NON-RATIONALITY OF A THREE-DIMENSIONAL FANO VARIETY OF INDEX 2 AND DEGREE 1.

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ABSTRACT. It is proved that a smooth three-dimensional Fano variety U of index 2 and degree 1 (the double cone over the Veronese surface) has only the following Mori structures: U itself, and a "two-dimensional family" of fibrations $U_l \to \mathbb{P}^1$ in del Pezzo surfaces of degree 1 arising from blow-ups of curves l of genus 1 and degree 1. In particular, $\mathrm{Bir}(U) = \mathrm{Aut}(U)$, U has no structures of conic bundle, and U is not rational.

1. PRELIMINARIES.

Let U be a non-singular three-dimensional Fano variety of index 2 and degree 1. It is often called "the double cone over the Veronese surface" by the reason given in section 3. We suppose

$$\mathrm{Pic}(U) = \mathbb{Z}H,$$

where H is the class of ample divisors with conditions $\dim |H| = 2$, $K_V = -2H$, and $H^3 = 1$. Let C be an effective curve of (arithmetical) genus 1 and degree 1, i.e., $p_a(C) = 1$ and $C \circ H = 1$, and l the class of C in $A^2(U)$. Then $A^2(U) = \mathbb{Z}l$, $[H]^2 = l$, and there exists a 2-dimensional family $\mathcal{P}$ of effective curves of this class. General elements of $\mathcal{P}$ are non-singular, but $\mathcal{P}$ contains a 1-dimensional sub-family of rational curves with nodes or cusps.

For any $C \in \mathcal{P}$, the linear system $|H - C|$ is a pencil consisting of del Pezzo surfaces of degree 1. Since C is the complete intersection of any 2 elements of $|H - C|$, the blow-up $\varphi_C : U_C \to U$ has a natural projection $\rho_C : U_C \to \mathbb{P}^1$, which is a Mori fibration in del Pezzo surfaces of degree 1. Let us note that if C has a node, the corresponding U_C has the simplest double point (given by the equation $x^2+y^2+z^2+w^2 = 0$ in some local coordinates), and if C has a cusp, U_C has a unique singular point of type $x^2 + y^2 + z^2 + w^3 = 0$.

We recall that, by definition, a birational map $\chi : V \dashrightarrow V'$ between two Mori fibrations $\rho : V \to S$ and $\rho' : V' \to S'$ defines the same Mori structure (or that V/S and V'/S' are birational over base), if

- in case $\dim S = \dim S' > 0$, there exists a birational map ψ such that $\rho' \circ \chi = \psi \circ \rho$,

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- in case $\dim S = \dim S' = 0$ (V and V' are $\mathbb{Q}$-Fano varieties), V and V' are isomorphic to each other (whence $\chi \in \mathrm{Bir}(V)$).

This determines an equivalence relation for Mori fibrations, and for any threefold X we define its set of Mori (fiber) structures $\mathcal{MS}(X)$ as

$$\mathcal{MS}(X) = \{\text{Mori fibrations birational to } X\} \,/\, \{\text{birational over base}\}$$

Remark that the set of Mori structures is a birational invariant. We are ready to formulate the main result.

Theorem 1.1. *Let U be a non-sungular Fano threefold of index 2 and degree 1. Then*

$$\mathcal{MS}(U) = \{U\} \cup \{U_C \to \mathbb{P}^1, \text{ where } C \in \mathcal{P}\}\,.$$

In particular, U has a unique model of Fano variety and has no structures of conic bundles, $\mathrm{Bir}(U) = \mathrm{Aut}(U)$, *and U is not rational. In general,* $\mathrm{Bir}(U) \cong \mathbb{Z}_2$.

Some attempts to study birational geometry of U were undertaken in [4], [7], and [2], using the maximal singularity method. In [4] it is shown that linear systems on U may have maximal singularities only at curves from $\mathcal{P}$, and some cases of maximal singularities over points are dealt. Note that $|H - C|$ for $C \in \mathcal{P}$ gives an example of a linear system without fixed components with a maximal singularity along C. The author of [7] tried to complete the work but failed in excluding maximal singurarities over singular points of curves from $\mathcal{P}$ (see [2] for details). In [2], the previous results were re-proved in a more regular way, and finally the following assertion was achieved: if a linear system $\mathcal{D}$ without fixed components has a maximal singularity at B_0, then either B_0 is a curve from $\mathcal{P}$, or B_0 is the singular point of some $C \in \mathcal{P}$ for the corresponding maximal valuation on U given by a weighted blow-up of type $(1, 1, N)$, $2 \leq N \leq 5$, in nodal case and $(1, 1, 2)$ in cuspidal case (theorem 3.1 in [2]).

In order to complete proving theorem 1.1, we have to exclude maximal singularities over points of U and, if $\mathcal{D}$ has a maximal singularity along $C \in \mathcal{P}$, jump onto $U_C/\mathbb{P}^1$. On U_C, one needs to show that the strict transform of $\mathcal{D}$ must not have super-maximal singularities, and then we are done.

The paper is organized as follows. In section 2, we recall some recent results of M.Kawakita that allow to simplify much the usual argumentation in the maximal singularities method. Section 3 contains the description of geometry of U and $U_C/\mathbb{P}^1$. Section 4 deals with cases $(1, 1, N)$ over points mentioned above. We exclude super-maximal singularities over non-sungular points of $U_C/\mathbb{P}^1$ in section 5, and over singular points in section 6. The last section contains the conclusion.

We assume that all required statements of the maximal singularities method (see [8] for reference) and the Sarkisov program ([1]) are known.

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2. Maximal singularities and weighted blow-ups.

Let $\mathcal{D}$ be a linear system without fixed components on a threefold V. We assume V to be in the Mori category. Suppose that $\mathcal{D}$ has a maximal singularity, i.e., there exists a discrete valuation $\mathfrak{v}$ centered at a point or a curve on V such that the log pair $K_V+\frac{1}{m}\mathcal{D}$ is not canonical with respect to $\mathfrak{v}$, where m is the adjunction threshold of $\mathcal{D}$. The argumentation of [1] shows that there exists a divisorial contraction $\varphi : V' \to V$ in the Mori category such that the exceptional divisor E' of φ also defines a maximal singularity of $\mathcal{D}$. The corresponding discrete valuation $\mathfrak{v}_{div}$ has two main properties. First, $\mathrm{Center}_V\, \mathfrak{v}_{div} \subset \mathrm{Center}_V\, \mathfrak{v}$. Second, we may suppose that the minimal discrepancy of the log pair $K_V+\frac{1}{m}\mathcal{D}$ corresponds to $\mathfrak{v}_{div}$. In other words, if $\psi : W \to V$ is a log resolution with exceptional divisors $\{E_1, E_2, \ldots\}$, and E_1 corresponds to $\mathfrak{v}_{div}$, then in the equation

$$K_W+\frac{1}{m}\psi_*^{-1}\mathcal{D}=\psi^*\left(K_V+\frac{1}{m}\mathcal{D}\right)+\sum a\left(W, K_V+\frac{1}{m}\mathcal{D}, E_i\right)E_i$$

we have $a(W, K_V+\frac{1}{m}\mathcal{D}, E_1) \le a(W, K_V+\frac{1}{m}\mathcal{D}, E_i)$ for all i. Brief explanations can be also found in section 2 of [2].

Recently the following remarkable results were achieved ([5], theorem 2.2, and [6], theorem 2.5):

Theorem 2.1. *Let X be the germ of a three-dimensional variety, and $\varphi : Y \to X$ is a divisorial contraction in the Mori category to a point $B_0 \in X$. Then:*
i) if B_0 is a smooth point of X, φ is the weighted blow-up with weights $(1, L, N)$ in suitable local coordinates $[x, y, z]$ on X, where $L \le N$ are coprime integers;
ii) if B_0 is a singular point of $X \in \mathbb{C}^4$ given by $xy+z^2+w^2=0$ in suitable local coordinates, φ is the weighted blow-up with $wt(x, y, z, w) = (1, 1, 1, 1)$;
iii) if B_0 is a singular point of $X \in \mathbb{C}^4$ given by $xy+z^3+w^2=0$ in suitable local coordinates, φ is the weighted blow-up with $wt(x, y, z, w) = (1, 1, 1, 1)$ or $wt(x, y, z, w) = (1, 5, 2, 3)$.

In other words, if $\mathcal{D}$ has a maximal singularity over a point $B_0 \in V$, we can always assume that there exists a maximal singularity over B_0 which is realized as a divisorial contraction in the Mori category, and if B_0 is one of the listed above, this contraction is inverse to the corresponding weighted blow-up.

The assertion *i)* of theorem 2.1 has the following interpretation. Let $\mathfrak{v}$ be any geometrical discrete valuation centered at a smooth point

$B_0 \in V$, and $\varphi_1 : V_1 \to V$ be the blow-up of B_0 with the exceptional divisor $E_1 \subset V_1$, $E_1 \cong \mathbb{P}^2$. Denote by $B_1 \subset E_1$ the center of $\mathfrak{v}$ on V_1. If B_1 is also a point, we blow up V_1 at B_1 with the exceptional divisor $E_2 \subset V_2$, and so on, until for some $L \geq 1$ we obtain B_L is a curve on $E_L \subset V_L$. In the maximal singularities method we can always suppose that B_L is a line on $E_L \cong \mathbb{P}^2$. Then blowing up B_L, we obtain $\varphi_{L+1} : V_{L+1} \to V_L$ with the exceptional divisor $E_{L+1} \cong \mathbb{F}_2$. Now B_{L+1} has to be a section of E_{L+1} that covers B_L, and we proceed with this way, each time blowing up sections of exceptional divisors that are ruled surfaces. Finally, we must stop at E_N which realizes $\mathfrak{v}$.

