# Hopf Algebras with Integrals

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## 1. Introduction

Integrals on Hopf algebras are motivated by the Haar integrals on compact groups. Hopf algebras possessing an integral are also called co-Frobenius Hopf algebras. Integrals on Hopf algebras were studied by several authors [13, 12, 9, 1, 11]. The main results obtained are the equivalence between the the existence of an integral and the existence of a finite dimensional injective comodule, and the uniqueness of the integral (up to a constant).

In this work, we study the injective hull of simple comodule over a Hopf algebra with integral. Among them there is an interesting class of splitting comodules, that are simple comodules, which coinside with their injective hull. These comodules split in any comodule.

The notion of splitting comodules comes from the representation theory of Lie super algebras, due to V. Kac [7]. Irreducible representations of a simple Lie super algebras split into two classes, the class of typical representations and the class of non-typical representations. One of the characterization of typical representations is that they split in any representation.

Using the integral, we give a criteria for a simple comodule over a Hopf algebra to be a splitting comodule. As an application, we classify simple comodules over the Hopf algebra, associated to a Hecke symmetry of birank $(1,1)$ [5], i.e., quantum (super) groups of type $A_{0,0}$.

## 2. Co-Frobenius Coalgebras and Hopf algebras

We work over a field $\mathbf{k}$. Every tensor product if not explicitly indicated means tensor product over $\mathbf{k}$.

Let $C$ be a coalgebra and $M$ be a right $C$-comodule, the coaction of $C$ on $M$ is denoted by $\rho$, $\rho : M \longrightarrow M \otimes C$, $\rho(v) = v_0 \otimes v_1$. Let $C^* := \mathrm{Hom}_{\mathbf{k}}(C, \mathbf{k})$ be the dual of $C$. Then $C^*$ is a algebra, acting on $M$ in the following way

$$\phi \rightharpoonup v := v_0 \phi(v_1).$$

Analogously, if $\lambda : N \longrightarrow C \otimes N$, $\lambda(v) = v_1 \otimes v_2$, is a left $C$-comodule, then it is a right $C^*$-module through the action $v \leftharpoonup \phi := \phi(v_1) \otimes v_2$.

Thus, we have a fuctor from the category of right (left) $C$-comodules into the category of left (right) $C^*$-modules, which is full, faithfull and exact. The following statement is due to Doi.

**2.1** *[1] Let $M$ be a right (left) $C$-comodule, which is finite dimensional. Then $M$ is injective (projective) if and only if it is injective (projective) as left (right) $C^*$-module.*

A $C^*$-module may not be a $C$-comodule by the above correspondence. Those $C^*$-module induced from $C$-comodules are called rational modules. Each left $C^*$-module $M$ contains a unique maximal rational submodule, denoted by ${}_{\mathrm{rat}}M$. Analogously, for a right $C^*$-module $M$, its rational submodule will be denoted by $M_{\mathrm{rat}}$.

Let $M$ be a $C$-comodule. The map $\rho : M \longrightarrow M \otimes C$ induce a map $M^* \otimes M \longrightarrow C$, which can be considered as a coalgebra homomorphism or a morphism of $C$-comodules, where $C$ coacts on $M^* \otimes M$ on the second tensor component. In the latter case, we shall use the notation $(M^*) \otimes M$ to indicate that $C$ coacts only on $M$. The image of $M^* \otimes M$ is called the coefficient space of $M$, denoted by $\mathcal{C}f(M)$.

Let $S$ be a simple (left or right) $C$-comodule. The fundamental theorem of comodule (saying that a finite generated comodule is finite dimensional) implies that $S$ is finite dimensional. Let $\mathcal{D} := \mathrm{End}^C(S)$. Then, by Schur lemma, $\mathcal{D}$ is a division algebra over $\mathbf{k}$ and $V$ is a vector space over $\mathcal{D}$. We have $\mathcal{C}f(S) \cong V^* \otimes_{\mathcal{D}} V$, as coalgebras [2]. Let $\{M_\alpha | \alpha \in \mathcal{A}\}$ be the set of all simple $C$-comodules. We define $\mathcal{D}_\alpha$ as above and $m_\alpha := \dim_{\mathcal{D}_\alpha}(M_\alpha)$, $d_\alpha^2 := \dim_{\mathbf{k}} \mathcal{D}_\alpha$. Note that $d_\alpha$ are positive integers.

If $\mathbf{k}$ is algebraically closed then $\mathcal{D}_\alpha = \mathbf{k}$, $\forall \alpha$, i.e., $d_\alpha = 1$.

By definition, the socle of a comodule $M$ is the sum of all its simple subcomodule. The sum is plainly direct, it is denoted by $\sigma(M)$. The injective hull (or cover) of $M$, is an injective comodule $\mathcal{J}(M)$ to gether with a morphism $M \longrightarrow \mathcal{J}(M)$ inducing an isomorphism $\sigma(M) \longrightarrow \sigma(\mathcal{J}(M))$. It is easy to see that the injective hull of a simple comodule, if it exists, is indecomposable. The following results are due ot J. Green.

**2.2** *[2] (i) The injective hull of any comodule exists uniquely.*

*(ii) $C$ itself decomposes in to indecomposable injective subcomodules as follows*

$$C \cong \bigoplus_{\alpha \in \mathcal{A}} \mathcal{J}(M_\alpha)^{\oplus m_\alpha}. \tag{1}$$

*(iii) If $C = \bigoplus_{\lambda \in \mathcal{L}} C_\lambda$ is another decomposition then for each $\alpha \in \mathcal{A}$, the set $\{\lambda \in \mathcal{L} | C_\lambda \cong \mathcal{J}(M_\alpha)\}$ contains exactly $m_\alpha$ elements.*

A bilinear form $b$ on $C$ is called balanced if, for all $\phi \in C^*$,

$$b(x \leftharpoonup \phi, y) = b(x, \phi \rightharpoonup y).$$

Balanced bilinear forms on $C$ are in 1-1 corespondence with right $C^*$ comodules homomorphism $r : C \longrightarrow C^*$ by the formula $r(x)(y) = b(x,y)$ and in 1-1 corespondece with left $C^*$ comodules homomorphism $r : C \longrightarrow C^*$ by the formula $l(x)(y) = b(y,x)$.

