}^{(n-1)}$ or $\mathbf{Q}^{(n-1)}$ or a del Pezzo manifold with $b_2(F) = 1$. Moreover if $n \geq 6$ then π is an elementary contraction by proposition (1.8).

Claim Let $n \geq 6$ and assume that the general fiber is $\mathbf{P}^{n-1}$, then X is a classical scroll and $E_{|F}$ is the same for all F.

(See also [Fu2]) Let $S = W \setminus U$ be the locus of points over which the fiber is not $\mathbf{P}^{n-1}$. Over U we have a projective fiber bundle. Since $H^2(U, \mathcal{O}^*) = 0$ we can associate this $\mathbf{P}$-bundle to a vector bundle $\mathcal{F}$ over U. Let $Y = \mathbf{P}(\mathcal{F})$ and H the tautological bundle; by abuse of language let H the extension of H to X. Since π is elementary H is an ample line bundle on X. Therefore by semicontinuity $\Delta(F, H_F) \geq \Delta(G, H_G)$, for any fiber G, where $\Delta(X, L)$ is Fujita delta-genus. In our case this yields $0 = \Delta(F, H_F) \geq \Delta(G, H_G) \geq 0$. Moreover by flatness $(H_G)^{n-1} = (H_F)^{n-1} = 1$ and Fujita classification allows to conclude. The possible vector bundle restricted to the fibers are all decomposables, hence they are rigid, that is $H^1(End(E)) = \oplus_i H^1(End(\mathcal{O}(a_i)) = \oplus_i H^1(\mathcal{O}(-a_i)) = 0$. Hence the decomposition is the same along all fibers of π.

Claim Let $n \geq 6$ and assume that the general fiber is $\mathbf{Q}^{n-1}$. Then X is a quadric bundle.

Let as above $S = W \setminus U$ be the locus of points over which the fiber is not a smooth quadric. Let $X^* = \pi^{-1}(U)$ then we can embed X^* in a fiber bundle of projective spaces over U, since it is locally trivial. Associate this P-bundle over U to a projective bundle and argue as before.

□

(5.3) Let now $dimW = 2$ and assume that $n \geq 5$; then π is an elementary contraction. This implies first, by [ABW2, Prop. 1.4.1], that W is smooth; secondly that π is equidimensional, hence flat and the general fiber is $\mathbf{P}^{n-2}$ or $\mathbf{Q}^{n-2}$, see [Fu2].

Claim Let $n \geq 5$ and the general fiber is $\mathbf{P}^{n-2}$ then for any fiber $F \simeq \mathbf{P}^{n-2}$ and $E_{|F}$ is the same for all F.

Let $S \subset W$ be the locus of singular fibers, then $dimS \leq 0$ since W is normal. Let $U \subset W$ be an open set, in the complex topology, with $U \cap S = \{0\}$ and let $V \subset X$ such that $V = \pi^{-1}(U)$. We are in the hypothesis of lemma 3.3 thus we get a "tautological" line bundle H on V and we conclude by [Fu1, Prop. 2.12].

There are two possible restriction of E to the fiber, namely $E_{|F} \simeq \mathcal{O}(2) \oplus (\oplus^{n-1}\mathcal{O}(1))$ or $E_{|F}$ is the tangent bundle. As observed by Fujita in [Fu2] this two restrictions have a different behavior in the diagram (1), in the former φ is birational while in the latter it is of fiber type. Hence the restriction has to be constant along all the fibers.

□

(5.4) Let finally $dimW = 3$; the general fiber is $\mathbf{P}^{n-3}$ (see for instance [Fu2]). Assume that $n \geq 5$, therefore π is elementary; we claim that all fibers are $\mathbf{P}^{n-3}$.

Since π is elementary any fiber G has $codG \geq 2$. Let $S \subset W$ be the locus of point over which the fiber is not $\mathbf{P}^{n-3}$; $dimS \leq 0$ since a generic linear space section can not intersect S, by the above. Let $U \subset W$ be an open set, in the complex topology, with $U \cap S = \{0\}$ and let $V \subset X$ such that $\pi(V) = U$. Then by lemma 3.3 we get a "tautological" line bundle H on V; $\pi : V \to U$ is supported by $K_V + (n-2)H$. Thus by [AW, Th 4.1] U is smooth and all the fibers are $\mathbf{P}^{n-3}$ (we use that $n \geq 5$).

□

Proof of the part 3) of the theorem

In the last part of the theorem we assume that $K_X + detE$ is nef and big but not ample. Then $K_X + detE$ is a supporting divisor of an extremal face, F; let R_i the extremal rays spanning this face. Fix one of this ray, say $R = R_i$ and let $\pi : X \to W$ be the elementary contraction associated to R.