We introduce the oriented graph $\Gamma(\mathfrak{v})$ as follows. It consists of N vertices $\{1, 2, \ldots, N\}$, and there is an arrow $j \to i$, if $j > i$ and $B_{j-1} \subset E_i^{j-1}$, where leading indices mean the strict transform on the corresponding V_i's. Note that there are always arrows $i+1 \to i$ since $B_i \subset E_i$ by choosing the resolution of $\mathfrak{v}$.

The statement i) of theorem 2.1 means that $\Gamma(\mathfrak{v}_{div})$ is a chain, i.e., only arrows $i+1 \to i$ exist. Moreover, the numbers L and N are the same as in the theorem. Finally, for $i > L$, B_i does not intersect the minimal section of the ruled surface E_i.

From the last assertion we can deduice the following proposition:

LEMMA 2.2. *For $L < i < N$, we have $E_i \cong \mathbb{F}_{i+1-L}$, $\mathcal{N}_{E_i|V_i} = \mathcal{O}_{E_i}(-s_i - f_i)$, $E_{i-1}^i|_{E_i} = s_i$, and $\mathcal{N}_{B_i|V_i} \cong \mathcal{O}(-1) \oplus \mathcal{O}(i+1-L)$, where s_i and f_i are the classes of the minimal section and a fiber of E_i. Viewing B_i as a curve on E_i, we obtain $B_i \sim s_i + (i+1-L)f_i$.*

Proof. Direct calculation, starting from E_{L+1}. Use the fact that B_i does not intersect the minimal section of E_i. □

The valuation $\mathfrak{v}_{div}$ defines the numbers $\nu_i = \mathrm{mult}_{B_{i-1}} \mathcal{D}^{i-1}$, $i = 1, \ldots, N$. Since $K_V + \frac{1}{m}\mathcal{D}$ is not canonical with respect to $\mathfrak{v}$, we obtain the Nöther-Fano inequality in the form

$$\nu_1 + \nu_2 + \ldots + \nu_N > m(L+N).$$

Recall that m is the adjunction threshold. For example, if $\mathcal{D} \subset |nH|$ is a linear system on the double cone over the Veronses surface U, then $m = \frac{n}{2}$, and we have $\nu_1 + \ldots + \nu_N > \frac{n}{2}(L+N)$.

3. Geometric constructions.

Equation and construction of U. It is easy to observe that U can be realized as a hypersurface of degree 6 in the weighted ptojective space $\mathbb{P} = \mathbb{P}(1,1,1,2,3)$. Let $[p,x,y,z,w]$ be the coordinates in $\mathbb{P}$ with $wt(p,x,y,z,w) = (1,1,1,2,3)$. Since U does not pass through the singular points $(0,0,0,1,0)$ and $(0,0,0,0,1)$ of $\mathbb{P}$, the equation of U has to contain monomials w^2 and z^3. Suppose that $B_0 = (1,0,0,0,0)$, and B_0 is the singular point of $l_0 \in \mathcal{P}$. We assume that the "hyperplanes" $T = \{x = 0\}$ and $\{y = 0\}$ cut out l_0, and T is tangent to U at

B_0. To avoid complicating the notation, we keep $[x, y, z, w]$ as the corresponding coordinates in the affine part $\{p \neq 0\}$ of $\mathbb{P}$.

Suppose that B_0 is the node of l_0. Then the equation of U can be chosen in the form

$$w^2 + z^3 + z^2 f_2(x, y) + z f_4(x, y) + x + f_6(x, y) = 0,$$

where $\deg f_i \leq i$, $f_2(0,0) \neq 0$, $f_4(0,0) = f_6(0,0) = 0$, and the minimal degree of monomials in f_6 is not less than 2 (i.e., f_6 does not contain a linear part).

If B_0 is the cusp on l_0, we may assume the equation of U to be in the form

$$w^2 + z^3 + z f_4(x, y) + x + f_6(x, y) = 0,$$

where f_4 and f_6 satisfy the same requirements as before.

Being restricted to U, the projection $\mathbb{P} \dashrightarrow \mathbb{P}(1,1,1,2)$ gives a morphism of degree 2 $U \xrightarrow{2:1} \mathbb{P}(1,1,1,2)$ branched over a surface of degree 6. We can view $\mathbb{P}(1,1,1,2)$ as the cone in $\mathbb{P}^6$ over the Veronese surface in $\mathbb{P}^5$. This is why U is often called "the double cone over the Veronese surface".

Construction of the corresponding del Pezzo fibrations. Let $C \in \mathcal{P}$, and $V = U_C \to U$ the blow-up of C. Recall that C is the complete intersection of any two elements from $|H - C|$. The natural projection $\rho : V \to \mathbb{P}^1$ realizes V as a Mori fibrations on del Pezzo surfaces of degree 1. We see that V is always Gorenstein, all fibers of $\rho : V \to \mathbb{P}^1$ are irreducible and reduced, and

$$\mathrm{Pic}(V) = \mathbb{Z}[-K_V] \oplus \mathbb{Z}F,$$

where F is the class of a fiber. V is non-singular if C is non-singular, V has an ordinary double point in the fiber corresponding to $T \in |H - C|$ (as before, T is the tangent section of U at B_0) if C has a node, and V has a double point with a local equation $ab + c^2 + d^3 = 0$ if C has a cusp.

Another construction of $\rho : V \to \mathbb{P}^1$ can be achieved as follows (see [3], section 2.1, the case $n_1 = 0$, $n_2 = 1$, $n_3 = 2$.). Let $X = \mathbf{Proj}_{\mathbb{P}^1}\, \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$ with the natural projection $\pi : X \to \mathbb{P}^1$. Denote by M the class of the tautological bundle on X, by L the class of a fiber, by t_0 the class of minimally twisted sections (i.e., t_0 has an effective representative and $t_0 \circ M = 0$), and by p the class of a line in a fiber of $\pi : X \to \mathbb{P}^1$. Consider a hypersurface $Q \sim 2M - 2L$ which is fibered over $\mathbb{P}^1$ into non-degenerated two-dimensional quadric cones. Denote by t_b the section of $X/\mathbb{P}^1$ that consists of the vertices of the cones. The class of t_b is exactly t_0. Let R_Q be the restriction of some effective divisor $R \sim 3M$ to Q. Finally, take the double cover $\varphi : V \xrightarrow{2:1} Q$ branched over R_Q. We obtain a variety fibered into del Pezzo surfaces of degree 1 over $\mathbb{P}^1$. It is not very difficult to see that

$V \to \mathbb{P}^1$ is isomorphic to U_C for some $C \in \mathcal{P}$ under the suitable choice of R_Q, and $\rho = \pi \circ \varphi$.

Let $s_b = \varphi^{-1}(t_b)$. Then s_b is a section of ρ, and s_b cuts out the base point of the anticanonical linear system on each fiber. It can be defined also as

$$s_b = \mathrm{Bas}\,|-K_V + lF|$$

for $l \gg 0$. We define $s_0 = \frac{1}{2}\varphi^*(t_0)$ and $f = \frac{1}{2}\varphi^*(p)$. It is easy to see that $s_b \sim s_0$. Note that

$$NE(V) = \overline{NE(V)} = \mathbb{R}_+ s_0 \oplus \mathbb{R}_+ f.$$

Denote by G_V the unique effective element of $|-K_V - 2F|$. The divisor G_V is exceptional for the blow-up $V = U_C \to U$ and isomorphic to $C \times \mathbb{P}^1$. In [3] it was shown that $K_V^2 \sim 2s_0 + 3f$ and $s_0 \circ (-K_V) = 1$. Clearly, $f \circ F = 0$ and $s_0 \circ F = f \circ (-K_V) = 1$.

Relations between divisors on U and $V = U_C \to \mathbb{P}^1$. Let $\varphi_C : V = U_C \to U$ be the blow-up of $C \in \mathcal{P}$. Suppose that $\mathcal{D} \subset |aH - \mu C|$. In other words, $\mathcal{D}$ consists of divisors of the class aH that have the multiplicity μ along C. Let $\mathcal{D}_V = (\varphi_C)_*^{-1} \subset |n(-K_V) + mF|$ be the strict transform of $\mathcal{D}_U$ on V. Then it is easy to compute that

$$\begin{aligned} n &= a - \mu, \\ m &= 2\mu - a, \end{aligned}$$

i.e., $\mathcal{D}_V \subset |(a-\mu)(-K_V) + (2\mu - a)F|$.

4. Excluding infinitely near singularities on U.

Let $l_0 \in \mathcal{P}$, B_0 the singular point of l_0. As it follows from theorem 3.1.2 of [2], we have to consider the case of infinitely near maximal singularities over B_0 that are realized by the weighted blow-up with weights either $(1,1,N)$, $2 \le N \le 5$, if B_0 is a node, or $(1,1,2)$, if B_0 is a cusp. In all these cases we assume that $\mu = \mathrm{mult}_{l_0} \mathcal{D} \le \frac{n}{2}$, where $\mathcal{D} \subset |nH|$ is a linear system without fixed components.

Following to section 2, weighted blow-ups of type $(1,1,N)$ can be resolved as follows. We blow up the point B_0, and the exceptional divisor (say, E_1) contains the next center B_1, which must be a line. We blow up B_1 with a ruled surface E_2 as the exceptional divisor, and this is all if $N = 2$. Otherwise, there exists a section $B_2 \subset E_2 \setminus E_1^2$ which is the next center. We blow up it, and so on. As in the end of section 2, we denote $\nu_i = \mathrm{mult}_{B_{i-1}} \mathcal{D}^{i-1}$, $\nu_1 \ge \nu_2 \ge \ldots > \frac{n}{2}$. The corresponding Nöther-Fano inequalities are given by

$$\nu_1 + \ldots + \nu_N > \frac{n}{2}(1+N). \tag{4.1}$$

The nodal case. As it is shown in section 3, we may choose local coordinates $[x, y, z, w]$ such that the local equation of U near $B_0 = (0,0,0,0)$ is

$$w^2 + z^3 + z^2 f_2(x,y) + z f_4(x,y) + x + f_6(x,y) = 0,$$

where $\deg f_i \leq i$, $f_2(0,0) \neq 0$, $f_4(0,0) = f_6(0,0) = 0$, f_6 does not contain a linear part, and $T = \{x = 0\}$ is the tangent space to U at B_0. We set $T_U = T|_U \in |H - B_0| = |H - l_0|$.

