UNBOUNDED BIVARIANT K-THEORY AND CORRESPONDENCES IN NONCOMMUTATIVE GEOMETRY

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Abstract. By adapting the algebraic notion of universal connection to the setting of unbounded KK-cycles, we show that the Kasparov product of such cycles can be defined directly, by an algebraic formula. In order to achieve this it is necessary to develop a framework of smooth algebras and a notion of differentiable C^*-module. The theory of operator spaces provides the required tools. Finally, the above mentioned KK-cycles with connection can be viewed as the morphisms in a category whose objects are spectral triples.

Keywords: KK-theory; Kasparov product; spectral triples; operator modules.

Contents

Date: 9-7-2008.
Key words and phrases. KK-theory, Kasparov product, spectral triples, operator modules.

Introduction

Spectral triples [8] are a central notion in Connes' noncommutative geometry. The data for a spectral triple consist of a $\mathbb{Z}/2$-graded C^*-algebra A, acting on a likewise graded Hilbert space $\mathcal{H}$, and a selfadjoint unbounded odd operator D in $\mathcal{H}$, with compact resolvent, such that the subalgebra

$$\mathcal{A} := \{a \in A : [D, a] \in B(\mathcal{H})\},$$

is dense in A. The above commutator is understood to be graded. The motivating example is the Dirac operator acting on the Hilbert space of L^2-sections of a compact spin manifold M. The C^*-algebra in question is then just $C(M)$. Over the years, many noncommutative examples of this structure have arisen, in particular in foliation theory [11] and examples dealing with non-proper group actions.

Shortly after Connes introduction of spectral triples as cycles for K-homology [9], Baaj and Julg [2] generalized this notion to a bivariant setting, by replacing the Hilbert space $\mathcal{H}$ by a C^*-module $\mathcal{E}$ over a second C^*-algebra B. The notion of unbounded operator with compact resolvent extends to C^*-modules, and the commutator condition is left unchanged. Such an object $(\mathcal{E}, D)$ can be thought of as a field of spectral triples parametrized by B. Baaj and Julg showed, moreover, that such objects can be taken as the cycles for Kasparov's KK-theory [19], and the external product in KK-theory simplifies in this picture. It is given by an algebraic formula.

The results in this paper are twofold. We show that by adapting the notion of universal connection as defined by Cuntz and Quilen [13] to the setting of unbounded KK-cycles, a category of such cycles arises. More precisely, if $(A, \mathcal{H}, D)$ and $(B, \mathcal{H}', D')$ are spectral triples then a *morphism* from D' to D is given by (the unitary ismorphism class of) an unbounded bimodule $(\mathcal{E}, S, \nabla)$ with universal connection

$$\nabla : E^1 \to E^1 \tilde{\otimes}_{\mathcal{B}} \Omega^1(\mathcal{B}),$$

such that

- $[\nabla, S]$ is a completely bounded operator,
- $\mathcal{H} \cong \mathcal{E} \tilde{\otimes}_B \mathcal{H}'$,
- $D = S \otimes 1 + 1 \otimes_\nabla D'$ as unbounded operators.

The operator $1 \otimes_\nabla D'$ is defined by

$$1 \otimes_\nabla D'(e \otimes f) := (-1)^{\partial e}(e \otimes D' f + \nabla_{D'}(e) f),$$

where

$$\nabla_{D'} : E^1 \to E^1 \tilde{\otimes}_{\mathcal{B}} \Omega^1_{D'},$$

is the connection induced by ∇ from the derivation

$$\begin{aligned}\delta : \mathcal{B} &\rightarrow B(\mathcal{H})\\ b &\mapsto [D', b].\end{aligned}$$

The module $E^1 \subset \mathcal{E}$ is a dense $\mathcal{B}$-submodule with properties analoguous to that of a C^*-module, in particular it has the property that

$$E^1 \tilde{\otimes}_{\mathcal{B}} \mathcal{H}' = \mathcal{E} \tilde{\otimes}_B \mathcal{H}'.$$

This type of module has been studied extensively by Blecher ([4], [5]) and turns out to provide the right framework for noncommutive geometry for dense subalgebras of C^*-algebras given by spectral triples. The Haagerup tensor product plays a crucial role in this theory, and we base the definition of $\Omega^1(A)$ on it, instead of the default Grothendieck projective tensor product. The notion of morphism thus obtained can be captured in a diagram:

$$\begin{array}{ccccc} A & \rightarrow & (\mathcal{H}, D) & \leftrightharpoons & \mathbb{C} \\ \downarrow & & & & \| \\ (\mathcal{E}, S, \nabla) & & & & \mathbb{C} \\ \downharpoonleft\upharpoonright & & & & \| \\ B & \rightarrow & (\mathcal{H}', D') & \leftrightharpoons & \mathbb{C}. \end{array}$$

We use the notation $\mathcal{E} \leftrightharpoons B$ to indicate that $\mathcal{E}$ is a C^*-module over B. This also emphasizes the asymmtery, and hence the direction, of the morphsims. The notion can be extended to unbounded bimodules by considering diagrams

$$\begin{array}{ccccc} A & \rightarrow & (\mathcal{F}, D) & \leftrightharpoons & A' \\ \downarrow & & & & \downarrow \\ (\mathcal{E}, S, \nabla) & & & & \mathcal{E}' \\ \downharpoonleft\upharpoonright & & & & \downharpoonleft\upharpoonright \\ B & \rightarrow & (\mathcal{F}', D') & \leftrightharpoons & B'. \end{array}$$

Here the C^*-algebra A' acts on the C^*-module $\mathcal{E}'$ by compact endomorphisms. In this setting one requires $\mathcal{F} \tilde{\otimes}_{A'} \mathcal{E}' \cong \mathcal{E} \tilde{\otimes}_B \mathcal{F}'$, and the operator identities suggested by the diagram. It seems appropriate to refer to a bimodule with connection $(\mathcal{E}, S, \nabla)$ as a *geometric correspondence*. We show these correspondences can be composed to yield a category. By developing a theory of smooth algebras and smooth modules, generalizing the notion of smoothness defined in [8], we show that the composition law can be viewed as a pullback construction on generalized Sobolev chains.

The second result is that the composition of geometric correspondences is the unbounded version of the Kasparov product in KK-theory. Recall that the Kasparov product ([19])

$$KK_i(A,B)\otimes KK_j(B,C)\to KK_{i+j}(A,C),$$

allows one to view the KK-groups as morphisms in a category whose objects are all C^*-algebras. KK is a triangulated category and is universal for C^*-stable, split-exact functors on the category of C^*-algebras [17]. The degree of a KK-cycle is determined by the action of a Clifford algebra. In particular spectral triples can be assigned a degree. If we denote the set of unitary isomorphism classes of geometric correspondences the above spectral triples, which we assume to have degrees i and j, respectively, by $\mathfrak{Cor}(D,D')$, then the main result of this paper states that the *bounded transform* $\mathfrak{b}:D\mapsto D(1+D^2)^{-1}$ defines a functor

$$\begin{aligned}\mathfrak{b}:\mathfrak{Cor}(D,D')&\to KK_{i-j}(A,B)\\ (\mathcal{E},S,\nabla)&\mapsto[(\mathcal{E},\mathfrak{b}(D))].\end{aligned}$$

In particular it follows that the map $K^j(B)\to K^i(A)$ defined by the correspondence maps the K-homology class of $(B,\mathcal{H}',D')$ to that of $(A,\mathcal{H},D)$. The construction of this category, and the bounded tranform functor to KK is one possible answer to a question raised in [10], and can as such be viewed as motivic.

The structure of the paper is as follows. In the first three sections we review the theory of C^*-modules, unbounded operators, KK-theory and operator modules. Although most of this material is well known, we include some results that are not stated explicitly in the literature, or emphasize the interconnection of the theories. This should make the second part of the paper an easier read. In section 4 we introduce a notion of smoothness for spectral triples that generalizes the definition used by Connes [8]. For theoretical purposes this notion is easier to work with and it allows for the definiton of a general notion of smooth C^*-module. In section 5 we adapt the Cuntz-Quillen theory of universal connections to the operatore module setting and obtain results on the structure of the graphs of unbounded operators twisted by such a connection. This is used in section 6 to show that the twisting construction is in fact the Kasparov product in disguise. That in turn leads to the definition of the category of spectral triples described above.

Acknowledgements. My appreciation goes out to Matilde Marcolli, both as a mathematician and as a human being. This paper was written and conceived during my Ph.D. studies at the Max Planck Institut fuer Mathematik in Bonn, Germany, under her supervision. I thank the Max Planck Geselschaft for the financial support of my Ph.D.studies. I also thank Florida State University and the California Institute of Technology for their hospitality and support. Many thanks as well to Nigel Higson, for useful and motivating correspondence and conversations. I thank Saad Baaj for his effort in clarifying to me part of the proof of theorem 2.3.2. I am indebted to Alain Connes for bringing lemma 1.3.4 to my attention, as well as pointing out its usefulness in dealing with resolvent operators. Finally I thank Javier Lopez and Nikolay Ivankov for several useful conversations we had in the course of this project.

1. C^*-MODULES

From the Gelfand-Naimark theorem we know that C^*-algebras are a natural generalization of locally compact Hausdorff topological spaces. In the same vein, the Serre-Swan theorem tells us that finite projective modules are analogues of locally trivial finite-dimensional complex vector bundles over a topological space.The subsequent theory of C^*-modules, pioneered by Paschke and Rieffel, should be viewed in the light of these theorems. They are like Hermitian vector bundles over a space.

1.1. C^*-modules and their endomorphism algebras.

In the subsequent review of the established theory, we will assume all C^*-algebras and Hilbert spaces to be separable, and all modules to be countably generated. This last assumption means that there exists a countable set of generators whose algebraic span is dense in the module.

Definition 1.1.1. Let B be a C^*-algebra. A *right C^*-B-module* is a complex vector space $\mathcal{E}$ which is also a right B-module, equipped with a bilinear pairing

$$\mathcal{E} \times \mathcal{E} \to B$$
$$(e_1, e_2) \mapsto \langle e_1, e_2 \rangle,$$

such that

- $\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle^*$,
- $\langle e_1, e_2 b \rangle = \langle e_1, e_2 \rangle b$,
- $\langle e, e \rangle \geq 0$ and $\langle e, e \rangle = 0 \Leftrightarrow e = 0$,
- $\mathcal{E}$ is complete in the norm $\|e\|^2 := \|\langle e, e \rangle\|$.

We use Landsman's notation ([22]) $\mathcal{E} \leftrightharpoons B$ to indicate this structure.

For two such modules, $\mathcal{E}$ and $\mathcal{F}$, one can consider operators $T : \mathcal{E} \to \mathcal{F}$. As opposed to the case of a Hilbert space ($B = \mathbb{C}$), such operators need not always have an adjoint with respect to the inner product. As a consequence, we consider two kinds of operator between C^*-modules.

Definition 1.1.2. Let $\mathcal{E}, \mathcal{F}$ be C^*-B-modules. The Banach algebra of continuous B-module homomorphims from $\mathcal{E}$ to $\mathcal{F}$ is denoted by $\mathrm{Hom}_B(\mathcal{E}, \mathcal{F})$. Furthermore let

$$\mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F}) := \{T : \mathcal{E} \to \mathcal{E} : \quad \exists T^* : \mathcal{E} \to \mathcal{E}, \quad \langle Te_1, e_2 \rangle = \langle e_1, T^* e_2 \rangle\}.$$

Elements of $\mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$ are called *adjointable operators.*

Similarly we let $\mathrm{End}_B(\mathcal{E})$ and $\mathrm{End}^*_B(\mathcal{E})$ denote the continuous, respectively adjointable endomorphisms of the C^*-module $\mathcal{E}$.

Proposition 1.1.3. *Let $T \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$. Then $\mathrm{End}^*_B(\mathcal{E})$ is a closed subalgebra of $\mathrm{End}_B(\mathcal{E})$, and it is a C^*-algebra in the operator norm and the involution $T \mapsto T^*$.*

The concept of unitary isomorphism of C^*-modules is the obvious one: Two C^*-modules $\mathcal{E}$ and $\mathcal{F}$ over B are *unitarily isomorphic* if there exists a unitary $u \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$. $\mathcal{E}$ and $\mathcal{F}$ are said to be merely *topologically isomorphic* if there exists an invertible element $S \in \mathrm{Hom}_B(\mathcal{E}, \mathcal{F})$. An *isometric isomorphism* is a topological isomorphism that is isometric. The following remarkable result is due to M.Frank.

Theorem 1.1.4 ([15])**.** *Two countably generated C^*-modules are unitarily ismorphic if and only if they are isometrically isomorphic if and only if they are topologically isomorphic.*

$\mathrm{End}^*_B(\mathcal{E})$ contains another canonical C^*-subalgebra. Note that the involution on B allows for considering $\mathcal{E}$ as a left B-module via $be := eb^*$. The inner product can be used to turn the algebraic tensor product $\mathcal{E} \otimes_B \mathcal{E}$ into a $*$-algebra:

$$e_1 \otimes e_2 \circ f_1 \otimes f_2 := e_1\langle e_2, f_1\rangle \otimes f_2, \quad (e_1 \otimes e_2)^* := e_2 \otimes e_1.$$

This algebra is denoted by $\mathrm{Fin}_B(\mathcal{E})$. There is an injective *-homomorphism

$$\mathrm{Fin}_B(\mathcal{E}) \to \mathrm{End}^*_B(\mathcal{E}),$$

given by $e_1 \otimes e_2(e) := e_1\langle e_2, e\rangle$. The closure of $\mathrm{Fin}_B(\mathcal{E})$ in the operator norm is the C^*-algebra of B-*compact operators* on $\mathcal{E}$. It is denoted by $\mathbb{K}_B(\mathcal{E})$.

A *grading* on a C^*-algebra B is a self-adjoint unitary $\gamma \in \mathrm{Aut}B$. If such a grading is present, B decomposes as $B^0 \oplus B^1$, where B^0 is the C^*-subalgebra of *even* elements, and B^1 the closed subspace of *odd* elements. We have $B^iB^j \subset B^{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. For $b \in B^i$, we denote the *degree* of b by $\partial b \in \mathbb{Z}/2\mathbb{Z}$. From now on, we assume all C^*-algebras to be graded, possibly trivially, i.e. $\gamma = 1$.

Definition 1.1.5. A C^*-module $\mathcal{E} \leftrightharpoons B$ is *graded* if it comes equipped with a selfadjoint unitary $\gamma \in \mathrm{Aut}^*_B(\mathcal{E})$ such that

- $\gamma(eb) = \gamma(e)\gamma(b)$,
- $\langle\gamma(e_1), \gamma(e_2)\rangle = \gamma\langle e_1, e_2\rangle$.

In this case $\mathcal{E}$ also decomposes as $\mathcal{E}^0 \oplus \mathcal{E}^1$, and we have $\mathcal{E}^iB^j \subset \mathcal{E}^{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. The algebras $\mathrm{End}_B(\mathcal{E}), \mathrm{End}^*_B(\mathcal{E})$ and $\mathbb{K}_B(\mathcal{E})$ inherit a natural grading from $\mathcal{E}$ by setting $\gamma\phi(e) := \phi(\gamma(e))$. For $e \in \mathcal{E}^i$, we denote the *degree* of e by $\partial e \in \mathbb{Z}/2\mathbb{Z}$.From now on we assume all C^*-modules to be graded, possibly trivially.

1.2. **Tensor products.** For a pair of C^*-modules $\mathcal{E} \leftrightharpoons A$ and $\mathcal{F} \leftrightharpoons B$, the vector space tensor product $\mathcal{E} \otimes \mathcal{F}$ (over $\mathbb{C}$, which will be always supressed in the notation) can be made into a C^*-module over the minimal C^*-tensor product $A\bar{\otimes}B$. The minimal or *spatial* C^*-tensor product is obtained as the closure of $A \otimes B$ in $\mathbb{B}(\mathcal{H} \otimes \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are graded Hilbert spaces that carry faithful graded representations of A and B respectively. In order to make $A\bar{\otimes}B$ into a graded algebra, the multiplication law is defined as

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\partial b_1 \partial a_2} a_1a_2 \otimes b_1b_2. \tag{1.1}$$

The completion of $\mathcal{E} \otimes \mathcal{F}$ in the inner product

$$\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle := \langle e_1, e_2\rangle \otimes \langle f_1, f_2\rangle,$$

is a C^*-module denoted by $\mathcal{E}\bar{\otimes}\mathcal{F}$. It inherits a grading by setting $\gamma := \gamma_{\mathcal{E}} \otimes \gamma_{\mathcal{F}}$.