A coalgebra is called left (right) co-Frobenius if there exist a left (right) monomorphism of $C^*$-modules $C \longrightarrow C^*$. The following results are due to B. Lin.

**2.3** *[9] If $C$ is a* left *co-Frobenius coalgebra then:*

*(i) The injective cover of every finite dimensional* right *$C$-comodule is finite dimensional.*

*(ii) Every injective* right *$C$-comodule is projective.*

*(iii) ${}_{\mathrm{rat}}C^*$ is dense in $C^*$.*

In the next section we shall need the following result.

**Lemma 2.4** *Let $M$ be a $C$-comodule of finite dimension. Then $M$ is projective (resp. injective) if and only if it is projective (resp. injective) in the category of finite dimensional $C$-comodules.*

*Proof.* It is sufficient to show the "if" part. Assume that $M$ is projective in the category of finite dimensional $C$-comodule. Consider a diagram

$$\begin{array}{ccc} & & M \\ & \exists \swarrow & \downarrow \pi \\ N & \xrightarrow{\nu} & P \longrightarrow 0. \end{array}$$

By replacing $P$ with $\mathrm{Im}(P)$ and $N$ with $\nu^{-1}(\mathrm{Im}(P))$, we can assume that $\pi$ is surjective, thus $P$ is finite dimensional. Let $\mathcal{P}$ be a basis of $P$ and $\mathcal{N}$ be a set of elements of $N$ such that $\nu(\mathcal{N}) = \mathcal{P}$. the submodule $N_1$ of $N$, generated by $\mathcal{N}$ is finite dimensional and we have $\nu(N_1) = P$. Hence, by assumption, there exists a morphism $\mu : M \longrightarrow N_1$: $\nu \circ \mu = \pi$.

Assume that $M$ is injective in the category of finite dimensional $C$-comodule. Consider the diagram

$$\begin{array}{ccc} 0 \longrightarrow P & \xrightarrow{\nu} & N \\ \pi \downarrow & \swarrow \exists & \\ M. & & \end{array}$$

By replacing $P, N$ with $P/\mathrm{Ker}(\pi), N/\mathrm{Ker}(\pi)$, we can assume that $\pi$ injective. Hence $P$ has finite dimension. If $N/P$ is finite dimensional, we are done by the assumption on $M$. Otherwise, consider the set

$$\mathcal{A} := \{(N_\alpha, \mu_\alpha) | N_\alpha \supset P, \mu_\alpha : N_\lambda \longrightarrow M, \mu_\lambda \circ \nu = \pi\}.$$

Define an order on this set, setting $\alpha \prec \beta$ iff $N_\alpha \subset N_\beta$ and $\mu_\beta|_{N_\beta} = \mu_\alpha$. The chain condition is satisfied hence there exists a maximal element, say, $N_1$. Since any submodule of $N$, containing $P$ and having finited dimension is contained in $\mathcal{A}$, $N_1$ is strictly bigger then $P$. Were $N_1 \neq N$, repeating the above process, we would get a submodule $N_1$, $N_2 \succ N_1$, which contradicts the maximality of $N_1$. Hence $N_1 = N$.∎

Let $H$ be a Hopf algebra. Then $\mathbf{k}$ is a left (right) $H$-comodule be means of the unit map. A left (right) integral on $H$ is an $H$-comodule morphism $H \longrightarrow \mathbf{k}$, where $H$ is considered as left (right) comodule on itself by means of the copoduct. Let $\int_r$ (resp. $\int_l$) denoted a left (resp. right) integral on $H$, then we have

$$a_1 \int_l (a_2) = \int_l (a), \tag{2}$$

$$\int_r (a_1)\, a_2 = \int_r (a), \tag{3}$$

$\forall a \in H$.

We need the following information on the integrals.

**2.5** *Let $H$ be a Hopf algebra. The following conditions are equivalent*
*(i) $H$ possesses a left integral.*
*(ii) $H$ is left co-Frobenius as a coalgebra.*
*(iii) $H$ possesses a right integral.*
*(iv) $H$ is right co-Frobenius as a coalgebra.*
*(v) The injective hull of every left comodule is finite dimension.*
*(vi) The injective hull of every right comodule is finite dimension.*
*(vii) $H$ possesses a finite dimension injective left comodule.*
*(viii) $H$ possesses a finite dimension injective right comodule.*
*(ix) $H^*_{\mathrm{rat}}$ is dense in $H^*$.*
*(x) ${}_{\mathrm{rat}}H^*$ is dense in $H^*$.*
*(xi) Every injective left $H$-comodule is projective.*
*(xii) Every injective right $H$-comodule is projective.*

The first 6 conditions are due to Larson-Sweedler-Sullivan, the conditions (vii)-(xii) follow from 2.3

Define bilinear form $b$: $b(x,y) := \int_l (xS(y))$. Using the identity

$$h_1 \int_l (h_2 S(g)) = \int_l (hS(g_1))\, g_2 \tag{4}$$

which follows immediately form the definition of $\int_l$, we can easily show that $b$ is balanced. The folowing results are due to Ḑ. Ştefan.

