HYPEREUCLIDEAN MANIFOLDS AND THE NOVIKOV CONJECTURE

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Abstract. We develop some elementary Lipschitz homotopy theory and apply it to manifolds with finite asymptotic dimension. In particular we show that the Higson compactification of a uniformly contractible manifold is mod p acyclic in the finite dimensional case. Also we give an alternative proof of the Novikov Conjecture for the groups with finite asymptotic dimension. Finally we define an asymptotically euclidean metric space as a space which admits an approximation by euclidean asymptotic polyhedra. We show that the Gromov-Lawson conjecture holds for the asymptotically euclidean groups.

§1 Introduction

The Novikov Conjecture states that the higher signatures of a manifold are homotopy invariants. The ordinary signature of a manifold M can be computed by the celebrated Hirzebruch formula $\sigma(M) = \langle L, [M]\rangle$ where the right hand side is the integration of the Hirzebruch polynomial over the fundamental homology class of M. If the fundamental group $\Gamma = \pi_1(M) \neq 0$, there are distinguished homology classes $\beta \in H_*(M)$ which come from the group Γ. The integration of the Hirzebruch polynomial over these classes β gives rise the higher signatures $\sigma_\beta(M)$ of M. In view of this it makes sense to speak about the Novikov Conjecture for a given finitely presented group Γ.

The most successful approach to the Novikov Conjecture in the last decade was the so called coarse approach. The coarse approach consists of studying the large scale geometry of a group Γ as a metric space with the word metric. Quite often it is more convenient to consider the universal cover $\tilde{M}$ with the induced metric, though in the coarse sense $\tilde{M}$ is equivalent to Γ. The most advanced result on the Novikov Conjecture is a theorem of G. Yu [Yu2] which asserts that the Novikov Conjecture holds true for the groups Γ that admit embedding in a coarse sense in the Hilbert space l_2. This Yu's theorem formally generalizes the other his theorem [Yu1] saying that the Novikov Conjecture holds for

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1991 *Mathematics Subject Classification.* Primary 20H15.

Key words and phrases. dimension, asymptotic dimension, absolute extensor, Higson corona, Novikov conjecture..

The author was partially supported by NSF grant DMS-9971709.

groups Γ with finite asymptotic dimension $as\dim\Gamma$. Both Yu's result are obtained by proving the coarse Baum-Connes conjecture which states that the Roe's index map

$$A : K_*(\tilde{M}) \to K_*(C^*(\tilde{M}))$$

is an isomorphism when $\tilde{M}$ is the universal cover of a closed aspherical manifold M [H-R],[Ro2]. The C^*-algebra $C^*(\tilde{M})$ here is the completion of the algebra of infinite matrices $A = (a_{x,y})_{x,y\in\Gamma}$ whose entries are compact operators on l_2 and $a_{x,y} = 0$ for $d_\Gamma(x, y) \geq r$ for some $r = r(A)$. Thus both Yu's works are heavily operator algebra theoretic. The descent from the coarse Baum-Connes conjecture to the Novikov Conjecture is shown in [Ro2]. It uses the homotopy fixed point theory and it proves the analytic Novikov conjecture. The analytic Novikov conjecture states that the analytic assembly map $A_\Gamma : K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ is injective. Here $C^*_r\Gamma$ is the reduced group C^*-algebra. The classical Novikov Conjecture is equivalent to the statement that the rational assembly map from the surgery exact sequence $H_*(B\Gamma; \mathbf{L}) \otimes \mathbf{Q} \to L_*(\mathbf{Z}[\Gamma]) \otimes \mathbf{Q}$ is injective [K-M], [F-R-R]. It is known that the analytic Novikov conjecture implies the original conjecture [F-R-R]. It is possible to show that the equivariant split monicity of the Roe's index map also enables to derive the Novikov Conjecture. As far as it known to the author that the monomorphism version of the coarse Baum-Connes conjecture is not disproved to this moment.

More geometric coarse reduction of the Novikov Conjecture was considered by Ferry and Weinberger [F-W]. They were looking for obstruction to make a homotopy between two closed aspherical manifold tangential. This approach can be traced back to Farrell-Hsiang [F-H]. The Ferry-Weinberger approach reduces the Novikov Conjecture to the question whether the boundary homomorphism $\delta : \check{H}^{n-1}(\nu\tilde{M}; \mathbf{Q}) \to H^n_c(\tilde{M}; \mathbf{Q})$ is an equivariant split epimorphism where $\nu\tilde{M}$ is the Higson corona [Ro1].

We recall that the Higson compactification $X \cup \nu X$ of X is generated by the algebra of functions with the gradient tending to zero at infinity. The exact sequence of pair $(\tilde{M} \cup \nu\tilde{M}, \nu\tilde{M})$ implies that δ is an epimorphism if $H^n(\tilde{M} \cup \nu\tilde{M}; \mathbf{Q}) = 0$. The assertion that the Higson compactification $\tilde{M} \cup \nu\tilde{M}$ is acyclic is called the Higson conjecture [Ro1]. It is known that for integral coefficients this conjecture false even for $\mathbf{R}^n$ [Ke],[D-F]. In §3 we prove the Higson conjecture for finite coefficients in the finite dimensional case. Still there are chances that the mod p Higson conjecture holds in full generality. Also we formulate at the end of §6 the stable version of the Higson conjecture which could be true for reasonable spaces.

The other (equivalent) approach to the Novikov Conjecture is due to Gromov. He introduced the notions of hyperspherical and hypereuclidean manifolds [G2]. Roe proved [Ro1] that an n-manifold $\tilde{M}$ is hypereuclidean if and only if the Higson corona $\nu\tilde{M}$ admits a map of degree one onto the sphere S^{n-1}. It is easy to show that if a group Γ acts on $\tilde{M}$ by isometries then that action can be extended to an action on the Higson corona [Dr1]. We introduces the notion of an equivariantly hypereuclidean n-manifold as a manifold $\tilde{M}$ that admits an equivariant map of degree one onto $(n-1)$-dimensional sphere. Then using Ferry-Weinberger approach it is possible to show that the Novikov Conjecture holds for manifolds M with the equivariantly hypereuclidean universal cover $\tilde{M}$. In §5, §6 we show that if $as\dim\pi_1(M) < \infty$ then $\tilde{M} \times \mathbf{R}^n$ is equivariantly hypereuclidean for some n. This allows to establish the Novikov Conjecture for M. The concepts of hypersphericity is formally weaker. There is an open question whether an integrally hyperspherical manifold is always hypereuclidean. The hypersphericity of $\tilde{M}$ implies the Gromov-Lawson conjecture for M [G-L]: *A closed aspherical manifold cannot carry a metric of a positive scalar curvature*. This conjecture is quite close to the Novikov Conjecture [Ros]. Perhaps the Novikov Conjecture can be derived from some equivariant version of the hypersphericity. In §4 of this paper we show that in finite dimensional case the manifold $\tilde{M} \times \mathbf{R}^k$ is hyperspherical. The first proof of that (perhaps with larger value of k) was given in [Dr2]. In this paper in §7 we extend the result about Gromov-Lawson conjecture to asymptotically euclidean manifolds $\tilde{M}$. All groups uniformly embeddable in the Hilbert space l_2 are asymptotically euclidean, although it is unclear whether the inclusion of these classes is strong. Thus, this is a generalization of the theorem of Yu [Yu2]. Like in the case with Yu's theorem [G4], this result is also limited by expanders. In §8 we show that an asymptotically euclidean metric space cannot contain an expander.

The connection between Ferry-Weinberger's (and Gromov's) and the coarse Baum-Connes conjecture approaches to the Novikov Conjecture is based on the fact that the topological K-theory $K_*(\nu\tilde{M})$ is an approximation to C^*-algebra K-theory $K_*(C^*(\tilde{M}))$ [Ro1].

Using Roe's coarse cohomology one can define an asymptotic cohomological dimension $\operatorname{asdim}_{\mathbf{Z}} X$ in the coarse category as it was done in [Dr1]. To do that properly one has to make a shift in grading of Roe's cohomology (it was not done in [Dr1]). Thus, Roe's cohomology of $\mathbf{R}^n$ should be nontrivial in dimension $n-1$, not n. In that case we will get $\operatorname{asdim}_{\mathbf{Z}} \mathbf{R}^n = n$. The Roe cohomology is dual in the macro-micro sense to the Steenrod homology. It is curious that a similar story happened with the Steenrod homology. It turns out that Steenrod defined his homology with shifted dimensions [St] and only later Sitnikov made the correction [Sit].

For the universal cover $\tilde{M}$ of a an aspherical manifold we always have the inequality $\operatorname{asdim}_{\mathbf{Z}} \tilde{M} < \infty$ [Dr1]. If the asymptotic cohomological dimension always agreed with the Gromov asymptotic dimension, then the Novikov Conjecture would follow. In the ordinary topology the problem about coincidence of the cohomological dimension and the Lebesgue dimension was known from late 20s as the Alexandroff problem. The Alexandroff problem was solved by a counterexample [Dr3]. In the large scale world a counterexample was constructed in [D-F-W] but that example does not have a bounded geometry. Recently a counterexample among finitely presented groups was found by means of expanders and random groups [G], [R-S]. Thus, the dimension theoretic approach to the Novikov Conjecture has got its limit.

The author wish to thank the Max Plank Institute fur Mathematik for the hospitality where this paper was written.

§2 Lipschitz homotopy theory

In this section we consider variations of the following question: *When does a null homotopic λ-Lipschitz map $f : X \to Y$ admit a μ-Lipschitz homotopy $H : X \times I \to Y$ to a constant map?* We recall that a map $f : X \to Y$ between metric spaces is λ-Lipschitz if $d_Y(f(x), f(x')) \le \lambda d_X(x, x')$ for all $x, x' \in X$. Denote by $L(f) = \sup\{\frac{d_Y(f(x),f(x'))}{d_X(x,x')}\}$. Then for a λ-Lipschitz map f we have $L(f) \le \lambda$.

First we give an answer to the question for finite simplicial complexes. Every simplicial complex K carries a metric where all simplexes are the standard of size one. We will call such metric *uniform* and usually we will denote corresponding metric space as K_U. If it is not specified, we will assume that a finite complex always supplied with the uniform metric.

Lemma 1. *Suppose that X and Y are finite simplicial complexes. Then for every λ there exists $\mu = \mu(\lambda)$ such that every null homotopic λ-Lipschitz map $f : X \to Y$ admits a μ-Lipschitz homotopy $H : X \times I \to Y$ to a constant map.*

Proof. Let r_n be the radius of the inscribed sphere in the standard n-simplex. We fix a subdivision T of X with the mesh $< \frac{r_n}{4\lambda}$. There are finitely many different simplicial maps $\phi : T \to Y$. We consider among them null homotopic maps ϕ_i. For every i we fix a homotopy $H_i : T \times I \to Y$ to a constant map. By the Simplicial Approximation Theorem we may assume that the map H_i is μ_i-Lipschitz. We take $\mu \ge \max\{2\mu_i\}$. Also we assume that $\mu \ge \frac{1}{2r_T}$ where r_T is the minimum of radii of inscribed spheres in simplices of T. According to the following Lemma 2 every λ-Lipschitz map $f : X \to Y$ is homotopic to a simplicial map $g : T \to Y$ for the above T and the homotopy is $(\mu/2)$-Lipschitz with the above μ. Then $g = \phi_i$ for some i and a μ-Lipschitz homotopy of f to a constant map will be the sum of the above homotopy deforming f to g and H_i. $\square$

By $B_r(x_0) = \{x \in X \mid d(x, x_0) \le r\}$ we denote a closed ball of the radius r in a metric space X with the center at $x_0 \in X$. The unit n-ball in $\mathbf{R}^n$ will be denoted as B^n.

Lemma 2. *Let $f : L \to K$ be a λ-Lipschitz map between uniform finite dimensional polyhedra. Then f is homotopic to a simplicial map g with respect to some subdivision of L by means of μ-Lipschitz homotopy where μ depends on λ and $\dim L, \dim K$ only.*

Proof. We apply the standard argument of the Simplicial Approximation theorem. Consider an open cover $\mathcal{U} = \{f^{-1}(OSt(v, K)) \mid v \in K^{(0)}\}$ of L where $OSt(v, K)$ denotes the open star of a vertex v in a complex K. Since f is λ-Lipschitz, the Lebesgue number of $\mathcal{U}$ is greater than $\frac{r_n}{2\lambda}$ where r_n is the radius of the inscribed sphere in the unit standard n-simplex and $n = \dim K$. To see that take a point $y \in L$ and consider a closest vertex $v \in K^{(0)}$ to $f(y)$. Then the ball $B_{\frac{r_n}{2}}(y)$ is contained in $OSt(v, K)$. Hence the ball $B_{\frac{r_n}{2\lambda}}$ is contained in $f^{-1}(OSt(v, K))$. Consider a triangulation T of L which is an iterated barycentric subdivision of K with mesh $< \frac{r_n}{4\lambda}$. Let r_T denote the minimum of radii of inscribed spheres in simplices of T. We take $\mu > \frac{1}{r_T}$. Then μ depends on λ, n and the dimension of L. We consider a simplicial approximation $g : T \to K$ defined by the standard rule: for every vertex $a \in T^{(0)}$ we take

$g(a) \in K^{(0)}$ such that $OSt(a, L) \subset f^{-1}(OSt(g(a), K))$. Let us show that for every $x \in L$ the image $g(x)$ belongs to σ whenever $f(x)$ lies in the interior of a simplex σ in K. It suffices to show that $g(\Delta) = \sigma$ where Δ is the unique simplex in L containing x as an interior point. Let $p \in \Delta^{(0)}$ be a vertex, then $x \in OSt(p, L)$. Therefore $f(x) \in f(OSt(p, L)) \subset OSt(g(p), K)$. This means that $g(p)$ is one of the vertices of σ. Then a Lipschitz homotopy between f and g is defined linearly by joining $f(x)$ with $g(x)$ in σ. $\square$

REMARK 1. Lemma 1 holds true if one consider a compact metric space X. In that case by the Ascoli-Arzela theorem the space Y_λ^X of λ-Lipschitz maps $f : X \to Y$ is compact. We can take a finite ϵ-net in Y_λ^X for small enough ϵ. Similarly for every map $\phi_i : X \to Y$ from this ϵ-net we fix a Lipschitz homotopy H_i to a constant map. Then any other λ-Lipschitz map is μ'-homotopic to one from the net.