Let $\varphi_1 : U_1 \to U$ be the blow-up of B_0, $E_1 \cong \mathbb{P}^2$ the exceptional divisor, l_0^1 the strict transform of l_0 on U_1 (leading indices usually mean the strict transform on the corresponding floor of the chain of blow-ups), $\{P, P'\} = l_0^1 \cap E_1$ (clearly, $P \neq P'$), $e_1 \subset E_1$ the line joining P and P'. The situation is given by figure 1. Note that except for T_U, the strict transform of any element $S \in |H - l_0|$ on U_1 cuts out the line e_1 on E_1.

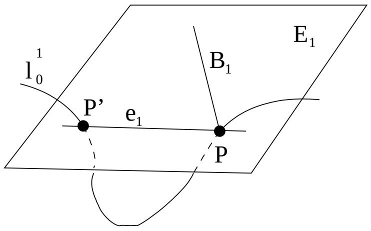

FIGURE 1

It is easy to observe from the equation of U that the singular point of T_U at B_0 is exactly A_k, $1 \leq k \leq 5$. Respectively, the restriction $T_U^1|_{E_1}$ is either a non-singular conic that passes through P and P', or a couple of lines that contain P and P' and have their common point outside of e_1 (thus they are different from e_1). In the latter case T_U^1 may have a singular point in the common point of the couple of the lines, but anyway T_U^1 is non-singular along l_0^1.

The possible situation is exhausted by the following cases:

A). neither P nor P' is contained in B_1;
B). $e_1 = B_1$;
C). $P \in B_1 \neq e_1$ and $B_1 \subset \operatorname{Supp} T_U^1|_{E_1}$;
D). $P \in B_1 \neq e_1$, $T_U^1|_{E_1}$ is a conic, and B_1 is tangent to $T_U^1|_{E_1}$ at P;
E). other cases when $P \in B_1 \neq e_1$.

Note that in all cases except for $e_1 = B_1$ we have the inequality

$$\nu_1 \geq \nu_2 + \mu, \tag{4.2}$$

which arises from the intersection of a general line in E_1 through the point P' and an element from $\mathcal{D}$. Now we deal separately with the listed cases.

The case C). Consider the family of divisors $\mathcal{L} = \{z = ay, a \neq 0\}$. Clearly, $\mathcal{L} \subset |2H|$ with non-singular $K3$ surfaces as general elements. Then, $\mathcal{L}^1$ has no base curves on U_1, so its elements cover E_1. Choose general elements $S \in \mathcal{L}$ and $D \in \mathcal{D}$, and consider the curve $C = T_U \cap S$.

Clearly, $C \sim 2l$, $C \not\subset D$, C has a double point at B_0, and $C^1 \cap B_1 \neq \emptyset$. Thus

$$2n = D \circ C \geq 2\nu_1 + \nu_2.$$

Since $\nu_1 + \ldots + \nu_N > n\left(1 + \frac{N}{2}\right)$ from the Nöther-Fano inequality, we see that $N = 2$ or $N = 3$ are impossible. In order to exclude the cases $N > 3$, apply the inequality 4.2. We have

$$2n \geq 2\nu_1 + \nu_2 \geq 2\mu + 3\nu_2 > 2\mu + \frac{3}{2}n,$$

hence $\mu < \frac{n}{4}$.

Now let us proceed with the way which we will use repeatedly in the sequel. Choose general elements $S \in |H - l_0|$ and $D \in \mathcal{D}$. S is a non-singular del Pezzo surface of degree 1. Set

$$D|_S = \mu l_0 + C,$$

where the components of the residual curve C do not contain l_0. We resolve the maximal valuation over B_0 in the way given in section 2. Thus, we have the blow-ups $\varphi_i : U_i \to U_{i-1}$, $U_0 = U$, of the centers $B_{i-1} \subset E_{i-1}$. In our case, B_0 is a point, the other B_i's are curves. As it has been mentioned, the corresponding oriented graph has no incidencies and $B_i \cap E^i_{i-1} = \emptyset$ for $i \geq 2$. We put $e_i = E_i|_{S^i}$, $\tilde{B}_i = B_i \cap S^i$ for $i \geq 1$, and $\tilde{\nu}_i = \mathrm{mult}_{\tilde{B}_{i-1}} D^{i-1}|_{S^{i-1}}$. We have

$$\begin{aligned}
&D|_S = \mu l_0 + C,\\
&D^1|_{S^1} = \mu l_0^1 + C^1 + m_1 e_1,\\
&\ldots\\
&D^{N-1}|_{S^{N-1}} = \mu l_0^{N-1} + m_1 e_1^{N-1} + \ldots + m_{N-1} e_{N-1},
\end{aligned}$$

where the numbers m_i satisfy the relations

$$\tilde{\nu}_i = n u_i + m_i. \tag{4.3}$$

Note that $\tilde{B}_i \notin e^i_{i-1}$. Set $\alpha_i = \mathrm{mult}_{\tilde{B}_i} C^i$, $\alpha_0 \geq \alpha_1 \geq \ldots$, and let

$$k = \max\{i : \tilde{B}_{i-1} \in l_0^{i-1}\}.$$

Then we have

$$\begin{aligned}
&\tilde{\nu}_1 = 2\mu + \alpha_0,\\
&\tilde{\nu}_2 = \mu + \alpha_1 + m_1,\\
&\ldots\\
&\tilde{\nu}_k = \mu + \alpha_{k-1} + m_{k-1}.
\end{aligned}$$

Moreover, one deduices that

$$2\alpha_0 + \alpha_1 + \ldots + \alpha_{k-1} \leq (D|_S - \mu l_0) \circ l_0 = n - \mu.$$

Suppose that $k < N$. Then $\tilde{\nu}_{k+1} = \alpha_k + m_k$. Thus

$$\tilde{\nu}_1 + \ldots + \tilde{\nu}_{k+1} = (k+1)\mu + \sum_{i=0}^{k} \alpha_i + \sum_{i=1}^{k} m_i.$$

Using (4.3) and the inequality

$$\sum_{i=0}^{k} \alpha_i \leq 2\alpha_0 + \alpha_1 + \ldots + \alpha_{k-1} \leq n - \mu,$$

we obtain

$$\nu_1 + \ldots + \nu_{k+1} \leq (k+1)\mu + n - \mu = n + k\mu.$$

Since $\mu \leq \frac{n}{2}$, we get

$$\nu_1 + \ldots + \nu_{k+1} \leq \frac{n}{2}(k+1),$$

which is impossible because $\nu_i > \frac{n}{2}$ for all i. Hence $k = N$. Arguing as before, we see that

$$\nu_1 + \ldots + \nu_N \leq (N+1)\mu + n - \mu = n + N\mu.$$

Using the estimation $\mu < \frac{n}{4}$, one has

$$\nu_1 + \ldots + \nu_N < n\left(1 + \frac{N}{4}\right).$$

It only remains to compare it with the Nöether-Fano inequality (4.1), and we get the contradiction.

The case B). In this case $e_1 = B_1$. We repeat the argumentation of the previous case with minor changes. Let $\mathcal{L} = |H - l_0|$, $\varphi_1 : U_1 \to U$ the blow-up of B_0. For general $S \in \mathcal{L}$, $S^1|_{E_1} = e_1$. We set

$$\begin{aligned} D|_S &= \mu l_0 + C, \\ D^1|_{S^1} &= \mu l_0^1 + C^1 + m_1 e_1, \end{aligned}$$

where $m_1 = \nu_2$ since $e_1 = B_1$. Let $\varphi_2 : U_2 \to U_1$ be the blow-up of B_1, $E_2 \cong \mathbb{F}_2$ the exceptional divisor with s_2 and f_2 as the classes of the minimal section and a fiber respectively. Note that $E_1^2|_{E_2}$ cuts out the minimal section of E_2, the pencil $\mathcal{L}^2|_{E_2} \subset |s_2 + 2f_2|$ has two base points, say, Q and Q', outside of the minimal section.

We resolve the valuation as in section 2. Let

$$k = \max\left\{i : B_{i-1} \cap l_0^{i-1} \neq \emptyset\right\}.$$

Clearly, $k \geq 3$. For $i \geq 2$, let $\tilde{B}_i$ be any of the intersection points $S^i \cap B_i$. We see that $\tilde{B}_k \notin l_0^k$. Then, for $i \geq 2$,

$$D^i|_{S^i} = \mu l_0^i + C^i + m_2 e_2^i + m_3 e_3^i + \ldots + m_i e_i,$$

where $e_i = E_i|_{S^i}$. The important observation is that $m_2 = 0$.

Suppose that $k < N$. We have

$$\begin{aligned}
&\nu_1 + \nu_2 = 2\mu + \alpha_0,\\
&\tilde{\nu}_3 = \mu + \alpha_2,\\
&\tilde{\nu}_4 = \mu + \alpha_3 + m_3,\\
&\dots\\
&\tilde{\nu}_k = \mu + \alpha_{k-1} + m_{k-1},\\
&\tilde{\nu}_{k+1} = \alpha_k + m_k,
\end{aligned}$$

where $\tilde{\nu}_i = \operatorname{mult}_{\tilde{B}_{i-1}} D^{i-1}|_{S^{i-1}} = \nu_i + m_i$ and $\alpha_i = \operatorname{mult}_{\tilde{B}_i} C^i$. Since $\mu \le \frac{n}{2}$ and $\alpha_0 + \alpha_2 + \ldots + \alpha_k \le n - \mu$, we get a contradiction:

$$\frac{n}{2}(k+1) < \nu_1 + \ldots + \nu_{k+1} \le n + (k-1)\mu \le \frac{n}{2}(k+1).$$

Suppose that $k = N$. Arguing as before, we see that

$$\nu_1 + \ldots + \nu_N \le N\mu + \alpha_0 + \alpha_2 + \ldots + \alpha_{N-1} \le n + (N-1)\mu \le \frac{n}{2}(N+1),$$

and this contradicts to the Nöther-Fano inequality (4.1).