We have $l(R) \geq n-2$; this implies first that the exceptional loci are disjoint if $n > 3$, proposition (1.7). Secondly, by the inequality (1.5), we have

$$dimE(R) + dimF(R) \geq 2n - 3.$$

Therefore $dimE(R) = n-1$ and either $dimF(R) = n-1$ or $dimF(R) = n-2$; if $Z := \rho(E)$ and $D = E(R)$ this implies that either $dimZ = 0$ or 1.

If $dimZ = 1$ then $dimF(\pi) = n-2$ for all fibers (note that since the contraction π is elementary there cannot be fiber of dimension $(n-1)$); thus we can apply proposition (3.1) with $r = (n-2)$. This will give the case 3-(iii) of the theorem.

Consider again the construction in section (2), in particular we refer to the diagram (1). Let S be the extremal ray contracted by φ; note that $l(S) \geq n-2$ and that the inequality (1.5) gives

$$dimE(S) + dimF(S) \geq 3n - 6;$$

in particular, since $dimF(S) \leq dimF(R)$, we have two cases, namely $dimE(S) = 2n-5$ and $dimF(S) = (n-1)$ or $dimE(S) = 2n-4$ and $dimF(S) = (n-1)$ or $(n-2)$.

The case in which $dimE(S) = 2n-5$ will not occur. In fact, after "slicing", (see 2), we would obtain a map $\varphi' = \varphi_{|Z}$ which would be a small contraction supported by a divisor of the type $K_Z + (n-2)L$ but this is impossible by the classification of [Fu1, Th 4] (see also [An]).

Hence $dimE(S) = 2n-4$, that is also φ is divisorial.

Suppose that the general fiber of φ, $F(S)$, has dimension $(n-2)$. After slicing we obtain a map $\varphi' = \varphi_{|Z} : Z \to T$ supported by $K_Z + (n-2)L$, where $L = \xi_{E|Z}$. This map contracts divisors D in Z to curves; by ([Fu1, Th 4]) we know that every fiber F of this map is $\mathbf{P}^{(n-2)}$ and that $D_{|F} = \mathcal{O}(-1)$ (actually this map is a blow up of a smooth curve in a smooth variety). In particular there are curves in Y, call them C, such that $-E(S).C = 1$. We will discuss this case in a while.

Suppose then the general fiber of φ, $F(S)$, has dimension $(n-1)$; therefore all fibers have dimension $(n-1)$. Slicing we obtain a map $\varphi' = \varphi_{|Z} : Z \to T$ supported by $K_Z + (n-2)L$, where $L = \xi_{E|Z}$. This map contracts divisors D in Z to points; by ([Fu1]) we know that these divisors are either $\mathbf{P}^{(n-1)}$ with normal bundle $\mathcal{O}(-2)$ or $\mathbf{Q}^{(n-1)} \subset \mathbf{P}^n$ with normal bundle $\mathcal{O}(-1)$. In the latter case we have as above that there are curves C in Y, such that $-E(S).C = 1$.

In these cases observe that $E(S) \cdot \tilde{C} = 0$, where $\tilde{C}$ is a curve in the fiber of p. Hence $E(S) = p^*(-M)$ for some $M \in Div(X)$. Let l be an extremal curve of $E(S)$. Then, by projection formula, we have $-1 = E(S) \cdot l = -M \cdot mC$ and thus M generates $Im[Pic(X) \to Pic(D)]$, hence M is π-ample; note that in general it does not generate $Pic(D)$. We study now the Hilbert polynomial of $M_{|D}$ to show that $\Delta(D, M_{|D}) = 0$, where $\Delta(X, L)$ is Fujita delta genus. Let $\mathcal{O}_D(-K_X) \simeq \mathcal{O}_D(pM)$, where $p = l(R) \geq n-2$, and $\mathcal{O}_D(-D) \simeq \mathcal{O}_D(qM)$ for some $p, q \in \mathbf{N}$. By adjunction formula $\omega_D \simeq \mathcal{O}_D(-(p+q)M)$. By [Ando, Lemma 2.2] or [BS, pag 179], Serre duality and relative vanishing we obtain that $q \leq 2$, the Hilbert polynomial is

$$P(D, M_{|D}) = \frac{a}{(n-1)!}(t+1)\cdots(t+(n-2))(t+c)$$

and the only possibilities are $a = 1, c = n-1, q = 1or2$ and $a = 2, c = (n-1)/2, q = 1$. In particular $\Delta(D, M_{|D}) = 0$ and, by Fujita classification, D is equal to $\mathbf{P}^{(n-1)}$ or to $\mathbf{Q}^{(n-1)} \subset \mathbf{P}^n$. Now the rest of the claim in 3) i) and ii) follows easily.