The following can be also obtained as a corollary of Lemma 5 which is due to Siegel and Williams [S-W].

Lemma 3. *Let Y be a finite simplicial complex with $\pi_n(Y)$ finite. Then for every λ there is μ such that every map $f : B^n \to Y$ with $L(f|_{S^{n-1}}) \le \lambda$ can be deformed to a μ-Lipschitz map $g : B^n \to Y$ by means of a homotopy $h_t : B^n \to Y$ with $h_t|_{S^{n-1}} = f|_{S^{n-1}}$.*

Proof. We consider a finite family of simplicial maps $\phi_i : S^{n-1} \to Y$ as in the proof of Lemma 1. Since the group $\pi_n(Y)$ is finite, there are finite number of homotopically different extensions $H_i^j : B^n \to Y$. We assume that every H_i^j is μ_i^j-Lipschitz. The rest of the proof is the same as in Lemma 1. $\square$

Lemma 4. *Let L be a finite dimensional complex and let K be a finite complex with finite homotopy groups $\pi_i(K)$ for $i \le \dim L + 1$. Let $f, g : L \to K$ be homotopic Lipschitz maps. Then every homotopy between f and g can be deformed to a Lipschitz homotopy $H : L \times [0, 1] \to K$.*

Proof. Let $F : L \times I \to K$ be a homotopy between f and g. By induction on n and using Lemma 3 we construct a μ_n-Lipschitz map $H_n : L^{(n)} \times I \cup L \times \{0, 1\} \to K$ which is a deformation of F restricted to the n-skeleton $L^{(n)}$ and H_n extends H_{n-1}. Then $H = H_m$ for $m = \dim L$. $\square$

Further generalization can be done by extending the target space Y in the above lemmas to locally nice metric compacta. They suppose to be sufficiently nice in order to have the Lipschitz approximation property.

We recall that a topological space Y is called *equilocally connected (ELC)* if there is a neighborhood U of the diagonal in $Y \times Y$ and a map $\lambda : U \times [0, 1] \to Y$ such that $\lambda(y_0, y_1, i) = y_i$, $i = 0, 1$ and $\lambda(y, y, t) = y$ for all $y, y_0, y_1 \in U$.

A *geodesic* between points x and y in a metric space (Y, d) is an isometric embedding $g : [0, c] \to Y$ with $g(0) = x$, $g(c) = y$ and $c = d(x, y)$. A metric space (Y, d) where every two points can be joined by a geodesic is called a *geodesic metric space*. A geodesic metric space (Y, d) is called *geodesically equilocally connected (GELC)* if additionally the map λ possesses the property that the restriction $\lambda|_{(y_0,y_1)\times I} : I \to Y$ is a geodesic in

Y considered with the metric d multiplied by $d(y_0, y_1)$. Examples of GELC spaces are Riemanian manifolds, metric simplicial complexes, path and loop spaces on a GELC space.

The length $|\phi|$ of a path $\phi : I \to Y$ in a metric space (Y, d) is a supremum over all partitions of I of the expression $\Sigma d(\phi(t_i), \phi(t_{i+1}))$. The length of a homotopy $H : X \times I \to Y$ is $|H| = \sup |H|_{\{x\}\times I}|$.

In [Mi] Milnor considered a uniform approximation in a Riemanian manifold of topological paths by smooth. The same approach works in GELC spaces. We formulate it in the following theorem (see [C-S]):

Approximation Theorem. *Let $H : X \times I \to Y$ be a homotopy of a compactum X in a GELC space Y. Then for every ϵ the homotopy H can be deformed modulo $X \times \{0, 1\}$ to a homotopy $F : X \times I \to Y$ which is an ϵ-approximation of H such that*

(1) $\sup_{x\in X} L(F|_{\{x\}\times I}) < \infty$*;*
(2) $|F| < \infty$*;*
(3) *the function* $|F|_{X\times[0,t]}|$ *is a continuous function of x and t.*

Briefly the argument goes as follows. The neighborhood U of the diagonal in $Y \times Y$ gives a finite cover $\mathcal{U}$ of $H(X \times I)$. We take the Lebesgue number of the cover $H^{-1}(\mathcal{U})$ and consider a partition of the interval with mesh smaller than the Lebesgue number and such that $H(X \times \{0, t_1, \dots, t_n, 1\})$ is sufficiently dense in $H(X \times I)$. Then for every x and every i the map F sends the interval $\{x\} \times [t_i, t_{i+1}]$ linearly along the geodesic $\lambda|_{(H(x,t_i),H(x,t_{i+1}))\times I}$.

DEFINITION. Let $\mathcal{C}(H)$ be a family of homotopies which can be obtained by deformations with fixed ends of given homotopy H. We introduce the homotopy norm of H as $\|H\| = \inf\{|F| \mid F \in \mathcal{C}(H)\}$. A *Calder-Siegel number* $B_n(M)$ of a metric space M is a supremum of $\|H\|$ for all homotopies $H : X \times I \to M$ of n-dimensional spaces X.

The cellular approximation theorem implies that every map $f : X \to Y$ of an n-dimensional space to a CW complex can be pushed by a homotopy to the n-skeleton $Y^{(n)}$. If Y is homotopy equivalent to a CW complex with a finite n-skeleton, then we can state that Y contains a compact subset C_n such that every map $f : X \to Y$ with $\dim X \le n$ can be pushed by a homotopy to $f' : X \to C_n$. Clearly this property holds fiberwise for a trivial fibration over a contractible compactum with a fiber homotopy equivalent to a CW complex with a finite n-skeleton. It can be extended simply for locally trivial fibrations over nice spaces such as finite complexes. Then it can be extended to fiber-homotopically trivial fibrations as defined in [Do].

Theorem [C-S]. *Let $p : E \to B$ be a fibration over a compact base, locally trivial in the sense of Dold. Suppose that the fiber F is homotopy equivalent to a complex of finite type, i.e. of finite type i.e. with all skeletons finite. Then E contains a compactum C_n such that every map $f : X \to E$ of an n-dimensional space can be deformed by a fiberwise homotopy to a map with the image in C_n.*

Corollary[C-S]. *Let Y be a GELC compact space with the finite fundamental group $\pi_1(Y)$. Then the Calder-Siegel numbers $B_n(Y)$ are finite for all n.*

Proof. We apply the Theorem to the fibration $p : Y^I \to Y \times Y$ where Y^I is the space of all paths $\phi : I \to Y$ in Y. A set C_n defines a homotopy $H : C_n \times I \to Y$. Then we apply the Approximation Theorem to H and define $B_n(Y) = \max\{|F|_{\{x\}\times I} \mid x \in C_n\}$. By the definition of the set C_n, every homotopy $G : X \times I \to Y$ of an n-dimensional space can be deformed modulo ends to a homotopy H' lying in C_n. The deformation of H to F defines a deformation of H' to $\bar{H}$ and the length of $\bar{H}$ does not exceed $B_n(Y)$. □

The Corollary says that every homotopy to the constant map of n-dimensional space in Y as above can be deformed to a 'short' homotopy. In particular, it means that one cannot wrap properly S^n, $n > 1$ by the Euclidean space $\mathbf{R}^n$ unlike in the case of the circle and $\mathbf{R}^1$. These homotopies $H : X \times [0,1] \to Y$ of the length bounded by $B_n(Y)$ can be considered as λ-Lipschitz on $\{x\} \times [0,1]$ for all $x \in X$ with λ slightly greater than $B_n(Y)$.

For a map $f : X \times Z \to Y$ we denote by $L_X(f) = \sup\{L(f|_{x\times Z}) \mid x \in X\}$.
One of the applications of the Calder-Siegel Theorem is the following [S-W]:

Lemma 5. *Let Y be a finite polyhedron with $\pi_n(Y)$ finite. Then for every λ and every r there is μ such that every map $f : X \times B^n \to Y$ with $L_X(f|_{X\times S^{n-1}}) \le \lambda$ where $\dim X \le r$ can be deformed to a map $g : X \times B^n \to Y$ with $L_X(g) < \mu$ by means of a homotopy $h_t : X \times B^n \to Y$ with $h_t|_{X\times S^{n-1}} = f|_{X\times S^{n-1}}$.*

Proof. The space $M = Y_\lambda^{S^{n-1}}$ of all λ-Lipschitz maps $\phi : S^{n-1} \to Y$ is a GLEC compactum. Then the component M_0 of the constant map is also a GLEC compactum. We consider the fibration $p : E \to M_0$ where E is the space of Lipschitz mappings $\psi : B^n \to Y$ with $L(\psi|_{S^{n-1}}) \le \lambda$. Note that the condition $|\pi_n(Y)| < \infty$ implies that the fiber of p has finite number of components. Then we consider a compactum $C_r \subset E$ from the Calder-Siegel Theorem. It defines a map $\xi : C_r \times B^n \to Y$. We take a Lipschitz approximation γ of ξ with $\gamma|_{C_r\times S^{n-1}} = \xi|_{C_r\times S^{n-1}}$. Then take $\mu = L(\gamma)$. □

Lemma 6. *Let Y be a finite complex with $\pi_k(Y)$ finite and $\pi_{k+1}(Y) = 0$. Then for any given λ and n there is μ such that every map $f : S^n \times B^k \to Y$ with $f(z_0 \times B^k) = y_0$ and $L_{S^n}(f|_{S^n\times\partial B^k}) < \lambda$, where z_0 and y_0 are base points in S^n and Y, can be deformed by a homotopy h_t to a map $g : S^n \times B^k \to Y$ with the properties:*

(1) $L_{S^n}(g) \le \mu$;
(2) $h_t|_{S^n\times\partial B^k} = f|_{S^n\times\partial B^k}$;
(3) $h_t(z_0 \times B^k) = y_0$ *for all* $t \in [0,1]$.

Proof. We may assume that $f(D_{2\epsilon} \times B^k) = y_0$ for the 2ϵ-ball in S^n centered at z_0. Also we assume that this ball is supplied with the standard metric from $\mathbf{R}^n$. Since $\pi_k(Y)$ is finite we can apply Lemma 5 to the map f to obtain a deformation $\tilde{h}_t : S^n \to Y^{B^k}$

satisfying the conditions (1), (2). To achieve the property (3) we do the following. We consider the map $\tilde{h}_1(z_0) : B^k \to Y$. Since $\tilde{h}_1(z_0)(\partial B^k) = y_0$, it defines a map $\xi : S^k \to Y$ with $L(\xi) < \mu'$. The homotopy $\tilde{h}_t$ produces a homotopy of the map ξ to the constant map to y_0. According to Lemma 1 there is a μ-Lipschitz extension $\bar{\xi} : S^k \times I / S^k \times \{1\} \to Y$ of ξ considered as a map of $S^k \times \{0\}$. We define a homotopy $\bar{h}_t : S^n \to Y^{B^k}$ by the formula:

$$\bar{h}_t(z)(x) = \begin{cases} \tilde{h}_1(z)(x) & if \quad z \notin D_{2\epsilon} \\ \tilde{h}_1((1-t)z + tz(\frac{\|z\|}{\epsilon} - 1))(x) & if \quad z \in D_{2\epsilon} \setminus D_\epsilon \\ \tilde{h}_1((1-t)z + tz(1 - \frac{\epsilon}{\|z\|}))(x) & if \quad z \in D_\epsilon \quad and \quad t \leq \frac{\|z\|}{\epsilon} \\ \bar{\xi}(x, (t - \frac{\|z\|}{\epsilon})) & if \quad z \in D_\epsilon \quad and \quad t > \frac{\|z\|}{\epsilon} \end{cases}$$

This homotopy basically contracts $\tilde{h}_1(z_0)(x)$ to constant map by means of $\bar{\xi}$. The disk $D_{2\epsilon}$ is used to get an extension over all S^n such that $\bar{h}_t(z)(x) = f(z,x)$ for $x \in \partial B^k$. It is easy to check that the homotopy $\bar{h}_t$ is performed in the space of μ-Lipschitz maps $\phi : B^k \to Y$.

The homotopy $\tilde{h}_t$ followed by $\bar{h}_t$ deforms f to a map g with the properties (1),(2) and with the additional condition $g(z_0 \times B^k) = y_0$. Therefore this deformation, let us denote it by h'_t, defines a map $\psi : S^{k+1} \to Y$. Now we construct a homotopy h_t between f and g satisfying the conditions (2) and (3). Since $\pi_{k+1}(Y) = 0$, there is an extension $\bar{\psi} : B^{k+2} \to Y$. Using this extension we define a contraction $\tilde{\psi}_t : [0,1] \to Y^{B^k/\partial B^k}$ with the properties $\tilde{\psi}_t(0) = \tilde{\psi}_t(1) = y_0$ for all t, $\tilde{\psi}_0(z) = h'_s(z_0)$ and $\tilde{\psi}_1 = y_0$. Let $X = S^n \times I \times B^k$ and let $A = S^n \times \{0,1\} \times B^k \cup S^n \times I \times \partial B^k \cup \{z_0\} \times I \times B^k$. We define a map $H : A \times [0,1] \to Y$ by the cases:

$H(z, 0, x, t) = f(z, x)$,

$H(z, 1, x, t) = q(z, x)$,

$H(z, s, x, t) = h'_s(z)(x) = f(z, x)$ if $x \in \partial B^k$,

$H(z_0, s, x, t) = \tilde{\psi}_t(z)(x)$.