**2.6** *[11] Let $H$ be a Hopf algebra with integral. Then the following facts hold.*
*(i) The bilinear form $b$ is non-degenerate.*
*(ii) For any finite dimensional $H$-comodule,* $\dim_{\mathbf{k}}(\mathrm{Hom}^H(H,M)) = \dim_{\mathbf{k}} M$.

In particular, we have

(i) the antipode is injective, and

(ii) there exists $h$ such that $\int_l (h) \neq 0$. Since $\int_l \circ S$ satisfies (3), it is a right integral on $H$.

Assume for a moment that the field $\mathbf{k}$ is algebraically closed. Let $R$ be the radical of $H$, i.e. $R = \oplus \mathcal{C}f(M_\alpha)$, where $\{M_\alpha, \alpha \in \mathcal{A}\}$ is the set of all simple left (or right) $H$-comodules. As we have seen in the previous subsection, $\mathcal{C}f(M_\alpha) \cong M(m_\alpha)^*$, $m_\alpha := \dim_{\mathbf{k}}(M_\alpha)$. Fix idempotents $\{e_{\alpha,i} | \alpha \in \mathcal{A}, 1 \leq i \leq m_\alpha\}$ of the algebras $M(m_\alpha)$. They can be considered as linear functional on $R$ by defining

$$e_{\alpha,i}(\mathcal{C}f(M_\beta)) = 0, \text{ whenever } \alpha \neq \beta.$$

A theorem of Sweeder-Sullivan [12], stating that there exists a coalgebra projection $H \longrightarrow R$, implies $e_{\alpha,i}$ can be extended on the whole $H$ and that

$$H \leftharpoonup e_{\alpha,i} \quad (\text{ resp. } e_{\alpha,i} \rightharpoonup H) \text{ is a right (resp. left) } H\text{-comodules.}$$

Consequently, we have a decompostion

$$H \cong \bigoplus_{\substack{\alpha\in\mathcal{A},\\ 1\le i\le m_\alpha}} H \leftharpoonup e_{\alpha,i} \left( \text{resp. } \cong \bigoplus_{\substack{\alpha\in\mathcal{A},\\ 1\le i\le m_\alpha}} e_{\alpha,i} \rightharpoonup H \right)$$

as right (resp. left) $H$-comodules.

On the other hand, it is easy to see that $M_\alpha \subset H \leftharpoonup e_{\alpha,i}$ as right $H$-comodules. Thus, comparing with the decomposition in 2.2, we have:

**2.7** *Assume that the field* $\mathbf{k}$ *is algebraically closed. Then*

$$\mathcal{J}(M_\alpha) \cong H \leftharpoonup e_{\alpha,i} \tag{5}$$

*as right* $H$*-comodules.*

## 3. The Injective Hull of a Simple Comodule

We define a new product on $H$:

$$g * h := h_1 \int_l (h_2 S(g)) = \int_l (hS(g_1))\, g_2 \text{ (by (4))}.$$

Using 4 we can easily check that $*$ is associative. Denote $\check{H} := (H, *)$. Then $\check{H}$ is a (non-unital) algebra. Let $V$ be a right $H$-comodule. Then $V$ is a left $H$-module by means of the action

$$h * v := v_0 \int_l (v_1 S(h)).$$

The verification again uses (4). Denote $\check{V} := (V, *)$.

Let $f : V \longrightarrow W$ be a homomorphism of right $H$-comodules, i.e., $f(v)_0 \otimes f(v)_1 = f(v_0) \otimes v_1$. We have

$$h * f(v) = f(v)_0 \int_l (f(v)_1 S(h)) = f(v_0) \int_l (v_1 S(h)) = f(h * v).$$

Thus $f$ is a homomorphism of left $\check{H}$-modules. Conversely, if $f$ is a homomorphism $\check{V} \longrightarrow \check{W}$, then we have, for all $h \in H$,

$$f(v)_0 \int_l (f(v)_1 S(h)) = f(v_0) \int_l (v_1 S(h)).$$

By the non-degeneracy of the integral (2.6, (i)), we have

$$f(v)_0 \otimes f(v)_1 = f(v_0) \otimes v_1,$$

which means that $f$ is a homomorphism of right $H$-comodules. Thus we have

$$\mathrm{Hom}_{\check{H}}(\check{V}, \check{W}) = \mathrm{Hom}^H(V, W). \tag{6}$$

In particular we have proved

**Lemma 3.1** *If* $M$ *is a simple right* $H$*-comodule then* $\check{M}$ *is a simple left* $\check{H}$*-module.*

Let now $V$ be a cyclic $\check{H}$-module, that is, there exists $\bar{v} \in V$, such that $V$ is generated by $\bar{v}$. We want to define a coaction of $H$ on $V$. Let $v \in V$. Then there exists (not uniquely) $h \in \check{H}$, such that $v = h * \bar{v}$. Set $\hat{\delta}(v) := h_1 * \bar{v} \otimes h_2$. We show that $\hat{\delta}$ is independent of the choice of $\bar{v}$ and $h$, and that it is in fact a coaction of $H$ on $V$.

The fact that $\hat{\delta}$ does not change when $\bar{v}$ is replaced by $\tilde{v}$ is represented by the equation

$$h_1 * \bar{v} \otimes h_2 = (h * g)_1 * \tilde{v} \otimes (f * g)_2,$$

where $g * \tilde{v} = \bar{v}$. This equation follows immediately form the definition of $*$. The independence on the choice of $h$ means that, whenever $h * \bar{v} = 0$, we have $h_1 * \bar{v} \otimes h_2 = 0$. Indeed, we have

$$\int_l (h_2 S(g))\, h_1 * \bar{v} = (g * h) * \bar{v} = g * (h * \bar{v}) = 0,$$

for all $g \in H$. By the non-degeneracy of $\int$, we conclude that $h_1 * \bar{v} \otimes h_2 = 0$.