The graded module so obtained is the *exterior tensor product* of $\mathcal{E}$ and $\mathcal{F}$. The *graded tensor product* of maps $\phi \in \mathrm{End}^*_A(\mathcal{E})$ and $\psi \in \mathrm{End}^*_B(\mathcal{F})$ is defined by

$$\phi \otimes \psi(e \otimes f) := (-1)^{\partial(e)\partial(\psi)}\phi(e) \otimes \psi(f),$$

gives a graded inclusion

$$\mathrm{End}^*_A(\mathcal{E})\bar{\otimes}\mathrm{End}^*_B(\mathcal{F}) \to \mathrm{End}^*_{A\bar{\otimes}B}(\mathcal{E}\bar{\otimes}\mathcal{F}),$$

which restricts to an isomorphism

$$\mathbb{K}_A(\mathcal{E})\overline{\otimes}\mathbb{K}_B(\mathcal{F}) \to \mathbb{K}_{A\overline{\otimes}B}(\mathcal{E}\overline{\otimes}\mathcal{F}).$$

A *-homomorphism $A \to \mathrm{End}^*_B(\mathcal{E})$ is said to be *essential* if

$$A\mathcal{E} := \{\sum_{i=0}^{n} a_i e_i : a_i \in A, e_i \in \mathcal{E}, n \in \mathbb{N}\},$$

is dense in $\mathcal{E}$. If a graded essential *-homomorphism $A \to \mathrm{End}^*_B(\mathcal{F})$ is given, one can complete the algebraic tensor product $\mathcal{E} \otimes_A \mathcal{F}$ to a C^*-module $\mathcal{E}\tilde{\otimes}_A\mathcal{F}$ over B. The norm in which to complete comes from the B-valued inner product

$$\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle := \langle e_1, \langle f_1, f_2\rangle e_2\rangle. \tag{1.2}$$

There is a *-homomorphism

$$\begin{array}{rcl} \mathrm{End}^*_A(\mathcal{E}) & \to & \mathrm{End}^*_B(\mathcal{E}\tilde{\otimes}_A\mathcal{F}) \\ T & \mapsto & T \otimes 1, \end{array}$$

which restricts to a homomorphism $\mathbb{K}_A(\mathcal{E}) \to \mathbb{K}_B(\mathcal{E}\tilde{\otimes}_A\mathcal{F})$. The standard module $\mathcal{H}_B$ absorbs any countably generated C^*-module. The direct sum $\mathcal{E} \oplus \mathcal{F}$ of C^*-B-modules becomes a C^*-module in the inner product

$$\langle (e_1, f_1), (e_2, f_2)\rangle := \langle e_1, e_2\rangle + \langle f_1, f_2\rangle.$$

Theorem 1.2.1 (Kasparov [19]). *Let $\mathcal{E} \leftrightharpoons B$ be a countably generated graded C^*-module. Then there exists a graded unitary isomorphism $\mathcal{E} \oplus \mathcal{H}_B \xrightarrow{\sim} \mathcal{H}_B$.*

1.3. **Unbounded operators.** Similar to the Hilbert space setting, there is a notion of unbounded operator on a C^*-module. Many of the already subtle issues in the theory of unbounded operators should be handled with even more care. This is mostly due to the fact that closed submodules of a C^*-module need not be orthogonally complemented. We refer to [1], [21] and [27] for detailed expositions of this theory.

Definition 1.3.1 ([2]). Let $\mathcal{E}, \mathcal{F}$ be C^*-B-modules. A densely defined closed operator $D : \mathfrak{Dom} D \to \mathcal{F}$ is called *regular* if

- D^* is densely defined in $\mathcal{F}$
- $1 + D^*D$ has dense range.

Such an operator is automatically B-linear, and $\mathfrak{Dom} D$ is a B-submodule of $\mathcal{E}$. There are two operators, $\mathfrak{r}(D), \mathfrak{b}(D) \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$ canonically associated with a regular operator D. They are the *resolvent* of D

$$\mathfrak{r}(D) := (1 + D^*D)^{-\frac{1}{2}}, \tag{1.3}$$

and the *bounded transform*

$$\mathfrak{b}(D) := D(1 + D^*D)^{-\frac{1}{2}}. \tag{1.4}$$

Proposition 1.3.2. *If $D : \mathfrak{Dom} D \to \mathcal{F}$ is regular, then D^*D is selfadjoint and regular. Moreover, $\mathfrak{Dom} D^*D$ is a core for D and $\mathfrak{Im}\mathfrak{r}(D) = \mathfrak{Dom} D$.*

It follows that D is completely determined by $\mathfrak{b}(D)$, as $\mathfrak{r}(D)^2 = 1 - \mathfrak{b}(D)^*\mathfrak{b}(D)$. Due to this fact, selfadjoint regular regular operators share many properties with selfdajoint closed operators on Hilbert space. In particular, they admit a functional calculus.

Theorem 1.3.3 ([1],[21]). *Let $\mathcal{E} \leftrightarrows B$ be a C^*-module, and D a selfadjoint regular operator in $\mathcal{E}$. There is a *-homomorphism $f \mapsto f(D)$, from $C(\mathbb{R})$ into the regular operators on $\mathcal{E}$, such that $(x \mapsto x) \mapsto D$ and $(x \mapsto x(1+x^2)^{-\frac{1}{2}}) \mapsto \mathfrak{b}(D)$. Moreover, it restricts to a *-homomorphism $C_0(\mathbb{R}) \to \mathrm{End}^*_B(\mathcal{E})$.*

This theorem allows us to derive a useful formula for the resolvent of D. We include it here for later reference.

Corollary 1.3.4. *Let D be a selfadjoint regular operator on a C^*-module $\mathcal{E}$. Then the equality*

$$\mathfrak{r}(D)^2 = (1+D^2)^{-1} = \int_0^\infty e^{-x(1+D^2)}dx,$$

holds in $\mathrm{End}^*_B(\mathcal{E})$.

Proof. We have to check convergence of the integral at $x = 0$ and for $x \to \infty$. To this end, let $s \leq t$ and compute:

$$\begin{aligned} \|\int_s^t e^{-x(1+D^2)}dx\| &\leq \int_s^t \|e^{-x(1+D^2)}\|dx \\ &\leq \int_s^t \sup_{y\in\mathbb{R}} |e^{-x(1+y^2)}|dx \\ &= \int_s^t e^{-x}dx \\ &= e^{-t} - e^{-s}. \end{aligned}$$

Hence the integral converges for both $t \to 0$ and $s \to \infty$. □

Recall that a submodule $\mathcal{F} \subset \mathcal{E}$ is *complemented* if $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{F}^\perp$, where

$$\mathcal{F}^\perp := \{e \in \mathcal{E} : \forall f \in \mathcal{F} \quad \langle e, f\rangle = 0\}.$$

Contrary to the Hilbert space case, closed submodules of a C^*-module need not be complemented.

The *graph of* D is the closed submodule

$$\mathfrak{G}(D) := \{(e, De) : e \in \mathfrak{Dom}(D)\} \subset \mathcal{E} \oplus \mathcal{F}.$$

There is a canonical unitary $v \in \mathrm{Hom}_B(\mathcal{E} \oplus \mathcal{F}, \mathcal{F} \oplus \mathcal{E})$, defined by $v(e, f) := (-f, e)$.Note that $\mathfrak{G}(D)$ and $v\mathfrak{G}(D^*)$ are orthogonal submodules of $\mathcal{E} \oplus \mathcal{F}$. The following algebraic characterization of regularity is due to Woronowicz .

Theorem 1.3.5 ([27]). *A densely defined operator $D : \mathcal{E} \to \mathcal{F}$ is regular if and only if $\mathfrak{G}(D) \oplus v\mathfrak{G}(D^*) \cong \mathcal{E} \oplus \mathcal{F}$.*

The isomorphism is given by coordinatewise addition. Moreover, the operator

$$p_D := \begin{pmatrix} \mathfrak{r}(D)^2 & D\mathfrak{r}(D)^2 \\ D\mathfrak{r}(D)^2 & D^2\mathfrak{r}(D)^2 \end{pmatrix} \tag{1.5}$$

satisfies $p_D^2 = p_D^* = p_D$, i.e. it is a projection, and $p_D(\mathcal{E} \oplus \mathcal{F}) = \mathfrak{G}(D)$. $\mathfrak{G}(D)$, which is naturally in bijection with $\mathfrak{Dom}(D)$, inherits the structure C^*-module from $\mathcal{E} \oplus \mathcal{F}$, and hence so does $\mathfrak{Dom}D$. We denote its inner product by $\langle \cdot, \cdot \rangle_1$. Since D commutes with $\mathfrak{r}(D)$, D maps $\mathfrak{r}(D)\mathfrak{G}(D)$ into $\mathfrak{G}(D)$. We denote this operator by D_1.

Proposition 1.3.6. *Let $D : \mathfrak{Dom} D \to \mathcal{E}$ be a selfdajoint regular operator. Then $D_1 : \mathfrak{r}(D)\mathfrak{G}(D) \to \mathfrak{G}(D)$ is a selfadjoint regular operator.*

Proof. From proposition 1.3.2 it follows that

$$\mathfrak{r}(D)\mathfrak{G}(D) = \mathfrak{r}(D)^2\mathcal{E} = \mathfrak{Dom} D^2.$$

D_1 is closed as an operator in $\mathfrak{G}(D)$ for if $\mathfrak{r}(D)^2 e_n \to \mathfrak{r}(D)^2 e$ and $D\mathfrak{r}(D)^2 e_n \to e'$ in the topology of $\mathfrak{G}(D)$, then it follows immediatley that

$$e' = D(D\mathfrak{r}(D)^2 e) = D^2\mathfrak{r}(D)^2 e.$$

It is straightforward to check that D_1 is symmetric for the inner product of $\mathfrak{G}(D)$. Hence it is regular, because $(1 + D^2)\mathfrak{r}(D)^4\mathcal{E} = \mathfrak{r}(D)^2\mathcal{E}$. To prove selfadjointness, suppose $y \in \mathfrak{Dom} D$ is such that there exists $z \in \mathfrak{Dom} D$ such that for all $x \in \mathfrak{r}(D)^2\mathcal{E}$ $\langle D_1 x, y\rangle_1 = \langle x, z\rangle_1$. Then $z = Dy$, because

$$\begin{aligned}
\langle Dx, y\rangle_1 &= \langle Dx, y\rangle + \langle D^2 x, Dy\rangle \\
&= \langle D\mathfrak{r}(D)^2 e, y\rangle + \langle D^2\mathfrak{r}(D)^2 e, Dy\rangle \\
&= \langle \mathfrak{r}(D)^2 e, Dy\rangle + \langle D^2\mathfrak{r}(D)^2 e, Dy\rangle \\
&= \langle e, Dy\rangle.
\end{aligned}$$

A similar computation shows that $\langle x, z\rangle_1 = \langle e, z\rangle$. Since $\mathfrak{r}(D)^2$ is injective this holds for all $e \in \mathcal{E}$, and hence $z = Dy$. Therefore

$$\mathfrak{Dom} D_1^* = \{y \in \mathfrak{Dom} D : Dy \in \mathfrak{Dom} D\} = \mathfrak{Dom} D^2 = \mathfrak{r}(D)^2\mathcal{E} = \mathfrak{Dom} D_1,$$

so D_1 is selfadjoint. □

Corollary 1.3.7. *A selfadjoint regular operator $D : \mathfrak{Dom} D \to \mathcal{E}$ induces a morphism of inverse systems of C^*-modules:*

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr, "D_{i+1}"] & \mathcal{E}_{i+1} \arrow[r] \arrow[dr, "D_i"] & \mathcal{E}_i \arrow[r] \arrow[dr, "D_{i-1}"] & \mathcal{E}_{i-1} \arrow[r] \arrow[dr, "D_{i-2}"] & \cdots \arrow[r] \arrow[dr, "D_1"] & \mathcal{E}_1 \arrow[r] \arrow[dr, "D"] & \mathcal{E} \\
\cdots \arrow[r] & \mathcal{E}_{i+1} \arrow[r] & \mathcal{E}_i \arrow[r] & \mathcal{E}_{i-1} \arrow[r] & \cdots \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E}
\end{tikzcd}$$

Proof. Set $\mathcal{E}_i = \mathfrak{G}(D_{i-1})$. Then the maps $\mathcal{E}_i \to \mathcal{E}_{i-1}$ are just projection on the first coordinate, whereas the maps $D_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$ are the projections on the second coordinates. These maps are adjointable,and we have

$$D_i^*(e_i) = (D_i\mathfrak{r}(D_i)^2 e_i, D_i^2\mathfrak{r}(D_i)^2 e_i), \quad \phi_i^*(e_i) = (\mathfrak{r}(D_i)^2, D_i\mathfrak{r}(D_i)^2).$$

These are exactly the components of the Woronowicz projection 1.5. □

We will refer to this inverse system as the *Sobolev chain* of D.

2. KK-theory

Kasparov's bivariant K-theory KK [19] has become a central tool in noncommutative geometry since its creation. It is a bifunctor on pairs of C^*-algebras, associating to (A, B) a $\mathbb{Z}/2\mathbb{Z}$-graded group $KK_*(A, B)$. It unifies K-theory and K-homology in the sense that

$$KK_*(\mathbb{C}, B) \cong K_*(B) \text{ and } KK_*(A, \mathbb{C}) \cong K^*(A).$$

Much of its usefulness comes from the existence of internal and external product structures, by which KK-elements induce homomorphisms between K-theory and K-homology groups. In Kasparov's original approach, the definition and computation of the products is very complicated. In order to simplify the external product, Baaj and Julg [2] introduced another model for KK, in which the external product is given by a simple algebraic formula. The price one has to pay is working with unbounded operators. We will describe both models, and their relation, together with some results on the structure of KK as a category.

2.1. **C^*-correspondences.** In view of our aim of constrcuting a category of spectral triples, we now describe a natural bimodule category of C^*-algebras. The Kasparov bimodules used to define KK-theory in the next two sections, especially in the unbounded case, should be viewed as refinements of the bimodules described here.

Noncommutative rings behave very differently from commutative rings in many ways. In particular, a given noncommutative ring can have very few ideals, or none at all. $M_n(\mathbb{C})$ for instance, is a simple algebra, and it is a not at all pathological object.The ordinary notion of homomorphism does not give an adequate categorical setting for noncommutative rings, because of the above mentioned lack of ideals. In pure algebra, a more flexible notion of morphism is given by bimodules, whose composition is the module tensor product. We now describe a category of such *correspondences* for C^*-algebras, taking into account the topology of these objects. The resulting category is slightly different from the usual category $\mathfrak{C}^*$, in which morphisms are essential $*$-homomorphisms.

Definition 2.1.1. Let A, B be C^*-algebras. A *C^*-correspondence from A to B* consists of a C^*-B-module $\mathcal{E}$ together with an essential $*$-homomorphism $\pi : A \to \mathrm{End}^*_B(\mathcal{E})$, written $A \rightarrowtail \mathcal{E} \leftrightharpoons B$.
Two such correspondences are called isomorphic when there exists a unitary in $\mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$ intertwining the A-representations.

We can compose correspondences $A \rightarrowtail \mathcal{E} \leftrightharpoons B$ and $B \rightarrowtail \mathcal{F} \leftrightharpoons C$ via the internal tensor product. Denote by $\mathfrak{Cor}_{C^*}(A, B)$ the set of isomorphism classes of correspondences from A to B. It is straightforward to check that the correpondences $A \rightarrowtail A \leftrightharpoons A$ are units for the composition operation modulo unitary equivalence.

Proposition 2.1.2. *Composition of correspondences as described above is associative on isomorphism classes of correspondences. Therefore the sets $\mathfrak{Cor}_{C^*}(A, B)$ are the morphism sets of a category $\mathfrak{Cor}_{C^*}$, whose objects are all C^*-algebras.*

The proof of this result is straightforward, as unitary equivalence provides enough freedom for associativity and identity to hold. There is a functor $\mathfrak{C}^* \to \mathfrak{Cor}_{C^*}$, which is the identity on objects. To a *-homomorphism $\pi \in \mathfrak{C}^*(A, B)$ it associates the correspondence $A \rightarrowtail B \leftrightharpoons B \in \mathfrak{Cor}_{C^*}(A, B)$.

Definition 2.1.3. Let A, B be C^*-algebras. A and B are said to be *strongly Morita equivalent* if there exists a correspondence $A \rightarrowtail \mathcal{E} \leftrightharpoons B$ such that $\pi : A \to \mathrm{End}^*_B(\mathcal{E})$ is an isomorphism onto $\mathbb{K}_B(\mathcal{E})$.

Strong Morita equivalence is amongst the most important equivalence relations for C^*-algebras. Two commutative C^*-algebras are strongly Morita equivalent if and only if they are isomorphic. As such the relation can be viewed as an extension (via the Gelfand-Naimark theorem) of the notion of homeomorphism for locally compact Hausdorff spaces. The following result supports that view.

Theorem 2.1.4. *Two C^*-algebras A, B are isomorphic in $\mathfrak{Cor}_{C^*}$ if and only if they are strongly Morita equivalent.*

The reader can consult [22] for a proof. KK-theory arises by equipping C^*-correspondences with some extra structure in the form of a generalized Fredholm operator, and taking homotopy classes of such correspondences. It yields a category with remarkable properties. The subsequent sections are devoted to the bounded and unbounded descriptions of KK-theory.

2.2. **The bounded picture.** The main idea behind Kasparov's approach to K-homology and KK-theory is that of a family of abstract elliptic operators. This was an idea pioneered by Atiyah, in his construction of K-homology for spaces and the family index theorem. We will consider bimodules $A \to \mathcal{E} \leftrightharpoons B$, without assuming the action of A to be essential, nor the inner product to be full.

Definition 2.2.1. For $p \in \mathbb{N}$, the *complex Clifford algebra* $\mathbb{C}_p$ is the complex unital graded C^*-algebra generated by symbols ε_j, $j = 1, \dots, n$, of degree 1, satisfying the following relations:

$$\varepsilon_j^* = -\varepsilon_j, \quad \varepsilon_j^2 = -1, \quad [\varepsilon_i, \varepsilon_j] = 0.$$

Here we assume $i \neq j$, and the commutator is graded.

The algebra $\mathbb{C}_p$ is generated by the 2^n monomials $\varepsilon_{j_1} \dots \varepsilon_{j_k}$, $0 \leq k \leq n$ and $j_1 < \cdots < j_k$. Considering these monomials as an orthonormal basis, the left regular representation of $\mathbb{C}_p$ on itself equips it with a C^*-norm. It is a well known fact that $\mathbb{C}_{p+2} \cong M_2(\mathbb{C}_p)$. This is sometimes referred to as *formal Bott periodicity.*

Definition 2.2.2. Let $A \to \mathcal{E} \leftrightharpoons B$ be a graded bimodule and $F \in \mathrm{End}_B^*(\mathcal{E})$ an odd operator. $(\mathcal{E}, F)$ is a *Kasparov (A, B)-bimodule* if, for all $a \in A$,

- $[F, a], a(F^2 - 1), a(F - F^*) \in \mathbb{K}_B(\mathcal{E})$.