It remains the case in which $\varphi' = \varphi_{|Z} : Z \to T$ contracts divisors $D = \mathbf{P}^{(n-1)}$ with normal bundle $\mathcal{O}(-2)$ to points. We can apply the above proposition (2.1) and show that the singularities of W are the same as those of T. Then, as in ([Mo1]), this means that we can factorize π with the blow up of the singular point. Let $X' = Bl_w(W)$, then we have a birational map $g : X \to X'$. Note that X' is smooth and that g is finite. Actually it is an isomorphism outside D and cannot contract any curve of D. Assume to the contrary that g contracts a curve $B \subset D$; let $N \in Pic(X')$ be an ample divisor then we have $g^*N \cdot B = 0$ while $g^*N \cdot C \neq 0$ contradiction. Thus by Zarisky's main theorem g is an isomorphism. This gives a case in 3)i).

References

[Ando] Ando, T., On extremal rays of the higher dimensional varieties, Inv.Math. 81 (1985), 347-357

[An] Andreatta,M., Contractions of Gorenstein polarized varieties with high nef value, to appear on Math. Ann. (1994).

[ABW1] Andreatta,M.- Ballico,E.-Wiśniewski, On contractions of smooth algebraic varieties, preprint UTM 344 (1991).

[ABW2] Andreatta,M.- Ballico,E.-Wiśniewski, Vector bundles and adjunction, International Journal of Mathematics,**3** (1992), 331-340.

[AW] Andreatta,M.- Wiśniewski, J., A note on non vanishing and its applications,Duke Math.J., **72**, (1993).

[BS] Beltrametti,M. Sommese,A.J. On the adjunction theoretic classification of polarized varieties, J. reine angew. Math **427** (1992), 157-192.

[CKM] Clemens,H. Kollár,J. Mori,S. Higher dimensional complex geometry Asterisque **166** 1988

[Fu1] Fujita, T., On polarized manifolds whose adjoint bundle is not semipositive, in Algebraic Geometry, Sendai, Adv. Studies in Pure Math. **10**, Kinokuniya–North-Holland 1987, 167—178.

[Fu2] Fujita, T., On adjoint bundle of ample vector bundles, in Proc. Alg. Geom. Conf. Bayreuth (1990), Lect. Notes Math., **1507**, 105-112.

[Fu3] Fujita,T. On Kodaira energy and reduction of polarized manifolds, Manuscr. Math **76** (1992), 59-84.

[KMM] Kawamata, Y., Matsuda, K., Matsuki, K., Introduction to the Minimal Model Program in Algebraic Geometry, Sendai, Adv. Studies in Pure Math. **10**, Kinokuniya–North-Holland 1987, 283-360.

[Io] Ionescu,P. Generalized adjunctionand applications, Math. Proc. Camb. Phil. Soc. **99** (1986), 457-472.

[Mo] Mori,S. Projective manifolds with ample tangent bundle, Ann.Math. **110** (1979), 593-606

[Mo1] Mori,S., Threefolds whose canonical bundles are not numerical effective, Ann. Math., **116**, (1982), 133-176.

[Mu] Mukai,S. Problems on characterizations of complex projective space, Birational Geometry of Algebraic varieties - Open problems, Katata Japan (1988), 57-60.

[OSS] Okonek,C. Schneider,M. Spindler,H. Complex vector bundle on projective space. (Prog.Math) Boston Basel Stuttgart: Birkhäuser 1981

[PSW] Peternell, T.- Szurek,M.- Wiśniewski,J.A., Fano Manifolds and Vector Bundles, Math.Ann., **294**, (1992), 151-165.

[So] Sommese, A.J., On the adjunction theoretic structure of projective varieties, Complex Analysis and Algebraic Geometry, Proceedings Göttingen, 1985(ed. H. Grauert), Lecture Notes in Math., **1194** (1986), 175—213.

[Wi1] Wiśniewski,J.A., On a conjecture of Mukai, Manuscr. Math. **68** (1990), 135-141.

[Wi2] Wiśniewski,J.A., Lenght of extremal ray and generalized adjunction, Math. Z. **200** (1989), 409-427.

[Wi3] Wiśniewski,J.A., On contraction of extremal rays of Fano manifolds, J. reine und angew. Math. **417** (1991), p. 141-157.

[YZ] Ye, Y.G. - Zhang,Q., On ample vector bundle whose adjunction bundles are not numerically effective, Duke Math. Journal, **60** n. 3 (1990), p. 671-687.

[Zh] Zhang,Q., A theorem of the adjoint system for vector bundles, Manuscripta Math., **70** (1991), p. 189-201.

[Zh2] Zhang,Q., Ample vector bundle, preprint