Since the restriction $H|_{A \times \{0\}}$ agrees with h'_t, the union of them defines a continuous map $H' : X \times \{0\} \cup A \times I \to Y$. By the Homotopy Extension Theorem there is an extension $\bar{H} : X \times I \to Y$. We take $h_t(z,x) = \bar{H}(z,t,x,1)$. Then the above four cases in the definition of H assert respectively that h_t joins f with g, h_t agrees with f on the boundary and h_t keeps the ball $z_0 \times B^k$ in y_0. □

We finish this section by some elementary metric observations about the Serre path fibration. Let Y be a space with a base point y_0 and let $E(Y)$ denote the space of all pathes in Y starting at y_0. Let $p : E(Y) \to Y$ be the path fibration. We note that every fiber $p^{-1}(y)$ is homotopy equivalent to the loop space $\Omega(Y)$. For a GLEC compactum Y the fibration p is locally trivial in Dold's sense. In this case compacta C_n can be viewed as the sets of the following type: $E_\lambda(Y) = \{\phi \in E(Y) \mid L(\phi) \leq \lambda\}$. In view of the argument of the Approximation Theorem one can choose the sets C_n even better.

Namely, the set $W_m = \{(y_0, y_1, \dots, y_m) \in Y^{m+1} \mid (y_i, y_{i+1}) \in U\}$ is compact if one takes a compact neighborhood U in the definition of GLEC structure. The set W_m is naturally embedded in $E(Y)$ by connecting by geodesic intervals the points y_0 with y_1, y_1 with y_2 and so on. For large enough m it absorbs the set $E_\lambda(Y)$ by means of a deformation having the ends fixed. Also this space W_n inherits a GLEC structure from Y^m when one makes a careful choice of U and the original GLEC structure on Y satisfies all the axioms from [Mi].

Similarly we denote by $E_{<\lambda} = \{\phi \in E(Y) \mid L(\phi) < \lambda\}$. Let $p_\lambda : E_\lambda(Y) \to Y$ and $p_{<\lambda} : E_{<\lambda}(Y) \to Y$ be corresponding restrictions of p. We note that for $\lambda > B_n(Y)$ the map $p_{<\lambda}$ is n-Serre fibration and even more, it is a locally n-soft map in Shchepin's sense [Sh].

Below we assume that the product $X \times I$ of a metric space X and the interval is supplied with the l_1-metric; $d_{X\times I}((x,t),(x',t')) = d_X(x,x') + |t - t'|$.

Assertion 1. *A map $f : X \to Y$ admits a λ-Lipschitz homotopy to the constant map to y_0 if and only if there is a λ-Lipschitz lift $s : X \to E_\lambda(Y)$ of f with respect to $p_\lambda : E_\lambda(Y) \to Y$.*

Proof. Let $H : X \times I \to Y$ be a λ-Lipschitz homotopy to the constant map. We define $s(y) = H|_{\{y\}\times I} \in E_\lambda(Y)$. The map s is λ-Lipschitz, since $d(s(x), s(x')) = \sup_t d_Y(H(x,t), H(x',t)) \le \lambda d_X(x,x')$.

Let s be given. We define $H(x,t) = s(x)(t)$. Then

$$d_Y(H(x,t),(x',t)) = d_Y(s(x)(t), s(x')(t)) \le d_{E(X)}(s(x), s(x')) \le \lambda d_X(x,x').$$

Since $s(x) \in E_\lambda(Y)$, we have $d_Y(H(x,t), H(x,t')) = d_Y(s(x)(t), s(x)(t')) \le \lambda|t-t'|$. Then $d_Y(H(x,t),(x',t')) \le d_Y(H(x,t), H(x',t')) + d_Y(H(x',t), H(x',t')) \le \lambda d_X(x,x') + \lambda|t-t'| = \lambda d_{X\times I}((x,t),(x',t'))$. $\square$

§3 Modulo p Higson conjecture

A countable simplicial complex L with a geodesic metric on it is called an *asymptotic polyhedron* if every simplex in L is isometric to a simplex Δ spanned in the Hilbert space and the radii of inscribed spheres r_Δ tend to infinity. In this definition any Banach space can be used moreover asymptotic polyhedra naturally appear with l_∞ metric. Since we are working here with finite dimensional complexes we can stick to the euclidean case.

We recall that by the definition the *asymptotic dimension* $\operatorname{asdim} X$ of a metric space X does not exceed n if for any arbitrary large number d there is a uniformly bounded open cover $\mathcal{U}$ of X with multiplicity $\le n+1$ and with the Lebesgue number $\ge d$ [G1]. This is equivalent that for arbitrary small λ a space X admits a uniformly cobounded λ-Lipschitz map onto an n-dimensional uniform simplicial complex [G1].

We note that for every n-dimensional asymptotic polyhedron L, $\operatorname{asdim} L = n$.

Lemma 7. *Suppose that* $\operatorname{asdim} X \le n$ *and* $f : X \to \mathbf{R}_+$ *is a given proper function. Then there are a compact set* $C \subset X$ *and a 1-Lipshitz map* $\phi : X \to L$ *to an* n*-dimensional asymptotic polyhedron with* $diam(\phi^{-1}(\Delta)) \le f(z)$ *for every* $z \in \phi^{-1}(\Delta) \setminus C$*. Moreover* L *can be presented as the union* $L_0 \cup J_0 \cup L_1 \cup J_1 \cup \dots$ *where each* L_i *is a uniform polyhedron of size* 2^i *and edges in each complex* J_i *have the length* 2^i *or* 2^{i+1}*.*

Proof. Let $\mathcal{U}$ be an open cover of X of multiplicity $\le n+1$ with the Lebesgue number $> 2\lambda$ and with $\max\{\operatorname{diam} U \mid U \in \mathcal{U}\} \le R$. We may assume that $\max_{x\in X} d(x, X \setminus U) > 2\lambda$ for all $U \in \mathcal{U}$. Denote $\phi(x) = d(x, X \setminus U)$. The family of maps $\{\phi_U\}_{U\in\mathcal{U}}$ defines an 1-Lipschitz map $\phi' : X \to L_\infty$ to the Banach space l_∞. Note that $\phi'(X) \subset l_\infty \setminus B_{2\lambda}(0)$. Also we note that the radial projection π of $l_\infty \setminus B_{2\lambda}(0)$ onto the sphere $S_\lambda(0)$ is 1-Lipschitz. Then the map $\phi = \pi \circ \phi' : X \to S_\lambda(0)$ is 1-Lipschitz. Since the multiplicity of the cover $\mathcal{U}$ does not exceed $n+1$, the image $\phi(X)$ lies in the n-skeleton of the spherical infinite dimensional simplex $\Delta^\infty = S_\lambda(0) \cap l^+_\infty$ of the size λ. In other words the nerve $N(\mathcal{U})$ of the cover $\mathcal{U}$ is contained as a subcomplex in the n-skeleton of Δ^∞. There is a constant c_n which depends on n only such that the identification of an n-simplex in Δ^∞ with the standard n-simplex of the size $c_n\lambda$ is 1-Lipschitz. Now we have that the map $\phi : X \to N(\mathcal{U})$ is 1-Lipschitz where the nerve $N(\mathcal{U})$ is given a uniform metric of size $c_n\lambda$. Note that $\phi^{-1}(\Delta) \subset \overline{St(x, \mathcal{U})} \subset B_R(x)$ for some $x \in \phi^{-1}(\Delta)$ for every simplex Δ.

We consider a sequence of such covers $\mathcal{U}_k$ with $\lambda_k > \frac{1}{c_n}2^k$; $k = 0, 1, \dots$ and define $C = f^{-1}([0, 2R_2])$. Then we construct compact sets $C_k \supset f^{-1}([0, 2R_{k+3}])$ such that $d(C_k \setminus C_{k-1}, C_{k-2}) \ge R_{k+2}$. Let $\tilde{\mathcal{U}}_k = \{U \in \mathcal{U}_k \mid U \cap (C_k \setminus C_{k-1}) \ne \emptyset\}$. We form a cover $\tilde{\mathcal{U}} = \cup_k \tilde{\mathcal{U}}_k$. Then the nerve $N(\tilde{\mathcal{U}})$ has a required type $L_0 \cup J_0 \cup L_1 \cup J_1 \cup \dots$. It can be given the metric where each L_i is a uniform complex with the size of simplices 2^i and every J_i is a complex with simplices of a mixed type: their edges which formed by two elements of the cover $\tilde{\mathcal{U}}_i$ have the length 2^i, and the edges which are formed by two elements of $\tilde{\mathcal{U}}_{i+1}$ or by one element from $\tilde{\mathcal{U}}_i$ and the other element from $\tilde{\mathcal{U}}_{i+1}$ have the length 2^{i+1}. Here L_i is the nerve of the cover $\tilde{\mathcal{U}}_k$ and J_i is the nerve of the restriction of $\mathcal{U}_k \cup \mathcal{U}_{k+1}$ over the boundary ∂C_k. If $x \in C_k \setminus C_{k-1}$, then $f(x) \ge 2R_{k+1}$. Note that $\phi^{-1}(\Delta) \subset B_{R_{k+1}}(x)$ for some $x \in \phi^{-1}(\Delta)$. Then for any $z \in \phi^{-1}(\Delta)$ we have $d(z, x) \le R_{k+1}$. Hence $z \notin C_{k-2}$ and therefore $f(z) \ge 2R_{k+1} \ge diam(\phi^{-1}(\Delta))$.

The complex $N(\tilde{\mathcal{U}})$ satisfies all the requirements except it is not necessarily n-dimensional. We note that he complexes J_i are at most $(2n+1)$-dimensional. The n-dimensionality can be achieved by some standard dimension theoretic trick with the choice of covers. We are not giving all details, since for the purpose of this paper the finite dimensionality is enough. □

A metric space X is called *uniformly contractible* if there is a function $S : \mathbf{R}_+ \to \mathbf{R}_+$ such that every ball $B_r(x)$ is contractible to a point in the ball $B_{S(r)}(x)$.

Let $x_0 \in X$ be a base point. We denote $\|x\| = d_X(x, x_0)$.

Lemma 8. *Let* X *be a uniformly contractible proper metric space with* $\operatorname{asdim} X = n$*. Then given a proper function* $g : X \to \mathbf{R}_+$ *there are an* n*-dimensional asymptotic polyhedron* N*, a proper 1-Lipschitz map* $\phi : X \to N$ *and a proper homotopy inverse map*

$\gamma : N \to X$ *with* $d(x, \gamma\phi(x)) < g(x)$ *for all* $x \in X$.

Proof. We define by induction on i a lift γ on the i-skeleton $N^{(i)}$ of the nerve of a cover of X given by Lemma 7 for an appropriate choice of f. We take $\gamma(v) \in \phi^{-1}(v)$ for every vertex v. Then using the uniform contractibility of X we can extend γ with control over the 1-skeleton $N^{(1)}$ and so on. Without loss of generality we may assume that X is a polyhedron of the dimension n supplied with a triangulation of mesh ≤ 1. By induction on i we define a homotopy $H : X^{(i)} \times I \to X$ joining the identity map with $\gamma \circ \phi$. We consider a function $\psi(x) = \|x\| - \max\{d(x, y) \mid y \in H(x \times I)\}$. If ψ tends to infinity, then the map H is proper. Therefore it suffices to show that ψ tends to infinity for an appropriate choice of f. Let S be a contractibility function. We define $\rho(t) = S^{-1}(t/2)$ where S^{-1} is the inverse function to S. Then we take $f = \rho^{2n+1} \circ g$, the composition of g and $2n + 1$ times iteration of ρ. We assume here that $g(x) \leq \|x\|/2$. Then it is easy to verify that $\psi(x) \geq \|x\|/2$. $\square$

A metric space X is called *proper* if every ball $B_r(x)$ in X is compact. We recall that the Higson compactification $\bar{X}$ of a proper metric space X is defined by the ring of bounded functions with the gradient tending to zero at infinity [Ro1]. The reminder of this compactification is called the *Higson corona* and it is denoted as νX. Thus, $\bar{X} = X \cup \nu X$. The defining property of the Higson corona is the following:

(*) a continuous map $f : X \to Z$ *to a compact metric space is extendable over the Higson corona* νX *if and only if for every* R *the diameter of the image* $f(B_R(x))$ *of the* R*-ball centered at* x *tends to zero as* x *approaches infinity.*

Note that a proper Lipschitz map $f : X \to Y$ induces a continuous mapping between the Higson coronas $\bar{f} : \nu X \to \nu Y$.

Theorem 1. *Let* X *be a uniformly contractible proper metric space with a finite asymptotic dimension and let* $\bar{X}$ *be the Higson compactification. Then* $\check{H}^n(\bar{X}; \mathbf{Z}_p) = 0$ *for all* n *and all* p.

Proof. We show that every map $\alpha : \bar{X} \to K(\mathbf{Z}_p, n)$ is null homotopic. Since $\bar{X}$ is compact, the image $\alpha(\bar{X})$ is contained in the k-skeleton $K = K(\mathbf{Z}_p, n)^{(k)}$, $k > as \dim X + 1$. We fix a geodesic metric on K. Let ϵ_K be an injectivity radius in K, i.e. every two points within a distance ϵ_K can be joined by a unique geodesic. Since the map $\alpha|_X \to K$ is extendable over the Higson corona the function $R_\alpha(t) = L(\alpha|_{X \setminus B_t(x_0)})$ tends to zero at infinity. We apply Lemma 8 with $g(x) \leq \min\{\epsilon_K / R_\alpha(\|x\|/2), \|x\|/4\}$ to obtain an asymptotic polyhedron N and maps $\phi : X \to N$ and $\gamma : N \to X$. Let $[u, v]$ be an edge in N, then $d_K(\alpha\gamma(u), \alpha\gamma(v)) \leq R_\alpha(t_0) d_X(\gamma(u), \gamma(v))$ where $t_0 = \min\{\|\gamma(u)\|, \|\gamma(v)\|\}$. We may assume that there are $x, y \in X$ such that $\phi(x) = u$ and $\phi(y) = v$. Then

$$d_X(\gamma(u), \gamma(v)) \leq d_X(x, y) + \epsilon_K / R_\alpha(\frac{1}{2}\|x\|) + \epsilon_K / R_\alpha(\frac{1}{2}\|y\|) \leq diam\phi^{-1}[u, v] + 2\epsilon_K / R_\alpha(\frac{1}{2}\|x\|)$$

if we assume that $\|x\| \leq \|y\|$. Because of the inequality $g(x) \leq \|x\|/4$ we have that $R_\alpha(t_0) \leq R_\alpha(\|x\|/2)$. We may assume that $f < g$ and then $diam(\phi^{-1}([u, v])) < g(x)$.