We denote by $\mathbb{E}_j(A, B)$ the set of Kasparov modules for $(A, B\tilde{\otimes}\mathbb{C}_j)$ modulo unitary equivalence. Unitary equivalence is defined by the existence of a unitary intertwining the action of the algebras and the operators. An *ungraded* C^*-module $\mathcal{E} \leftrightharpoons B$ equipped with a left action of A and an operator F satisfying the relations from definition 2.2.2 defines an element $[(\mathcal{E}', F)] \in \mathbb{E}_1(A, B)$. This is done by setting

$$\mathcal{E}' := \mathcal{E} \oplus \mathcal{E}, \quad \gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F' := \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}, \quad \varepsilon_1 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.6}$$

Here ε_1 is the generator of the Clifford algebra $\mathbb{C}_1$. Ungraded modules of this kind are therefore referred to as *odd* Kasparov modules.

The set of *degenerate* elements consists of bimodules for which

$$\forall a \in A : [F, a] = a(F^2 - 1) = a(F - F^*) = 0.$$

Denote by $e_i : C[0,1]\overline{\otimes}B \to B$ the evalution map at $i \in [0,1]$. Two Kasparov (A, B)-bimodules $(\mathcal{E}_i, F_i) \in \mathbb{E}_j(A, B)$, $i = 0, 1$ are *homotopic* if there exists a Kasparov $(A, C[0,1]\overline{\otimes}B)$-module $(\mathcal{E}, F) \in \mathbb{E}_j(A, C[0,1] \otimes B)$ for which $(\mathcal{E} \otimes_{e_i} B, F \otimes 1)$ is unitarily equivalent to $(\mathcal{E}_i, F_i)$, $i = 0, 1$. It is an equivalence relation, denoted $\sim$. Define

$$KK_j(A, B) := \mathbb{E}_j(A, B)/\sim.$$

KK_j is a bifunctor, contravariant in A, covariant in B, taking values in abelian groups. It is not hard to show that $KK_*(\mathbb{C}, A)$ and $KK_*(A, \mathbb{C})$ are naturally isomorphic to the K-theory and K-homology of A, respectively. Moreover, Kasparov proved the following deep theorem.

Theorem 2.2.3 ([19]). *For any C^*-algebras A, B, C there exists an associative bilinear pairing*

$$KK_i(A, B) \otimes_{\mathbb{Z}} KK_j(B, C) \xrightarrow{\otimes_B} KK_{i+j}(A, C).$$

Therefore, the groups $KK_*(A, B)$ are the morphism sets of a category KK whose objects are all C^*-algebras.

The standard module A^n, viewed as an $(M_n(A), A)$-bimodule, defines an invertible element for the Kasparov product. Hence, in both variables, the KK-groups of A and $M_n(A)$ are isomorphic. In fact, the KK-groups of A and $\mathbb{K}\tilde{\otimes}A$ are naturally isomorphic, which is referred to as the *stability property* of KK-theory. Combining this with formal Bott periodicity yields a natural isomorphism $KK_i(A, B) \xrightarrow{\sim} KK_{i+2}(A, B)$. It follows that KK-theory can be defined using just $\mathbb{E}_0$ and $\mathbb{E}_1$. Moreover KK_1 can be defined using just odd (that is, ungraded) Kasparov modules. Because of this result we will refer to elements of $\mathbb{E}_0(A, B)$ as *even* Kasparov modules. There also is a notion of external product in KK-theory.

Theorem 2.2.4 ([19]). *For any C^*-algebras A, B, C, D there exists an associative bilinear pairing*

$$KK_i(A, C) \otimes_{\mathbb{Z}} KK_j(B, D) \xrightarrow{\overline{\otimes}} KK_{i+j}(A\overline{\otimes}B, C\overline{\otimes}D).$$

The external product makes KK into a symmetric monoidal category

The category KK has more remarkable properties. Although we will not use them in this paper, we do believe they deserve a brief mention. It was shown by Cuntz and Higson ([12],[17]) that the category KK is universal in the sense that any split exact stable functor from the category of C^*-algebras to, say, that of abelian groups, factors through the category KK. Altough it fails to be abelian, KK is a triangulated category. This allows for the development of homological algebra in it, which has special interest in relation to the Baum-Connes conjecture, an approach pursued by Nest and Meyer [24].

2.3. **The unbounded picture.** One can define KK-theory using unbounded operatros on C^*-modules. As the bounded definition corresponds to abstract order zero elliptic pseudodifferential operators, the unbounded version corrseponds to order one operators.

Definition 2.3.1 ([2]). Let $A \to \mathcal{E} \leftrightharpoons B$ be a graded bimodule and $D : \mathfrak{Dom} D \to \mathcal{E}$ an odd regular operator. $(\mathcal{E}, D)$ is an *unbounded* (A, B)*-bimodule* if, for all $a \in \mathcal{A}$, a dense subalgebra of A

- $[D, a]$, extends to an adjointable operator in $\mathrm{End}^*_B(\mathcal{E})$
- $a\mathfrak{r}(D) \in \mathbb{K}_B(\mathcal{E})$.

Denote the set of unbounded bimodules for $(A, B\tilde{\otimes}\mathbb{C}_i)$ modulo unitary equivalence by $\Psi_i(A, B)$. An ungraded module equipped with an operator satisfying the relations from definition 2.3.1 is called an *odd* unbounded (A, B)-bimodule. As in the bounded case, they define elements in $\Psi_1(A, B)$, by replacing F with D in

2.6. As in the bounded case, we will refer to elements of Ψ_0 as *even* unbounded bimodules. In [2] it is shown that $(\mathcal{E}, \mathfrak{b}(D))$ is a Kasparov bimodule, and that every element in $KK_*(A,B)$ can be represented by an unbounded bimodule. The motivation for introducing unbounded modules is the following result.

Theorem 2.3.2 ([2]). *Let $(\mathcal{E}_i, D_i)$ be unbounded bimodules for (A_i, B_i), $i = 1,2$. The operator*

$$D_1 \otimes 1 + 1 \otimes D_2 : \mathfrak{Dom} D_1 \otimes \mathfrak{Dom} D_2 \to \mathcal{E} \otimes \mathcal{F},$$

extends to a selfadjoint regular operator with compact resolvent. Moreover, the diagram

$$\begin{tikzcd}
\Psi_i(A_1,B_1) \times \Psi_j(A_2,B_2) \arrow[r] \arrow[d, "\mathfrak{b}"] & \Psi_{i+j}(A_1 \overline{\otimes} A_2, B_1 \overline{\otimes} B_2) \arrow[d, "\mathfrak{b}"] \\
KK_i(A_1,B_1) \times KK_j(A_2,B_2) \arrow[r, "\overline{\otimes}"] & KK_{i+j}(A_1 \overline{\otimes} A_2, B_1 \overline{\otimes} B_2)
\end{tikzcd}$$

commutes.

Consequently, we can *define* the external product in this way, using unbounded modules, and this is what we will do. Note that lemma 1.3.4 can be used to show that the resolvent of the operator $D_1 \otimes 1 + 1 \otimes D_2$ is compact. Indeed, writing $s = D_1 \tilde{\otimes} 1$ and $t = 1 \tilde{\otimes} D_2$, we have $[s,t] = 0$, i.e. s and t anticommute, and hence

$$\mathfrak{Dom}(s+t) = \mathfrak{Dom} s \cap \mathfrak{Dom} t, \quad 1 + (s+t)^2 = 1 + s^2 + t^2, \quad [s^2, t^2] = 0.$$

Now

$$(2 + s^2 + t^2)^{-1} = \int_0^\infty e^{-x(2+s^2+t^2)} dx = \int_0^\infty e^{-x(1+s^2)} e^{-x(1+t^2)} dx,$$

and $e^{-x(1+s^2)} e^{-x(1+t^2)} = e^{-x(1+D_1^2)} \otimes e^{-x(1+D_2^2)}$ is compact because both the $e^{-x(1+D_i^2)}$ are. Hence by lemma 1.3.4, $(2 + s^2 + t^2)^{-1}$ is a limit of compact operators, which is compact.

In [20], Kucerovsky gives sufficient conditions for an unbounded module $(\mathcal{E} \tilde{\otimes}_A \mathcal{F}, D)$ to be the internal product of $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$. For each $e \in \mathcal{E}$, we have an operator

$$\begin{aligned} T_e : \mathcal{F} &\to \mathcal{E} \tilde{\otimes}_A \mathcal{F} \\ f &\mapsto e \otimes f. \end{aligned}$$

Its adjoint is given by $T_e^*(e' \otimes f) = \langle e, e' \rangle f$. We also need the concept of *semi-boundedness* which carries over from the Hilbert space setting.

Definition 2.3.3 ([20]). Let D be a symmetric operator in a C^*-module $\mathcal{E} \leftrightarrows B$. D is *semi-bounded below* if there exists a real number κ such that $\langle De, e \rangle \geq \kappa \langle e, e \rangle$. If $\kappa \geq 0$, D is *form-positive.*

It is immediate that D is semibounded below if and only if it is the sum of an operator in $\mathrm{End}_B^*(\mathcal{E})$ and a form positive operator. Kucerovsky's result now reads as follows.

Theorem 2.3.4 ([20]). *Let $(\mathcal{E} \tilde{\otimes}_A \mathcal{F}, D) \in \Psi_0(A, C)$. Supppose that $(\mathcal{E}, S) \in \Psi_0(A, B)$ and $(\mathcal{F}, T) \in \Psi_0(B, C)$ are such that*

- *For e in some dense subset of $A\mathcal{E}$, the operator*
$$\left[\begin{pmatrix} D & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix}\right]$$
is bounded on $\mathfrak{Dom}(D \oplus T)$;
- $\mathfrak{Dom} D \subset \mathfrak{Dom} S\tilde{\otimes}1$ *;*
- $\langle Sx, Dx\rangle + \langle Dx, Sx\rangle \geq \kappa\langle x, x\rangle$ *for all x in the domain.*

Then $(\mathcal{E}\tilde{\otimes}_A\mathcal{F}, D) \in \Psi_0(A, C)$ represents the internal Kasparov product of $(\mathcal{E}, S) \in \Psi_0(A, B)$ and $(\mathcal{F}, T) \in \Psi_0(B, C)$.

This theorem only gives sufficient conditions, and tells us very little about the actual form of the product of two given cycles. By equipping unbounded bimodules with some extra differential structure, we will obtain an algebriac description of the product cycle. To this end, we need to extend our scope from C^*-modules to a class of similar objects, defined over a larger class of topological algebras.

3. Operator modules

When dealing with unbounded operators, it becomes necessary to deal with dense subalgebras of C^*-algebras and modules over these. The theory of C^*-modules, which is the basis of Kasparov's approach to bivariant K-theory for C^*-algebras, needs to be extended in an appropriate way. The framework of operator spaces and the Haagerup tensor product provides with a category of modules over operator algebras which is sufficiently rich to accomodate for the phenomana occurring in the Baaj-Julg picture of KK-theory.

3.1. **Operator spaces.**

We will frequently deal with algebras and modules that are not C^*, and with operators that are not adjointable. In this section we discuss the basic notions of the theory of operator spaces, in which all of our examples will fit. The intrinsic approach presented here was taken from [16]. In the classic literature, operator spaces are described using matrix norms. These are globalized to yield the approach involving compact operators given here.

Definition 3.1.1. An *operator space* is a linear space X together with a norm $\|\cdot\|$ on the algebraic tensor product $\mathbb{K} \otimes X$ such that

- For all $b \in \mathbb{B}(\mathcal{H})$ and $v \in \mathbb{K} \otimes X$, $\max\{\|bv\|, \|vb\|\} \leq \|b\|\|v\|$,
- For all orthogonal projections $p, q \in \mathbb{K}$ and $v, w \in \mathbb{K} \otimes X$, $\|pxp + qyq\| = \max\{\|pxp\|, \|qyq\|\}$,
- For each rank one projection $p \in \mathbb{K}$, X is complete in the norm $\|x\| := \|p \otimes x\|$.

A linear map $\phi : X \to Y$ between operator spaces is called *completely bounded*, resp. *completely contractive*, resp. *completely isometric* if the induced map
$$1 \otimes \phi : \mathbb{K} \otimes X \to \mathbb{K} \otimes Y,$$
is bounded, resp. contractive, resp isometric.

The following theorem is very important in identifying operator spaces in practice.

Theorem 3.1.2 ([25])**.** *For every operator space X there exists a Hilbert space $\mathcal{H}$ and and a complete isometry $\phi : X \to \mathbb{B}(\mathcal{H})$.*

Hence an alternative definition of an operator space is that of a complete normed space X that is isometrically isomorphic to a closed subspace of a C^*-algebra. The (unique) C^*-tensor norm on $\mathbb{K}\otimes X$ would then equip X with the structure of an operator space in the sense of definition 3.1.1.

Example 3.1.3. Any C^*-module $\mathcal{E}$ over a C^*-algebra B is an operator space, as it is isometric to $\mathbb{K}(\mathcal{E},B)$, which is a closed subspace of $\mathbb{K}(B\oplus\mathcal{E})$.

Example 3.1.4. Let $(\mathcal{E},D)$ be an unbounded cycle for (A,B) and $\delta:\mathcal{A}\to\mathrm{End}^*_B(\mathcal{E})$ the closed densely defined derivation $a\mapsto[D,a]$. Then $\mathcal{A}$ can be made into an operator space via

$$\begin{aligned}\pi:\mathcal{A}&\to M_2(\mathrm{End}^*_B(\mathcal{E}))\\ a&\mapsto\begin{pmatrix}a&0\\ \delta(a)&a\end{pmatrix}.\end{aligned}$$

Note that, actually $\mathcal{A}\subset\mathrm{End}_B(\mathfrak{G}(D))$, but that π is not *-homomorphism. This example is tantamount in our discussion of the Kasparov product, and it is also the main example of a non-selfadjoint operator algebra.

Definition 3.1.5. For operator spaces X,Y,Z, a bilinear map $\phi:X\times Y\to Z$ is called completely bounded, resp. completely contractive, resp. completely isometric if the operator

$$\begin{aligned}\mathbb{K}\otimes X\times\mathbb{K}\otimes Y&\quad\to\quad\mathbb{K}\otimes Z\\ (m\otimes x,n\otimes y)&\quad\mapsto\quad(mn\otimes\phi(x,y)),\end{aligned}\tag{3.7}$$

is bounded, resp. contractive, resp. isometric.
An *operator algebra* is an operator space $\mathcal{A}$ with a completely contractive multiplication $m:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$. An *operator module* over an operator algebra $\mathcal{A}$ is an operator space X with a completely contractive $\mathcal{A}$-module structure $X\times\mathcal{A}\to X$.

Of course, C^*-algebras and -modules are examples that fit this definition. The module $\mathfrak{G}(D)\subset\mathcal{E}\oplus\mathcal{E}$ from example 3.1.4 is a (left)-operator module over $\mathcal{A}$.
The natural choice of morphisms between operator modules are the completely bounded module maps. If E and F are operator modules over an operator algebra $\mathcal{A}$, we denote the set of these maps by $\mathrm{Hom}^c_{\mathcal{A}}(E,F)$.

3.2. **The Haagerup tensor product.** For operator spaces X and Y, one can define their spatial tensor product $X\otimes Y$ as the norm closure of the algebraic tensor product in the spatial tensor product of some containing C^*-algebras. This gives rise to an exterior tensor product of operator modules.

The internal tensor product of C^*-modules is an example of the Haagerup tensor product for operator spaces. This tensor product will be extremely important in what follows.

Definition 3.2.1. Let X,Y be operator spaces. The *Haagerup norm* on $\mathbb{K}\otimes X\otimes Y$ is defined by

$$\|u\|_h:=\inf\{\sum_{i=0}^n\|x_i\|\|y_i\|:u=m(\sum x_i\otimes y_i),x_i\in\mathbb{K}\otimes X,y_i\in\mathbb{K}\otimes Y\}.$$

Here $m:\mathbb{K}\otimes X\otimes\mathbb{K}\otimes Y\to\mathbb{K}\otimes X\otimes Y$ is the linearization of the map 3.7.

Theorem 3.2.2. *The norm on $X \otimes Y$ induced by the Haagerup norm is given by*

$$\|u\|_h = \inf\{\|x\|\|y\| : x \in X^{n+1}, y \in Y^{n+1}, u = \sum_{i=0}^{n} x_i \otimes y_i\},$$

and the completion of $X \otimes Y$ in this norm is an operator space.

This completion is denoted $X\tilde{\otimes}Y$ and is called the *Haagerup tensor product* of X and Y. By construction, the multiplication in operator algebra $\mathcal{A}$ induces a continuous map $\mathcal{A}\tilde{\otimes}\mathcal{A} \to \mathcal{A}$. A similar statement holds for operator modules.

Now suppose M is a right operator $\mathcal{A}$-module, and N a left operator $\mathcal{A}$-module. Denote by $I_{\mathcal{A}} \subset M\tilde{\otimes}N$ the closure of the linear span of the expressions $(ma \otimes n - m \otimes an)$. The *module Haagerup tensor product* of M and N over $\mathcal{A}$ is

$$M\tilde{\otimes}_{\mathcal{A}}N := M\tilde{\otimes}N/I_{\mathcal{A}},$$

equipped with the quotient norm, in which it is obviously complete. Moreover, if M also carries a left $\mathcal{B}$ operator module structure, and N a right $\mathcal{C}$ operator module structure, then $M\tilde{\otimes}_{\mathcal{A}}N$ is an operator $\mathcal{B},\mathcal{C}$-bimodule. Graded operator algebras and -modules can be defined by the same conventions as in definition 1.1.5 and the discussion preceding it. If the modules and operator algebras are graded, so are the Haagerup tensor products, again in the same way as in the C^*-case, as in the discussion around equation 1.1. The following theorem resolves the ambiguity in the notation for the interior tensor product of C^*-modules and the Haagerup tensor product of operator spaces.