The case A). As it has been mentioned above, in this case we have $\nu_1 \ge \nu_2 + \mu$. Denote $\tilde{B}_1 = B_1 \cap e_1$, $\tilde{B}_1$ is different from both P and P'. We set

$$\begin{aligned}
&D|_S = \mu l_0 + C,\\
&D^1|_{S^1} = \mu l_0^1 + C^1 + m_1 e_1.
\end{aligned}$$

Since $\tilde{B}_1 \notin l_0^1$, we have

$$\begin{aligned}
&\tilde{\nu}_1 = 2\mu + \alpha_0,\\
&\tilde{\nu}_2 = \alpha_1 + m_1,
\end{aligned}$$

and $\alpha_0 + \alpha_1 \le n - \mu$ (we keep the notation of the previous cases). Thus,

$$2\nu_2 + \mu \le \nu_1 + \nu_2 \le 2\mu + \alpha_0 + \alpha_1 \le n + \mu,$$

whence $n < 2\nu_2 \le n$, a contradiction.

The case E). It is easy to compute that

$$\mathcal{N}_{l_0^1|U_1} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-3).$$

Let $\varphi_2 : U_2 \to U_1$ be the blow-up of l_0^1 (not B_1!!), $E_2 \cong \mathbb{F}_2$ the exceptional divisor with s_2 and f_2 as the classes of the minimal section and a fiber. We see that

$$D^2|_{E_2} \sim \mu s_2 + (3\mu + n - 2\nu_1) f_2.$$

By the condition, $T_U^1|_{E_1}$ and B_1 are transversal to each other at P. Since T_U^2 cuts out the minimal section of E_2, we deduice that B_1^2 intersects E_2 outside of the minimal section, and at this point $D^2|_{E_2}$ has the multiplicity not less than ν_2. Thus

$$3\mu + n - 2\nu_1 \ge \nu_2,$$

and since $\nu_1 \geq \nu_2 + \mu$, we get

$$n + \mu \geq 3\nu_2 > \frac{3}{2}n,$$

i.e., $\mu > \frac{n}{2}$, a contradiction.

The case D). As in the case E), we first blow-up l_0^1, but now $B_1^2 \cap E_2 \in s_2$. Set $\mu' = \operatorname{mult}_{s_2} D^2$, $\mu' \leq \mu$. Then,

$$\mathcal{N}_{s_2|U_2} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2),$$

and let $\varphi_3 : U_3 \to U_2$ be the blow-up of s_2, E_3 the exceptional divisor with the minimal section s_3 and a fiber f_3. We obtain the situation like in the previous case: $B_1^3 \cap T_U^3 = \emptyset$, $T_U^3|_{E_3} = s_3$, so $D^3|_{E_3}$ has the multiplicity not less than ν_2 at the point $B_1^3 \cap E_3$. Since

$$D^3|_{E_3} \sim \mu' s_3 + (2\mu' + \mu + n - 2\nu_1) f_3,$$

and $\mu' \leq \mu$, we get the inequality $3\mu + n - 2\nu_1 \geq \nu_2$, i.e., the same as in the case E).

The cuspidal case. Let B_0 be the cuspidal point of $l_0 \in \mathcal{P}$. As it is shown in section 3, we can define U locally by

$$w^2 + z^3 + zf_4(x, y) + x + f_6(x, y) = 0,$$

where $\deg f_i \leq i$, $f_i(0, 0) = 0$, f_6 has no the linear part, $B_0 = (0, 0, 0, 0)$, $l_0 = \{x = y = 0\}$, and $T_U = \{x = 0\}$. We see that T_U has at B_0 one of the following du Val singularities: A_1, A_2, D_4, E_6, E_7, or E_8.

As in the nodal case, let $\varphi_1 : U_1 \to U$ be the blow-up of B_0, E_1 the exceptional divisor, and $P = l_0^1 \cap E_1$. The strict transforms of divisors from $|H - l_0| \setminus \{T_U\}$ cut out a line e_1 on E_1. Then, T_U^1 is non-singular along l_0^1, and $T_U^1|_{E_1}$ is one of the following:

- a non-singular conic that is tangent to e_1 at P, if B_0 is A_1-point on T_U;
- a couple of lines that are different from e_1 and have their common point at P, in the case A_2;
- a double line that is different from e_1 and pass through P, in the other cases.

We have the following possibilities:

A). $B_1 \neq e_1$ and $P \notin B_1$;
B). $B_1 = e_1$;
C). $P \in B_1$ and $B_1 \subset \operatorname{Supp} T_U^1|_{E_1}$;
D). $P \in B_1 \neq e_1$, $B_1 \not\subset \operatorname{Supp} T_U^1|_{E_1}$, and $T_U^1|_{E_1}$ consists of lines;
E). $P \in B_1 \neq e_1$ and $T_U^1|_{E_1}$ is a non-singular conic.

The case A) was excluded in [2]. The argumentation of the cases B)–E) of the nodal situation works in the caspidal situation as well, we only give some remarks. One keeps the notation of the corresponding cases.

First, there are no changes in the cases C), D), and E). Then, in the case B), let $Q = l_0^2 \cap E_2$; Q is not contained in the minimal section.

Elements from $\mathcal{L}^2|_{E_2}$ (except for T_U^2) are all tangent to each other at Q. Suppose that B_2 is also tangent to them at Q (if not, the argumentation of the case $k < N$ works). Then, for a general $S \in \mathcal{L}$, S^3 cuts out a double fiber of E_3, along which it has a double point. We can always assume that B_3 does not pass through the singular point of S^3. Since B_3 is also tangent to S^3, we will have the analoguos situation on U_4, and so on. Anyway, we may always assume S^i to be non-singular at the point $\tilde{B}_i = S^i \cap B_i$. There are no other changes in this case.

So, taking into account the results of [2], we proved the following proposition:

PROPOSITION 4.1. *Suppose that a linear system $\mathcal{D}$ on U has no maximal curves. Then $\mathcal{D}$ has no maximal singularities over points.*

5. Excluding super-maximal singularities over non-singular points.

In this section we assume the reader to be more or less familiar with the "super-maximal" modification of the method of maximal singularities (see [9]). All denotations are close to those in [3].

Let V be the blow-up of a curve from $\mathcal{P}$ on U, $\rho : V \to \mathbb{P}^1$ the corresponding fibered structure, s_0 and f are the classes of effective generators of 1-dimensional cycles on V, and $\{G_V\} = |-K_V - 2F|$ (see section 3).

Consider a linear system $\mathcal{D} \subset |n(-K_V)+mF|$, $m \geq 0$, without fixed components, and general elements $D_1, D_2 \in \mathcal{D}$. Then

$$D_1 \circ D_2 \sim n^2 s_0 + (3n^2 + 2mn)f,$$

and we set

$$D_1 \circ D_2 = Z^h + \sum_{t \in \mathbb{P}^1} Z_t^v,$$

where Z^h and Z_t^v are horizontal and vertical (over $t \in \mathbb{P}^1$) effective 1-cycles.

We recall that $\mathcal{D}$ has no maximal singularities over curves ([9], [3]). Assume $\mathcal{D}$ has a super-maximal singularity. Then we have

$$m < \sum_{t \in \mathbb{P}^1} \max_{\substack{\mathfrak{v} \in \mathcal{M} \\ \text{Center } \mathfrak{v} \in F_t}} \frac{e(\mathfrak{v})}{\mathfrak{v}(F_t)}, \tag{5.1}$$

where $e(\mathfrak{v}) = \mathfrak{v}(\mathcal{D}) - \delta_{\mathfrak{v}}$, $\delta_{\mathfrak{v}}$ is the canonical multiplicity with respect to $\mathfrak{v}$, and $\mathcal{M} = \{\mathfrak{v} : e(\mathfrak{v}) > 0\}$ is the (finite) set of maximal valuations of $\mathcal{D}$.