The coassociativity and counitary of $\hat{\delta}$ are also checked directly using (4). Moreover, denoting by $\hat{V}$ the resulting $H$-comodule, we also have $\check{\hat{V}} \cong V$.

Since simple modules are cyclic, we have

**Lemma 3.2** *Let $M$ be a simple $\check{H}$-module, then there exists a right $H$-comodule $\hat{M}$, such that $M \cong \check{\hat{M}}$.*

We also need another action of $\check{H}$ on a right comodule $V$ of $H$, given by

$$h \circ v := v_0 \int_l (hS(v_1))\,.$$

Indeed, we have

$$\begin{aligned} g \circ (h \circ v) &= g \circ v_0 \int_l (hS(v_1)) = v_0 \int_l (gS(v_1)) \int_l (hS(v_2)) \\ &= v_0 \int_l \left( hS\left( \int_l (gS(v_1))\, v_2 \right)\right) = v_0 \int_l (hS(g_1)) \int_l (g_2 S(v_1)) \\ &= (g * h) \circ v. \end{aligned}$$

It is again easy to checke that $\check{V} := (V, \circ)$ is a left $\check{H}$-module, and that, if $V$ is simple then $\check{V}$ is simple. Lemma 3.2 holds only in case the antipode is bijective. In fact, if $V$ is a cyclic $\check{H}$-module. Then we can define the following coaction of $H$ on $V$:

$$\dot{\delta}(v) := h_1 \circ \bar{v} \otimes S^{-2}(h_2),$$

where $\bar{v}$ is a generating element and $h * \bar{v} = v$. Denote by $\dot{V}$ the comodule of $H$ induced from $V$, we have $\check{\dot{V}} \cong V$.

Composing the operation $\check{}$ and $\hat{}$ on a simple comodule $V$ we obtain a new simple comodule $\hat{\check{V}}$, denoted by $V^\bullet$. The coaction of $H$ on $V^\bullet$ is given by

$$\delta^\bullet(v) = v_0 \int_l (h_1 S(v_1)) \otimes h_2,$$

with $h$ given by condition $v_0 \int_l (hS(v_1)) = v$.

Now, assume that $\mathbf{k}$ is algebraically closed. Let $M_\alpha$ be a simple right $H$-comodule. Then $\check{M}_\alpha$ is a simple left $\check{H}$-comodule. The action of $\check{H}$ on $\check{M}_\alpha$ induces a $\check{H}$-module homomorphism $\pi : \check{H} \longrightarrow \check{M}_\alpha \otimes (\check{M}_\alpha^*)$, where $\check{M}_\alpha \otimes (\check{M}_\alpha^*) \cong \check{M}_\alpha^{\oplus \dim_{\mathbf{k}} M_\alpha}$ as $\check{H}$-modules. The isomorphism (6) shows that $\pi$ is a homomorphism of $H$-comodules.

On the other hand, $H$ decomposes into the direct sum of its indecomposable injective subcomodules as in Lemma 2.2. For $h \in H \leftharpoondown e_{\beta,j}$, i.e., $h = e_{\beta,j}(g_1)g_2$ for some $g \in H$, and for $v \in \check{M}_\alpha$, we have

$$h * v = v_0 e_{\beta,j}(g_1) \int_l (g_2 S(v_1)) = v_0 \int_l (g(S(v_1))\, e_{\beta,i}(v_2).$$

Thus, if $\alpha \neq \beta$, $h * M_\alpha = 0$, therefore, $\pi(h) = 0$. Thus

$$\mathrm{Hom}^H(\mathcal{J}(M_\alpha), M_\alpha^\bullet) \neq 0. \tag{7}$$

According to 2.6,

$$\dim_{\mathbf{k}}(\mathrm{Hom}^H(H, M_\alpha^\bullet)) = \dim_{\mathbf{k}} M_\alpha^\bullet. \tag{8}$$

Since $H$ contains precisely $m_\alpha = \dim_{\mathbf{k}} M_\alpha$ copies of $\mathcal{J}(M_\alpha)$, we conclude that

$$\dim_{\mathbf{k}} \mathrm{Hom}^H(\mathcal{J}(M_\alpha), M_\alpha^\bullet) = 1, \tag{9}$$
$$\dim_{\mathbf{k}} \mathrm{Hom}^H(\mathcal{J}(M_\beta), M_\alpha^\bullet) = 0 \quad \text{if } \alpha \neq \beta. \tag{10}$$

**Theorem 3.3** *Let $H$ be a Hopf algebra with integral. Then for any simple comodules $M_\alpha, M_\beta$*

$$\dim_{\mathbf{k}} \mathrm{Hom}(\mathcal{J}(M_\alpha), M_\beta^\bullet) = \delta_\beta^\alpha d_\beta^2,$$

*where $d_\beta^2$ is the dimension over $\mathbf{k}$ of $\mathcal{D}_\beta = \mathrm{End}^H(M_\beta)$.*