Theorem 3.2.3 ([5])**.** *Let $\mathcal{E}, \mathcal{F}$ be C^*-modules over the C^*-algebras A and B respectively, and $\pi : A \to \mathrm{End}^*_B(\mathcal{F})$ a nondegenrate *-homomorphism. Then the interior tensor product and the Haagerup tensor product of $\mathcal{E}$ and $\mathcal{F}$ are completely isometrically isomorphic.*

This result provides us with a convenient description of algebras of compact operators on C^*-modules. The *dual module* of a C^*-module $\mathcal{E}$ is equal to $\mathcal{E}$ as a linear space, but we equip it with a left C^*-A-module structure using the involution:

$$ae := ea^*, \quad (e_1, e_2) \mapsto \langle e_1, e_2\rangle^*.$$

Theorem 3.2.4 ([5])**.** *There is a complete isometric isomorphism*

$$\mathbb{K}_A(\mathcal{E}\tilde{\otimes}\mathcal{F}) \xrightarrow{\sim} \mathcal{E}\tilde{\otimes}_A\mathbb{K}_B(\mathcal{F})\tilde{\otimes}_A\mathcal{E}^*.$$

In particular $\mathbb{K}_A(\mathcal{E}) \cong \mathcal{E}\tilde{\otimes}_A\mathcal{E}^*$.

The notion of direct sum of operator modules turns out to be a problematic issue [6]. In the C^*-module case, the existence of a canonical inner product on direct sums prevents us from running into problems. This is one of the reasons to work with a more restricted class of modules, resembling C^*-modules in many ways.

3.3. **Approximate projectivity of C^*-modules.** The work of Blecher [5] provides a metric description of C^*-modules which is useful in extending the theory to non C^*-algebras. We will discuss some of his work on these extensions in the next section. The motivating observation for this generalization is the characterization of C^*-modules as "approximately projective" modules, which we now describe.

For a countably generated C^*-A-module $\mathcal{E}$, the algebra $\mathbb{K}_A(\mathcal{E})$ has a countable approximate unit $\{u_\alpha\}_{\alpha\in\mathbb{N}}$ consisting of elements in $\mathrm{Fin}_A(\mathcal{E})$. Replacing u_α by $u_\alpha^*u_\alpha$ if necessary, we may assume

$$u_\alpha=\sum_{i=1}^{n_\alpha}x_i^\alpha\otimes x_i^\alpha.$$

For each n_α we get operators $\phi_\alpha\in\mathbb{K}_A(\mathcal{E},A^{n_\alpha})$, defined by

$$\phi_\alpha:e\mapsto\sum_{i=1}^{n_\alpha}e_i\langle x_i^\alpha,e\rangle, \tag{3.8}$$

where e_i denotes the standard basis of A^{n_α}.We have

$$\phi_\alpha^*:x\mapsto\sum_{i=1}^{n_\alpha}x_i^\alpha\langle e_i,x\rangle, \tag{3.9}$$

and hence $\phi_\alpha^*\circ\phi_\alpha\to\mathrm{id}_\mathcal{E}$ pointwise. This structure determines the $\mathcal{E}$ completely as a C^*-module.

Theorem 3.3.1 ([5]). *Let A be a C^*-algebra and $\mathcal{E}$ be a Banach, (operator) space which is also a right (operator) module over A. $\mathcal{E}$ is (completely) isometrically isomorphic to a countably generated C^*-module if and only if there exists a sequence $\{n_\alpha\}$ of positive integers and contractive module maps*

$$\phi_\alpha:\mathcal{E}\to A^{n_\alpha},\quad\psi_\alpha:A^{n_\alpha}\to\mathcal{E},$$

such that $\psi_\alpha\circ\phi_\alpha$ converges pointwise to the identity on $\mathcal{E}$. In this case the inner product on $\mathcal{E}$ is given by

$$\langle e,f\rangle=\lim_{\alpha\to\infty}\langle\phi_\alpha(e),\phi_\alpha(f)\rangle.$$

For this reason we can think of C^*-modules as approximately finitely generated projective modules. Also note that the maps ϕ_α,ψ_α are by no means unique, and that different maps can thus give rise to the same inner product on $\mathcal{E}$.

3.4. **Rigged modules.** Blecher's characterization of C^*-modules as approximately finitely generated projective modules allows for a generalization of C^*-modules to non-selfadjoint operator algebras. The resulting theory is only slightly more involved than that for the C^*-case, and is exposed in [4]. The following definition is modelled on theorem3.3.1.

Definition 3.4.1. Let $\mathcal{A}$ be an operator algebra and E a right $\mathcal{A}$-operator module. E is an $\mathcal{A}$-*rigged module* if there exists a sequence of positive integers $\{n_\alpha\}$ and completely contractive $\mathcal{A}$-module maps

$$\psi_\alpha:E\to\mathcal{A}^{n_\alpha},\quad\phi_\alpha:\mathcal{A}^{n_\alpha}\to E,$$

such that

- ψ_α and ψ_α are completely contractive ;
- $\psi_\alpha\circ\phi_\alpha\to\mathrm{id}_E$ strongly on E ;
- ψ_α is $\mathcal{A}$-essential ;
- $\forall\beta:\phi_\beta\circ\psi_\alpha\circ\phi_\alpha\to\phi_\beta$ uniformly.

Subsequently define the *dual rigged module* of E by

$$E^*:=\{e^*\in\mathrm{Hom}^c_\mathcal{A}(E,\mathcal{A}):e^*\circ\psi_\alpha\circ\phi_\alpha\to e^*\},$$

and the algebra of $\mathcal{A}$-*compact operators* as $\mathbb{K}_\mathcal{A}(E):=E\tilde{\otimes}_\mathcal{A}E^*$.

It is immediate from this definition that $E^* = \mathbb{K}_{\mathcal{A}}(E, , \mathcal{A})$. A rigged module can be viewed as the direct limit of the spaces $\mathcal{A}^{n_\alpha}$, by letting the transition maps $t_{\alpha\beta} : A^{n_\beta} \to A^{n_\alpha}$ be defined as $t_{\alpha\beta} := \psi_\alpha \circ \phi_\beta$. As such it has the following universal property: If completely contractive module maps $g_\alpha : \mathcal{A}^{n_\alpha} \to W$ into some operator space are given, satisfying $g_\alpha t_{\alpha\beta} \to g_\beta$, then there is a unique completely contractive morphism $g : E \to W$.

Emphasizing both the absence of a genuine inner product and the similarites with C^*-modules, Blecher choose to revive Rieffel's terminology of rigged modules. Instead of an inner product, we do have at our disposal the duality pairing $E \times E^* \to \mathcal{A}$. By abuse of notation, we will denote this pairing by $(e, e^*) \mapsto \langle e, e^* \rangle$. Rigged modules can be characterized using this pairing, yielding a description that is closer to the direct definition of a C^*-module.

Theorem 3.4.2. *Suppose $\mathcal{A}, \mathcal{B}$ are operator algebras, E a $(\mathcal{B}, \mathcal{A})$-operator bimodule and $\tilde{E}$ an $(\mathcal{A}, \mathcal{B})$-operator bimodule. Suppose there exist completely contractive pairings $E \times \tilde{E} \to \mathcal{B}$ and $\tilde{E} \times E \to \mathcal{A}$, such that $\langle e, \tilde{e} \rangle f = e \langle \tilde{e}, f \rangle$ and $\langle \tilde{e}, e \rangle \tilde{f} = \tilde{e} \langle e, \tilde{f} \rangle$. If $\mathcal{B}$ has an approximate identity of the form*

$$u_\beta = \sum_{i=0}^{n_\beta} \langle x_i^\beta, \tilde{x}_i^\beta \rangle, \quad \|(x_i^\beta)\| \leq 1, \quad \|(\tilde{x}_i^\beta)\| \leq 1.$$

Then E is a right $\mathcal{A}$-rigged module, $\mathcal{B} \cong \mathbb{K}_{\mathcal{A}}(E)$, and $\tilde{E} \cong E^*$. Moreover every right $\mathcal{A}$-rigged module arises in this way.

This description will be the one useful for us in dealing with unbounded bivariant K-theory. There is an analogue of adjointable operators on rigged modules. Their defintion is straightforward.

Definition 3.4.3. A completely bounded operator $T : E \to F$ between rigged modules is called *adjointable* if there exists an operator $T^* : F^* \to E^*$ such that

$$\forall e \in E, f^* \in F^* : \quad \langle Te, f^* \rangle = \langle e, T^* f^* \rangle.$$

The space of adjointable operators from E to F is denoted $\mathrm{End}^*_{\mathcal{A}}(E, F)$.

The compact and adjointable operators satisfy the usual relation $\mathrm{End}^*_{\mathcal{A}}(E) = \mathscr{M}(\mathbb{K}_{\mathcal{A}}(E))$, where $\mathscr{M}$ denotes the multiplier algebra. The direct sum of rigged modules is canonically defined. If $(E, \psi_\alpha^E, \phi_\alpha^E)$ and $(F, \psi_\alpha^F, \phi_\alpha^F)$ are rigged modules, $(E \oplus F, \psi_\alpha^E \oplus \psi_\alpha^F, \phi_\alpha^E \oplus \phi_\alpha^F)$ equips $E \oplus F$ with the structure of a rigged module. For the constrcution of general infinite direct sums, see [4]. As can be expected from theorem 3.2.3, the Haagerup tensor product of rigged modules behaves like the interior tensor product of C^*-modules.

Theorem 3.4.4. *Let E be a right $\mathcal{A}$-rigged module and F an $(\mathcal{A}, \mathcal{B})$ rigged bimodule. Then $E \tilde{\otimes}_{\mathcal{A}} F$ is a $\mathcal{B}$-rigged module and $\mathbb{K}_{\mathcal{B}}(E \tilde{\otimes}_{\mathcal{A}} F) \cong E \tilde{\otimes}_{\mathcal{A}} \mathbb{K}_{\mathcal{B}}(F) \tilde{\otimes} E^*$.*

If $\mathcal{B} = B$ happens to be a C^*-algebra, then $E \tilde{\otimes}_{\mathcal{A}} \mathcal{F}$ is a C^*-module. The rigged structure on $E \tilde{\otimes}_{\mathcal{A}} \mathcal{F}$ can be implemented by the approximate unit

$$\sum_{i,j=1}^{n_\alpha, n_\beta} e_i^\alpha \otimes f_j^\beta \otimes f_j^\beta \otimes \tilde{e}_i^\alpha,$$

where
$$\sum_{i=1}^{n_\alpha} e_i^\alpha \otimes \tilde{e}_i^\alpha \quad \text{and} \quad \sum_{j=1}^{n_\beta} f_j^\beta \otimes f_j^\beta,$$
are approximate units for $\mathbb{K}_{\mathcal{A}}(E)$ and $\mathbb{K}_B(\mathcal{F})$, respectively. The inner product on $E\tilde{\otimes}_{\mathcal{A}}\mathcal{F}$ is then given by

$$\begin{aligned}\langle e\otimes f, e'\otimes f'\rangle : &= \lim_{\alpha,\beta}\sum_{i,j=1}^{n_\alpha,n_\beta}\langle\langle \tilde{e}_i^\alpha, e\rangle f, f_j^\beta\rangle\langle f_j^\beta, \langle\tilde{e}_i^\alpha, e\rangle f\rangle \\ &= \lim_{\alpha}\sum_{i=1}^{n_\alpha}\langle\langle \tilde{e}_i^\alpha, e\rangle f, \langle\tilde{e}_i^\alpha, e\rangle f\rangle. \end{aligned} \tag{3.10}$$

In this way one constructs C^*-modules from noninvolutive representations $\mathcal{A}\to\mathrm{End}_B^*(\mathcal{F})$.

Example 3.4.5 (The standard module)**.** Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $\mathcal{A}$ an operator algebra. Then the $\mathcal{H}_{\mathcal{A}} := \mathcal{H}\tilde{\otimes}\mathcal{A}$ is the *standard rigged module* over $\mathcal{A}$.

The Haagerup tensor product can be used to define a notion of projective rigged module, which in the finitely generated case coincides with the usual algebraic notion of projectivity. This notion is different from Connes topological projective modules [9], but the definition is completely analoguous.

Definition 3.4.6. Let E be a rigged module over an operator algebra $\mathcal{A}$. E is a *projective* rigged module if there exists a Hilbert space space $\mathcal{H}$ such that E is completely isometrically isomorphic to a direct summand in $\mathcal{H}\tilde{\otimes}\mathcal{A}$.

Such an E has the usual properties of a projective object in a category. We will state one of them.

Proposition 3.4.7. *An $\mathcal{A}$-rigged module P is projective if and only if any diagram of completely bounded $\mathcal{A}$-module maps*

$$\begin{tikzcd} & P \arrow[d, "\phi"'] \\ M \arrow[r, "\psi"] & N \end{tikzcd}$$

such that ψ admits a completely bounded linear splitting, can be completed to a diagram

$$\begin{tikzcd} & P \arrow[dl, "\chi"'] \arrow[d, "\phi"'] \\ M \arrow[r, "\psi"] & N. \end{tikzcd}$$

Proof. $\Rightarrow$ Let Q be such that $P\oplus Q\cong\mathcal{H}\tilde{\otimes}\mathcal{A}$ and replace ϕ,ψ by $\phi\oplus\mathrm{id}: P\oplus Q\to M\oplus Q$ and $\psi\oplus\mathrm{id}: M\oplus Q\to N\oplus Q$. Then the hypotheses on these maps are still valid, and we can define

$$\mathscr{H}_{\mathcal{A}} \to M \oplus Q$$
$$e_\alpha \mapsto \psi^{-1} \circ \phi(e_\alpha),$$

where e_α is a basis for $\mathscr{H}$. This fills in the diagram.
$\Leftarrow$ If any such diagram can be filled in, we chose $N = P$ and $M = \mathscr{H}\tilde{\otimes}\mathcal{A}$, where $\mathscr{H} = \ell^2(X)$, where X is a generating set for P. $\square$

4. Smoothness

There are several definitions of smoothness to be found in the literature. We adopt the philosophy that a smooth structure on a C^*-algebra should come from a spectral triple (or, equivalently, from an unbounded bimodule). The most important feature of a smooth subalgebra is stability under holomorphic functional calculus, implying K-equivalence. We will show our smooth algebras satisfy this property. Moreover, we give sufficient conditions for an unbounded module to define a smooth structure. Subsequently, we turn to the notion of a smooth C^*-module over a C^*-algebra equipped with a smooth structure. In this and subsequent sections, unless otherwise stated, all operator algebras will be assumed to have a unit.

4.1. Smooth algebras.

The following notion of smoothness will be used. It is slightly more general than Connes' notion of smoothness for spectral triples.

Definition 4.1.1. Let $\mathcal{E}$ be an unbounded (A, B)-bimodule and view the Sobolev modules $\mathcal{E}_i$ as submodules of $\mathcal{E}$. $\mathcal{E}$ is said to be *(left) smooth* if the subalgebra

$$\mathscr{A} := \bigcap_{i=0}^{\infty} \{a \in A : [D_i, a] \in \mathrm{End}^*_B(\mathcal{E}_i)\},$$

is dense in A.

If $\mathscr{A} \subset \mathfrak{Dom}^\infty(\mathrm{ad}D)$, the bimodule will be referred to as being *naively smooth.* Recall that Connes calls a spectral triple $(\mathscr{A}, \mathcal{H}, D)$ smooth if both $\mathscr{A}$ and $[D, \mathscr{A}]$ are in $\mathfrak{Dom}^\infty\mathrm{ad}|D|$.

Lemma 4.1.2. *Let $(\mathcal{E}, D)$ be an unbounded (A, B)-bimodule. Then for $(\mathcal{E}, D)$ to be smooth it suffices that it be naively smooth or smooth in the sense of Connes.*

Proof. Since naive smoothness implies smoothness in the sense of Connes, we show that the latter implies smoothness in the sense of definition 4.1.1. Note that for any unbounded regular operator S, $\mathfrak{Dom}S = \mathfrak{Dom}|S|$ as C^*-modules, since $\mathfrak{r}(S) = \mathfrak{r}(|S|)$. Connes conditions assure that $\mathscr{A} \to \mathrm{End}^*_B(\mathfrak{Dom}|D|^i)$, and hence the module is smooth in our sense. $\square$

Denote by $\pi_i : \mathscr{A} \to \mathrm{End}^*_B(\mathcal{E}_i \oplus \mathcal{E}_i)$ the representations

$$a \mapsto \begin{pmatrix} a & 0 \\ [D_i, a] & a \end{pmatrix}.$$

Let $\mathcal{A}_i$ be the closure of $\mathscr{A}$ in the norm inherited via π_i. It is clear that the $\mathcal{A}_i$ are operator algebras and that $\mathscr{A}$ equals their inverse limit. Hence it carries a Fréchet topology.