Summarizing all this we obtain the inequality $d_K(\alpha\gamma(u), \alpha\gamma(v)) \leq 3\epsilon_K$. This means that the map $\alpha \circ \gamma \circ u^{-1}$ is $3\epsilon_K$-Lipschitz where $u : K \to K_U$ is the projection to the uniform metric.

Since X is contractible, the map $\alpha \circ \gamma$ is null homotopic. Note that the homotopy groups $\pi_i(K)$ are finite for $i \leq \dim N + 1$. We apply Lemma 4 to obtain a λ-Lipschitz homotopy $H : N_U \times I \to K$ of $\alpha \circ \gamma$ to a constant map. This homotopy defines a Lipschitz map $\tilde{H} : N_U \to K^I_\lambda$ to the space of λ-Lipschitz mappings of the unit interval I to K. Then the composition $\tilde{H} \circ u : N \to K^I_\lambda$ satisfies the Higson extendibility condition (*). Let $\bar{h} : \bar{N} \to K^I_\lambda$ be the extension over the Higson corona. This extension defines a map $\bar{H} : \bar{N} \times I \to K$. The map $\bar{H}$ is a homotopy between the extension $\xi = \overline{\alpha \circ \gamma}$ and a constant map. To complete the proof we show that α is homotopic to $\xi \circ \bar{\phi}$ where $\bar{\phi}$ is the extension of the Lipschitz map ϕ to the Higson compactifications. Note that

$$d_K(\alpha(x), \alpha\gamma\phi(x)) \leq R_\alpha(t_0) d(x, \gamma\phi(x)) \leq R_\alpha(t_0)\epsilon_K / R_\alpha(\|x\|/2) \leq \epsilon_K$$

where $t_0 = \min\{\|x\|, \|\gamma\phi(x)\|\} \geq \|x\|/2$. Then for every $x \in X$ we join the points $\alpha(x)$ and $\alpha\gamma\phi(x)$ by the unique geodesic $\psi_x : I \to K$. This defines a map $\tilde{\psi} : X \to K^I_\mu$. Since both α and $\alpha \circ \gamma \circ \phi$ satisfy the condition (*), the map $\tilde{\psi}$ has the property (*). Let $\bar{\psi} : \bar{X} \to K^I_\mu$ be the extension of $\tilde{\psi}$ to the Higson corona. The map $\bar{\psi}$ defines a homotopy $\Psi : \bar{X} \times I \to K$ between α an $\xi \circ \bar{\phi}$. $\square$

Corollary. *Let X be a uniformly contractible n-manifold with a finite asymptotic dimension. Then there exists a Higson dominated metrizible corona N of X such that the boundary homomorphism $H^{lf}_n(X; \mathbf{Z}) = \mathbf{Z} \to H^{st}_{n-1}(N; \mathbf{Z})$ is a monomorphism.*

Proof. First we apply Theorem 1. Then the Schepin Spectral theorem [Dr1] implies that there is a metrizable Higson dominated corona N of X such that $\check{H}^*(X \cup N; \mathbf{Z}_p) = 0$ for all prime p. Then for the Steenrod homology the inclusion of the boundary induces an isomorphism $\tilde{\partial}_p : H^{lf}_n(X; \mathbf{Z}_p) \to H^{st}_{n-1}(N; \mathbf{Z}_p)$ for all p. Consider an $(n-2)$-connected and locally $(n-2)$-connected compactum $Y = N \cup W$ with $\dim W \leq n - 1$. Consider a diagram generated by exact sequence of pairs $(X \cup N, N) \subset (X \cup Y, Y)$. If the image in $H^{st}_{n-1}(Y)$ of the generator $1 \in \mathbf{Z} = H^{lf}_n(X)$ is not divisible by p, then it is not divisible by p in $H^{st}_{n-1}(N)$. Since W is $(n-1)$-dimensional, the inclusion homomorphism $H^{st}_{n-1}(N; \mathbf{Z}_p) \to H^{st}_{n-1}(Y; \mathbf{Z}_p)$ is a monomorphism. Hence, the boundary homomorphism $H^{lf}_n(X; \mathbf{Z}_p) \to H^{st}_{n-1}(Y; \mathbf{Z}_p)$ is a monomorphism. Since Y is locally $(n-2)$-connected, the Steenrod homology in the dimension $n - 1$ agrees with the singular homology and we can use the universal coefficient formula. Since Y is $(n-2)$-connected, the universal coefficient formula implies that $H^{st}_{n-1}(Y; \mathbf{Z}_p) = H^{st}_{n-1}(Y) \otimes \mathbf{Z}_p$. Therefore the homomorphism $H^{lf}_n(X) \otimes \mathbf{Z}_p \to H^{st}_{n-1}(Y) \otimes \mathbf{Z}_p$ is a monomorphism. Hence the image of the generator is not divisible by p in $H^{st}_{n-1}(Y)$ and as well in $H^{st}_{n-1}(N)$. Therefore $\mathbf{Z} = H^{lf}_n(X) \to H^{st}_{n-1}(N)$ is a monomorphism. $\square$

REMARK 3. In [Ke] Keesling established that 1-dimensional cohomology of the Higson compactification of $\mathbf{R}^n$ is nonzero. Also it was shown in [D-F] that $\check{H}^n(\nu\mathbf{R}^n; \mathbf{Z}) \neq 0$. Then Theorem 1 implies that these groups are p-divisible for all p.

§4 Hyperspherical manifolds

Let $f : K \to Y$ be a map, by $1\hat{\times}f : B^n \times K \to \Sigma^n Y$ we denote the through map in the composition $B^n \times K \to (B^n/\partial B^n) \times K \to S^n \times Y \to S^n \wedge Y = \Sigma^n Y$.

Lemma 9. *Let $f : K \to S^m$ be a map of a simplicial complex of dimension l. Then there is λ and a map $g : B^n \times K \to S^{l+2}$ with $n = l - m + 2$ such that*

(1) $L_{B^n}(g) \le \lambda$,
(2) $L_\Delta(g|_{B^n\times\Delta}) \le L_\Delta$ *for every simplex Δ in K,*
(3) *there is a homotopy h_t which deforms $1\hat{\times}f$ to g such that $h_t(\partial B^n \times K) = y_0$ for all t.*

Proof. We assume that $l \ge m$ and $f(K^{(m-1)}) = s_0 \in S^m$. By induction on k we construct a sequence of maps $g_k : B^n \times K^{(k)} \to S^{l+2}$ with the properties:

(1) $L_{B^n}(g_k) \le \lambda_k$,
(2) $L_\Delta(g_k|_{B^n\times\Delta}) < L_\Delta$ for all simplices $\Delta \subset K^{(k)}$,
(3) there exists a homotopy $h_t^k : B^n \times K \to S^{l+2}$ whose restriction onto $B^n \times K^{(k)}$ joins the restriction $1\hat{\times}f$ with g_k in such a way that $h_t^k(\partial B^n \times K) = y_0$ for all t.

We take g_k to be constant maps to y_0 for $k < m$. Let us assume that g_{k-1} and h_t^{k-1} are constructed. We take an arbitrary k-dimensional simplex Δ in K and consider the map $f = h_1^{k-1}|_{B^n\times\Delta} : B^n \times \Delta \to S^{l+2}$. Since $h_1^{k-1}(\partial B^n \times \Delta) = y_0$, we may assume that f is a map of $S^n \times \Delta$ to S^{l+2} with $f(z_0 \times \Delta) = y_0$ and $L_{S^n}(f|_{S^n\times\partial\Delta}) \le \lambda_{k-1}$. First we note that $\pi_k(S^{l+2}) = \pi_{k+1}(S^{l+2}) = 0$ for $k \le l$. Then we can apply Lemma 6 to obtain a homotopy h_t^Δ of the map f to a map g with the properties (1)-(3). Additionaly we may assume that h_1^Δ is a Lipschitz map with some probably very large constant L_Δ. By the condition (2) we can glue together all homotopies h_t^Δ in a homotopy $H : B^n \times K^{(k)} \times I \to S^{l+2}$ with the property $H(\partial B^n \times K^{(k)} \times I) = y_0$. We extend H to $\bar{H}$ over $\partial B^n \times K \times I$ as the constant map to the point y_0. Let $\tilde{h}_t^k$ be any extension over $B^n \times K \times I$ of the union of the maps $\bar{H}$ and h_1^{k-1}. We define g_k as the restriction of $\tilde{h}_1^k$ onto $B^n \times K$ and define a homotopy h_t^k as h_t^{k-1} followed by $\tilde{h}_t^k$. Then $\lambda_k = \mu(\lambda_{k-1}, n)$ where μ is defined in Lemma 6.

Then $\lambda = \lambda_l$, $g = g_l$ and $h_t = h_t^l$. □

DEFINITION. A map (not necessarily continuous) $f : X \to Y$ between metric spaces is said to be *asymptotically Lipschitz* if there are constants λ and R such that $d_Y(f(x), f(x')) \le \lambda d_x(x, x') + R$.

Lemma 10. *Let X be a metric space with* asdim $X \le m$ *and let $Y \subset \mathbf{R}^n$ be a finite complex. Then for every $\epsilon > 0$ there is $\delta > 0$ such that every asymptotically Lipschitz map $f : X \to Y$ with the constants $\lambda, R < \delta$ is ϵ-close to an ϵ-Lipschitz map $g : X \to Y$.*

Proof. Let $r : N_a(Y) \to Y$ be a retraction of an a-neighborhood such that $\|r(x) - x\| < \frac{\epsilon}{4}$ for all x and $L(r) < 2$. We fix a finite cover $\mathcal{U} = \{U_i\}$ of Y by open balls in $\mathbf{R}^n$ lying in $N_a(Y)$ with centers c_i. Denote by $c = \max\{\|c_i\|\}$. Let λ_0 be the Lebesgue number of $\mathcal{U}$. By the definition of the asymptotic dimension, there is a uniformly bounded cover

$\mathcal{V} = \{V_\alpha\}$ of X of multiplicity $\leq m+1$ with the Lebesgue number $L > \frac{16(m+1)^2 c}{\epsilon}$. Let $\phi_\alpha(x) = d(x, M \setminus V_\alpha)$, let $\Phi(x) = \Sigma_\alpha \phi_\alpha(x)$ and let $\psi_\alpha(x) = \frac{\phi_\alpha(x)}{\Phi(x)}$. We note that $\Phi \geq L$. Let R be the upper bound on diameters of elements of the cover $\mathcal{V}$. We take $\delta < \min\{\frac{1}{R}, \frac{\lambda_0}{3}\}$. Since $\delta < \frac{\lambda_0}{3}$, for every asymptotically Lipschitz map $f : X \to Y$ with the constants $< \delta$ the image $f(V_\alpha)$ has the diameter less than λ_0. Hence for every α there is $i = i(\alpha)$ such that $f(V_\alpha)$ is contained in U_i. We define a map g as the composition $r \circ q$ where q is defined by the formula: $q(x) = \Sigma_\alpha \psi_\alpha(x) c_{i(\alpha)}$.

First we check that g is ϵ-close to f. By the triangle inequality we have

$$\|f(x) - g(x)\| \leq \|f(x) - q(x)\| + \|q(x) - rq(x)\| \leq \|f(x) - \Sigma_\alpha \psi_\alpha(x) c_{i(\alpha)}\| + \frac{\epsilon}{4}.$$

Since $\{\psi_\alpha\}$ is a partition of unity, $q(x)$ lies in the convex hull H in $\mathbf{R}^n$ of the set of those $c_{i(\alpha)}$ for which $f(x) \in U_{i(\alpha)}$. Then $\|f(x) - q(x)\| \leq diam(H) + \max\{radius(U_i(\alpha))\} \leq \frac{3}{4}\epsilon$. Therefore $\|f(x) - g(x)\| \leq \epsilon$ for all x.

Now we check that $L(g) \leq \epsilon$. For any two pints $x, y \in M$ we have

$$\|g(x) - g(y)\| \leq 2\|q(x) - q(y)\| = 2\|\Sigma_\alpha(\psi_\alpha(x) - \psi_\alpha(y)) c_{i(\alpha)}\| \leq$$

$$4c(m+1)\max\{|\psi_\alpha(x) - \psi_\alpha(y)|\} < \frac{\epsilon L}{4(m+1)} \max\{|\psi_\alpha(x) - \psi_\alpha(y)|\}.$$

We note that

$$|\psi_\alpha(x) - \psi_\alpha(y)| = |\tfrac{\phi_\alpha(x)}{\Phi(x)} - \tfrac{\phi_\alpha(y)}{\Phi(y)}| \leq \tfrac{1}{\Phi(x)}|\phi_\alpha(x) - \phi_\alpha(y)| + \tfrac{1}{\Phi(x)}\tfrac{\phi_\alpha(y)}{\Phi(y)}|\Phi(y) - \Phi(x)| \leq$$

$$\tfrac{1}{\Phi(x)}(d(x,y) + 2(m+1)d(x,y)) \leq \tfrac{4(m+1)}{L} d(x,y).$$

Then we put these inequalities together to obtain $\|g(x) - g(y)\| < \epsilon d(x,y)$. $\square$

DEFINITION [GL]. A m-manifold M is called *integrally hyperspherical* if for arbitrary small ϵ there is a proper ϵ-Lipschitz map $f : M \to S^m$ onto the unit m-sphere of degree one. Here the properness of f means that $f(M \setminus C) = pt$ for some compact set C.