*Proof.* The case $\mathbf{k}$ is algebraically closed is already proved. Assume that $\mathbf{k} \neq \overline{\mathbf{k}}$. Then $\mathcal{D}_\beta = \mathrm{End}^H(M_\beta)$ splits over $\overline{\mathbf{k}}$: $\mathcal{D}_\beta \otimes_{\mathbf{k}} \overline{\mathbf{k}} \cong \mathrm{M}_{\overline{\mathbf{k}}}(d_\beta)$. For the extension $\overline{H} := H \otimes_{\mathbf{k}} \overline{\mathbf{k}}$, the comodule $\overline{M_\beta} := M_\beta \otimes_{\mathbf{k}} \overline{\mathbf{k}}$ decomposes into $d_\beta$ exemplars of the simple $\overline{H}$-comodule $M'_\beta$. Since $\overline{\mathcal{J}(M_\beta)} := \mathcal{J}(M_\beta) \otimes_{\mathbf{k}} \overline{\mathbf{k}}$ remains a direct summand of $\overline{H}$, it is an injective $\overline{H}$-comodule. Therefore $\overline{\mathcal{J}(M_\beta)}$ is a direct sum of $d_\beta$ exemplars of $\mathcal{J}(M'_\beta)$. Since, for $\alpha \neq \beta$,

$$\mathrm{Hom}_{\overline{H}}(\mathcal{J}(M'_\alpha), M_\beta^{\bullet\prime}) = 0,$$

we have

$$\mathrm{Hom}_H(\mathcal{J}(M_\alpha), M_\beta^\bullet) = 0.$$

Therefore, by virtue of Equation (8) (which is valid on any field),

$$\dim_{\mathbf{k}} \mathrm{Hom}_H(\mathcal{J}(M_\alpha)^{\oplus m_\alpha}, M_\alpha) = \dim_{\mathbf{k}} M_\alpha = d_\alpha^2 m_\alpha.$$

Consequently

$$\dim_{\mathbf{k}} \mathrm{Hom}_H(\mathcal{J}(M_\alpha), M_\alpha) = d_\alpha^2.$$

∎

Let $M$ be a finite dimensional right $H$-comodule then $M^* := \mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$ is also a right comodule with the coaction give by the equation

$$\rho(\phi)(x) := \phi_0(x)\phi_1 = \phi(x_0)S(x_1), \quad x \in M, \phi \in M^*.$$

The map $\mathrm{ev} : M^* \otimes M \longrightarrow \mathbf{k}$, $\phi \otimes x \longmapsto \phi(x)$ is a morphism of $H$-comodules. The pair $(M^*, \mathrm{ev})$ is called left dual to $M$, it is defined uniquely upto isomorphism. There exists a monomorphism $\mathrm{db} : \mathbf{k} \longrightarrow S \otimes S^*$, defined by the conditions $(\mathrm{ev} \otimes \mathrm{id}_{S^*})(\mathrm{id}_{S^*} \otimes \mathrm{db}) = \mathrm{id}_{S^*}$ and $(\mathrm{id}_s \otimes \mathrm{ev})(\mathrm{db} \otimes \mathrm{id}_S) = \mathrm{id}_S$, which is also a comodule morphism. Dually, $(M, \mathrm{ev})$ is called the right dual to $M^*$.

Thus, we see that the left dual to a finite dimensional comodule always exists. If the antipoded is bijective then the right dual to any finite dimensional comodule also exists. We shall need the following isomorphism, given by manipulating the morphism ev and db: for any finite dimensional comodule $N$,

$$\mathrm{Hom}^H(M \otimes N, P) \cong \mathrm{Hom}^H(M, P \otimes N^*) \tag{11}$$
$$\mathrm{Hom}^H(M, N \otimes P) \cong \mathrm{Hom}^H(N^* \otimes M, P). \tag{12}$$

As an immediate corollary of Lemma 2.4 and these equations, we have

**Lemma 3.4** *Let $M$ be a finite dimensional comodule. Then we have:*

*(i) If $M^*$ is projective (resp. injective) then $M$ is injective (resp. projective).*

*(ii) If the antipode is bijective, then $M$ is injective (resp. projective) iff $M^*$ is projective (resp. injective).*

*Proof.* Equations (11) and (12) imply

$$\mathrm{Hom}^H(M^*, N^*) \cong \mathrm{Hom}^H(N, M). \tag{13}$$

Thus, if $M$ is projective (resp. injective) then $M^*$ is injective (resp. projective) in the category of finite dimensional comodules.∎

**Corollary 3.5** *Let $H$ be a Hopf algebra wiht integral. Assume that the antipode is injective. Then for any simple comodule $M$*

$$\mathcal{J}((M^\bullet)^*) \cong \mathcal{J}(M)^*.$$

**Lemma 3.6** *Assume that the Hopf algebra $H$ as a left-right integral. Then $V^\bullet \cong V^{**}$. If $H$ is moreover coquasitriangular then $V^\bullet \cong V$.*

*Proof.* Assume that $\int$ is a left-right integral. Thus we can define $V^\bullet$ as above. We want to show that

$$v_0 \int_l (h_1 S(v_1)) \otimes h_2 = c \cdot v_0 \otimes S^2(v_1),$$

for certain constant $c$, depending only on $\int$. By the non-degeneracy of integral, this equation is equivalent to

$$v_0\int(v_1S(h))\int(h_2S(g)) = c\cdot v_0\int(S^2(v_1)S(g)).$$

We have

$$\begin{aligned}&v_0\int(h_1S(v_1))\int(h_2S(g))\\&= v_0\int((g*h)S(v_1)) = (g*h)\circ v = g*(h\circ v) = g\circ v = v_0\int(gS(v_1)).\end{aligned}$$

By the uniqueness of integral, we can choose $c$ such that $c\cdot\int(S^2(v_1)S(g)) = \int(gS(v_1))$.

If $H$ is coquasitriangular then $V\cong V^{**}$. ∎

**Corollary 3.7** *With the assumption of Lemma 3.6, we have*

$$\mathcal{J}(M^*)\cong\mathcal{J}(M)^*.$$

**Remark 3.8** If the left and the right integrals do not coincide then in general, $M^\bullet\not\cong M^{**}$. An example is Sweedler's Hopf algebras, see, e.g., [8].