Proposition 4.1.3. *The inclusions $\mathcal{A}_{i+1} \to \mathcal{A}_i$ are completely contractive.*

Proof. We have to show that for all $a \in \mathcal{A}_i$, for all n, $\|\pi_i^n(a)\| \leq \|\pi_{i+1}^n(a)\|$, where $\pi_i^n : M_n(\mathcal{A}_i) \to \mathrm{End}_B^*(\bigoplus_{j=1}^n \mathcal{E}_i)$ and

$$\pi_{i+1}^n : M_n(\mathcal{A}_{i+1}) \to \mathrm{End}_B^*(\bigoplus_{j=1}^{2n} \mathcal{E}_i)$$
$$(a_{kn}) \mapsto (\pi_i(a_{km})).$$

Denote by $\iota : \mathcal{E}_i \to \mathcal{E}_i \oplus \mathcal{E}_i$ the inclusion in the first coordinate, and by $p : \mathcal{E}_i \oplus \mathcal{E}_i \to \mathcal{E}_i$ the projection on the first coordinate. Set

$$\iota_n := \oplus_{j=1}^n \iota : \bigoplus_{j=1}^{n} \mathcal{E}_i \to \bigoplus_{j=1}^{2n} \mathcal{E}_i, \quad p_n := \oplus_{j=1}^n p : \bigoplus_{j=1}^{2n} \mathcal{E}_i \to \bigoplus_{j=1}^{n} \mathcal{E}_i.$$

Then we have

$$\|\pi_i^n(a)\| = \|p_n \pi_{i+1}^n(a) \iota_n\| \leq \|p\| \|\pi_{i+1}^n(a)\| \|\iota_n\| = \|\pi_{i+1}^n(a)\|,$$

as desired. □

Now we turn to spectral invariance of the $\mathcal{A}_i$. The following definition is a modification of [3], definition 3.11:

Definition 4.1.4. Let $\mathscr{A}$ be an algebra with Banach norm $\|\cdot\|$. A norm $\|\cdot\|_\alpha$ on $\mathscr{A}$ is said to be *analytic* with respect to $\|\cdot\|$ if for each $x \in \mathcal{A}$, with $\|x\| < 1$ we have

$$\limsup_{n\to\infty} \frac{\ln \|x^n\|_\alpha}{n} \leq 0.$$

The reason for introducing the concept of analyticity is that analytic inclusions are spectral invariant.

Proposition 4.1.5 ([3])**.** *Let $\mathcal{A}_\beta \to \mathcal{A}_\alpha$ be a continuous dense inclusion of unital Banach algebras. If $\|\cdot\|_\beta$ is analytic with respect to $\|\cdot\|_\alpha$, then for all $a \in \mathcal{A}_\beta$ we have $\mathrm{Sp}_\beta(a) = \mathrm{Sp}_\alpha(a)$.*

Proof. It suffices to show that if $x \in \mathcal{A}_\beta$ is invertible in $\mathcal{A}_\alpha$, then $x^{-1} \in \mathcal{A}_\beta$. To this end choose $y \in \mathcal{A}_\beta$ with $\|x^{-1} - y\| < \frac{1}{2\|x\|_\alpha}$. Then $\|2 - 2xy\|_\alpha < 1$. By analyticity, there exists n such that $\|(2-2xy)^n\|_\beta < 1$, and hence $2 \notin \mathrm{Sp}_\beta(2-2xy)$. But then $0 \notin \mathrm{Sp}_\beta(2xy)$, hence $2xy$ has an inverse $u \in \mathcal{A}_\beta$. Therefore $x^{-1} = 2yu$. □

In order to prove spectral invariance of the inclusions $\mathcal{A}_{i+1} \to \mathcal{A}_i$ we need the following straightforward result, whose proof we include for the sake of completeness.

Lemma 4.1.6. *Let $\mathcal{A}$ be a Banach algebra and $\delta : \mathcal{A}_\alpha \to M$ a densely defined closed derivation into a Banach $\mathcal{A}$-module M. Then $\|a\|_\alpha := \|a\| + \|\delta(a)\|$ is analytic with respect to $\|\cdot\|$.*

Proof. Let $\|x\|<1$. We have $\|\delta(x^n)\|\leq n\|\delta(x)\|$, by an obvious induction. Then

$$\begin{aligned}\limsup_{n\to\infty}\frac{\ln\|x^n\|_\alpha}{n}&=\limsup_{n\to\infty}\frac{\ln(\|x^n\|+\|\delta(x^n)\|)}{n}\\&\leq\limsup_{n\to\infty}\frac{\ln(1+n\|\delta(x)\|)}{n}\\&\leq\limsup_{n\to\infty}\frac{\ln n}{n}+\frac{\ln(1+\|\delta(x)\|)}{n}\\&=0.\end{aligned}$$

□

Theorem 4.1.7. *Let $(\mathcal{E},D)$ be a smooth unbounded (A,B) bimodule. Then all inclusions $\mathcal{A}_{i+1}\to\mathcal{A}_i$ are spectral invariant, and hence $\mathscr{A}$ and all the $\mathcal{A}_i$ are stable under holomorphic functional calculus in A.*

Proof. Observe that

$$\|\pi_{i+1}(a)\|\leq\|\pi_i(a)\|+\|\pi_{i+1}(a)-\begin{pmatrix}a&0\\0&a\end{pmatrix}\|\leq 2\|\pi_i(a)\|+\|\pi_{i+1}(a)\|\leq 3\|\pi_{i+1}(a)\|;$$

and $\|\pi_{i+1}(a)-\begin{pmatrix}a&0\\0&a\end{pmatrix}\|=\|[D_i,a]\|$. Thus, by lemma 4.1.6, $\|\cdot\|_{i+1}$ is equivalent to a norm analytic with respect to $\|\cdot\|_i$. □

In the sequel, by a *smooth structure* on a C^*-algebra A we shall mean an inverse system of operator algebras

$$\cdots\to\mathcal{A}_{i+1}\to\mathcal{A}_i\to\cdots\to A$$

where the maps are spectral invariant complete contractions with dense range. In that, denote $\mathscr{A}=\lim_{\leftarrow}\mathcal{A}_i$. A *smooth C^*-algebra* shall be a C^*-algebra with a smooth structure coming from a smooth unbounded bimodule.

4.2. **Smooth C^*-modules.** In differential geometry, a finite dimensional topological vector bundle over a smooth manifold M can always be smoothened, i.e. it can be equipped with a smooth structure. From the algebro-analytic perspective this can be understood in terms of the spectral invariance of the algebra of smooth functions $C^\infty(M)\subset C(M)$. This spectral invariance passes to the matrix algebras, and any projection $p\in M_k(C(M))$ is close to a smooth projection q. By a now standard result, the bundles defined by p and q are isomorphic, giving the smooth structure. For infinite dimensional bundles the situation is more complicated, and I am not aware of any results of this kind in this setting. Thefore we will demand our modules to be smooth.

Definition 4.2.1. Let B be a smooth C^*-algebra, with smooth structure $\{\mathcal{B}_i\}$. A C^*-B-module $\mathcal{E}$ is a C^k-B-module, if there is an approximate unit

$$u_\alpha:=\sum_{i=0}^{n_\alpha}x_i^\alpha\otimes x_i^\alpha\in\mathrm{Fin}_B(\mathcal{E}),$$

such that for each α and $0\leq i,j\leq n_\alpha$, $\langle x_i^\alpha,x_j^\alpha\rangle\in\mathcal{B}_k$, and $\|\langle x_i^\alpha,x_j^\alpha\rangle\|_k\leq 1$. It is a *smooth* C^*-module if there is such an approximate unit that makes it a C^k-module for all k.

Proposition 4.2.2. *Let B be a smooth C^*-algebra and $\mathcal{E}$ a smooth C^*-B-module, with corresponding approximate unit $u_\alpha := \sum_{i=0}^{n_\alpha} x_i^\alpha \otimes x_i^\alpha$. Then*

$$E^k := \{e \in \mathcal{E} : \langle x_i^\alpha, e\rangle \in \mathcal{B}_k, \quad \sup_\alpha \| \sum_{i=1}^{n_\alpha} e_i \langle x_i^\alpha, e\rangle\|_k < \infty\},$$

is a rigged $\mathcal{B}_k$-module. Moreover, the inclusions $E_{k+1} \to E_k$ are completely contractive with dense range, and $E_{k+1}\tilde{\otimes}_{\mathcal{B}_{k+1}}\mathcal{B}_k \cong E_k$.

Proof. Recall the discussion before theorem 3.3.1. The maps $\phi_\alpha, \phi_\alpha^*$ of 3.8,3.9 restrict to maps

$$\phi_\alpha^k : \mathcal{B}_i^{n_\alpha} \to E^k, \quad \psi_\alpha^k : E^k \to \mathcal{B}_k^{n_\alpha}.$$

These are completely contractive for the matrix norms on E^k given by

$$\|(e_{ij})\| := \sup \|(\psi_k^\alpha(e_{ij}))\|,$$

and E^k is (by definition) complete in these matrix norms. It is straightforward to check that E_k is a rigged-$\mathcal{B}_k$-module in this way. For the last statement, the isomorphism will be implemented by the multiplication map

$$\begin{aligned} m : E^{k+1}\tilde{\otimes}_{\mathcal{B}_{k+1}}\mathcal{B}_k &\to E^k \\ e\otimes b &\mapsto eb. \end{aligned}$$

The inverse to this map is constructed via the direct limit property of E^k. Via the identification $\mathcal{A}_k^{n_\alpha} \cong \mathcal{A}_{k+1}^{n_\alpha}\tilde{\otimes}_{\mathcal{A}_{k+1}}\mathcal{A}_k$ define maps

$$\begin{aligned} m_\alpha^{-1} : \mathcal{B}_k^{n_\alpha} &\to E^{k+1}\tilde{\otimes}_{\mathcal{B}_{k+1}}\mathcal{B}_k \\ e_i &\mapsto \phi_\alpha^{k+1}(e_i)\otimes 1. \end{aligned}$$

They obviously satisfy the compatibility condition mentioned after definition 3.4.1 and induce a map $m^{-1} : E^k \to E^{k+1}\tilde{\otimes}_{\mathcal{B}_{k+1}}\mathcal{B}_k$, inverting m. $\square$

If B is a smooth C^*-algebra and $(\mathcal{E}, D)$ a left smooth unbounded (A,B)-bimodule $(\mathcal{E}, D)$, the appropriate notion of smoothness is the following. For each k, the Sobolev module $\mathcal{E}_i$ is smooth over B, and, denoting the associated inverse system by $\{E_i^k\}$, the adjointable operator $D_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$ restricts to an adjointable operator $D_{i,k} : E_{i+1}^k \to E_i^k$. We then require the algebra

$$\mathscr{A}_i := \bigcap_{i=0}^{\infty}\{a \in \mathcal{A}_i : [D_{i,j}, a] \in \operatorname{End}^*_{\mathcal{B}_i}(E_i^j)\},$$

to be dense in $\mathcal{A}_i$, for each i. We will call an unbounded bimodule *smooth* if it is smooth in this sense. This can be visualized by a diagram

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & E^{j+1}_{i+1} \arrow[r] \arrow[d] & E^{j+1}_{i} \arrow[r] \arrow[d] & E^{j+1}_{i-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{j+1} \arrow[d] \\
\cdots \arrow[r] & E^{j}_{i+1} \arrow[r] \arrow[d] & E^{j}_{i} \arrow[r] \arrow[d] & E^{j}_{i-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{j} \arrow[d] \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathscr{E}_{i+1} \arrow[r] & \mathscr{E}_{i} \arrow[r] & \mathscr{E}_{i-1} \arrow[r] & \cdots \arrow[r] & \mathscr{E}.
\end{tikzcd}$$

Here each E^j_i is a rigged $(\mathcal{A}^j_i, \mathcal{B}_j)$-bimodule. The bottom row is just the Sobolev chain of D.

4.3. **Inner products and stabilization.** For any operator algebra $\mathcal{A}$, a dual algebra $\mathcal{A}^*$ is defined, obtained via its realization as a non selfadjoint subalgebra of some C^*-algebra. $\mathcal{A}^*$ is the algebra of adjoints in this C^*-algebra. In general, $\mathcal{A}$ and $\mathcal{A}^*$ are not completely isometrically isomorphic. C^k-algebras do have this property, which makes working with rigged modules over them very similar to working with C^*-modules.

Proposition 4.3.1. *Let A be a smooth C^*-algebra with smooth structure $\{\mathcal{A}_i\}$. For i, $\mathcal{A}_i \cong \mathcal{A}_i^*$ completely isometrically. In particular, the involution on $\mathcal{A}_i$ induces an anti-isomorphism of $\mathcal{A}_i$ with itself.*

Proof. The operator algebra structure of $\mathcal{A}_i^*$ is given by

$$a \mapsto \begin{pmatrix} a & -[D,a] \\ 0 & a \end{pmatrix}.$$

This means that $\mathcal{A}^* = v\mathcal{A}v^* \subset \mathrm{End}^*_B(\mathscr{E})$, where v is the unitary $(x,y) \mapsto (-y,x)$. This isomorphism clearly extends to matrix algebras over $\mathcal{A}^*$. $\square$

Since the $\mathcal{A}_i$ are anti isomorphic to themselves, any right rigged $\mathcal{A}_i$-module has a canonically associated left rigged $\mathcal{A}_i$-module $\overline{E}$. As a linear space, this is just E with the left module structure $ae := ea^*$. The rigged structure comes from considering the modules $\mathcal{A}^{n_\alpha}$ as left modules via the same trick. The structural maps ϕ_α, ψ_α then become left-module maps having the desired properties.

Corollary 4.3.2. *Let $\mathcal{E}$ be a smooth C^*-module over a smooth C^*-algebra B with smooth structure $\{\mathcal{B}_i\}$. There is an isomorphism of rigged modules $\overline{E} \cong E^*$ given by restrcition of the inner product pairing on $\mathcal{E}$.*

Proof. The inner product on $\mathcal{E}$ induces an injection $\overline{E_i} \to E_i^*$. Conversely, for $f^* \in E_i^*$ we have

$$f^*(e) = \lim_\alpha \sum_{i=0}^{n_\alpha} f^*(x_i^\alpha \langle x_i^\alpha, e\rangle).$$

Thus, if we define $f := \lim_\alpha \sum_{i=0}^{n_\alpha} x_i^\alpha f^*(x_i^\alpha)$, it satifies $f^*(e) = \langle f, e\rangle$. □

As a consequence, C^k-modules over a C^k-algebra can be constructed similarly to C^*-modules, by defining a nondegenerate innerproduct pairing satisfying all the properties of definition 1.1.1 and then completing. Stability under holomorphic functional calculus assures us that many properties of C^*-modules carry over to the smooth setting. In particular we can think of adjointable operators in the same way as we do in C^*-modules, and also the notion of unbounded regular operator makes perfect sense. Kasparov's stabilization theorem is a key tool in C^*-modules and KK-theory. There is no such result for general rigged modules over operator algebras, see [4], but in the case of smooth C^*-algebras the result does hold.

Theorem 4.3.3. *Let B be a smooth graded C^*-algebra, and $\mathcal{E}$ a countably generated smooth graded C^*-module. Then $\mathcal{E} \oplus \mathcal{H}_B$ is smoothly isomorphic to $\mathcal{H}_B$. That is, there is an isomorphism of graded inverse systems*

$$\begin{tikzcd}
\cdots \arrow[r] & E_{i+1} \oplus \mathcal{H}_{\mathcal{B}_{i+1}} \arrow[r] \arrow[d] & E_i \oplus \mathcal{H}_{\mathcal{B}_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{E} \oplus \mathcal{H}_B \arrow[d] \\
\cdots \arrow[r] & \mathcal{H}_{\mathcal{B}_{i+1}} \arrow[r] & \mathcal{H}_{\mathcal{B}_i} \arrow[r] & \cdots \arrow[r] & \mathcal{H}_B
\end{tikzcd}$$

Proof. The proof is based on the method of almost orthogonalization as described in [14]. We incorporate it in the proof. For simplicity we ignore the gradings, but note that the proof can be adapted as to respect all gradings involved. Let $u_\alpha := \sum_{i=0}^{n_\alpha} x_i^\alpha \otimes x_i^\alpha$ be an approximate unit for $\mathbb{K}_B(\mathcal{E})$ implementing the smooth structure. The x_i^α form a generating set for $\mathcal{E}$. Denote by $\{e_i\}$ the standard basis of $\mathcal{H}_B$. Let $\{x_n\} \subset \{e_n\} \cup \{x_i^\alpha\}$ be a sequence which meets all the e_n, and all the x_i^α infinitely many times. We proceed by induction. Suppose that orthonormal ellements $h_1, ..., h_n$ and the number $m(n)$ have been constructed in such a way that

- $\{h_1, ..., h_n\} \subset \mathrm{span}_{\mathscr{A}}\{x_1, ..., x_n, e_1, ..., e_{m(n)}\}$
- $d(x_k, \mathrm{span}_{\mathscr{A}}\{h_1, ..., h_n\}) \leq \frac{1}{k}, k = 1, ..., n.$

There exists $m' > m(n)$ such that $e_{m'} \perp \{x_{n+1}, h_1, ..., h_n\}$. Let

$$x' := x_{n+1} - \sum_{i=1}^{n} h_i \langle h_i, x_{n+1}\rangle, \quad x'' = x' + \frac{1}{n+1} e_{m'}.$$

Then $\langle x'', x''\rangle = \langle x', x'\rangle + \frac{1}{(n+1)^2} > 0$, and hence this element is invertible in $\mathscr{A}$, and $\langle x'', x''\rangle^{-\frac{1}{2}} \in \mathscr{A}$ by 4.1.7. Set $h_{n+1} := x''\langle x'', x''\rangle^{-\frac{1}{2}}$. Then

$$h_{n+1} \in \mathrm{span}_{\mathscr{A}}\{x', e_{m'}\} \perp \{h_1, ..., h_n\}.$$

Thus $\{h_1, ..., h_{n+1}\}$ is an orthonormal set. Moreover,

$$x + \frac{1}{n+1} e_{m'} \in \mathrm{span}_{\mathscr{A}}\{h_1, ..., h_{n+1}\},$$

so

$$d(x_{n+1}, \mathrm{span}_{\mathscr{A}}\{h_1, ..., h_{n+1}\}) \leq \frac{1}{n+1}.$$

Thus, by setting $m' = m(n+1)$ we complete the induction step. The sequence $\{h_i\}$ thus constructed is orthonormal and its $\mathscr{A}$ span is dense in each of the modules $E_i \oplus \mathscr{H}_{\mathcal{B}_i}$. □

5. Universal connections

Connections on Riemannian manifolds are a vital tool for differentiating functions and vector fields over the manifold. Cuntz and Quillen [13] developed a purely algebraic theory of connections on modules, which is gives a beautiful characterization of projective modules. They are exactly those modules that admit a universal connection. We review their results, but will recast everything in the setting of operator modules. This is only straightforward, because the Haagerup tensor product linearizes the multiplication in an operator algebra in a continuous way. We then proceed to construct a category of modules with connection, and finally pass to inverse systems of modules.