Let A be the number of elements in $\mathcal{M}$, and define $\deg Z^h = Z^h \circ F$, $\deg Z_t^v = 2 Z_t^v \circ (-K_V)$. Then

$$\sum_{t \in \mathbb{P}^1} \deg Z_t^v \leq 6n^2 + 4mn < \sum_{t \in \mathbb{P}^1} \left(\frac{6n^2}{A} + 4n \max_{\substack{\mathfrak{v} \in \mathcal{M} \\ \text{Center } \mathfrak{v} \in F_t}} \frac{e(\mathfrak{v})}{\mathfrak{v}(F_t)} \right).$$

Thus, there exist $t \in \mathbb{P}^1$ and $\mathfrak{v} \in \mathcal{M}$ such that

$$\deg Z_t^v < \frac{6n^2}{A} + 4n\frac{e(\mathfrak{v})}{\mathfrak{v}(F_t)}.$$

Consider the graph $\Gamma = \Gamma(\mathfrak{v})$ (see section 2). Suppose it consists of N vertices, the vertices $\{1, \ldots, L\}$ correspond to points (i.e., $B_0, \ldots, B_{L-1}$ are points), and the others correspond to curves. We denote by r_i the number of all paths in Γ from N to i, and define $\Sigma_0 = \sum_{i=1}^{L} r_i$, $\Sigma_1 = \sum_{i=L+1}^{N} r_i$. Let

$$L' = \max\{1 \leq i \leq L : B_{i-1} \in F_t^{i-1}\}$$

and $\Sigma_0' = \sum_{i=1}^{L'} r_i$; clearly, $\Sigma_0' \leq \Sigma_0$. Then $\Sigma_0' \leq \mathfrak{v}(F_t)$, and we have

$$\Sigma_0' \deg Z_t^v < \frac{6n^2}{A} + 4ne(\mathfrak{v}).$$

Put $m_i^h = \mathrm{mult}_{B_{i-1}} \left(Z^h\right)^{i-1}$ and $m_i^v = \mathrm{mult}_{B_{i-1}} \left(Z_t^v\right)^{i-1}$. Then we have the quadratic inequality

$$\sum_{i=1}^{L} r_i m_i^h + \sum_{i=1}^{L'} r_i m_i^v \geq \frac{(2n\Sigma_0 + n\Sigma)1 + e(\mathfrak{v}))^2}{\Sigma_0 + \Sigma_1} \geq 4n^2\Sigma_0 + 4ne(\mathfrak{v}).$$

Note that $m_i^h \leq \deg Z^h \leq n^2$. Taking into account (5.1), we obtain

$$n^2\Sigma_0 + \frac{6n^2}{A}\Sigma_0' > 4n^2\Sigma_0,$$

hence

$$\frac{2}{A} > \frac{\Sigma_0}{\Sigma_0'} \geq 1,$$

which yields $A = 1$ and $\Sigma_0' > \frac{1}{2}\Sigma_0$.

Thus, it is shown that $\mathcal{D}$ only consists of valuations centered over points in F_t. This allows us to re-write the super-maximal condition for $\mathfrak{v}$ in the form

$$e(\mathfrak{v}) > m\mathfrak{v}(F_t).$$

Let $B_0 = \mathrm{Center}_V\, \mathfrak{v}$. We change the denotation F_t for the "central" fiber (i.e., that contains the super-maximal singularity), denoting it by F_0. The anticanonical linear system on F_0 contains a unique element, say, l_0, that passes through B_0.

We summarize the known results as follows (we assume B_0 to be a non-singular point of V):

- $\mathcal{D}$ may only have maximal singularities over points in F_0, and at least one of the maximal singularities is super-maximal;
- B_0 is necessarily contained in a section of the class s_0, thus $B_0 \in G_V$ and $l_0 = G_V \cap F_0$ ([3], §2.2);
- B_0 is a non-singular point of F_0, $B_0 \neq \mathrm{Bas}\,|-K_{F_0}|$, and l_0 has a double point (node or cusp) at B_0 ([3], lemma 2.8);

- we have the following inequality for $\mathfrak{v}$: $\sum_{i=1}^{N} r_i\nu_i > 2n\Sigma_0 + n\Sigma_1 + m\Sigma_0'$, where $\nu_i = \operatorname{mult}_{B_{i-1}} \mathcal{D}^{i-1}$;
- always $\Sigma_0' > \frac{1}{2}\Sigma_0$.

Note that if $V \to U$ is the blow-up of a curve $C \in \mathcal{P}$, then $l_0 \cong C$ and $G_V = \mathbb{P}^1 \times C$. Whence C, and thus l_0 and V itself, must be singular.

We can say a bit more about the situation. Since $B_0 = \operatorname{Center}_V \mathfrak{v}$, $\mathcal{M}$ has to contain a discrete valuation that defines a maximal singularity over B_0 and can be realized by a divisorial extraction (i.e., a wheighted blow-up, see section 2). We denote such a discrete valuation by $\mathfrak{v}_{div}$.

We have three possible cases:

A. $\mathfrak{v}_{div}$ is super-maximal (for example, $\mathfrak{v}_{div} = \mathfrak{v}$) and B_0 is a non-singular point of V;
B. $\mathfrak{v}_{div}$ is not super-maximal (thus $\mathfrak{v}_{div} \neq \mathfrak{v}$), B_0 is non-singular;
C. B_0 is the singular point of V.

The latter case is studied in the next section. Here we show that the first two cases are impossible. One can always assume $e(\mathfrak{v}_{div}) \geq e(\mathfrak{v})$, so in the case **B** $\mathfrak{v}_{div}(F_0) > \mathfrak{v}(F_0)$, and we have the weak super-maximal condition for $\mathfrak{v}_{div}$:

$$e(\mathfrak{v}_{div}) > m\mathfrak{v}(F_0).$$

The case A: $\mathfrak{v}_{div}$ is super-maximal. We may assume that $\mathfrak{v}_{div} = \mathfrak{v}$.

Suppose first that B_0 is a nodal point of l_0. As it is mentioned in section 2, $\mathfrak{v}$ can be realized as the weighted blow-up with weights $(1, L, N)$, $L < N$, or as a chain of blow-ups with centers $B_0, \dots, B_{N-1}$, where $B_0, \dots, B_{L-1}$ are points and $B_L, \dots, B_{N-1}$ are curves, with the conditions $B_i \not\subset E_{i-1}^i$ for all i and $B_i \cap E_{i-1}^i = \emptyset$ for $i > L$. One uses the latter realization. We set

$$L' = \max\{1 \leq i \leq L : B_{i-1} \in F_0^{i-1}\}$$

and

$$k = \max\{1 \leq i \leq L : B_{i-1} \in l_0^{i-1}\}.$$

Clearly, $L' \leq \mathfrak{v}(F_0)$ and $k \leq L'$. Since $r_i = 1$ for all i, we have the super-maximality condition in the form

$$\sum_{i=1}^{N} \nu_i > n(N + L) + mL'.$$

Choose general elements $D_1, D_2 \in \mathcal{D}$, and let $D_1 \circ D_2 = Z^h + Z^v$, where Z^h and Z^v are horizontal and vertical 1-cycles. Z^v contains the 1-cycle Z_0^v that lies in F_0, and let

$$Z_0^v = \alpha l_0 + \tilde{Z}_0^v,$$

where $l_0 \not\subset \operatorname{Supp} \tilde{Z}_0^v$. Define the multiplicities $m_i^h = \operatorname{mult}_{B_{i-1}}(Z^h)^{i-1}$ and $\tilde{m}_i^v = \operatorname{mult}_{B_{i-1}}(\tilde{Z}_0^v)^{i-1}$ for $1 \leq i \leq L$. We obtain the quadratic inequality in the form

$$\sum_{i=1}^{L} m_i^h + \alpha(k+1) + \sum_{i=1}^{L'} \tilde{m}_i^v > \frac{(n(L+N)+mL')^2}{N} \geq 4n^2L + 4mnL'.$$

Since $m_i^h \leq \deg Z^h \leq n^2$ and $\alpha + \tilde{m}_i^v \leq \deg Z^v \leq 3n^2 + 2mn$, we deduice

$$n^2L + \alpha + 3n^2L' + 2mnL' > 4n^2L + 4mnL',$$

whence

$$\alpha > 3n^2(L - L') + 2mnL'.$$

But $\alpha \leq 3n^2 + 2mn$, hence $L = L'$. Moreover, the same arguments show that $k = L'$. Thus, in what follows, we suppose that $k = L' = L$. In particular, $B_i \cap F_0^i \neq \emptyset$ for all i.

Now we show that $B_L \not\subset F_0^L$. Assume the converse. Set $e_i = E_i \cap F_0^i$, and choose a point $\tilde{B}_L \in e_L$ outside of $e_{L-1}^L \cup l_0^L$. Clearly, $\tilde{\nu}_{L+1} \stackrel{\text{def}}{=} \operatorname{mult}_{\tilde{B}_L}(D^L|_{F_0^L}) \geq \nu_{L+1}$. For a general $D \in \mathcal{D}$ we set

$$D|_{F_0} = \mu l_0 + C \in |n(-K_{F_0})|,$$

where $\mu \leq n$ and $l_0 \not\subset \operatorname{Supp} C$. Define $\alpha_i = \operatorname{mult}_{B_i} C^i$ for $i < L$, $\tilde{\alpha}_L = \operatorname{mult}_{\tilde{B}_L} C^L$, $m_i = \operatorname{ord}_{e_i}(D^i|_{F_0^i})$, and $\tilde{\nu}_i = \operatorname{mult}_{B_{i-1}}(D^{i-1}|_{F_0^{i-1}})$. Then we have

$$\begin{aligned}
&\tilde{\nu}_1 = 2\mu + \alpha_0,\\
&\tilde{\nu}_2 = \mu + \alpha_1 + m_1,\\
&\dots\\
&\tilde{\nu}_L = \mu + \alpha_{L-1} + m_{L-1},\\
&\tilde{\nu}_{L+1} = \tilde{\alpha}_L + m_L.
\end{aligned}$$

Since

$$\alpha_0 + \ldots + \alpha_{L-1} + \tilde{\alpha}_L \leq 2\alpha_0 + \alpha_1 + \ldots + \alpha_{L-1} \leq (D|_S - \mu l_0) \circ l_0 = n - \mu$$

and $\tilde{\nu}_i = \nu_i + m_i$ for $1 \leq i \leq L$, we obtain

$$\nu_1 + \ldots + \nu_{L+1} \leq (L+1)\mu + n - \mu \leq (L+1)n,$$

which contradicts to the condition $\nu_i > n$ for all i.