## 4. Splitting Comodules

Let $S$ be a simple comodule over $H$. $S$ is called splitting comodule, or typical comodule, if $S=\mathcal{J}(S)$. Since $\mathcal{J}(S)$ is injective and hence projective, we see that $S$ splits in any comodule. This explain the name splitting. The name typical was used by V. Kac for modules over a super Lie algebra [7].

By virtue of conditions in 2.5, if a Hopf algebra possesses a splitting comodule then it possesses a non-zero integral. The converse statement is not true. The aim of this section is to give a criteria for a simple comodule to be typical.

Let $M$ is a right $H$-comodule then the coaction of $H$ on $M^{**}$ – the double left dual to $M$ is given by (identifying $M^{**}$ with $M$ as vector spaces)

$$\rho_{M^{**}}(v) = x_0\otimes S^2(v_1).$$

**Lemma 4.1** *Let $M$ be a simple $H$-comodule. Then $M$ is splitting iff $M^*$ is splitting.*

*Proof.* Assume that $M$ is a typical. Then $M$ is injective. By a theorem of Doi [1], $M^*\otimes M$ is injective too. By definition of $M^*$, we have an epimorphism $\mathrm{ev}: M^*\otimes M\longrightarrow\mathbf{k}$. Among indecomposable injective subcomodule of $M^*\otimes M$ there exists one, say $J$, such that the restriction of ev on $J$ is not zero. On the other hand, since $J$ is indecomposable and injective, it should appear in the decomposition (1), and by Theorem 3.3, the only comodule with this property is $\mathcal{J}(\mathbf{k})$, the injective hull of $\mathbf{k}$. Thus we show that $M^* \otimes M$ contains $\mathcal{J}(\mathbf{k})$ as a subcomodule, consequently, it contains $\mathbf{k}$ as subcomodule, i.e. $\mathrm{Hom}^H(\mathbf{k}, M^* \otimes M) \neq 0$. According to (12), we have

$$\mathrm{Hom}^H(M^{**}, M) \neq 0.$$

Therefore $M^{**} \cong M$ and hence is splitting. Consequently $M^*$ is also splitting, by Lemma 3.4.

Assume now that $M^*$ is splitting. The discussion above shows that $M^{**}$ is also splitting and $M^{**} \cong M^{****}$. Since

$$\mathrm{Hom}^H(M^*, N^*) \cong \mathrm{Hom}^H(N, M),$$

we conclude that $S \cong S^{**}$. Thus $S$ is splitting.∎

**Theorem 4.2** *Let $M$ be a simple right $H$-comodule. Then $M$ is splitting if and only if the bilinear form $c$, $c(x,y) = \int_r (yS(x))$, is not identically zero on $\mathcal{C}f(M)$. In this case, $c$ is also non-degenerate on $\mathcal{C}f(M)$.*

*Proof.* "if".

For each $g \in H$, define a linear functional $\phi_g \in \mathcal{H}^*$: $\phi_g(h) := \int_r (hS(g))$. By assumption, there exist $g \in \mathcal{C}f(M)$ such that $\phi_g$ is not identically zero on$\mathcal{C}f(M)$. Since $M$ is simple, the right coideal generated by $g$ is isomorphic to $M$. Define a linear map $\eta = \eta_g$:

$$\eta : \mathcal{C}f(M) \longrightarrow M^{**}, \quad h \longmapsto g_1 \int_r (hS(g_2))\,, h \in M. \tag{14}$$

Since $\varepsilon(\eta(h)) = \phi_q(h)$, $\eta$ is not trivial.

We have the following identity, which is an immediately consequence of (3) and the injectivity of the antipode

$$\int_r (h_1 S(g))\, h_2 = S^2(g_1) \int_r (hS(g_2))\,. \tag{15}$$

It follows form (15) that $\eta$ is a morphism of $H$-comodules. Since $\eta$ is non-trivial on $\mathcal{C}f(M)$, which is a direct sum of copies of $M$, we conclude that $\eta$ should induce a morphism $M \longrightarrow M^{**}$, which is non-trival. Since $M$ is simple and $\dim_{\mathbf{k}} M = \dim_{\mathbf{k}} M^{**}$, this morphism is an isomorphism. As a consequence, $\mathcal{C}f(M^{**}) = S(\mathcal{C}f(M)) = \mathcal{C}f(M)$ and $\phi_g$ is $*$-invertible on $\mathcal{C}f(M)$. Let $\psi$ be the $*$-inverse to $\phi$, define on $\mathcal{C}f(M)$, thus $\phi(h_1)\psi(h_2) = \psi(h_1)\phi(h_2) = \varepsilon(h)$.

Let now $M \hookrightarrow N$ be an inclusion of $H$-comodules. Let $f : N \longrightarrow M$ be a linear projection on $M$. We define a new map $F : N \longrightarrow M$ as follows,

$$F(v) := f(v_0)_0 \psi(f(v_0)_1) \int_r (v_1 S(f(v_0)_2))\,.$$

$F$ is well defined by the assumption that $\mathrm{Im}(f) = M$, which implies $f(v_0)_1 \in \mathcal{C}f(M)$. For $v \in M$, $f(v) = v$, hence

$$F(v) = v_0 \psi(v_1) \int_r (v_3 S(v_2)) = v_0 \psi(v_1) q(v_2) = v.$$

Thus, $F$ is again a projection of $M$. We want to show that $F$ is a morphim of $H$-comodule, then we will be done.