5.1. Universal forms.

The notion of universal differential form is widely used in noncommutative geometry, especially in connection with cyclic homology [9]. For topological algebras, their exact definition depends on a choice of topological tensor product. The default choice is the Grothendieck projective tensor product, because it linearizes the multiplication in a topological algebra continuously. However, when dealing with operator algebras, the natural choice is the Haagerup tensor product.

Definition 5.1.1. Let $\mathcal{A}$ be an operator algebra. The module of *universal* 1*-forms* over $\mathcal{A}$ is defined as

$$\Omega^1(\mathcal{A}) := \ker(m : \mathcal{A}\tilde{\otimes}\mathcal{A} \to \mathcal{A}).$$

By definition, there is an exact sequence of operator bimodules

$$0 \to \Omega^1(\mathcal{A}) \to \mathcal{A}\tilde{\otimes}\mathcal{A} \xrightarrow{m} \mathcal{A} \to 0.$$

When $\mathcal{A}$ is graded, $\Omega^1(\mathcal{A})$ inherits a grading from $\mathcal{A}\tilde{\otimes}\mathcal{A}$. The map

$$\begin{array}{rcl} d : \mathcal{A} & \to & \Omega^1(\mathcal{A}) \\ a & \mapsto & 1 \otimes a - (-1)^{\partial a} a \otimes 1 \end{array}$$

is a graded bimodule derivation. $\Omega^1(\mathcal{A})$ carries a natural involution, defined by

$$(adb)^* := -(-1)^{\partial b} db^* a^*. \tag{5.11}$$

Lemma 5.1.2. *The derivation d is universal. For any completely bounded graded derivation $\delta : \mathcal{A} \to M$ into an $\mathcal{A}$ operator bimodule, there is a unique completely* *bounded bimodule homomorphism* $j_\delta : \Omega^1(\mathcal{A}) \to M$ *such that the diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\delta"] \arrow[dr, "d"'] & & M \\
& \Omega^1(\mathcal{A}) \arrow[ur, "j_\delta"'] &
\end{tikzcd}$$

commutes.

Proof. Set $j_\delta(da) = \delta(a)$. This determines j_δ because da generates $\Omega^1(\mathcal{A})$ as a bimodule. $\square$

Any derivation $\delta : \mathcal{A} \to M$ has its associated module of forms

$$\Omega^1_\delta := j_\delta(\Omega^1(\mathcal{A})) \subset M.$$

The inner product on $\mathcal{E}$ induces a pairing

$$\begin{aligned}\mathcal{E} \times \mathcal{E}\tilde{\otimes}_{\mathcal{A}}\Omega^1(\mathcal{A}) &\to \Omega^1(\mathcal{A}) \\ (e_1, e_2 \otimes \omega) &\mapsto \langle e_1, e_2\rangle \otimes \omega.\end{aligned}$$

By abuse of notation we write $\langle e_1, e_2 \otimes \omega\rangle$ for this pairing. A pairing

$$\mathcal{E}\tilde{\otimes}_{\mathcal{A}}\Omega^1(\mathcal{A}) \times \mathcal{E} \to \Omega^1(\mathcal{A}),$$

is obtained by setting $\langle e_1 \otimes \omega, e_2\rangle := \langle e_2, e_1 \otimes \omega\rangle^*$.

Definition 5.1.3. Let $\delta : \mathcal{A} \to M$ be a derivation as above, and E a right operator $\mathcal{A}$-module. A δ-*connection* on E is a completely bounded linear map

$$\nabla_\delta : E \to E\tilde{\otimes}_{\mathcal{A}}\Omega^1_\delta,$$

satifying the Leibniz rule

$$\nabla(ea) = \nabla(e)a + e \otimes \delta(a).$$

If $\delta = d$, the connection will be denoted ∇, and referred to as a *universal connection.* If moreover $\mathcal{E}$ is a C^*-module, a connection is *Hermitian* if

$$\langle e_1, \nabla(e_2)\rangle - \langle \nabla(e_1), e_2\rangle = d\langle e_1, e_2\rangle.$$

Note that a universal connection ∇ on a module E gives rise to δ-connections for any completely bounded derivation δ, simply by setting $\nabla_\delta := 1 \otimes j_\delta \circ \nabla$. If δ is of the form $\delta(a) = [S, a]$, for $S \in \mathrm{End}_{\mathbb{C}}(X, Y)$, where X and Y are left $\mathcal{A}$-operator modules, we write simply ∇_S for ∇_δ.

Not all modules admit a universal connection. Cuntz and Quillen showed that universal connections characterize algebraic projectivity. Their proof shows that projective rigged modules admit universal connections, but the class of modules admitting a connection might be larger. for our purposes however, this is sufficient.

Proposition 5.1.4 ([13])**.** *A right* $\mathcal{A}$ *operator module* E *admits a universal connection if and only if the multiplication map* $m : E\tilde{\otimes}\mathcal{A} \to E$ *is* $\mathcal{A}$*-split.*

Proof. Consider the exact sequence

$$0 \longrightarrow E\tilde{\otimes}_{\mathcal{A}}\Omega^1(\mathcal{A}) \xrightarrow{j} E\tilde{\otimes}\mathcal{A} \xrightarrow{m} E \longrightarrow 0,$$

where m is the multiplication map and $j(s\otimes da) = sa\otimes 1 - s\otimes a$. A linear map

$$s : E \to E\tilde{\otimes}\mathcal{A}$$

determines a linear map

$$\nabla : E \to E\tilde{\otimes}_{\mathcal{A}}\Omega^1(\mathcal{A})$$

by the formula $s(e) = e\otimes 1 - j(\nabla(e))$, since j is injective. Now

$$s(ea) - s(e)a = j(\nabla(e)a + e\otimes da - \nabla(ea)),$$

whence s being an $\mathcal{A}$-module map is equivalent to ∇ being a connection. □

Corollary 5.1.5. *A C^*-module $\mathcal{E} \leftrightharpoons A$ admits a Hermitian connection.*

Proof. By the stabilization theorem 1.2.1 $\mathcal{E}$ is an orthogonal direct summand in $\mathcal{H}_A = \mathcal{H}\tilde{\otimes}A$, i.e. $\mathcal{E} = p\mathcal{H}_a$, with $p^2 = p^* = p \in \mathrm{End}^*(\mathcal{H}_A)$. Observe that $\mathcal{H}_A\tilde{\otimes}_A\Omega^1A \cong \mathcal{H}\tilde{\otimes}\Omega^1(A)$.The Levi-Cevita connection

$$\begin{aligned}\nabla : \mathcal{H}_A &\to \mathcal{H}\tilde{\otimes}\Omega^1(A)\\ h\otimes a &\mapsto h\otimes da,\end{aligned}$$

is clearly Hermitian, and since p is a projection, so is $p\nabla p : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_A\Omega^1(A)$. □

5.2. **Inverse systems and smoothness.** As we have seen in corollary 1.3.7, an unbounded operator can be viewed as a morphism of inverse systems of C^*-modules, namely its Sobolev chain.

Definition 5.2.1. Let $\{E_i, \phi_i\}$ be an inverse system of $\mathcal{A}$ rigged modules. A *connection* on $\{E_i, \phi_i\}$ is a family of connections $\nabla_i : E_i \to E_i\tilde{\otimes}\Omega^1(\mathcal{A})$ such that $\phi_{i+1}\otimes 1\circ\nabla_{i+1} = \nabla_i\circ\phi_{i+1}$.

Definition 5.2.2. Let $(\mathcal{E}, D)$ be an unbounded bimodule and $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ a Hermitian connection. ∇ is said to be a *D-connection* if $[\nabla, D]$ extends to a completely bounded operator $\mathcal{E} :\to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$. ∇ is said to be a smooth D -connection if it is a D_i-connection, for all i, where we view the Sobolev modules $\mathcal{E}_i$ as dense submodules of $\mathcal{E}$.

Proposition 5.2.3. *Let $(\mathcal{E}, D)$ be an unbounded (A, B)-bimodule and $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ a smooth D-connection. Then ∇ induces a connection $\{\nabla_i\}$ on the Sobolev chain of D.*

Proof. We apply the usual trick. Define $\{\nabla_i\}$ inductively by

$$\nabla_{i+1} := \begin{pmatrix}\nabla_i & 0\\ [D_i,\nabla_i] & \nabla_i\end{pmatrix}.$$

By definition of smoothness this defines a connection on the Sobolev chain of D. □

Using the smooth stabilization theorem 4.3.3, we get:

Corollary 5.2.4. *A smooth C^*-module $\mathcal{E} \leftrightharpoons A$ admits a Hermitian connection.*

A smooth D-connection is said to be simply a smooth connection if it restricts to a $D_{i-1,k}$-connection $\nabla_{i,k} : E_i^k \to E_i^k\tilde{\otimes}_{\mathcal{B}_k}\Omega^1(\mathcal{B}_k)$.

5.3. **Product connections.** We now proceed to connections on tensor products of projective modules. Anticipating the use of connections on unbounded bimodules, a category of modules with connection is constructed.

Proposition 5.3.1. *Let P be a right projective rigged $\mathcal{A}$-module with a universal connection ∇, P' a right projective rigged $(\mathcal{A}-\mathcal{B})$-bimodule with universal connection ∇'. Then $P\otimes_{\mathcal{A}}P'$ is $\mathcal{B}$-projective, and ∇ and ∇' determine a universal $\mathcal{B}$-connection on $P\otimes_{\mathcal{A}}P'$. If both connections are Hermitian, then so is the induced connection.*

Proof. Let, Q,Q' be such that $P\oplus Q\cong\mathcal{H}\tilde{\otimes}\mathcal{A}$, $P'\oplus Q'\cong\mathcal{H}'\tilde{\otimes}\mathcal{B}$. Then:

$$P\tilde{\otimes}_{\mathcal{A}}P'\oplus Q\tilde{\otimes}_{\mathcal{A}}P'\oplus\mathcal{H}\tilde{\otimes}Q'\cong\mathcal{H}\tilde{\otimes}\mathcal{H}'\tilde{\otimes}\mathcal{B}.$$

Thus $P\tilde{\otimes}_{\mathcal{A}}P'$ is projective. Consider the derivation

$$\begin{aligned}\delta:\mathcal{A}&\to\mathrm{End}_{\mathcal{B}}(P',P'\tilde{\otimes}_{\mathcal{B}}\Omega^1(\mathcal{B}))\\ a&\mapsto[\nabla',a].\end{aligned}$$

By universality there is a unique map

$$j_\delta:\Omega^1(\mathcal{A})\to\Omega^1_\delta,$$

intertwining d and δ. Thus, ∇ induces a connection

$$\nabla_\delta:P\to P\tilde{\otimes}_{\mathcal{A}}\Omega^1_\delta,$$

by composing with j_δ. Subsequently define

$$\begin{aligned}\nabla\tilde{\otimes}_{\mathcal{A}}\nabla':P\tilde{\otimes}_{\mathcal{A}}P'&\to P\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}_{\mathcal{B}}\Omega^1(\mathcal{B})\\ p\otimes p'&\mapsto\nabla'(p')+\nabla_\delta(p)p',\end{aligned}$$

which is a connection. It is a straightforward calculation to check that this connection is Hermitian if ∇ and ∇' are. □

We will refer to the connection of proposition 5.3.1 as the *product connection*. Taking product connections is associative up to isomorphism.

Theorem 5.3.2. *Let P,P',P'' be right projective rigged $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$-modules respectively, with universal connections ∇,∇',∇''. Suppose P',P'' are left $\mathcal{A}$ and $\mathcal{B}$ modules, respectively. The natural isomorphism*

$$P\tilde{\otimes}_{\mathcal{A}}(P'\tilde{\otimes}_{\mathcal{B}}P'')\xrightarrow{\sim}(P\tilde{\otimes}_{\mathcal{A}}P')\tilde{\otimes}_{\mathcal{B}}P''$$

intertwines the product connections $\nabla\tilde{\otimes}_{\mathcal{A}}(\nabla'\tilde{\otimes}_{\mathcal{B}}\nabla'')$ and $(\nabla\tilde{\otimes}_{\mathcal{A}}\nabla')\tilde{\otimes}_{\mathcal{B}}\nabla''$

Proof. The two product connections on $M = P\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}_{\mathcal{B}}P''$ correspond to splittings of the universal exact sequence given by the follwing diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker(1\otimes_{\mathcal{B}} m) \arrow[r] \arrow[d] & M_{\mathcal{A}}\tilde{\otimes}P''\otimes\mathcal{C} \arrow[r, shift left, "1\otimes_{\mathcal{B}} m"] \arrow[d] & M_{\mathcal{A}}\tilde{\otimes}_{\mathcal{B}}P'' \arrow[l, shift left] \arrow[r] & 0 \\
0 \arrow[r] & M\tilde{\otimes}_{\mathcal{C}}\Omega^1(\mathcal{C}) \arrow[r, "j"] & M\tilde{\otimes}\mathcal{C} \arrow[r, "m"] & M \arrow[r] \arrow[u, "\sim"'] \arrow[d, "\sim"'] & 0 \\
0 \arrow[r] & \ker(m\otimes_{\mathcal{A}} 1) \arrow[r] \arrow[u] & P\tilde{\otimes}M_{\mathcal{B}}\tilde{\otimes}\mathcal{C} \arrow[r, shift left, "1\otimes_{\mathcal{A}} m"] \arrow[u] & P\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \arrow[l, shift left] \arrow[r] & 0.
\end{tikzcd}$$

Here $M_{\mathcal{A}} = P\tilde{\otimes}_{\mathcal{A}}P'$ and $M_{\mathcal{B}} = P'\tilde{\otimes}_{\mathcal{B}}P''$. To show that this diagram commutes, observe that the given connections induce natural splittings for the maps

$$P\tilde{\otimes}P'\tilde{\otimes}P'' \to P\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \quad \text{and} \quad P\tilde{\otimes}P'\tilde{\otimes}P'' \to M_{\mathcal{A}}\tilde{\otimes}_{\mathcal{B}}P''.$$

They correspond to the decompositions

$$P\tilde{\otimes}P'\tilde{\otimes}P'' \cong P\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \oplus Q\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \oplus P\tilde{\otimes}Q'\tilde{\otimes}_{\mathcal{B}}P'',$$

and

$$P\tilde{\otimes}P'\tilde{\otimes}P'' \cong M_{\mathcal{A}}\tilde{\otimes}_{\mathcal{B}}P'' \oplus Q_{\mathcal{A}}\tilde{\otimes}_{\mathcal{B}}P \oplus Q\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}P'',$$

where Q, Q' and $Q_{\mathcal{A}}$ are such that

$$P\oplus Q \cong P\tilde{\otimes}\mathcal{A}, \quad P'\oplus Q' \cong P'\tilde{\otimes}\mathcal{B}, \quad M_{\mathcal{A}}\oplus Q_{\mathcal{A}} \cong M_{\mathcal{A}}\tilde{\otimes}\mathcal{B}.$$

That is, Q and Q' come from ∇ and ∇' respectively, and $Q_{\mathcal{A}}$ from $\nabla\tilde{\otimes}_{\mathcal{A}}\nabla'$.

Therefore, the given connections induce natural splittings for the maps

$$P\tilde{\otimes}P'\tilde{\otimes}P'' \to P\tilde{\otimes}_{\mathcal{A}}M_{\mathcal{B}} \quad \text{and} \quad P\tilde{\otimes}P'\tilde{\otimes}P'' \to M_{\mathcal{A}}\otimes_{\mathcal{B}}P''.$$

These splittings correspond to the factorizations

$$\begin{tikzcd}
 & P\tilde{\otimes}P'\tilde{\otimes}P'' \arrow[dl] \arrow[dr] & \\
M_{\mathcal{A}}\tilde{\otimes}P'' \arrow[dr] & & P\tilde{\otimes}M_{\mathcal{B}} \arrow[dl] \\
 & P\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}_{\mathcal{B}}P'' &
\end{tikzcd}$$

of the map $P\tilde{\otimes}P'\tilde{\otimes}P'' \to P\tilde{\otimes}_{\mathcal{A}}P'\tilde{\otimes}_{\mathcal{B}}P''$. These factorizations are exactly the ones that give rise to the product connections $\nabla\tilde{\otimes}_{\mathcal{A}}(\nabla'\tilde{\otimes}_{\mathcal{B}}\nabla'')$ and $(\nabla\tilde{\otimes}_{\mathcal{A}}\nabla')\tilde{\otimes}_{\mathcal{B}}\nabla''$. Therefore the different splittings in the first diagram coincide under the intertwining isomorphisms. $\square$

The upshot of theorems 5.3.1 and 5.3.2 is that there is a category whose objects are operator algebras, and whose morphisms $\mathrm{Mor}(\mathcal{A},\mathcal{B})$ are given by pairs (P,∇) consisting of a right projective $(\mathcal{A},\mathcal{B})$-bimodule P with a universal $\mathcal{B}$ connection. The identiy morphisms are the pairs $(1_{\mathcal{A}},d)$ consisiting of the trivial bimodule $1_{\mathcal{A}}$ and the universal derivation $d:\mathcal{A}\rightarrow\Omega^1(\mathcal{A})$. Of course this category is described equivalently as the category of pairs (P,s) of bimodules together with a splitting s of the universal exact sequence.

5.4. **Induced operators and their graphs.** One can proceed to enrich the category described above by considering triples (P,T,∇) consisting of right projective bimodules with connection and a distinguished endomorphism $T\in\mathrm{End}_{\mathcal{B}}(P)$. Denote by $1\tilde{\otimes}_{\nabla}T$ the operator

$$1\otimes_{\nabla}T(p\otimes p'):=(-1)^{\partial T\partial p}(p\otimes T(p')+\nabla_T(p)p'),$$

which is well defined on $P\tilde{\otimes}_{\mathcal{A}}P'$. The composition law then becomes

$$(P,S,\nabla)\circ(P',T,\nabla'):=(P\tilde{\otimes}_{\mathcal{B}}P',S\tilde{\otimes}1+1\tilde{\otimes}_{\nabla}T,\nabla\tilde{\otimes}_{\mathcal{B}}\nabla').$$

Associativity of this composition is implied by the following proposition.