Lemma 11. *Let M be a uniformly contractible m-manifold. Assume that for every λ there exists a compact $S \subset M$ and a λ-Lipschitz map $f : S \to S^{m-1}$ of the degree one. Then M is integrally hyperspherical. Here the degree of f is the degree of the following homomorphism*

$$\mathbf{Z} = H^{m-1}(S^{m-1}) \to H^{m-1}(S) \to H^{m-1}(W_\infty) \to H^m(M, W_\infty) \to H^m_c(M) = \mathbf{Z},$$

where W_∞ is the closure of the infinity component of $M \setminus S$.

Proof. Let ϵ be given. We take δ as in Lemma 10 applied for M and ϵ. Let $f : S \to S^{m-1}$ be a $(\delta^2/4)$-Lipschitz map of the degree one in the above sense. We define a not necessarily continuous map $\bar{f} : M \to \Sigma S^{m-1} = S^{m-1} \times [-1,1]/\sim$ by the formula:

$$\bar{f}(x) = \begin{cases} (f(x'), \min\{1, \frac{\delta}{2} d(x,S)\}) & if \quad x \in W_\infty \\ (f(x'), -\min\{1, \frac{\delta}{2} d(x,S)\}) & if \quad x \notin W_\infty \end{cases}$$

Here $x' \in S$ is any point with the property $d(x,S) = d(x,x')$.

We check the asymptotically Lipschitz condition for $\bar{f}$. Let $x, y \in M$ be given points. If $d(x,S) \geq \frac{2}{\delta}$, then $\bar{f}(x)$ is the pole in the suspension ΣS^{m-1}. If also $d(y,S) \geq \frac{2}{\delta}$, then $d_{\Sigma S^{m-1}}(\bar{f}(x), \bar{f}(y))$ is either 0 or 2. In both cases we have the Lipschitz inequality with the coefficient $\delta/2$. If only $d(x,S) \geq \frac{2}{\delta}$, then $d_{\Sigma S^{m-1}}(\bar{f}(x), \bar{f}(y)) = |1 \pm \frac{\delta}{2} d(y,S)|$ depending on whether both x, y are in W_∞ or in $M \setminus W_\infty$ or not. In all these cases we have again the pure Lipschitz condition with the coefficient $\delta/2$.

Now we assume that $d(x,S) \leq \frac{2}{\delta}$ and $d(y,S) \leq \frac{2}{\delta}$. By the triangle inequality we have that

$$d_{\Sigma S^{m-1}}(\bar{f}(x), \bar{f}(y)) \leq d_{S^{m-1}}(f(x'), f(y')) + \frac{\delta}{2}|d(x,S) - d(y,S)| \leq$$

$$\frac{\delta^2}{4} d_M(x', y') + \frac{\delta}{2} d_M(x,y) \leq (\delta^2/4 + \delta/2) d_M(x,y) + \frac{\delta^2}{4}(d(x,S) + d(y,S)) \leq$$

$$\delta d(x,y) + \frac{\delta^2}{4}(\frac{2}{\delta} + \frac{2}{\delta}) = \delta d(x,y) + \delta.$$

We apply Lemma 10 to obtain a map $g : M \to \Sigma S^{m-1}$ which is ϵ-Lipschitz and ϵ-closed to $\bar{f}$. If ϵ is sufficiently small, the restriction $g|_S$ is homotopic to f in $S^{m-1} \times (-\epsilon, \epsilon) \subset \Sigma S^{m-1}$. Therefore the map g has the degree one. $\square$

Theorem 2. *Let M be a uniformly contractible m-manifold of asymptotic dimension* $\operatorname{asdim} M = l$. *Then the product $M \times \mathbf{R}^{l-m+1}$ is integrally hyperspherical.*

Proof. Let $\alpha : M \setminus \{x_0\} \to S^{m-1}$ be a map representing a generator in the cohomology group $H^{m-1}(M \setminus \{x_0\}; \mathbf{Z}) = H_c^m(M; \mathbf{Z}) = \mathbf{Z}$. Let $\phi : M \to N$ and $\gamma : N \to M$ be as in Lemma 8. Let $H : M \times I \to M$ be a proper homotopy joining the identity map 1_M with $\gamma \circ \phi$. Then there exists a compact subcomplex $C \subset N$ such that $H((M \setminus \phi^{-1}(C)) \times I) \subset M \setminus \{x_0\}$. Let $S_i \subset M$ be a sequence of compact sets tending to infinity such that the restriction $\alpha|_{S_i}$ induces an isomorphism in $(m-1)$-dimensional cohomologies. Let K_i be a sequence of compact subcomplexes in N with $\phi(S_i) \subset K_i$. We note that $\alpha \circ \gamma|_{K_i}$ defines a nontrivial element in the $(m-1)$-dimensional cohomology which goes to the element $[\alpha|_{S_i}] \in H^{m-1}(S_i; \mathbf{Z})$ under the homomorphism $(\phi|_{S_i})^*$ for all i. We assume that all complexes K_i are disjoint and define $K = \cup_i K_i$. We define a map $f : K \to S^{m-1}$ as the union of maps $\alpha \circ \gamma|_{K_i}$. Here we assume here that K is supplied with the uniform metric d_U. According to Lemma 9 there is a map $g : B^n \times K \to S^{l+2}$ with $n = l - m + 1$ satisfying the conditions (1)-(3) with $\lambda = \lambda_0$.

Now we are going to apply Lemma 11. Let λ be given. Denote by r_i the minimum of radii of inscribed spheres for simplices in K_i, considered with the metric induced from N. We take i such that $r_i > \frac{2\lambda_0}{\lambda}$. Let $L = \max\{L_\Delta \mid \Delta \subset K_i\}$ and let $\mu = \frac{2L}{\lambda}$. We consider the map

$$q = g \circ (\frac{1}{\mu} \times 1_{K_i}) : B^n_\mu \times K_i \to S^{l+2}.$$

Since the map $1_{K_i} : (K_i, d_N|_{K_i}) \to (K_i, d_U)$ is $\frac{1}{r_i}$-Lipschitz and $\frac{1}{r_i} < \frac{\lambda}{2\lambda_0}$, we have $L_{B^n_\mu}(q) \le \lambda/2$. Since the map $\frac{1}{\mu} : \mathbf{R}^n \to R^n$ is $\frac{1}{\mu}$-Lipschitz and $\frac{1}{\mu} \le \frac{\lambda}{2L}$, the map q is λ-Lipschitz. It is factored through the map $q' : \Sigma^n K_i \to S^{l+2}$ with $(q')^*(1) = \Sigma^n[\alpha \circ \gamma|_{K_i}]$. The set S_i bounds the set B_i in M such that the inclusion $S_i \subset B_i$ is cohomologically trivial. We define $\bar{S}_i = B^n_\mu \times S_i \cup \partial B^n_\mu \times B_i$. Then the map $g \circ (\frac{1}{\mu} \times K_i) \circ (1_{B^n_\mu} \times \phi|_{S_i})$ can be extended from $B^n_\mu \times S_i$ over $\bar{S}_i$ to a λ-Lipschitz map $\bar{f}$ of the degree one. $\square$

The following is the corollary of Theorem 2 and the Gromov-Lawson theorem [G-L],[G3]. It was first proven in [Yu1], see also [Dr2] for a different proof.

Corollary. *If the fundamental group of a closed aspherical manifold N has finite asymptotic dimension, then N cannot carry a metric of a positive scalar curvature.*

REMARK 4. According to the anticipated formula $\operatorname{asdim} X = \operatorname{asdim}_{\mathbf{Z}} X$ we have $l = m$ and then Theorem 2 would imply that the manifold $M \times \mathbf{R}$ is hyperspherical.

§5 Hypereuclidean manifolds

Lemma 12 (Trading Lemma). *For every n there is a number c_n such that for every L-Lipschitz map $f : B^n \to Y$ which is λ-Lipschitz on the boundary ∂B^n the map $1_{B^n} \hat{\times} f : B^n \times B^n \to \Sigma^n Y$ can be deformed by a homotopy h_t fixed on the boundary $\partial(B^n \times B^n)$ to a map g with the properties: $L(g|_{x \times B^n}) \le c_n \lambda$ and $L(g|_{B^n \times y}) \le c_n L$ for all $x, y \in B^n$.*

Proof. For every n there is a c'_n-Lipschitz isotopy $H : B^n \times B^n \times I \to B^n \times B^n$ that exchanges the factors. There is a $\bar{c}_n$-Lipschitz homeomorphism

$$\xi : (B^n \times B^n) \setminus Int\frac{1}{2}(B^n \times B^n) \to \partial(B^n \times B^n) \times I$$

such that the restriction of ξ on the boundary $\partial(B^n \times B^n)$ identifies it with $\partial(B^n \times B^n) \times \{0\}$ and the restriction of ξ on $\partial\frac{1}{2}(B^n \times B^n)$ is the multiplication by 2. We define

$$h_t(x,y) = \begin{cases} (1\hat{\times} f)H((1+t)x, (1+t)y, t) & if \quad (x,y) \in \frac{1}{2}(B^n \times B^n) \\ (1\hat{\times} f)(tH(\xi(x,y)) + (1-t)(x,y)) & otherwise \end{cases}$$

We note that $h_0 = 1\hat{\times} f$. If $(x,y) \in \partial(B^n \times B^n)$ we have that $h_t(x,y) = (1\hat{\times} f)(tH(\xi(x,y)) + (1-t)(x,y)) = (1\hat{\times} f)(t(x,y) - (1-t)(x,y)) = (1\hat{\times} f)(x,y)$. So the homotopy h_t is fixed on $\partial(B^n \times B^n)$. Let $g = h_1$, then

$$g(x,y) = \begin{cases} (1\hat{\times} f)(2y, 2x) & if \quad (x,y) \in \frac{1}{2}(B^n \times B^n) \\ (1\hat{\times} f)H(\xi(x,y)) & otherwise \end{cases}$$

It easy to see that $L(g|_{x \times B^n}) \le \max\{L(1\hat{\times} f)c'_n\bar{c}_n, 2\} \le c_n L(1\hat{\times} f) = c_n\lambda$ and

$$L(g|_{B^n \times y}) \le \max\{L(1\hat{\times} f)c'_n\bar{c}_n, 2L\} \le c_n L.$$

$\square$

Since the homotopy h_t depends continuously on f the Trading Lemma admits a parametrized version:

Parametrized Trading Lemma. *For every map $f : B^n \times Z \to Y$ with $L_Z(f) < L$ and $L_Z(f|_{\partial B^n \times Z}) < \lambda$ the map $1_{B^n}\hat{\times} f : B^n \times B^n \times Z \to \Sigma^n Y$ can be deformed modulo $\partial(B^n \times B^n) \times Z$ to a map g such that*

(1) *$L(g_z|_{x\times B^n}) \le c_n\lambda$ for all x and $g_z = g|_{B^n\times B^n\times z}$;*
(2) *$L(g_z|_{B^n\times y}) \le c_n L$ for all y;*
(3) *if f_z is a constant then g_z is a constant.*

Let $f : K \to Y$ be a map of an infinite simplicial complex to a metric space which is Lipschitz on every finite complex and let $x_0 \in K$ be a base point. We define a function $L(f) : \mathbf{R}_+ \to \mathbf{R}_+$ as $L(f)(t) = \max\{L(f|_{B_t(x_0)})\}$ where $B_t(x_0)$ is a ball of radius t centered at x_0.

Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a function, by $B^n_{\phi(t)} \times K$ we denote the set $\{(z, x) \in \mathbf{R}^n \times K \mid \|z\| \le \phi(d_K(x, x_0))\}$.

Lemma 13. *Let K be a finite dimensional uniform polyhedron and let $f : K \to Y$ be a continuous map to a GLEC compact space Y and let $L_f(t) \le \psi(t)$. Then there are numbers n, a and λ and a map $g : B^n \times K \to \Sigma^n Y$, satisfying the inequalities $L_{B^n}(g) < \lambda$ and $L_K(g)(t) < a\psi(t)$, which is homotopic to the map $(1\hat{\times} f)$ by means of a homotopy h_t with the property $h_t(\partial B^n \times K) = y_0$.*

Proof. We apply induction on $\dim K = i$. If $\dim K = 0$, then we assume that all points in K are at least one apart. Then the statement follows from compactness of Y with $n = 0$, $a = 1$ and $\lambda = 2diam(Y)$.

Assume that the lemma holds for i and let $\dim K = i+1$. Let $f : K \to Y$ be given with $L(f)(t) < \psi(t)$. By the induction assumption there is a map $g_i : B^r \times K^{(i)} \to \Sigma^r Y$ with $L_{B^r}(g_i) < \lambda_i$, $L_{K^{(i)}}(g_i)(t) < b\psi(t)$ and a homotopy h^i_t joining $1\hat{\times} f$ with g_i and leaving $\partial B^n \times K^{(i)}$ fixed. We consider an arbitrary $(i+1)$-simplex $\Delta \subset K$ and consider the map $1\hat{\times} f$ restricted on $B^r \times \Delta$. Then the homotopy h^i_t on $B^r \times \partial\Delta$ can be extended to a homotopy $\bar{h}^i_t$ of $B^r \times \Delta$ by means of an isotopy $H : B^r \times \Delta \times I \to B^r \times \Delta \cup B^r \times \partial\Delta \times I$, generated by an isotopy $G : \Delta \times I \to \Delta \times \{0\} \cup \partial\Delta \times I$, which joins the embedding Δ as $\Delta \times \{0\}$ to a homemomorphism between Δ and $\Delta \times \{0\} \cup \partial\Delta \times I$. We may assume that H is c-Lipschitz where c depends on i only.