By defintion of $\phi$, we can consider $F$ as a composition of the map $g : N \longrightarrow M^{**}$:

$$g(v) = f(v_0)_0 \int_r (v_1 S(f(v_0)_1),$$

and the morphism $\eta^{-1} : M^{**} \longrightarrow M$. Thus, it is sufficient to show that $g$ is a morphism of $H$-comodules, which means

$$f(v_0)_0 \int_r (v_1 S(f(v_0)_1) \otimes v_2 = f(v_0)_0 \int_r (v_1 S(f(v_0)_2) \otimes S^2(f(v_0)_1).$$

We have, according to (15),

$$\begin{aligned} \text{the left-hand side} &= f(v_0)_0 \otimes S^2(f(v_0)_1) \int_r (v_1 S(f(v_0)_2)) \\ &= \text{the right-hand side}. \end{aligned}$$

Therefore, $F$ is a morphism of $H$-comodules, consequently, $M$ is injective and $\mathcal{J}(M) = M$.

"only if" Assume now that $M$ is splitting, then, by Lemma 4.1, $M^*$ is also splitting, hence $M \otimes M^*$ is injective. By definition of $M^*$, there exists a monomorphism db : $\mathbf{k} \longrightarrow M \otimes M^*$, inducing a monomorphism $\mathcal{J}(\mathbf{k}) \hookrightarrow M \otimes M^*$. The latter inclusion induces the following inclusion

$$J(\mathbf{k}) \subset \mathcal{C}f(\mathcal{J}(\mathbf{k})) \hookrightarrow \mathcal{C}f(M \otimes M^*) = \mathcal{C}f(M) \cdot \mathcal{C}f(M^*).$$

Since the right integral does not vanish identically on $\mathcal{J}(\mathbf{k})$ (by 3.3), we conlude that the set $\int_r \left(a^i_j S(a^k_l)\right)$ is not identically zero, as $a^i_j S(a^k_l)$ span $\mathcal{C}f(M \otimes M^*)$. ∎

## 5. Simple Representations of The Quantum Groups $\mathrm{GL}_q(1|1)$

Let $V$ be a finite dimensional vector space over $\mathbf{k}$, a field of characteristic zero, and $R$ be a Hecke symmetry of birank $(1,1)$. We are interested in simple comodules of the Hopf algebra $H$, associated to $R$. The reader is refered to [5] for definitions and known results.

Let $E$ be the bialgebra associated to $R$. Then simple $E$-comodules are known from the birank of $R$. In particular, if birank of $R$ is $(1,1)$, simple $E$-comodules can be labeled by pairs $(m,n), m \geq 1, n \geq 0$ and $(0,0)$. On the other hand, simple $E$-comodules are also simple as $H$-comodules with the natural action induced from the inclusion $E \longrightarrow H$ [4].

Let $I_{m,n}$ be the simple $E$-comodules associated to the pair $(m,n)$. They are given by the following rule. $I_{n,0} = S_n$ is the $n$-th quantum symmetric tensor over $V$, $I_{1,n-1} = \Lambda_n$ is the $n$-th quantum anti-symmetric tensor over $V$, $I_{0,0} := \mathbf{k}$,

$$I_{p,q} \otimes I_{m,n} = I_{m+p,n+q} \oplus I_{m+p-1,n+q+1}, \tag{16}$$

for $m, p \geq 1, n, q \geq 0$.

The Koszul complex plays the central role in studying simple $E$-comodules. It is shown that, if $k-l\neq -\mathrm{rank}_q R$, then the sequence

$$\cdots \longrightarrow \Lambda_k\otimes S_l^* \xrightarrow{d_{k,l}} \Lambda_{k+1}\otimes S_{l+1}^* \xrightarrow{d_{k+1,l+1}} \Lambda_{k+2}\otimes S_{l+2}^* \longrightarrow \cdots$$

with the differential induced from the dual basis map $\mathrm{ev}:\mathbf{k}\longrightarrow V\otimes V^*$, is exact [3, 10].

We now show that $\mathrm{rank}_q R=0$. First, notice that, according to (16),

$$\begin{aligned}&\mathrm{Hom}^H(I_{p,q}, I_{m+p,n+q}\otimes {I_{m,n}}^*)=\mathbf{k}\\ &\mathrm{End}^H(I_{m+p,n+q}\otimes {I_{m,n}}^*)=\mathrm{End}^H(I_{m+p,n+q}\otimes I_{m,n})=\mathbf{k}^{\oplus 2}.\end{aligned}$$

Therefore, denoting $I_{-m,-n}:={I_{m,n}}^*$, we have

$$I_{p,q}\oplus I_{p+1,q-1}\subset I_{m+p,n+q}\otimes I_{-m,-n}, \tag{17}$$

and $I_{m+p,n+q}\otimes I_{-m,-n}$ does not contain any other simple $E$-comodule.

Assume that $\mathrm{rank}_q R\neq 0$. Then the complex

$$0\to\mathbf{k}\to I_{1,0}\otimes I_{-1,0}\to I_{1,1}\otimes I_{-2,0}\to\cdots \tag{18}$$

is exact. We have

$$\begin{aligned} I_{n,1}\otimes I_{1,m-1}\otimes I_{-m,0} &= (I_{n+1,m}\oplus I_{n,m+1})\otimes I_{-m,0}\\ &\supset {I_{n-m+1,m}}^{\oplus 2}\oplus I_{n-m,m+1}\oplus I_{n-m+2,m-1}. \end{aligned}\tag{19}$$

Thus, multiplying (18) with $I_{n,1}$, we have two sequences

$$\begin{array}{ccccccc} 0\to I_{n,1}\to & {I_{n,1}}^{\oplus 2}\oplus I_{n-1,2}\oplus I_{n+1,0} & & {I_{n-1,2}}^{\oplus 2}\oplus I_{n-2,3}\oplus I_{n,1} & \cdots\\ & \bigcap & & \bigcap & \\ 0\to I_{n,1}\to & I_{n,1}\otimes I_{1,0}\otimes I_{-1,0} & \to & I_{n,1}\otimes I_{1,1}\otimes I_{-2,0} & \to\cdots \end{array}\tag{20}$$