Thus, we assume that $B_L \not\subset F_0^L$. We define the points $\tilde{B}_i = B_i \cap F_0^i$, $0 \leq i \leq N-1$. Suppose that $\tilde{\nu}_i = \operatorname{mult}_{\tilde{B}_{i-1}}(D^{i-1}|_{F_0^{i-1}})$, $1 \leq i \leq N$, the numbers m_i, α_i, μ are defined as before. Set

$$k' = \max\{1 \leq i \leq N : \tilde{B}_{i-1} \in l_0^{i-1}\}.$$

Clearly, $k' \geq k = L' = L$. Assume that $k' < N$. Then we have

$$\begin{aligned}
&\tilde{\nu}_1 = 2\mu + \alpha_0,\\
&\tilde{\nu}_2 = \mu + \alpha_1 + m_1,\\
&\dots\\
&\tilde{\nu}_{k'} = \mu + \alpha_{k'-1} + m_{k'-1},\\
&\tilde{\nu}_{k'+1} = \tilde{\alpha}_{k'} + m_{k'},
\end{aligned}$$

and since

$$\alpha_0 + \ldots + \alpha_{k'} \leq 2\alpha_0 + \alpha_1 + \ldots + \alpha_{k'-1} \leq n - \mu,$$

we get a contradiction:

$$\nu_1 + \ldots + \nu_{k'+1} \leq (k'+1)\mu + n - \mu \leq (k'+1)n.$$

Thus, $k' = N$. We repeat the arguments to obtain

$$\nu_1 + \ldots + \nu_N \leq (N+1)\mu + n - \mu \leq (N+1)n,$$

but this is impossible because of the super-maximality condition

$$\nu_1 + \ldots + \nu_N > n(L+N) + mL' \geq (N+1)n.$$

The important remark is that we have used the Nöther-Fano inequality, not the super-maximality condition.

So, B_0 has to be a cusp of l_0. It is easy to observe that nothing changes in the cuspidal case, the above argumentation does work. In fact, this case is even simpler since B_2 avoids l_0^2 (note that $l_0^2 \cap E_1^2 \neq \emptyset$).

The case B: $\mathfrak{v}_{div}$ is not super-maximal. Since $\mathfrak{v}_{div}$ is not super-maximal, one has $e(\mathfrak{v}_{div}) \leq m\mathfrak{v}_{div}(F_0)$. We may always assume that $e(\mathfrak{v}_{div}) > e(\mathfrak{v})$. Since $\mathfrak{v}$ is super-maximal, that is, $e(\mathfrak{v}) > m\mathfrak{v}(F_0)$, we obtain two important inequalities:

$$\begin{aligned} &e(\mathfrak{v}_{div}) > m\mathfrak{v}(F_0), \\ &\mathfrak{v}_{div}(F_0) > \mathfrak{v}(F_0). \end{aligned}$$

Suppose that $\mathfrak{v}_{div}$ is realized by the weighted blow-up with weights $(1, L, N)$. Taking into account the scheme of resolution of weighted blow-up, we have the centers $B_0, \ldots, B_{N-1}$, where $B_0, \ldots, B_{L-1}$ are points and $B_L, \ldots, B_{N-1}$ are curves. Set

$$k = \max\{1 \leq i \leq L : B_{i-1} \in l_0^{i-1}\}.$$

We repeat the arguments of the case **A** to argue that $B_L \not\subset F_0^L$. Moreover, due to the remark at the end of the case **A**, that is, that we have not needed the super-maximality condition, we also argue that $k < L$. We show that $B_k \notin F_0^k$ for $k < L$. Assume the converse, that is, $B_k \in F_0^k \setminus l_0^k$. Restricting $\mathcal{D}$ to F_0 and using the denotations of $\tilde{\nu}_i$, μ, α_i, and so on, in the same way as above, we obtain

$$\begin{aligned} &\tilde{\nu}_1 = 2\mu + \alpha_0, \\ &\tilde{\nu}_2 = \mu + \alpha_1 + m_1, \\ &\cdots \\ &\tilde{\nu}_k = \mu + \alpha_{k-1} + m_{k-1}, \\ &\tilde{\nu}_{k+1} = \alpha_k + m_k, \end{aligned}$$

whence

$$\nu_1 + \ldots + \nu_{k+1} \leq (k+1)\mu + \alpha_0 + \ldots + \alpha_k \leq (k+1)\mu + n - \mu \leq (k+1)n,$$

which is impossible. Thus we have

$$\mathfrak{v}_{div}(F_0) = k < L,$$

and since $\mathfrak{v}_{div}(F_0) > \mathfrak{v}(F_0) \geq 1$, then $k \geq 2$.

Let $Z_0^v = \alpha l_0 + \tilde{Z}_0^v$, and let m_i^h and $\tilde{m}_i^v$ be defined as above. One uses the weak super-maximality condition

$$\sum_{i=1}^{N} \nu_i > n(L+N) + m\mathfrak{v}(F_0)$$

to obtain the quadratic inequality in the form

$$\sum_{i=1}^{L} m_i^h + (k+1)\alpha + \sum_{i=1}^{k} \tilde{m}_i^v > \frac{(n(L+N)+m\mathfrak{v}(F_0))^2}{N}.$$

Remark that

$$2\tilde{m}_1^v + \tilde{m}_2^v + \ldots \tilde{m}_k^v \leq \tilde{Z}_0^v \circ l_0 \leq 3n^2 + 2mn - \alpha,$$

and the more so

$$\tilde{m}_1^v + \tilde{m}_2^v + \ldots \tilde{m}_k^v \leq 3n^2 + 2mn - \alpha.$$

Moreover,

$$m_1^h + \ldots + m_k^h \leq Z^h \circ F_0 \leq n^2.$$

Thus we get

$$\frac{L}{k}n^2 + k\alpha + 3n^2 + 2mn > 4n^2L + 4mn\mathfrak{v}(F_0) + n^2\frac{\left((N-L) - \frac{m}{n}\mathfrak{v}(F_0)\right)^2}{N},$$

hence

$$k\alpha > n^2\left(4L - \frac{L}{k} - 3\right) + 2mn\left(2\mathfrak{v}(F_0) - 1\right) + n^2\frac{\left((N-L) - \frac{m}{n}\mathfrak{v}(F_0)\right)^2}{N}.$$

We observe that $\nu_1 + \ldots + \nu_k \leq (k+1)\mu + \alpha_0 + \ldots + \alpha_{k-1} \leq (k+1)n$, and then

$$\theta_k \stackrel{\text{def}}{=} \frac{\nu_1 + \ldots + \nu_k}{kn} \leq \frac{k+1}{k}.$$

Clearly,

$$n\theta_k \geq \frac{1}{N}\sum_{i=1}^{N} \nu_i,$$

and we deduice from the weak super-maximal condition that

$$N - L > \frac{2-\theta_k}{\theta_k - 1}L + \frac{1}{\theta_k - 1} \cdot \frac{m}{n}\mathfrak{v}(F_0).$$

Combining this and the estimation for θ_k with the quadratic inequality, one gets

$$\alpha > n^2\left(\frac{(k+2)L}{k} - \frac{3}{k}\right) + mn\left(\frac{(k+1)^2}{k^2}\mathfrak{v}(F_0) - \frac{2}{k}\right).$$

Recall that $L \geq k+1$. Thus we have

$$\alpha > n^2\left(\frac{k^2+3k+1}{k}\right) + mn\left(\mathfrak{v}(F_0) + \frac{2\mathfrak{v}(F_0)}{k} + \frac{\mathfrak{v}(F_0)}{k^2} - \frac{2}{k}\right),$$

and the condition $\alpha \leq 3n^2 + 2mn$ yields

$$\mathfrak{v}(F_0) = 1.$$

Note that the argumentation is the same in the nodal and in the cuspidal cases.

Now we turn back to the super-maximal valuation $\mathfrak{v}$. All denotations (ν_i, B_i, E_i, and so on) are related to $\mathfrak{v}$. Since $\Sigma_0' \leq \mathfrak{v}(F_0) = 1$ and $\Sigma_0 < 2\Sigma_0'$, we have $\Sigma_0 = \Sigma_0' = 1$. Thus $L = 1$ and $B_1, \ldots, B_{N-1}$ are curves. It is known that $B_{L+1} \cap E_L^{L+1} = \emptyset$ and $B_i \not\subset E_{i-1}^i$ for all $i > L$. In our case $L = 1$, thus $B_2 \cap E_1^2 = \emptyset$, $E_2 \cong \mathbb{F}_2$. Denote by s_i and f_i the minimal sections and fibers of the corresponding ruled surfaces E_i. We see that $B_2 \sim s_2 + 2f_2$ and $B_i \neq s_i$ for all $i > 1$. Set

$$l = \max\{3 \leq i \leq N : B_{i-1} \cap E_{i-2}^{i-1} = \emptyset\}.$$

Assume that $l < N$. Then $B_l \sim s_l + bf_l$, where $b > l$. Indeed, from lemma 2.2 for the case $L = 1$ it follows that $E_i \cong \mathbb{F}_i$ for $i \leq l$. Take into account that $E_{l-1}^l|_{E_l} = s_l$, thus $B_l \cap s_l \neq \emptyset$ and $b > l$. It is easy to compute that

$$D^l|_{E_l} \sim \nu_l s_l + (\nu_1 + \ldots + \nu_l) f_l.$$

Since $\nu_{l+1} B_l \subset D^l|_{E_l}$, we get

$$(l+1)\nu_{l+1} \leq b\nu_{l+1} \leq \nu_1 + \ldots + \nu_l.$$

Set

$$k = \max\{1 \leq i \leq N-1 : B_{i-1} \cap l_0^{i-1} \neq \emptyset\}.$$

Assume that $k < l$. Restricting $\mathcal{D}$ to F_0, we have

$$\begin{aligned} \tilde{\nu}_1 &= 2\mu + \alpha_0, \\ \tilde{\nu}_2 &= \mu + \alpha_1 + m_1, \\ &\ldots \\ \tilde{\nu}_{k+1} &= \alpha_k + m_k. \end{aligned}$$