Let $f_1 = \bar{h}^i_1 : B^r \times \Delta \to \Sigma^n Y$. Note that $L_{B^n}(f_1) < cL(f)(d(\Delta, x_0))$ and $L_{B^r}(f_1|_{\partial\Delta}) = L_{B^r}(g_i) < \lambda_i$. Note that Δ is c'-Lipschitz equivalent to B^{i+1}. Then by the Parametrized Trading Lemma we obtain a map $g^\Delta : B^{i+1} \times \Delta \times B^r \to \Sigma^{r+i+1} Y$ with

$$L_{B^r}(g^\Delta|_{x\times\Delta}) \le cc'\lambda_i \text{ and } L_{B^r}(g^\Delta|_{B^{i+1}\times y}) \le c'cL(f)(d(\Delta, x_0)) \le \bar{a}\psi(d(\Delta, x_0)).$$

The union of maps g^Δ defines a map $g : B^{i+1} \times B^r \times K \to \Sigma^{r+i+1} Y$. We note that

$$L_{B^{i+1}\times B^r}(g) \le cc'\lambda_i \text{ and } L_K(g)(t) \le L_{B^r\times K}(g)(t) + L_{B^{i+1}\times K}(g)(t) \le \bar{a}\psi(t) + b\psi(t).$$

Let $u : B^{r+i+1} \to B^{i+1} \times B^r$ be a natural homeomorphism. We define $\lambda = cc'\lambda_i L(u)$ and $a = (\bar{a} + b)L(u)$.

Let h_t^{Δ} be a homotopy supplied by the Trading Lemma. The union of all these homotopies defines a homotopy $\tilde{h}_t$ joining $\bar{h}_1^i$ with g. Then we define h_t as $\bar{h}_t^i$ followed by $\tilde{h}_t$. It is easy to check that the boundary condition holds for h_t. $\square$

Corollary. *Let K and $f : K \to Y$ be as above with a Lipschitz function ψ. Then there is n and a Lipschitz map $q : B^n_{\psi(t)} \times K \to \Sigma^n Y$ such that q is homotopic to $(1 \hat{\times} f) \circ (\frac{1}{\psi(t)} \times 1_K)$ by means of a homotopy h_t with $h_t(\partial B^n_{\psi(t)} \times K) = y_0$.*

Proof. Let $g : B^n \times K \to Y$ be given by Lemma 13. We define $q = g \circ (\frac{1}{\psi(t)} \times 1_K)$. We show that q is Lipschitz. First we consider a pair of points $(x_1, y), (x_2, y) \in B^n_{\psi} \times K \subset \mathbf{R}^n \times K$. Note that

$$d(q(x_1, y), q(x_2, y)) \leq d(g(x_1/\psi(d_K(y, x_0)), y), g(x_2/\psi(d_K(y, x_0)), y)) \leq$$

$$L_K(g)(d_K(y, x_0))\|x_1/\psi(d_K(y, x_0)) - x_2/\psi(d_K(y, x_0))\|.$$

Then by the conclusion of Lemma 13 we have

$$d(q(x_1, y), q(x_2, y)) \leq a\psi(d_K(y, x_0))\|\tfrac{x_1}{\psi(d_K(y,x_0))} - \tfrac{x_2}{\psi(d_K(y,x_0))}\| \leq a\|x_1 - x_2\|.$$

Next, we consider a pair of type (x, y_1) and (x, y_2). Then

$$d(q(x, y_1), q(x, y_2)) = d(g(\frac{x}{\psi(d_K(y_1, x_0))}, y_1), g(\frac{x}{\psi(d_K(y_2, x_0))}, y_2)) \leq$$

$$d(g(\frac{x}{\psi(d_K(y_1, x_0))}, y_1), g(\frac{x}{\psi(d_K(y_1, x_0))}, y_2)) + d(g(\frac{x}{\psi(d_K(y_1, x_0))}, y_2), g(\frac{x}{\psi(d_K(y_2, x_0))}, y_2))$$

$$\leq \lambda d(y_1, y_2) + a\psi(d_K(y_2, x_0))\|x/\psi(d_K(y_1, x_0)) - x/\psi(d_K(y_2, x_0))\| \leq (\lambda + a\mu)d(y_1, y_2)$$

where μ is a Lipschitz constant for ψ. $\square$

Proposition 1. *Let Y be an m-connected GLEC compact space. Then there is a constant C_Y such that for every monotone function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ with $\psi \geq C_Y$ there is a monotone function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ such that every map $f : K \to Y$ of an m-dimensional complex with $L(f)(t) \leq \phi(t)$ is homotopic to a constant map by means a homotopy $H : K \times [0, 1] \to Y$ with $L(H)(t) \leq \psi(t)$.*

Proof. We use induction on m.

If $m = 0$, then there is a homotopy H with $L(H) \leq diam(Y)$. We take $C^0_Y = 2diam(Y)$.

Let us assume that the proposition holds for m with a constant C^m_Y. We prove it for $m+1$ assuming that Y is $(m+1)$-connected.

Let $\mu(\lambda)$ be the function from Lemma 1 for $X = \partial(\Delta \times I)$ where Δ is the $(m+1)$-dimensional standard unit simplex. We define $C^{m+1}_Y = \mu(C^m_Y)$. Let ψ and $f : K \to Y$ be given. Let μ^{-1} denote the inverse of the function μ. We consider a function $\gamma(t) =$

$\max\{\mu^{-1}(\psi(t)), C_Y^m\}$ and apply the proposition for this function in the dimension m. Then we will obtain a function $\phi_{m,\gamma}$. We take ϕ equal to $\phi_{m,\gamma}$.

Let $f : K \to Y$ with $L(f)(t) \le \phi(t)$. Let $H' : K^{(m)} \times I \to Y$ be a homotopy to the constant map with $L(H') < \gamma$ given by the induction assumption. We consider an arbitrary $(m+1)$-dimensional simplex Δ in K. Then we extend the restriction $H'|_{\partial\Delta\times I}$ over $\Delta \times \{0\}$ by f and over $\Delta \times \{1\}$ by the constant map. We obtain a map $g_\Delta : \partial(\Delta \times I) \to Y$. Note that

$$L(g_\Delta) \le \max\{\gamma(d(\Delta, x_0)), \phi(d(\Delta, x_0))\} \le \gamma(d(\Delta, x_0)).$$

Since Y is $(m+1)$-connected, the map g_Δ is null homotopic, and hence by Lemma 1 there is a $\mu(L(g_\Delta))$-Lipschitz extension $\bar{g}_\Delta : \Delta \times I \to Y$. Hence

$$L(\bar{g}_\Delta) \le \max\{\psi(d(\Delta, x_0)), \mu(C_Y^m)\} \le \psi(d(\Delta, x_0)).$$

We define the homotopy H as the union of maps $\bar{g}_\Delta$ for all Δ. Then the inequality $L(H)(t) \le \psi(t)$ holds. □

Proposition 2. *Let M be a contractible m-manifold. Then there is a Lipschitz map of degree one $\alpha : M \times \mathbf{R} \setminus C \to S^m$ for some compact subset $C \subset M \times \mathbf{R}$.*

Proof. Let B_r be a small m-ball in M which admits a Lipschitz relative homeomorphism $\gamma : (B_r, \partial B_r) \to (B_r/\partial B_r, \{\partial B_r\}) \to (S^m, y_0)$ to the unit sphere. Let $C = B \times [-1, 1]$. We define

$$\alpha(x,t) = \begin{cases} \gamma(x) & if \quad x \in B_r \quad and \quad t \ge 1 \\ y_0 & otherwise \end{cases}$$

□

Let (X, d) be a metric space with a base point x_0. Then we denote $\|x\|_d = d(x, x_0)$. Generally $x_0 \notin X$. If this is the case then it means that $X \subset Y$, $x_0 \in Y$ and a metric on X is induced from the metric on Y.

Let (K, d) be an asymptotic polyhedron and let $u_K : K \to K_U$ denote the identity map which switch from the metric d to the uniform metric. Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be a tending to infinity function, we say that K has the growth less than f if $\|x\|_d < f(\|x\|_U)$. We say that K has a *quadratic growth* if $\|x\|_d < \|x\|_U^2$.

Lemma 14. *Let X be the universal cover of an aspherical manifold M_0 , $p : X \to M_0$, supplied with the induced metric, and let* as dim $X = n$. *Then given a tending to infinity function $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ there are an n-dimensional asymptotic polyhedron N, a proper 1-Lipschitz map $\phi : X \to N$ and a proper homotopy inverse map $\gamma : N \to X$ with $L(\gamma \circ u_K^{-1})(t) < \beta(t)$ where $K \subset N$ and the closure of $N \setminus K$ is compact.*

Proof. 1). First we show that for the projection $\phi_{\mathcal{U}} : X \to N(\mathcal{U})$ to the (uniform) nerve of a uniformly bounded cover $\mathcal{U}$ admits a Lipschitz homotopy inverse $\gamma : N(\mathcal{U}) \to X$. We define γ on the vertices v by taking points $\gamma(v) \in \phi_{\mathcal{U}}^{-1}(v)$ and define γ on every edge

$[v, u]$ by sending it with a constant speed along a minimal geodesic joining $\gamma(v)$ and $\gamma(u)$. Assume that a λ_i-Lipschitz map $\gamma : N^{(i)}(\mathcal{U}) \to X$ is already constructed for $i \geq 1$. For every $(i+1)$-dimensional simplex Δ we consider the map $f = p \circ \gamma|_{\Delta}$. Since this map is null homotopic and $\pi_{i+1}(M_0) = 0$, Lemma 3 implies that there exists λ_{i+1}-Lipschitz extension ξ_{Δ}, $\lambda_{i+1} = \mu(\lambda_i)$. Then there is a unique lift ξ' of ξ that extends $\gamma|_{\partial\Delta}$. Since p is a local isometry, the lift ξ' is also λ_{i+1}-Lipschitz. The union of all ξ' defines a λ_{i+1}-Lipschitz map on the $(i+1)$-skeleton.

2). For a homotopy lift $\gamma : N \to X$ from Lemma 8 we define a function $\psi_\gamma(t) = L(\gamma|_{B_t(x_0)})$. Because of the above remark we always may assume that the function ψ_γ takes finite values. Moreover for any given tending to infinity function $\beta : X \to \mathbf{R}_+$ there is an approximation $\phi : X \to N$ by an asymptotic n-dimensional polyhedron and the proper homotopy inverse map $\gamma : N \to X$ with $\psi_\gamma < \beta$.

3). For every approximation by an asymptotic polyhedron one can find better approximation by an asymptotic polyhedron of the quadratic growth. To do that we enlarge the sets C_i in the proof of Lemma 7 to such that C_i contains the set $\phi_i^{-1}(B_{2^i}(x_0))$ where $\phi_i : X \to N(\mathcal{U}_i)$ is the projection to the nerve with the uniform metric. Then for every $x \in B_{2^{i+1}}(x_0) \setminus B_{2^i}(x_0)$ we have $\|x\|_U = \Sigma_{k=1}^i 2^i + c$ where c is the length of a segment in $B_{2^{i+1}}(x_0) \setminus B_{2^i}(x_0)$. We note that

$$\|x\|_d \leq \Sigma_{k=1}^i 2^i 2^i + 2^{i+1}c \leq (\Sigma_{k=1}^i 2^i + c)^2 = \|x\|_U^2.$$

By 3) we can find an arbitrarily close approximation $\phi : X \to N$ by asymptotic polyhedra of the quadratic growth. According to 2) we can take such approximation that there exists a homotopy inverse map $\gamma : N \to X$ with $L(\gamma)(t) < \beta(\sqrt{t})$. Then

$$L(\gamma \circ u_K^{-1})(t) = L(\gamma \circ u_K^{-1}|_{B_t^U(x_0)}) \leq L(\gamma|_{u_K^{-1}(B_t^U(x_0))}) \leq L(\gamma|_{B_{t^2}(x_0)}) = L(\gamma)(t^2) < \beta(t)$$

□

DEFINITION. An m-manifold M is integrally *hypereuclidean* if there is a proper Lipschitz map $f : M \to \mathbf{R}^m$ of degree one.

Theorem 3. *Let M be the universal cover of a closed aspherical m-manifold with* $\operatorname{asdim}(M) = k$. *Then the product $M \times R^n$ is integrally hypereuclidean for* $n = \frac{(k+2)(k+1)-(m+1)m}{2} + 2$.

Proof. First we prove the theorem with some nonspecified n.

Let $\alpha : (M \times \mathbf{R}) \setminus C \to S^m$ as in Proposition 2 and let $L(\alpha) < a$. Let $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ be a Lipschitz map of a sublinear growth. Let $\phi : M \times \mathbf{R} \to N$ be an approximation by an asipsymptotic polyhedron of quadratic growth as in Lemma 14 with a homotopy inverse map $\gamma : N \to M \times \mathbf{R}$ with $L(\gamma)(t) \leq \max\{\psi(\sqrt{t})/a, c\}$. We can find a subpolyhedron $K \subset L$ with a relatively compact complement such that $L(\gamma|_K)(t) \leq \psi(\sqrt{t})/a$ and the map $\alpha \circ \gamma|_K \circ \phi|_{\gamma^{-1}(K)}$ induces the same element $[\alpha|_{\gamma^{-1}(K)}]$ in the m-dimensional cohomology. Let $f = \alpha \circ \gamma|_K \to S^m$ where K considered as a uniform complex. Then $L(f)(t) \leq \psi(t)$.

Then by the Corollary there is n and a Lipschitz map $q : B^n_{\psi(t)} \times K \to S^{n+m}$ such that q is homotopic to $(1\hat{\times} f) \circ (\frac{1}{t} \times 1_K)$ by means of a homotopy h_t with $h_t(\partial B^n_t \times K) = y_0$.