Proposition 5.4.1. *Let P be a right projective rigged $\mathcal{A}$-module, P' a right projective rigged $(\mathcal{A},\mathcal{B})$-bimodule and ∇,∇' universal connections. Furthermore let E,F be $(\mathcal{B},\mathcal{C})$-bimodules, and $D\in\mathrm{End}_{\mathcal{C}}(E,F)$. Then*

$$1\tilde{\otimes}_{\nabla}1\tilde{\otimes}_{\nabla'}D=1\tilde{\otimes}_{\nabla\tilde{\otimes}_{\mathcal{A}}\nabla'}D,$$

under the intertwining isomorphism.

Proof. Recall the formula for the product connection

$$\nabla\tilde{\otimes}_{\mathcal{A}}\nabla'(p\otimes p'):=p\otimes\nabla'(p')+\nabla_{\delta}(p)p'.$$

Morevoer, write ∇_D for $\nabla_{\nabla'_D}$. It is straightforward to check that

$$(\nabla\tilde{\otimes}_{\mathcal{A}}\nabla')_D(p\otimes p')=p\otimes\nabla'_D(p')+\nabla_D(p)p'.$$

Therefore we have

$$\begin{aligned}1\otimes_{\nabla\tilde{\otimes}_{\mathcal{A}}\nabla'}D(p\otimes p'\otimes e)&=p\otimes p'\otimes De+\nabla\otimes\nabla'(p\otimes p')e\\&=p\otimes p'\otimes De+p\otimes\nabla'_D(p')e+\nabla_D(p)(p'\otimes e).\end{aligned}$$

On the other hand

$$\begin{aligned}1\tilde{\otimes}_{\nabla}1\tilde{\otimes}_{\nabla'}D(p\otimes p'\otimes e)&=p\otimes(1\tilde{\otimes}_{\nabla'}D)(p'\otimes e)+\nabla_{1\tilde{\otimes}_{\nabla'}D}(p)(p'\otimes e)\\&=p\otimes p'\otimes De+p\otimes\nabla'_D(p')e+\nabla_{1\tilde{\otimes}_{\nabla'}}D(p)(p'\otimes e),\end{aligned}$$

thus, it suffices to show that $\nabla_D=\nabla_{1\tilde{\otimes}_{\nabla'}D}$. To this end, observe that

$$[1\tilde{\otimes}_{\nabla'}D,a]=[\nabla'_D,a]:P\otimes_{\mathcal{A}}P'\rightarrow P\otimes_{\mathcal{A}}P',$$

which gives a natural isomorphism $\Omega^1_{\nabla'_D}\xrightarrow{\sim}\Omega^1_{1\tilde{\otimes}_{\nabla'}D}$ intertwining the derivations. By universality this gives a commutative diagram

$$\begin{tikzcd} & \Omega^1(\mathcal{A}) \arrow[dl] \arrow[dr] & \\ \Omega^1_{1\tilde{\otimes}_{\nabla'}D} \arrow[rr, "\sim"] & & \Omega^1_{\nabla'_D}, \end{tikzcd}$$

which shows that $\nabla_D = \nabla_{1\tilde{\otimes}_{\nabla'} D}$. $\square$

As we have seen, a connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B \Omega^1(B)$ can be used to transfer operators on $\mathcal{F}$ to $\mathcal{E} \otimes_B \mathcal{F}$. We now show that this algebraic procedure is well behaved for selfadjoint regular operators T in $\mathcal{F}$, and describe the graph $\mathfrak{G}(1\tilde{\otimes}_\nabla T) \subset \mathcal{E}\tilde{\otimes}_B \mathcal{F} \oplus \mathcal{E}\tilde{\otimes}_B \mathcal{F}$ as a topological C^*-module, in terms of the graph of T.

Lemma 5.4.2. *Let $\mathcal{E}, \mathcal{F}$ be C^*-modules over B and C respectively, and $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}\Omega^1(B)$ a Hermitian connection. Suppose $\mathcal{F}$ is a left B-module and $T : \mathfrak{Dom}(T) \to \mathcal{F}$ a selfadjoint regular operator such that $[T, b] \in \mathrm{End}^*_B(\mathcal{F})$ for all $b \in \mathcal{B}_1 \subset B$, a dense subalgebra of B. If ∇ and $\mathcal{E}$ are C^1 with respect to $\mathcal{B}_1$, then the operator $1\tilde{\otimes}_\nabla T$ is selfadjoint and regular. The map*

$$\begin{aligned} E^1\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T) &\to \mathfrak{G}(1\tilde{\otimes}_\nabla T) \\ e\otimes(f, Tf) &\mapsto (e\otimes f, 1\tilde{\otimes}_\nabla T(e\otimes f)) \end{aligned}$$

is a topological isomorphism of C^*-modules.

Proof. Observe that $E^1\tilde{\otimes}_{\mathcal{B}_1}\mathcal{F} \cong \mathcal{E}\tilde{\otimes}_B\mathcal{F}$, since $\mathcal{E} = E^1\tilde{\otimes}_{\mathcal{B}_1} B$. To see that $t := 1\otimes_\nabla T$ is selfadjoint regular, stabilize $\mathcal{E}$, and denote by $\tilde{\nabla}$ the Levi-Civita connection on $\mathcal{H}_B$. Then, via the stabilization isomorphism $\nabla' := \nabla \oplus \tilde{\nabla}$ defines a Hermitian connection on $\mathcal{H}_B \cong \mathcal{E} \oplus \mathcal{H}_B$. Since the difference $\nabla' - \tilde{\nabla}$ is a completely bounded module map, it suffices to prove regularity of t when ∇ is the Levi-Civita connection on $\mathcal{H}_B$. But in that case, for $e = \sum_{i=1}^\infty e_i \otimes b_i$,

$$\begin{aligned} t : \mathcal{H}_{\mathcal{B}_1}\tilde{\otimes}_{\mathcal{B}_1}\mathcal{F} &\to \mathcal{H}_{\mathcal{B}_1}\tilde{\otimes}_{\mathcal{B}_1}\mathcal{F} \\ e\otimes f &\mapsto \sum_{i=1}^\infty e_i \otimes T(b_i f), \end{aligned}$$

which is clearly selfadjoint regular. For the statement on the topological type of $\mathfrak{G}(t)$, it again suffices to consider the Levi-Civita connection $\mathcal{H}_{\mathcal{B}}$. We have $R := \nabla'_T - \tilde{\nabla}_T \in \mathrm{End}^*_C(\mathcal{E}\tilde{\otimes}_B\mathcal{F})$ and hence

$$\begin{aligned} \mathfrak{G}(t) &\xrightarrow{\sim} \mathfrak{G}(t+R) \\ (x, tx) &\mapsto (x, (t+R)x) \end{aligned}$$

topologically, due to the fact that $(i+t+R)(i+t)^{-1} \in \mathrm{Aut}^*_C(\mathcal{E}\tilde{\otimes}_B\mathcal{F})$. Note that the standard orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}_B$ defines a C^1-approximate unit for $\mathbb{K}_B(\mathcal{H}_B)$. The inner product on $\mathcal{H}_{\mathcal{B}_1}\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T)$ is thus given by

$$\begin{aligned} \langle e\otimes(f, Tf), e'\otimes(f', Tf')\rangle &:= \lim_{n\to\infty}\sum_{i=1}^n \langle\langle e_i, e\rangle(f, Tf), \langle e_i, e'\rangle(f', Tf)\rangle \\ &= \sum_{i=1}^\infty \langle(b_i f, Tb_i f), (b'_i f', Tb_i f)\rangle \\ &= \sum_{i=1}^\infty \langle b_i f, b'_i f'\rangle + \langle Tb_i f, Tb'_i f'\rangle. \end{aligned}$$

Therefore the map

$$\begin{aligned} \mathcal{H}_{\mathcal{B}_1}\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T) &\to \mathfrak{G}(t) \\ e\otimes(f, Tf) &\mapsto (e\otimes f, t(e\otimes f)), \end{aligned}$$

is unitary. $\square$

If the module $\mathcal{E}$ comes equipped with a regular operator S, the operators $S\tilde{\otimes}1$ and $1\tilde{\otimes}_\nabla T$ almost anticommute. That is, they anticommute up to a bounded operator. This implies that their sum is well defined as a regular operator.

Proposition 5.4.3. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$ be (A, B)- and (B, C)-bimodules equipped with selfadjoint regular operators S and T, respectively, such that $[T, b] \in \mathrm{End}^*_B(\mathcal{F})$ for all $b \in \mathcal{B}_1 \subset B$, a dense subalgebra of B. If $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ is a C^1- connection, then the operator*

$$S \otimes 1 + 1 \otimes_\nabla T$$

is selfadjoint and regular.

Proof. It is a well known fact that $s := S \otimes 1$ is a regular operator on $\mathcal{E} \otimes_B \mathcal{F}$ and we saw that $t := 1 \otimes_\nabla T$ is regular. Thus, s and t are selfadjoint regular operators whose domains intersect densely, and the graded commutator $[s, t]$ is an adjointable operator. To show that $s+t$ is regular we have to show that $\mathfrak{G}(s+t) \oplus v\mathfrak{G}(s+t) \cong E\tilde{\otimes}_B\mathcal{F} \oplus \mathcal{E}\tilde{\otimes}_B\mathcal{F}$. Write $\mathfrak{r}_i(s) := (s+i)^{-1}$ and consider the endomorphism

$$g := \begin{pmatrix} \mathfrak{r}_i(s)\mathfrak{r}_i(t) & -(s+t)\mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^* \\ (s+t)\mathfrak{r}_i(s)\mathfrak{r}_i(t) & \mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^* \end{pmatrix} \in M_2(\mathrm{End}_B(\mathcal{E} \otimes_B \mathcal{F})).$$

The maps $(s+t)\mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^*$ and $(s+t)\mathfrak{r}_i(s)\mathfrak{r}_i(t)$ are well defined because $[s, \mathfrak{r}_i(t)^*]$ and $[t, \mathfrak{r}_i(s)]$ are bounded. This follows from the fact that $[s, t]$ is bounded:

$$\begin{aligned} 0 &= [s, 1] \\ &= [s, \mathfrak{r}_i(t)^*(t-i)] \\ &= \mathfrak{r}_i(t)^*[s, (t-i)] + [s, \mathfrak{r}_i(t)^*](t-i) \\ &= \mathfrak{r}_i(t)^*[s, t] + [s, \mathfrak{r}_i(t)^*](t-i), \end{aligned}$$

hence $[s, \mathfrak{r}_i(t)^*](t-i)$ is bounded and so is $[s, \mathfrak{r}_i(t)^*]$. Computing g^*g gives

$$g^*g = r + \begin{pmatrix} \mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^*(s+t)^2\mathfrak{r}_i(s)\mathfrak{r}_i(t) & 0 \\ 0 & \mathfrak{r}_i(s)\mathfrak{r}_i(t)(s+t)^2\mathfrak{r}_i(t)^*\mathfrak{r}_i(s)^* \end{pmatrix}.$$

The operator

$$r = \begin{pmatrix} \mathfrak{r}_i(t)^*\mathfrak{r}(s)^2\mathfrak{r}_i(t) & 0 \\ 0 & \mathfrak{r}_i(s)\mathfrak{r}(t)^2\mathfrak{r}_i(s)^* \end{pmatrix}$$

is positive and has dense range, and $r \leq g^*g$, so g^*g has dense dense range, and hence g does so too. Since g maps $\mathcal{E} \otimes_B \mathcal{F}^2$ into $\mathfrak{G}(s+t) \oplus v\mathfrak{G}(s+t)$, this is a dense and closed submodule, hence all of $\mathcal{E} \otimes_B \mathcal{F} \oplus \mathcal{E} \otimes_B \mathcal{F}$. Selfadjointness follows from the fact that s and t anticommute up to an adjointable operator. Hence $\mathfrak{Dom}(s+t) = \mathfrak{Dom} s \cap \mathfrak{Dom}(t)$, and s and t are selfadjoint. $\square$

Of course the case of unbounded bimodules is contained in this theorem. It will be the case we focus on the next section.

Corollary 5.4.4. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S)$ an unbounded (A, B) bimodule and $(\mathcal{F}, T)$ an unbounded (B, C)-bimodule. Let $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ be a C^1-connection on $\mathcal{E}$. Then the operator*

$$S \otimes 1 + 1 \otimes_\nabla T$$

is selfadjoint and regular.

The product construction preserves selfadjointness and regularity. On the level of the graphs of the operators, we now show it can be viewed as a pull-back construction of topological C^*-modules. By Frank's theorem 1.1.4, this yields a unitary isomorphism of the modules involved. This suggests the product construction might be defined intrinsically, without reference to the connection. However, the pull back need not be the graph of an operator.

Theorem 5.4.5. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$ be (A, B)- and (B, C)-bimodules equipped with selfadjoint regular operators S and T, respectively. Suppose that $[T, b] \in \mathrm{End}^*_B(\mathcal{F})$ for all $b \in \mathcal{B}_1 \subset B$, a dense subalgebra of B. Let $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ be a C^1- connection, and $\mathcal{G}$ the universal solution to the diagram*

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathfrak{G}(S)\tilde{\otimes}_B\mathcal{F} \arrow[d] \\ E^1\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T) \arrow[r] & \mathcal{E}\tilde{\otimes}_B\mathcal{F}. \end{tikzcd}$$

Then the natural map

$$\mathfrak{G}(S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T) \to \mathcal{G},$$

is a topological isomorphism of C^*-modules.

Proof. As usual, write $s = S\tilde{\otimes}1$ and $t = 1\tilde{\otimes}_\nabla T$. Since $\mathfrak{G}(S)\tilde{\otimes}_B\mathcal{F} \cong \mathfrak{G}(\sqrt{2}s)$ and $E^1\tilde{\otimes}_{\mathcal{B}_1}\mathfrak{G}(T) \cong \mathfrak{G}(\sqrt{2}t)$, we may replace $\mathcal{G}$ by the pull back of the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathfrak{G}(\sqrt{2}s) \arrow[d] \\ \mathfrak{G}(\sqrt{2}t) \arrow[r] & \mathcal{E}\tilde{\otimes}_B\mathcal{F}. \end{tikzcd}$$

Thus

$$\mathcal{G} = \{((x, \sqrt{2}sx), (x, \sqrt{2}tx)) : x \in \mathfrak{Dom}(s + t)\},$$

and

$$\langle ((x, \sqrt{2}sx), (x, \sqrt{2}tx)), ((x', \sqrt{2}sx'), (x', \sqrt{2}tx')) \rangle = 2(\langle x, x' \rangle + \langle sx, sx' \rangle + \langle tx, tx' \rangle).$$

Using

$$g := \begin{pmatrix} \mathfrak{r}(s+t)^2 & -(s^2+t^2)\mathfrak{r}(s+t)^2 \\ (s^2+t^2)\mathfrak{r}(s+t)^2 & \mathfrak{r}(s+t)^2 \end{pmatrix} \in M_2(\mathrm{End}_B(\mathcal{E} \otimes_B \mathcal{F})),$$

it follows that $s^2 + t^2$ is selfadjoint regular by the same reasoning as in proposition 5.4.3. Since the operator $(i + |s^2 + t^2|^{\frac{1}{2}})(i + s + t)^{-1} \in \mathrm{Aut}^*(\mathcal{E}\tilde{\otimes}_B\mathcal{F})$, the result follows. □

If the module $\mathcal{F}$ is smooth, i.e. induces a smooth structure $\{\mathcal{B}_i\}$ on B, and $\mathcal{E}$ is a smooth C^*-module for this smooth structure, the Sobolev chain of $D = S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T$

can be computed from the Sobolev chains of S and T by the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathfrak{G}(D_2) \arrow[r] \arrow[d] & \mathfrak{G}(t)_{D_1} \arrow[r] \arrow[d] & \mathfrak{G}(t_1)_D \arrow[r] \arrow[d] & E^3\tilde{\otimes}\mathcal{F}_3 \arrow[d] \\
\cdots \arrow[r] & \mathfrak{G}(s)_{D_1} \arrow[r] \arrow[d] & \mathfrak{G}(D_1) \arrow[r] \arrow[d] & \mathfrak{G}(t)_D \arrow[r] \arrow[d] & E^2\tilde{\otimes}\mathcal{F}_2 \arrow[d] \\
\cdots \arrow[r] & \mathfrak{G}(s_1)_D \arrow[r] \arrow[d] & \mathfrak{G}(s)_D \arrow[r] \arrow[d] & \mathfrak{G}(D) \arrow[r] \arrow[d] & E^1\tilde{\otimes}\mathcal{F}_1 \arrow[d] \\
\cdots \arrow[r] & \mathcal{E}_3\tilde{\otimes}\mathcal{F} \arrow[r] & \mathcal{E}_2\tilde{\otimes}\mathcal{F} \arrow[r] & \mathcal{E}_1\tilde{\otimes}\mathcal{F} \arrow[r] & \mathcal{E}\tilde{\otimes}\mathcal{F}
\end{tikzcd}$$

The Sobolev chain of D is on the diagonal, that of $s = S\tilde{\otimes}1$ is the bottom row, and that of $t = 1\tilde{\otimes}_\nabla T$ is the right vertical row. The upper triangular part consists of the Sobolev chains of t viewed as an operator in $\mathfrak{G}(D_i)$, and the lower triangular part of the Sobolev chains of s viewed as an operator in $\mathfrak{G}(D_i)$. Moreover, all squares all pull back squares.