Since $2\alpha_0 + \alpha_1 + \ldots + \alpha_{k-1} \leq n - \mu$, we get

$$\nu_1 + \ldots + \nu_{k+1} \leq (k+1)\mu + n - \mu \leq (k+1)n,$$

which is impossible because of the inequalities $\nu_i > n$ for all i. Thus $k \geq l$ and

$$\tilde{\nu}_l = \mu + \alpha_{l-1} + m_{l-1}.$$

We obtain

$$\nu_1 + \ldots + \nu_l \leq (l+1)\mu + n - \mu \leq (l+1)n.$$

Combining this estimation with $(l+1)\nu_{l+1} \leq \nu_1 + \ldots + \nu_l$, we get a contradiction:

$$\nu_{l+1} \leq n.$$

Thus, we obtain $l = N$. In other words, the valuation $\mathfrak{v}$ corresponds to a weighted blow-up. In the case **A** it is shown that this is impossible. We proved the following result:

PROPOSITION 5.1. *The linear system $\mathcal{D} \subset |n(-K_V) + mF|$, $m \geq 0$, has no super-maximal singularities over the non-singular part of V.*

6. Excluding super-maximal singularities over the singular point.

It remains to exclude super-maximal singularities over the point B_0 which is the singular point of $V/\mathbb{P}^1$. Let $\mathfrak{v}$ be a maximal valuation with the center at B_0 which is a super-maximal at the same time, that is, the inequality

$$e(\mathfrak{v}) > m\mathfrak{v}(F_0)$$

holds, where F_0 is the fiber that contains B_0. As before, there exists a maximal valuation $\mathfrak{v}_{div}$ which is a divisorial extraction in the Mori category, $\mathrm{Center}_V\, \mathfrak{v}_{div} = B_0$. We may always assume that $e(\mathfrak{v}_{div}) \geq e(\mathfrak{v})$. Hence $\mathfrak{v}_{div}$ satisfies the weak super-maximal condition $e(\mathfrak{v}_{div}) > m\mathfrak{v}(F_0)$. In particular, one has always

$$e(\mathfrak{v}_{div}) > m.$$

As it follows from Kawakita's theorem 2.1, only the following cases are possible:

- B_0 is the simplest double point, that is, it is locally defined by $xy + zw = 0$, and $\mathfrak{v}_{div}$ is realized by the blow-up of B_0;
- B_0 is defined by $xy + z^3 + w^2 = 0$ and $\mathfrak{v}_{div}$ is realized by the blow-up of B_0;
- B_0 is defined by $xy + z^3 + w^2 = 0$ and $\mathfrak{v}_{div}$ is realized by the weighted blow-up with weights $(1, 5, 2, 3)$.

Now we argue that the first two cases are impossible. Let $\gamma_1 : U_1 \to U$ be the blow up of B_0, E_1 the exceptional divisor. In the first two cases E_1 realizes $\mathfrak{v}_{div}$, and either $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ if B_0 is the simplest double point, or E_1 is isomorphic to a quadric cone in $\mathbb{P}^3$ (in the second case). Remark that U_1 is non-singular. For a general $D \in \mathcal{D} \subset |n(-K_V + mF)|$ we set

$$D^1 = \gamma_1^*(D) - \nu_1 E_1.$$

The Nöther-Fano inequality has the form $\nu_1 > n$. Choose a general curve $C \in |2(-K_{F_0}) - B_0|$. Then C is singular at B_0 and $C^1 \circ E_1 = 2$. We get a contradiction immediately: $0 \leq D^1 \circ C^1 = 2n - 2\nu_1 < 0$.

In what follows we assume that B_0 is locally defined by $xy+z^3+w^2 = 0$, $\mathfrak{v}_{div}$ corresponds to the third case and satisfies $e(\mathfrak{v}_{div}) > m$.

The valuation $\mathfrak{v}_{div}$ can be geometrically realized as follows (see [6], lemma 5.2). Let $\psi_1 : V_1 \to V$ be the blow-up of the point B_0. The exceptional divisor E_1 is isomorphic to a quadric cone in $\mathbb{P}^3$. The center of $\mathfrak{v}_{div}$ on V_1 is a point $B_1 \in E_1$ lying outside of the vertex of the cone. By $e_1 \subset E_1$ we denote the generator of the cone that passes through B_1 (see figure 2). Let $\psi_2 : V_2 \to V_1$ be the blow-up of B_1, $E_2 \cong \mathbb{P}^2$ the exceptional divisor. Then $B_2 = \mathrm{Center}_{V_2}\, \mathfrak{v}_{div}$ is a line in E_2 that passes through the point $e_1^2 \cap E_2$ and does not contained in E_1^2. The exceptional divisor E_3 of the blow-up $\psi_3 : V_3 \to V_2$ of B_2 realizes $\mathfrak{v}_{div}$.

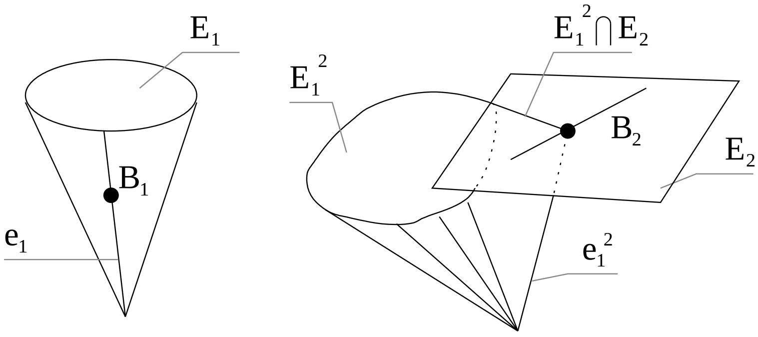

FIGURE 2

We set $\nu_i = \operatorname{mult}_{B_{i-1}} \mathcal{D}^{i-1}$. Observe that $2\nu_1 \geq \nu_2 \geq \nu_3$. We assume that $\nu_1 \leq n$, otherwise we get the situation of the first two cases. The weak super-maximal condition for $\mathfrak{v}_{div}$ has the form

$$\nu_1 + \nu_2 + \nu_3 > 4n + m.$$

We assume that V is obtained by the blow-up $\varphi : V \to U$ of a curve $l \in \mathcal{P}$ on U, and l has a cusp at a point P_0. The birational morphism φ can be also realized as follows (see figure 3). Let $\gamma_1 : U_1 \to U$ be the blow-up of P_0, $L_1 \cong \mathbb{P}^2$ the exceptional divisor. The strict transform l^1 of the curve l on V_1 is tangent to L_1 at a point P_1. The tangent direction to l^1 at P_1 defines a line $r_1 \subset L_1$. Let $\gamma_2 : U_2 \to U_1$ be the blow-up of l^1, L_2 the exceptional divisor. The strict transform L_1^2 of L_1 on V_2 becomes isomorphic to a quadric cone that is blown up at some point outside of the vertex of the cone, and r_1^2 is the (-1)-curve of L_1^2. Note that L_1^2 is tangent to L_2 along the fiber v^+.

Let us turn back to V for instance. We recall that the exceptional divisor G_V of the morphism φ is isomorphic to $l \times \mathbb{P}^1$ (see section 3). Hence G_V is singular along a section s of $V \to \mathbb{P}^1$. This section has the class s_0. Now let $\psi_1 : V_1 \to V$ be the blow-up of B_0, E_1 the exceptional divisor. As it is mentioned above, E_1 is isomorphic to a quadric cone in $\mathbb{P}^3$. The divisor G_V^1 is tangent to E_1 along the generator v. Remark that s^1 intersects E_1 outside of the vertex of the cone.

Finally, there exists a flop $\varepsilon : V_1 \dashrightarrow U_2$ centered at s^1. We observe that the flop ε^{-1} is centered at r_1^2. The birational map ε maps E_1 onto L_1^2, v onto v^+, and G_V^1 onto L_2.

It is worth to give an explicite equation for V in a neighborhood of the point B_0. As in section 3, we assume that U is defined by the equation

$$w^2 + z^3 + zf_4(x, y) + x + f_6(x, y) = 0$$

in a neighborhood of the point $P_0 = (0, 0, 0, 0)$, the polynomials f_i satisfy the same conditions as before. The blow-up φ is given by $x = ty$,

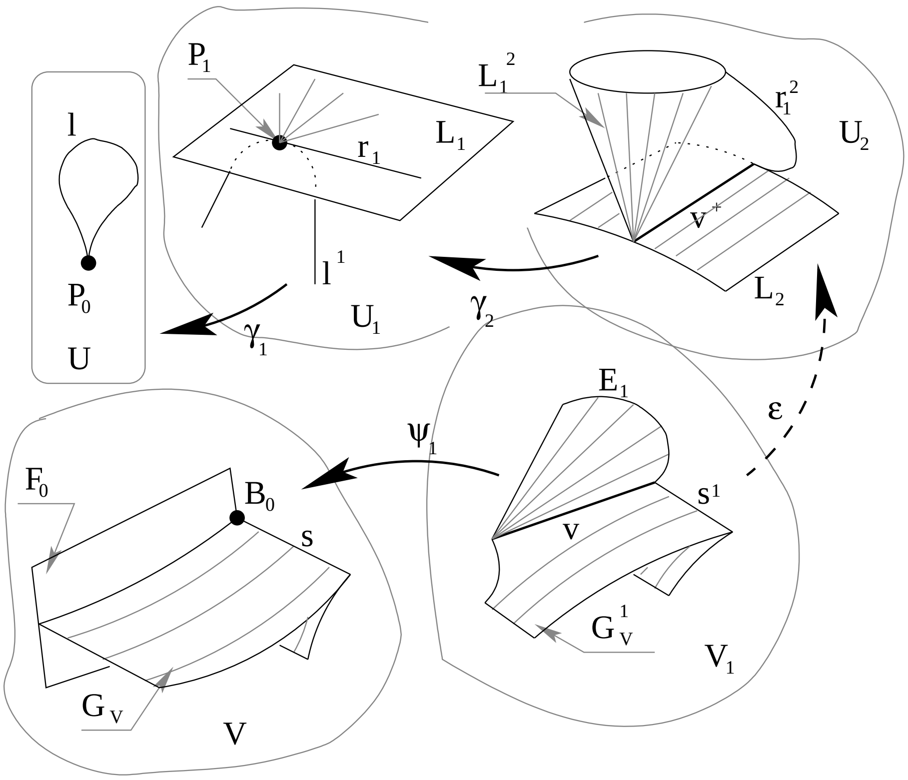

FIGURE 3

so we have for V

$$w^2 + z^3 + zf_4(ty, y) + ty + f_6(ty, y) = 0.$$

We observe that $B_0 = (0, 0, 0, 0)$, $G_V = \{y = 0\}$, the fiber $F_0 = \{t = 0\}$, and $s = \{w = z = y = 0\}$.