The exactness of the lower complex and the remark following (17) imply that $I_{n+1,0}=0$, contradiction. Thus we have $\mathrm{rank}_q R=0$. As a consequence, the Hopf algebra $H$ possesses an integral [5] and the formula for the integral in [6] implies that $I_{m,n}$ are all splitting, except for $I_{0,0}=\mathbf{k}$. Therefore, the inclusion in (17) is in fact an isomorphism, by means of the two isomorphism preceeding (17)

$$I_{p,q}\oplus I_{p+1,q-1}=I_{m+p,n+q}\otimes I_{-m,-n}. \tag{21}$$

On the other hand, $\mathrm{rank}_q R=0$ implies that all the sequences

$$\cdots\longrightarrow I_{1,k-1}\otimes I_{-l,0}\xrightarrow{d_{k,l}} I_{1,k}\otimes I_{-l-1,0}\xrightarrow{d_{k+1,l+1}}\otimes I_{-l-2,0}\longrightarrow\cdots \tag{22}$$

$$\cdots\longrightarrow I_{1,k}\otimes I_{-k-1,0}\xrightarrow{d_{l+1,k+1}}\otimes I_{-k-2,0}\xrightarrow{d_{k+2,l+2}}\otimes I_{-k-3,0}\longrightarrow\cdots, \tag{23}$$

are exact for $k>l\geq 0$.

Setting $I_{1-l,k-1}:=\mathrm{Im}d_{k,l}=\mathrm{Ker}d_{k+1,l+1}$, we have a short exact sequence

$$0\longrightarrow I_{1-l,k-1}\longrightarrow I_{1,k}\otimes I_{-l-1,0}\longrightarrow I_{-l,k}\longrightarrow 0 \tag{24}$$

For $l = 0$, the second and third term in (24) are injective, hence, so is the fourth term. Using induction we can easily show that $I_{-l,k}$ are injective for all $k > l \geq 0$. Similar discussion shows that $I_{-k,l} := \mathrm{Im}d_{l,k}$ are also injective for all $k > l \geq 0$. Thus, we have

$$I_{1,k} \otimes I_{-l-1,0} = I_{1-l,k-1} \oplus I_{-l,k} \tag{25}$$

$$I_{1,l} \otimes I_{-k-1,0} = I_{-k,l} \oplus I_{-k-1,l+1}. \tag{26}$$

Note that these modules may not be simple.

Denote, for $k > l \geq 0$, $I_{k,-l} := I_{k,-l}{}^*$, $I_{l,-k} := I_{l,-k}{}^*$.

The next step is to define the comodules $I_{-1,1}$ and $I_{1,-1}$.

Since $\mathrm{rank}_q R = 0$, $I_{1,0} \otimes I_{-1,0} = V \otimes V^*$ contains two examplars of $\mathbf{k}$ in its composition series. Let $M := (V \otimes V^*)/\mathbf{k}$, then the map $V \otimes V^* \longrightarrow \mathbf{k}$ factorizes through $\mathbf{k}$ to a map $M \longrightarrow \mathbf{k}$. Dualizing this we get a sequence $\mathbf{k} \longrightarrow \mathcal{M}^* \longrightarrow V \otimes V^*$. Since $\mathrm{Hom}^H(\mathbf{k}, I_{1,1} \otimes I_{-2,0}) \cong \mathrm{Hom}^H(I_{2,0}, I_{1,1}) = 0$, $\mathrm{Im}d_{1,1} \neq \mathbf{k}$. From (20), we see that $\mathrm{Ker}d_{2,2} \supsetneq \mathbf{k}$. Let $N := V \otimes V^*/\mathrm{Ker}d_{2,2}$. Then $N$ is a factor comodule of $M$, which is different from $\mathbf{k}$ and $M$. Therefore $V \otimes V^*$ contains at least 4 simple comodules in its composition series. It cannot be larger then 4, as on the left hand side of (19), there are 4 simple comodules.

Tensoring (18) with $I_{1,0}$, we see that there exists a simple comodule in $V \otimes V^*$, denoted by $I_{-1,1}$, such that $I_{-1,1} \otimes I_{1,0}$. Using induction we can show that

$$I_{-1,1} \otimes I_{m,n} = I_{m-1,n+1}, \quad m \geq 1, n \geq 0. \tag{27}$$

Since $V \otimes V^*$ is selfdual, the dual to $I_{-1,1}$ is also contained in $V \otimes V^*$. We have $I_{-1,1} \otimes I_{1,-1} = \mathbf{k}$, and

$$I_{1,-1} \otimes I_{m,n} = I_{m+1,n-1}, \quad m \geq 1, n \geq 0. \tag{28}$$

Denote $I_{-m,m} := I_{-1,1}{}^{\otimes m}$, $m \in \mathbb{Z}$. Thus, we have associated to each pair $(m, n)$ of integers a comodule $I_{m,n}$ of $H$. Note that $I_{-1,1}$ is not splitting, whence so are all its powers. Therefore $V \otimes V^*$ is the injective hull of $\mathbf{k}$. Tedious computations show that

$$I_{-1,1} \otimes I_{m,n} = I_{m-1,n+1}, m, n \in \mathbb{Z}. \tag{29}$$

From this isomorphism, it follows that the comodules $I_{m,n}$ are all simple. There are no more simple comodules, for each simple comodule should appear in the decomposition series of $V^{\otimes n} \otimes V^{*\otimes m}$, some $m, n \in \mathbb{N} \cap 0$.

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