We define a map $\xi : \mathbf{R}^n \to \mathbf{R}^n$ as $\xi(x) = x/\psi(\|x\|)$ and consider the map $g = q \circ (\xi|_{B^n_t} \times u_K) : B^n_t \times K \to S^{n+m}$ where K is taken with its metric of an asymptotic polyhedron. It is easy to see that the map g has the Lipschitz constant vanishing at infinity. We extend the map g to a map $\bar{g} : \mathbf{R}^n \times K \to S^{n+m}$ by the constant map. Then the map

$$w = \bar{g} \circ (1_{\mathbf{R}^n} \times \phi|) : \mathbf{R}^n \times ((M \times \mathbf{R}) \setminus C) \to S^{n+m}$$

can be continuously extended to $\bar{w}$ by the constant map over $(\mathbf{R}^n \setminus B^n_R) \times C$ for a large enough R. Then $\bar{w}$ has the gradient tending to zero at infinity. Therefore w can be extended over the Higson corona $\nu(\mathbf{R}^{n+1} \times M)$. We note that the map $\bar{w}$ has degree one. Then it follows that $\mathbf{R}^{n+1} \times M$ is integrally hypereuclidean [Ro1].

The value n can be chosen as it is claimed after a careful application of Proposition 1 and the argument of Lemma 13. $\square$

We note here in the presence of the equality $\operatorname{asdim} M = \operatorname{asdim}_{\mathbf{Z}} M$ we would get $n \le m + 3$.

§6 Application to the Novikov Conjecture

There are several different coarse type conditions on a manifold M that apply the Novikov Higher Signature Conjecture. A comparison of some of them was attempted in [Dr1]. We will use here the following refinement of Carlson-Pedersen's conditions [C-P], [F-W] which is due to Ferry-Weinberger:

Ferry-Weinberger Descent Principle [D-F]. *Let $p : X \to M$ be the universal cover of a closed aspherical n-manifold with the fundamental group $\Gamma = \pi_1(M)$. Suppose that X admits a Γ-equivariant metrizable compactification $\bar{X} = X \cup \partial X$ such that the boundary homomorphism $H^{lf}_n(X; \mathbf{Q}) \to H^{st}_{n-1}(\partial X; \mathbf{Q})$ is a Γ-equivariant split monomorphism for the Steenrod homology. Then the Novikov conjecture holds for M.*

G. Yu proved the coarse Baum-Connes conjecture for all geometrically finite groups with finite asymptotic dimension [Yu1]. In particular his result implies the Novikov conjecture for these groups [Ro2]. Here we give an alternative prove of the Novikov conjecture based on the Ferry-Weinberger Descent Principle.

Theorem 4. *Suppose that the fundamental group $\Gamma = \pi_1(M)$ of a closed aspherical manifold M has a finite asymptotic dimension as a metric space with the word metric. Then the Novikov conjecture holds for M.*

We note that an action of a group Γ by isometries on a metric space X can be extended over the Higson corona.

DEFINITION. An n-manifold X with an isometric action of a group Γ on it is called *equivariantly hypereuclidean* if its Higson corona νX admits a map $f : \nu X \to S^{n-1}$ of the degree one which induces a Γ-equivariant homomorphism of the cohomologies. Here we assume that the action on S^{n-1} is trivial. The degree of f is the degree of the homomorphism

$$\mathbf{Z} \to H^{n-1}(S^{n-1}) \xrightarrow{f^*} \check{H}^{n-1}(\nu X) \xrightarrow{\delta} H_c^n(X) = \mathbf{Z}.$$

The proof of Theorem 4 is based on the following two lemmas.

Lemma 15. *Suppose that the universal cover X of a closed aspherical m-manifold M is equivariantly hypereuclidean. Then the Novikov conjecture holds for M.*

Proof. Let $f : X \to \mathbf{R}^m$ be a proper Lipschitz map of degree one such that the extension of f to the map $\bar{f}$ of the Higson corona to S^{m-1} is equivariant in the cohomologies. Consider the γ-equivariant commutative diagram induced by this extension and the boundary maps

$$\begin{array}{ccc}
\check{H}^{m-1}(\nu X; \mathbf{Q}) & \xrightarrow{\delta} & H_c^m(X; \mathbf{Q}) \\
{\scriptstyle \bar{f}^*}\uparrow & & {\scriptstyle f^*}\uparrow \\
H^{m-1}(S^{m-1}; \mathbf{Q}) & \xrightarrow{\tilde{\delta}} & H_c^m(\mathbf{R}^m; \mathbf{Q})
\end{array}$$

Then the homomorphism $\delta \circ \bar{f}^*$ is an equivariant isomorphism. Therefore the homomorphism δ is a Γ-equivariant split epimorphism.

Then by Shchepin spectral theorem one can find a Γ-equivariant metrizable compactification $\bar{Y} = Y \cup \partial Y$ such that the boundary homomorphism $\bar{\delta} : \check{H}^{m-1}(\partial Y; \mathbf{Q}) \to H_c^m(Y; \mathbf{Q})$ is a Γ-equivariant split monomorphism (see the proof of Lemma 9.3 in [Dr1]). Then the boundary homomorphism for the Steenrod homology

$$\bar{\partial} : H_m^{lf}(Y; \mathbf{Q}) \to H_{m-1}^{st}(\partial Y; \mathbf{Q})$$

is a Γ-equivariant split monomorphism. Then the Ferry-Weinberger Descent principle completes the proof. $\square$

Lemma 16. *Let $\Gamma = \pi_1(M)$ be the fundamental group of an aspherical m-manifold and let X be its universal cover. Assume that* asdim $X < \infty$. *Then there is a number N such that $X \times \mathbf{R}^N$ is Γ-equivariantly hypereuclidean*

Proof. We fix an isometry $h : X \to X$ that preserves the orientation. We consider a Lipschitz map $\alpha : X \times \mathbf{R} \setminus C \to S^m$ from Proposition 2. Note that $\alpha|_{X \times \mathbf{R} \setminus C'}$ is Lipschitz homotopic to $\alpha \circ (h \times 1)|_{X \times \mathbf{R} \setminus C'}$ where C' is a compact set in $X \times \mathbf{R}$ that contains C and $h^{-1}(C)$. The proof of Theorem 3 produces a map of the degree one $w : \mathbf{R}^n \times (X \times \mathbf{R} \setminus C') \to S^{n+m}$ extendable over the Higson corona and a Lipschitz homotopy H, connecting $1 \hat{\times} \alpha$ with w. Then $H \circ (1_{\mathbf{R}^n} \times h \times 1_{\mathbf{R}})$ is a Lipschitz homotopy connecting $1 \hat{\times} \alpha \circ (h \times 1)$ with $w \circ (1_{\mathbf{R}^n} \times h \times 1_{\mathbf{R}})$. Then the maps w and $w \circ (1_{\mathbf{R}^n} \times h \times 1_{\mathbf{R}})$ are Lipschitz homotopic. Then like in the proof of Theorem 3 we can show that there is a Lipschitz homotopy H' between $1_{\mathbf{R}^k} \hat{\times} w$ and $1_{\mathbf{R}^k} \hat{\times} w \circ (1_{\mathbf{R}^n} \times h \times 1_{\mathbf{R}})$ for some k such that for every $t \in [0,1]$ it is extendable over the Higson corona. This homotopy defines a map

$$G : \mathbf{R}^{k+n} \times (X \times \mathbf{R} \setminus C') \to (S^{k+n+m})^I_\lambda$$

to the space of λ-Lipschitz mappings of the interval I to the sphere S^{k+n+m} for some λ. Then it follows that G is extendible over the Higson corona. Therefore G defines a homotopy $\bar{G} : \nu(X \times \mathbf{R}^{n+k+1}) \times I \to S^{k+n+m}$ between the extensions $\tilde{w}$ and $\tilde{w}_h$ of the maps $1\hat{\times}w$ and $1\hat{\times}(w \circ (1 \times h)$ over the Higson corona. This means that the map $\tilde{w}$ is cohomologically h-invariant.

Let $h_i : X \to X$, $i = 1, \dots, l$ be isometries on X defined by the generators of Γ. We apply the above argument to each of h_i and take maximal k. Then $N = k + n + 1$. $\square$

Since the action of $\mathbf{Z}^N$ on $X \times \mathbf{R}^N$ induces the trivial action on $\nu(X \times \mathbf{R}^N)$, the product $X \times \mathbf{R}^N$ is Γ'-equivariantly hypereuclidean for $\Gamma' = \Gamma \times \mathbf{Z}^N$. Then by Lemma 15 the Novikov Conjecture holds for Γ' and hence for Γ.

REMARK 5. One can extend the above proof of the Novikov Conjecture for all groups Γ with finite $B\Gamma$ and finite asymptotic dimension by playing with the assembly map diagram and using the following trick.

Davis' trick. *Let K be a finite aspherical complex with* $\operatorname{asdim} \pi_1(K) < \infty$. *Then K is a retract of a closed aspherical manifold M with* $\operatorname{asdim} \pi_1(M) < \infty$.

Proof. We embed K in a euclidean space of large enough dimension and consider its regular neighborhood N. We may assume that the inclusion homomorphism $\pi_1(\partial N) \to \pi_1(N) = \pi_1(K)$ is an isomorphism. We take a flag triangulation on N and consider a rightangled Coxeter group G with the complex ∂N as the nerve of G. Thus G is generated by vertices of ∂N and all relations are of the type $uv = vu$, one for every edge $[u, v]$ in ∂N. Next we consider the Davis complex and replace all the chambers there by copies of N attached to the boundaries of chambers along ∂N. Then we obtain an open manifold U on which the group G acts properly and cocompactly. The manifold U can be defined as the qoutient space $G \times N/\sim$ by the relation $(a, x) \sim (b, x)$ if and only if $a^{-1}b$ lies in the group G_Δ generated by the vertices of the only simplex Δ that contains x in its interior. Then the orbit space U/G is homeomorphic to N and the projection onto it is the retraction. Let G' be a torsion free subgroup of G of a finite index. We define $M = U/G'$. Then the projection of M onto $U/G = N$ is a retraction. Therefore K is a retract of M. On the other hand one can present U as the connected sum of infinitely many copies of N [Dav]. This presentation implies that U is aspherical and $\pi_1(U)$ is the infinite free product $*\pi_1(K)$. Since the action of G' is free, the orbit space M is an aspherical manifold. Since G' is of a finite index, the manifold M is closed.

Let $\tilde{M}$ be the universal cover of M. Then $\tilde{M}$ is the universal cover for U. By a recent computations [Be] we have that $\operatorname{asdim} *_{i=1}^{\infty} \pi_1(K) = \max\{1, \operatorname{asdim} \pi_1(K)\} < \infty$. Note that the space U is coarsely isomorphic to the group G. By the result of [D-J] all Coxeter groups are asymptotically finite dimensional. Thus the space $\tilde{M}$ is presented as a fiber product with a finite dimensional base U (asymptotically) and with a finite dimensional fiber $\pi_1(U)$. Then the standard dimension theory technique applied in the coarse situation implies that $\operatorname{asdim} \tilde{M} < \infty$. $\square$

The proof of Remark 5 follows from the diagram:

$$\begin{array}{ccc} H_*(B\pi_1(K);\mathbf{Q}) & \xrightarrow{A_K} & L_*(\mathbf{Z}[\pi_1(K)])\otimes\mathbf{Q} \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ H_*(B\pi_1(M);\mathbf{Q}) & \xrightarrow{A_M} & L_*(\mathbf{Z}[\pi_1(M)])\otimes\mathbf{Q} \end{array}$$

It is well-known that the Novikov Conjecture for $\pi_1(K)$ is equivalent to the monicity of the assembly map A_K [F-R-R]. Since the inclusion $i : K \to M$ admits a retraction, it follows that i^* is a monomorphism. Since A_K is a left divisor of a monomorphism $A_M \circ i_*$, it is a monomorphism itself.

In the conclusion of this section we formulate a problem which can be considered as a stable version of the Higson conjecture [Ro1]. We recall that the original unstable version is incorrect even in the cases where the Novikov conjecture is true [Ke].

Let

$$p : \overline{X \times \mathbf{R}} = X \times \mathbf{R} \cup \nu(X \times \mathbf{R}) \to X \times \mathbf{R} \cup \Sigma\nu X = \Sigma\bar{X}$$

be the natural projection of compactifications of $X \times \mathbf{R}$. We define a group

$$SH^i(X) = \lim_{\to}\{\check{H}^i(\bar{X}) \xrightarrow{\Sigma} \check{H}^{i+1}(\Sigma\bar{X}) \xrightarrow{p^*} \check{H}^i(\overline{X \times \mathbf{R}}) \to \dots\}.$$

Problem. *Is $SH^*(X) = 0$ for the universal cover X of a closed aspherical manifold M?*

We note that the affirmative answer to this problem implies the Novikov conjecture for M.

§7 Asymptotically Euclidean spaces and the Gromov-Lawson conjecture

We recall that a metric space X has a *bounded geometry* if for every $\epsilon > 0$ and $R > 0$ there is a number C such that for every $x \in X$ every ϵ-net in the ball $B_R(x)$ contains no more than C points. A finitely generated group with the word metric or the universal cover of a finite complex with the lifted metric are typical examples of metric spaces of bounded geometry. Every metric space X of bounded geometry for arbitrarily large λ admits a uniformly bounded cover of a finite multiplicity with the Lebesgue number $\geq \lambda$. Like in the proof of Lemma 7 we can conclude that every metric space of bounded geometry has the following property:

(**) *For arbitrarily large λ there is $n = n(\lambda)$, n-dimensional uniform simplicial complex K_λ of size λ, supplied with l_∞-metric, and a 1-Lipschitz uniformly cobounded map $\phi_\lambda : X \to K_\lambda$.*

A map of a metric space to a simplicial complex $f : X \to K$ is called uniformly cobounded if there is a constant C such that $\operatorname{diam} \phi^{-1}(\Delta) \leq C$ for all simplices Δ.

The property (**) can be organized in so called an anti-Čech approximation of X. An *anti-Čech approximation* of X is a direct system of finite dimensional uniform simplicial complexes with 1-Lipschitz bonding maps $\{K_{\lambda_n}, g_n^m\}$ with $\lambda_n \to \infty$ together with a sequence of 1-Lipschitz uniformly cobounded maps $\phi_{\lambda_n} : X \to K_{\lambda_n}$ such that $\phi_{\lambda_n} = g_n^m \phi_{\lambda_m}$ for all $n > m$. We call an anti-Čech approximation *euclidean* if all complexes K_{λ_n} are given the euclidean metric.