We assume that $\mathcal{D}_V = \mathcal{D} \subset |n(-K_V + mF)|$. Denote by $\mathcal{D}_U$ the strict transform of $\mathcal{D}_V$ on U, by $\mathcal{D}_{U_1}$ the strict transform on U_1, and so on. Suppose that $\mathcal{D}_U \subset |aH|$, and let $\mu = \operatorname{mult}_l \mathcal{D}_U$ and $\nu^* = \operatorname{mult}_{P_0} \mathcal{D}_U$. It is easy to compute that

$$\begin{aligned} &a = 2n + m, \\ &\mu = n + m, \\ &\nu^* = \nu_1 + \mu = \nu_1 + n + m. \end{aligned}$$

There are the following possibilities:

A. $e_1 = v$ and $B_1 = s^1 \cap E_1$ (see figures 2 and 3);
B. $e_1 = v$ and $B_1 \notin s^1$;
C. $B_1 \in E^1 \setminus G^1_V$ (that is, $e_1 \neq v$).

We deal with them separately.

The case A: $e_1 = v$ and $B_1 = s^1 \cap E_1$. The divisor G_V is cut out by the hypersurface $\{y = 0\}$. In the local coordinates $[w, z, t]$ the equation $\{w^2 + z^3 = 0\}$ defines G_V. According to the construction of

V via double covering, we see that G_V is a double cover of a quadric $G \cong \mathbb{P}^1 \times \mathbb{P}^1$ with the ramification divisor $3p+p'$, where p and p' have the class $(1,0)$ on G and $p \neq p'$. The quadric G has the local coordinates $[z,t]$. We assume that p is defined by the equation $p = \{z = 0\}$. Remark that s lies over p. Consider a curve $\tilde{C} \subset G$ of the class $(1,2)$ defined by the equation $\{z = at^2\}$ for some $a \neq 0$. Let C be the pre-image of $\tilde{C}$ on G_V. Thus, C is locally defined by the equations $\{y = w^2 + a^3t^6 = 0\}$. Using the equation for V, it is easy to check that C^1 has a double point at the point $s^1 \cap E_1$, and C^2 also has a double point at $s^2 \cap E^2$ (see figure 2 and note that $e_1 = v$ and $B_1 = s^1 \cap E_1$ in the considered case). Taking into account that C has the class $2s + 2f$ on V, for a general $D \in \mathcal{D}_V$ we obtain

$$D^2 \circ C^2 = 4n + 2m - 2\nu_1 - 2\nu_2 \geq 0,$$

hence $\nu_1 + \nu_2 \leq 2n + m$. Since $\nu_3 \leq 2n$, we get a contradiction with the super-maximal condition:

$$4n + m \geq \nu_1 + \nu_2 + \nu_3 > 4n + m.$$

The case B: $e_1 = v$ **and** $B_1 \notin s^1$**.** For the curves v and v^+, we define

$$\eta = \operatorname{mult}_v \mathcal{D}_{V_1} = \operatorname{mult}_{v^+} \mathcal{D}_{U_2}.$$

Since $B_1 \notin s^1$, the flop $\varepsilon : V_1 \dashrightarrow U_2$ is a local isomorphism in a neighborhood of B_1, and we denote the image of B_1 on U_2 by B_1 to avoid complicating the notation. Thus, the picture of the blow-up of B_1 given by figure 2 can be transfered to U_2.

It is easy to compute that $\mathcal{N}_{v^+|U_2} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$. We blow up the point B_1 and then the strict transform of v^+, and denote by $\varphi' : U' \to U_2$ the composition of these blow-ups. Let E' be the exceptional divisor that corresponds to the blow-up of the strict transform of v^+. We see that $E' \cong \mathbb{F}_1$, and denote by s' and f' the minimal section and a fiber of E'. We have

$$\mathcal{D}_{U'}|_{E'} \subset |\eta s' + (2\eta + \mu - \nu_2) f'|.$$

Moreover, the strict transform of L_2 on U' cut out the minimal section s'. Thus, comparing the situation with figure 2, we see that the strict transform of the curve B_2 intersects E' at a point lying outside of the minimal section. Consequently, $\mathcal{D}_{U'}|_{E'}$ has the multiplicity not less than ν_3 at this point. We obtain

$$2\eta + \mu - \nu_2 \geq \nu_3,$$

or $2\eta + \mu \geq \nu_2 + \nu_3$. Since $\nu_1 \leq n$ and $\nu_1 + \nu_2 + \nu_3 > 4n + m$, we get $\nu_2 + \nu_3 > 3n + m$. Taking into account that $\mu = n + m$, we obtain $2\eta > 2n$, that is, $\eta > n$.

Hence the linear system $\mathcal{D}_{V_1}$ has the multiplicity $\eta > n$ along one of the generators of the cone E_1 (this generator is v in our case), which yields $E_1 \subset \operatorname{Bas} \mathcal{D}_{V_1}$ because $\nu_1 \leq n$. Thus we get a contradiction.

The case C: $B_1 \notin E_1 \backslash G_V^1$**.** In this case we have $e_1 \neq v$. Consequently, V_1 and U_1 are isomorphic in a neighborhood of the point B_1. To avoid complicating the notation, by B_1 and e_1 we denote the image of B_1 and the strict transform of e_1 on U_1 respectively. Note that e_1 becomes the line in L_1 that passes through B_1 and P_1, and $e_1 \neq r_1$. We set $\eta = \mathrm{mult}_{e_1} \mathcal{D}_{U_1}$ and $\tilde{\nu} = \mathrm{mult}_{P_1} \mathcal{D}_{U_1}$.

It is easy to compute that $\mathcal{N}_{e_1|U_1} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$. On U_1, let us first blow up the points B_1 and P_1 and then the strict transform of e_1. We denote by $U' \to U_1$ the composition of these blow-ups, by E' the exceptional divisor lying over e_1, and by s' and f' the minimal section and a fiber of E'.

We see that $E' \cong \mathbb{F}_2$, the strict transform of L_1 cuts out the minimal section s' on E'. The strict transforms of the curves l^1 and B_2 intersect E' at two points that lie outside of the minimal section and does not lie in the same fiber. Thus at these points $\mathcal{D}_{U'}|_{E'}$ has the multiplicities not less than μ and ν_3 respectively. Taking into account that

$$\mathcal{D}_{U_1}|_{E'} \subset |\eta s' + (3\eta + \nu^* - \tilde{\nu} - \nu_2)|,$$

where $\nu^* = \nu_1 + n + m = \mathrm{mult}_{P_0} \mathcal{D}_U$, we obtain

$$3\eta + \nu^* - \tilde{\nu} - \nu_2 \geq \mu + \nu_3.$$

But $\tilde{\nu} \geq \mu$, $\nu_2 + \nu_3 > 3n + m$, and $\mu = n + m$, thus we get $\eta > m$. We get the contradiction exactly as in **the case B**.

So, the following proposition is proved:

Proposition 6.1. *Linear systems on V have no super-maximal singularities over the singular point of V.*

7. Conclusion.

We are ready to finish proving theorem 1.1. Let $\rho' : V' \to S'$ be a Mori fibration, $\chi : U \dashrightarrow V'$ a birational map. Consider the complete linear system $\mathcal{D}' = |n'(-K_{V'}) + \rho'^*(L')|$ on V'. We assume that $\mathcal{D}'$ is very ample and $L' \in \mathrm{Pic}(S')$ is an ample divisor.

Denote by $\mathcal{D}$ the strict transform of $\mathcal{D}'$ on U. Suppose that $\mathcal{D} \subset |aH|$. It follows from proposition 4.1 that either $K_V + \frac{2}{a}\mathcal{D}$ is canonical, and then χ is an isomorphism by [1], theorem 4.2, or $\mathcal{D}$ has a maximal singularity over a curve $l \in \mathcal{P}$. In the latter case, let $\mu = \mathrm{mult}_l \mathcal{D}$ and $\varphi : V = U_l \to U$ the blow up of l. We have

$$\mathcal{D}_V = \varphi_*^{-1}\mathcal{D} \subset |(a-\mu)(-K_V) + (2\mu - a)F|.$$

Observe that $2\mu - a > 0$. Propositions 5.1 and 6.1 together with the results of [3], section 2.2, show that $\mathcal{D}_V$ has no maximal singularities over curves or super-maximal singularities over points. Thus $a - \mu = n'$, and from [1], theorem 4.2, it follows that the composition $\chi \circ \varphi : V \dashrightarrow V'$ is birational over the base. Theorem 1.1 is proved.

Remark that theorem 1.1 completes studying the birational classification problem for non-singular Mori fibrations in del Pezzo surfaces of degree 1. Indeed, theorem 2.6 together with corollary 2.10 in [3], proposition 2.12 in [3], and theorem 1.1 of the present paper exhaust all possible sets of Mori structures $\mathcal{MS}$ for varieties of the indicated type. It only remains to recall that $\mathcal{MS}$ is a birational invariant that uniquely defines the birational class.

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