We say that X admits a Čech approximation by a certain class $\mathcal{C}$ of asymptotic polyhedra if given a proper function $f : X \to \mathbf{R}_+$ there is an asymptotic polyhedron $N \in \mathcal{C}$ and an 1-Lipschitz map $\phi : X \to N$ with $\operatorname{diam} \phi^{-1}(\Delta) \leq \min_{x \in \Delta} f(x)$ for every simplex Δ lying in $N \setminus C$ for some compact set C. We recall that Lemma 7 asserts that if $\operatorname{asdim} X < \infty$ then X can be approximated by n-dimensional asymptotic polyhedra K with the property that

1) every simplex in K is either the standard Δ_{2^i} of size 2^i for some i or it isomorphic to the simplex $\tilde{\Delta}_{2^i}$ with edges of two types: of the length 2^i and 2^{i+1};

2) if two simplices in K have nonempty intersection then they either of the same type or they have the types Δ_{2^i} and $\tilde{\Delta}_{2^i}$ or they have the types Δ_{2^i} and $\tilde{\Delta}_{2^{i-1}}$.

In the finite dimensional case we always may assume that these polyhedra are euclidean.

The class of euclidean asymptotic polyhedra with the above property we denote by $\mathcal{C}_0$. The argument of Lemma 7 shows that in infinite dimensional case if X admits an euclidean anti-Čech approximation, then it can be approximated by asymptotic polyhedra from the class $\mathcal{C}_0$. Moreover the argument of Lemma 8 also can be extended to the infinite dimensional case. As the result we have the following

Proposition 4. *Every coarsely euclidean uniformly contractible proper metric space X admits an approximation by euclidean asymptotic polyhedra $\phi : X \to K$ from the class $\mathcal{C}_0$ which admit proper homotopy inverse maps $\gamma : X \to K$.*

DEFINITION. A metric space that admits a (Cech) approximation by euclidean asymptotic polyhedra is called *asymptotically euclidean.*

We note that if X admits an approximation by euclidean asymptotic polyhedra then it admits an approximation by asymptotic polyhedra from the class $\mathcal{C}_0$.

We recall that a *uniform embedding* in the coarse sense is a map $q : X \to Y$ between metric spaces such that

$$\rho_1(d_X(x,y)) \leq d_Y(q(x), q(y)) \leq \rho_2(d_X(x,y))$$

for two tending to infinity functions $\rho_1, \rho_2 : \mathbf{R}_+ \to \mathbf{R}_+$ and all $x, y \in X$.

Proposition 5. *Every euclidean asymptotic polyhedron $K \in \mathcal{C}_0$ admits a uniform embedding in the Hilbert space l_2.*

Proof. The uniform simplicial complex K_U can be naturally embedded into the infinite simplex $\Delta^\infty = \{(t_1, t_2, \dots) \in l_2 \mid \Sigma t_i = 1, t_i \geq 0\}$. Let Δ be a simplex in K and let $l(\Delta)$ denote the minimum of the length of edges in Δ. We define a map $l : K \to \mathbf{R}_+$ by the rule: $l(x) = \min\{l(\Delta) \mid x \in \Delta\}$. Let $\rho : K \to \mathbf{R}_+$ be an 1-Lipschitz function tending to infinity and $l/2 \le \rho \le l$. We define a map $q : K \to l_2$ by the formula $q(x) = \rho(x)u(x)$ and show that it is a uniform embedding.

First we show that q is 3-Lipschitz. Since a metric on K is geodesic, it suffices to show that q is 3-Lipschitz on every simplex. For every two points x and y from the same simplex Δ we have

$$\|q(x)-q(y)\| = \|\rho(x)u(x)-\rho(y)u(y)\| \le \|\rho(x)u(y)-\rho(y)u(y)\|+\|\rho(x)u(x)-\rho(x)u(y)\| \le$$
$$d_K(x,y)\|u(y)\| + \rho(x)\|u(x)-u(y)\| \le d_K(x,y) + \rho(x)d_U(x,y).$$

Since the projection of the simplices Δ_l and $\tilde{\Delta}_l$ onto the standard unit simplex is $(1/l)$-Lipschitz, we have

$$d_U(x,y) \le (2/l(\Delta))d_K(x,y) \le (2/l(x))d_K(x,y) \le (2/\rho(x))d_K(x,y).$$

Then $\|q(x) - q(y)\| \le 3d_K(x,y)$.

Now assume that q^{-1} is not uniform. It means that there is a number $c > 0$ and a sequence of pairs of points $\{(x_k, y_k)\}$ in K such that $d_K(x_k, y_k) \to \infty$ and $\|q(x_k) - q(y_k)\| \le c$. We may assume that $\rho(x_k) \le \rho(y_k)$ for all k . Then $c/\rho(x_k) \ge \|u(x_k) - u(y_k)\|$. Since $\rho(x_k) \to \infty$, for large enough k we have the inequality $\|u(x_k) - u(y_k)\| \ge (1/2)d_U(x_k, y_k)$. Moreover we may assume that x_k and y_k lie in two simplices Δ_1, Δ_2 in K with $\Delta_1 \cap \Delta_2 \ne \emptyset$. Then $2\max\{l(\Delta_1), l(\Delta_2)\}d_U(x_k, y_k) \ge d_K(x_k, y_k)$. Therefore $4\max\{l(x), l(y)\}d_U(x_k, y_k) \ge d_K(x_k, y_k)$. Then $16\rho(x)d_U(x_k, y_k) \ge d_K(x_k, y_k)$ and hence we get a contradiction: $32c \ge 16\rho(x)d_U(x_k, y_k) \ge d_K(x_k, y_k)$ for all k. □

Clearly every discrete subspace of the Hilbert space is asymptotically euclidean.

Theorem 5. *For an asymptotically euclidean geometrically finite groups Γ the Roe's index map is a split monomorphism.*

Proof. Let $X = E\Gamma$ and let $\phi : X \to K$ and $\gamma : K \to X$ as in Proposition 4. According to Proposition 5 the asymptotic polyhedron K is uniformly embeddable into l_2. By the theorem of Yu [Yu2] the coarse Baum-Connes conjecture holds for K, i.e. the Roe index map $A_K : K_*^(K) \to K_*(C * K)$ is an isomorphism. Then the following diagram implies the splitting of A_X:

$$\begin{array}{ccc}
K_*(X) & \xrightarrow{A_X} & K_*(C^*X) \\
\phi_*\downarrow & & \phi'\downarrow \\
K_*(K) & \xrightarrow{A_K} & K_*(C^*K) \\
\gamma_*\downarrow & & \\
K_*(X) & &
\end{array}$$

□

It is known [Ro1] that the monomorphism version of the coarse Baum-Connes conjecture implies the Gromov-Lawson conjecture. This together with Theorem 5 gives the following

Corollary. *The Gromov-Lawson conjecture holds for manifolds with asymptotically euclidean universal covers.*

In view of the main result of the next section an affirmative answer to the following question would illiminate an approach to the Novikov Conjecture via hypersphericity (see a remark on page 35 of [G4]).

Problem. *Is every hypereuclidean (hyperspherical) manifold asymptotically euclidean?*

§8 Expanders are not asymptotically euclidean

Let X be a finite graph, we denote by V the set of vertices and by E the set of edges in X. We will identify the graph X with its set of vertices V. Every graph is a metric space with respect to the natural metric where every edge has the length one. For a subset $A \subset X$ we define the boundary $\partial A = \{x \in X \setminus A \mid dist(x, A) = 1\}$. Let $|A|$ denote the cardinality of A.

DEFINITION [Lu]. An *expander* with a conductance number c and the degree d is an infinite sequence of finite graphs $\{X_n\}$ with the degree d such that $|X_n|$ tends to infinity and for every $A \subset X_n$ with $|A| \le |X_n|/2$ there is the inequality $|\partial A| \ge c|A|$.

Let X be a finite graph, we denote by K all nonordered pairs of distinct points in X. For every nonconstant map $f : X \to l_2$ to the Hilbert space we introduce the number

$$D_f^2 = \frac{\frac{1}{|K|}\Sigma_{\{x,y\}\in K}\|f(x) - f(y)\|^2}{\frac{1}{|E|}\Sigma_{\{x,y\}\in E}\|f(x) - f(y)\|^2}.$$

If X is a graph with the degree d and with $|X| = n$, then $|K| = n(n-1)/2$ and $|E| = dn/2$.

The following Lemma is well-known. It can be derrived from [M, Proposition 3]. It also can be obtain from the equality

$$\lambda_1(X) = \inf\{\frac{\|df\|^2}{\|f\|^2} \mid \Sigma f(x) = 0\}$$

for the first positive eigenvalue of the Laplacian on X and the Cheeger's inequality (see Proposition 4.2.3 in [Lu]).

Lemma 17. *Let $\{X_n\}$ be an expander. Then there is a constant c_0 such that $D_{f_n}^2 \le c_0$ for all n for every sequence $f_n : X_n \to l_2$ of maps to the Hilbert space.*

Corollary. *For every sequence of 1-Lipschitz maps $f_n : X_n \to l_2$ there is the inequality $\frac{1}{|K_n|}\Sigma_{K_n}\|f(x) - f(y)\|^2 \le c_0$ for every n.*

Proof. In the case of 1-Lipschitz map we have

$$\frac{1}{|E|}\Sigma_{\{x,y\}\in E}\|f(x) - f(y)\|^2 \le 1.$$

□

We say that a metric space X contains an expander $\{X_n\}$ if there is a sequence of isometric embeddings $X_n \to X$.

Theorem 6. *An asymptotically euclidean metric space X cannot contain an expander.*

Proof. Assume the contrary, i.e. assume that X contains an expander of degree d. We enumerate all the graphs in the expander by a subsequence of $\mathbf{N}$ in such a way that $|X_n| = n$. Let $x_0 \in X$ be a base point. Let $n(t) = \min\{n \mid X_n \cap \partial B_t(x_0) \neq \emptyset\}$. We note that $n(t)$ tends to infinity as t approaches infinity. We consider a proper function $f : X \to \mathbf{R}_+$ such that $f(x) < \log_d n(\|x\|)/4$. Then like in the proof of Lemma 7 one can take an 1-Lipschitz map $\phi : X \to N$ to an asymptotic polyhedron such that for every $r > 0$ there is a compact set $C_r \subset X$ with $diam(\phi^{-1}(B_r(y))) \le f(z)$ for every $z \in \phi^{-1}(B_r(y)) \setminus C_r$. Since X is asymptotically euclidean, we may assume that N is an euclidean asymptotic polyhedron. By Proposition 5 there is a uniform embedding $g : N \to l_2$. By the definition, there is a monotone function $\rho : \mathbf{R}_+ \to \mathbf{R}_+$ tending to infinity such that

$$\rho(d_N(x', y')) \le \|g(x') - g(y')\| \le d_N(x', y')$$

for all $x', y' \in N$.

We take r such that $\rho(r) > 2\sqrt{c_0}$ and we take n sufficiently large. Denote by $K'_n \subset K_n$ the set of pairs $\{x, y\}$ in $X_n \setminus C_r$ with $d(x, y) \ge \log_d(n/4)$. Since the degree of X_n is d, we have

$$|B_k(v) \cap X_n| \le 1 + d + d^2 + \cdots + d^k \le 2d^k.$$

Therefore at least $n(n - 2\frac{n}{4})/2 = n^2/4$ such pairs are contained in X_n. Then for sufficiently large n we have at least $n^2/8$ such pairs in $X_n \setminus C_r$. Show that every such pair $\{x, y\}$ in $X_n \setminus C_r$ satisfies the inequality $d_N(\phi(x), \phi(y)) > r$. Indeed, if we assume that $d_N(\phi(x), \phi(y)) \le r$, then we will obtain that

$$diam\phi^{-1}(B_r(\phi(y))) \ge d(x, y) \ge \log_d(n/4).$$

On the other hand, we have that

$$diam\phi^{-1}(B_r(\phi(y))) \le f(x) < \log_d(n(\|x\|)/4).$$

Since $n \ge n(\|x\|)$, we obtain the contradiction

$$diam\phi^{-1}(B_r(\phi(y))) < \log_d(n/4).$$

Let $f = g \circ \phi$. By the Corollary of Lemma 17 we have

$$c_0 \ge \frac{1}{|K_n|}\Sigma_{K_n}\|f(x) - f(y)\|^2 \ge \frac{1}{|K_n|}\Sigma_{K'_n}\|f(x) - f(y)\|^2 \ge \frac{n^2}{8|K_n|}\min_{K'_n}\|f(x) - f(y)\|^2 =$$

$$\frac{n^2}{4n(n-1)}\|f(\tilde{x}) - f(\tilde{y})\|^2 \ge \frac{1}{4}\|f(\tilde{x}) - f(\tilde{y})\|^2 = \frac{1}{4}\|g(\phi(\tilde{x})) - g(\phi(\tilde{y}))\|^2 \ge$$

$$\frac{1}{4}\rho^2(d_N(\phi(\tilde{x}), \phi(\tilde{y}))) \ge \frac{1}{4}\rho^2(r) > c_0.$$

This contradiction completes the proof. □

REMARK 6. One can show (see [M]) that for every $p \geq 1$ the number

$$D_f^p = \frac{\frac{1}{|K|}\Sigma_{\{x,y\}\in K}||f(x) - f(y)||^p}{\frac{1}{|E|}\Sigma_{\{x,y\}\in E}||f(x) - f(y)||^p} \leq c_0$$

is bounded from above by the same number c_0 for every map $f : X \to l_p$ to the Banach space l_p. This implies in particular that an expander is not uniformly embeddable in l_p for any p.

Problem. *Given $p > 0$, does the Novikov conjecture holds for a group Γ which admits a uniform embedding in l_p ?*

It is known that the answer is 'yes' for $p = 2$ [Yu2] and for $p = 1$ [D-G-L-Y].

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