6. Correspondences

Universal connections can be employed to give a transparent construction of the Kasparov product, on the level of unbounded bimodules. This observation leads to the construction of a category of spectral triples and even of unbounded bimodules themselves. They give a notion of morphism of noncommutative geometries, in such a way that the bounded transform induces a functor from correspondences to KK-groups. By considering several levels of differentiability and smoothness on correspondences, one gets subcategories of correspondences of C^k- and smooth C^*-algebras.

6.1. **The Trotter-Kato formula.** When dealing with addition of noncommuting unbounded operators s and t, on a Hilbert space $\mathcal{H}$, several subtleties arise. First of all one needs to check closability and density of the domain of $s+t$. If these things are in order, one would like to obtain information about $e^{-x(s+t)}$, the one parameter group generated by $s+t$, and its resolvent. These questions can be tantalizingly difficult and involve some deep analysis. Under favourable conditions, though, a satisfactory description of $e^{-x(s+t)}$ can be given in terms of e^{-xs} and e^{-xt}. It is quite striking that one might as well use other functions of s and t instead of exponentials. For our purposes it is enough to consider the function $f(s) = (1+s)^{-1}$.

Theorem 6.1.1 ([26]). *Let f be either one of the functions $s \mapsto e^{-s}$ or $s \mapsto (1+s)^{-1}$. Suppose s and t are nonnegative selfadjoint operators on a Hilbert space $\mathcal{H}$, such that their sum $s+t$ is selfadjoint on $\mathfrak{Dom}(s) \cap \mathfrak{Dom}(t)$. Then*

$$\lim_{n\to\infty}(f(\frac{xs}{2n})f(\frac{xt}{n})f(\frac{xs}{2n}))^n = e^{-x(s+t)},$$

in norm for x in compact intervals in $(0,\infty)$. If $s+t$ is strictly positive, the convergence holds for $x \in [\epsilon,\infty)$ for any $\epsilon > 0$.

We now argue that a similar result holds for unbounded operators in C^*-modules. Let s and t be nonnegative regular operators in a C^*-B-module $\mathcal{E}$, such that their sum $s+t$ is densely defined and regular. By representing B faithfully and nondegenerate on a Hilbert space $\mathcal{H}'$, one obtains a second Hilbert space $\mathcal{H} := \mathcal{E}\tilde{\otimes}_B\mathcal{H}'$ and operators $s\otimes 1, t\otimes 1$ and $(s+t)\otimes 1 = s\otimes 1 + t\otimes 1$. Moreover, $\mathrm{End}^*_B(\mathcal{E})$ is faithfully represented on $\mathcal{H}$, and $f(s)\otimes 1 = f(s\otimes 1)$ for any $f \in C_0(\mathbb{R})$. Also, $s\otimes 1, t\otimes 1$ and $(s+t)\otimes 1$ are positive whenever $s, t, s+t$ are. Therefore we have

Corollary 6.1.2. *Let f be either one of the functions $s \mapsto e^{-s}$ or $s \mapsto (1+s)^{-1}$. Suppose s and t are nonnegative selfadjoint regular operators on a C^*-module $\mathcal{E}$, such that their sum $s+t$ is selfadjoint and regular on $\mathfrak{Dom}(s) \cap \mathfrak{Dom}(t)$. Then*

$$\lim_{n\to\infty}(f(\frac{xs}{2n})f(\frac{xt}{n})f(\frac{xs}{2n}))^n = e^{-x(s+t)},$$

in norm for x in compact intervals in $(0,\infty)$. If $s+t$ is strictly positive, the convergence holds for $x \in [\epsilon,\infty)$ for any $\epsilon > 0$.

The Trotter-Kato formula in C^*-modules will be a crucial tool in what follows.

6.2. **The KK-product.** Everything is in place now to establish that compact resolvents are preserved under taking products. Then we will see that the product operator satisfies Kucerovsky's conditions for an unbounded Kasparov product. Thus, if two unbounded bimodules are compatible in the sense that there exists a C^1-connection for them, the KK-product of these modules is given by an explicit algebraic formula. Let us put the pieces together.

Lemma 6.2.1. *Let s, t be selfadjoint regular operators on a C^*-module $\mathcal{E}$, and $R \in \mathrm{End}^*_B(\mathcal{E})$ be a selfadjoint element. If $(1+s^2)^{-1}(t+i)^{-1} \in \mathbb{K}_B(\mathcal{E})$, then $(1+s^2)^{-1}(t+R+i)^{-1} \in \mathbb{K}_B(\mathcal{E})$.*

Proof. One has the identity

$$(1+s^2)^{-1}(i+t+R)^{-1} = (1+s^2)^{-1}(i+t)^{-1}(1-R(t+i)^{-1}),$$

which is a compact operator. □

We now employ the Trotter-Kato formula from the previous section to show that the product of cycles is a cycle. Note that this result is a generalization of the stability property of spectral triples proved in [7]. There it was shown that tensoring a given spectral triple by a finitely generated projective module yields again a spectral triple.

Proposition 6.2.2. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S)$ an unbounded (A,B) bimodule and $(\mathcal{F}, S)$ an unbounded (B,C)-bimodule. Let $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ be a C^1-connection on $\mathcal{E}$. Then the operator*

$$S\otimes 1 + 1\otimes_\nabla T$$

has compact resolvent.

Proof. The operator s^2+t^2 is selfadjoint and regular, as we saw in the proof of theorem 5.4.5. Moreover, since s^2+t^2 is positive we have

$$s^2+t^2 = |s^2+t^2|.$$

Since $(s+t)^2$ is a bounded perturbation of s^2+t^2, for $s+t$ to have compact resolvent it is sufficient that $(1+s^2+t^2)^{-1}$ be compact. By applying lemma 1.3.4 to the operator $|s^2+t^2|^{\frac{1}{2}}$, we get the identity

$$(2+s^2+t^2)^{-1} = \int_0^\infty e^{-x(2+s^2+t^2)}dx.$$

By this same lemma it suffices to show that the integrand $e^{-x(2+s^2+t^2)}$ is compact for $x>0$. The Trotter-Kato formula 6.1.2 gives the equality

$$e^{-x(2+s^2+t^2)} = \lim_{n\to\infty}((1+\frac{x}{2n}s^2)^{-1}(1+\frac{x}{n}t^2)^{-1}(1+\frac{x}{2n}s^2)^{-1})^n.$$

Therefore it suffices to show that $(1+\frac{1}{2}s^2)^{-1}(i+t)^{-1}$ is compact. By the previous lemma, we only have to check this in case ∇ is the Levi-Civita connection on $\mathcal{H}_B$. In that case

$$(1+\frac{1}{2}s^2)^{-1}(i+t)^{-1} = \sum_{i=0}^{\infty}(1+\frac{1}{2}S^2)^{-1}e_i\otimes(i+T)^{-1}\otimes e_i,$$

which is a norm convergent series in $\mathbb{K}_C(\mathcal{H}_B\tilde{\otimes}\mathcal{F}) = \mathcal{H}_B\tilde{\otimes}\mathbb{K}_C(\mathcal{F})\tilde{\otimes}\mathcal{H}_B$. $\square$

At this point, we would like to note that for a given pair of cycles $(\mathcal{E},S)$ and $(\mathcal{F},T)$, the existence of a C^1-connection is not guaranteed. In the presence of such a connection, we have the follwoing result.

Theorem 6.2.3. *The diagram*

$$\begin{tikzcd}
\Psi_0(A,B)\times\Psi_0(B,C) \arrow[r, "(S,T)\mapsto S\otimes 1+1\otimes_\nabla T"] \arrow[d, "\mathfrak{b}"] & \Psi_0(A,C) \arrow[d, "\mathfrak{b}"] \\
KK_0(A,B)\otimes KK_0(B,C) \arrow[r, "\otimes_B"] & KK_0(A,C)
\end{tikzcd}$$

commutes, whenever the composition in the top row is defined.

Proof. We just need to check that the unbounded bimodules $(\mathcal{E},S)$, $(\mathcal{F},T)$ and $(\mathcal{E}\tilde{\otimes}_B\mathcal{F}, S\tilde{\otimes}1+1\otimes_\nabla T)$ satisfy the conditions of theorem 2.3.4. If we write D for $S\otimes +1\otimes_\nabla T$, we have to check that

$$J := \left[\begin{pmatrix} D & 0\\ 0 & T\end{pmatrix},\begin{pmatrix} 0 & T_e\\ T_e^* & 0\end{pmatrix}\right]$$

is bounded on $\mathfrak{Dom}(D\oplus T)$. This is a straightforward calculation:

$$\begin{aligned}
J\begin{pmatrix} e'\otimes f'\\ f\end{pmatrix} &= \begin{pmatrix} Se\otimes f+(-1)^{\partial e}\nabla_T(e)f\\ \langle e,Se'\rangle f+[T,\langle e,e'\rangle]f+(-1)^{-\partial e'}\langle e,\nabla_T(e')\rangle f\end{pmatrix}\\
&= \begin{pmatrix} Se\otimes f+(-1)^{\partial e}\nabla_T(e)f\\ \langle Se,e'\rangle f+\langle\nabla_T(e),e'\rangle f\end{pmatrix}.
\end{aligned}$$

This is valid whenever $e \in E_1^1$, which is dense in $\mathcal{E}$.

The second condition $\mathfrak{Dom}(D) \subset \mathfrak{Dom}(S\tilde{\otimes}1)$ is obvious, so we turn the semiboundedness condition

$$\langle S\tilde{\otimes}1x, Dx\rangle + \langle Dx, S\tilde{\otimes}1x\rangle \geq \kappa\langle x, x\rangle, \tag{6.12}$$

must hold for all x in the domain. The expression 6.12 is equal to

$$\langle [D, S\tilde{\otimes}1]x, x\rangle = \langle [s,t]x, x\rangle \geq -\|[s,t]\|\langle x, x\rangle,$$

and the last estimate is valid since $[s,t]$ is in $\mathrm{End}^*_C(\mathcal{E}\tilde{\otimes}_B\mathcal{F})$. □

6.3. **Formal Bott periodicity.** We obtained a description of the KK-product of even unbounded bimodules, in the presence of a connection. This construction can be lifted to $\Psi_i(A,B)$ for each i. The way to go is indicated by the following result of Kasparov.

Theorem 6.3.1 ([19])**.** *For all j, the map*

$$\begin{aligned}\Psi_i(A,B) &\to \Psi_{i+j}(A\tilde{\otimes}\mathbb{C}_j, B)\\ (\mathcal{E}, D) &\mapsto (\mathcal{E}\tilde{\otimes}\mathbb{C}_j, D\tilde{\otimes}1),\end{aligned}$$

induces an isomorphism $KK_j(A,B) \to KK_{i+j}(A\tilde{\otimes}\mathbb{C}_j, B)$.

Using this, we can define the composition of two unbounded bimodules with connection as the composition

$$\Psi_i(A,B)\times\Psi_j(B,C) \to \Psi_i(A,B)\times\Psi_{i+j}(B\tilde{\otimes}\mathbb{C}_i, C) \to \Psi_{i+j}(A,C).$$

From theorem 6.2.3 we directly obtain the analoguous result in all degrees, whenever a connection for two given cycles exists.

Theorem 6.3.2. *The diagram*

$$\begin{tikzcd} \Psi_i(A,B)\times\Psi_j(B,C) \arrow[r, "(S,T)\mapsto S\otimes 1 + 1\otimes_\nabla T"] \arrow[d, "\mathfrak{b}"'] & \Psi_{i+j}(A,C) \arrow[d, "\mathfrak{b}"] \\ KK_i(A,B)\otimes KK_j(B,C) \arrow[r, "\otimes_B"] & KK_{i+j}(A,C) \end{tikzcd}$$

commutes, whenever the composition in the top row is defined.

In order to obtain a useful formula in the case of odd modules, we only have to delve a little deeper into formal Bott periodicity. Recall that elements of $\Psi_i(A,B)$ by definition equals $\Psi_0(A, B\tilde{\otimes}\mathbb{C}_i)$. Hence its elements are given by unbounded $(A, B\tilde{\otimes}\mathbb{C}_i)$ bimodules $(\mathcal{E}, D)$. Thus, D is an operator that commutes with the action of B and the action of $\mathbb{C}_i$. From $(\mathcal{E}, D)$ we can construct $(\mathcal{E}', D')$ in the following way (cf.[18], appendix A.3):

$$\mathcal{E}' := \mathcal{E}\oplus\mathcal{E}, \quad D' := \begin{pmatrix} 0 & D \\ D & 0\end{pmatrix},$$

as in 2.6. The action of $\mathbb{C}_{i+2}$ is given by

$$\varepsilon_j := \begin{pmatrix} \varepsilon_j & 0 \\ 0 & \varepsilon_j\end{pmatrix}, \quad \varepsilon_{i+1} := \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \quad \varepsilon_{i+1} := \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}.$$

Here $j = 1, \ldots, i$. In view of 2.6 we could denote the set of odd Kasparov modules by $\Psi_{-1}(A, B)$. The map $\Psi_i(A, B) \to \Psi_{i+2}(A, B)$ so defined is the *formal periodicity map*, and induces an isomorphism $KK_i(A, B) \to KK_{i+2}(A, B)$. It's inverse, on the level of unbounded bimodules $(\mathcal{E}, D) \in \Psi_{i+2}(A, B)$, with $i \geq 1$, is given by compressing the operator D to the $+1$ eigenspace of the involution $-i\varepsilon_{i+1}\varepsilon_{i+2}$. For $i = -1$, one compresses to the $+1$ eigenspace of ϵ_1. Applying this procedure to the case of two composable odd modules $(\mathcal{E}, S, \nabla)$ and $(\mathcal{F}, T)$ yields that the product operator is

$$\begin{pmatrix} 0 & S\tilde{\otimes}1 - i\tilde{\otimes}_\nabla T \\ S\tilde{\otimes}1 + i\tilde{\otimes}_\nabla T & 0 \end{pmatrix},$$

on the module $\mathcal{E}\tilde{\otimes}_B\mathcal{F} \oplus \mathcal{E}\tilde{\otimes}_B\mathcal{F}$.

6.4. **The nonunital case.** So far, we have only been working with unital C^*-algebras. In this section we show that this restriction, imposed for the sake of clarity, is harmless. In [23] it is shown that any operator algebra is contained in a unital operator algebra, and that the operator norms on the unization are uniquely defined.

Definition 6.4.1. Let $\mathcal{A}$ be an operator algebra and $A \to B(\mathcal{H})$ a complete isometry. Its *unitization* $\mathcal{A}_+$ is the algebraic unitization $\mathcal{A} \oplus \mathbb{C}$ with product $(a, z)(b, w) = (ab + aw + zb, zw)$. Identifying $\mathcal{A}_+$ with

$$\{a + \lambda \cdot 1 : a \in \mathcal{A}, \lambda \in \mathbb{C}\},$$

$\mathcal{A}_+$ becomes an operator algebra.

This definition is independent of the choice of complete isometry [23].The standard C^*-unitization is a special case of this. Now note that a rigged module over $\mathcal{A}$ is automatically a rigged module over $\mathcal{A}_+$. Hence for a smooth C^*-algebra A, with smooth structure $\{\mathcal{A}_i\}$, any smooth C^*-module $\mathcal{E}$ is a direct summand in $\mathcal{H}_{A_+}$. Hence a smooth Hermitian connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_A\Omega^1(A_+)$ always exists. Since

$$E^{i+1}\tilde{\otimes}_{\mathcal{A}_{(i+1)+}}\mathcal{A}_{i+} = E^i,$$

and the modules $E^i \subset \mathcal{E}$ are $\mathcal{A}_i$-essential, the property $E^{i+1}\tilde{\otimes}_{\mathcal{A}_{i+1}}\mathcal{A}_i = E^i$ of proposition 4.2.2 remains valid:

$$E^i = E^i\mathcal{A}_i \cong E^{i+1}\tilde{\otimes}_{\mathcal{A}_{i+1+}}\mathcal{A}_{i+}\mathcal{A}_i = E^{i+1}\tilde{\otimes}_{\mathcal{A}_{i+1}}\mathcal{A}_i.$$

A smooth connection on a smooth KK-cycle $(\mathcal{E}, D)$ for (A, B) is now a connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_{B_+}\Omega^1(B_+)$, such that $[\nabla, D_i]$ is completely bounded for all i. If B was already unital, then B_+ decomposes as $B \oplus \mathbb{C}$, so $\Omega^1(B)$ is a direct summand in $\Omega^1(B_+)$ in this case. A connection $\nabla^+ : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_{B_+}\Omega^1(A_+)$ therefore induces a connection $\nabla : \mathcal{E} \to \mathcal{E}\tilde{\otimes}_B\Omega^1(B)$ and vice versa. Although this is not a bijective correspondence, the ambiguity is irrelevant, as the subsequent discussion shows.

Let $(\mathcal{E}, S, \nabla)$ and $(\mathcal{F}, T, \nabla')$ be nonunital unbounded bimodules, with connections. The tensor product $\mathcal{E}\tilde{\otimes}_B\mathcal{F}$ equals $\mathcal{E}\tilde{\otimes}_{B_+}\mathcal{F}$, by definition. To show that the product of nonunital cycles is again a cycle, only the compact resolvent property needs some care. Thus we have to show that $a(S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T)$ is compact, using compactness of $a(1 + S^2)^{-1}$ and $b(1 + T^2)^{-1}$, for $a \in \mathcal{A}$, $b \in \mathcal{B}$. The argument in proposition 6.2.2 carries through up to the reduction of the necessity to show that $a(1 + \frac{1}{2}s^2)^{-1}(i + t)^{-1}$ is compact for $a \in \mathcal{A}$. To achieve this, one employs an approximate unit e_α for $\mathcal{B}$. The operators $a(1+\frac{1}{2}s^2)^{-1}e_\alpha(i+t)^{-1}$ are shown to be compact in the the same way as before. Then, $a(1+\frac{1}{2}s^2)^{-1}(i+t)^{-1}$ is their norm limit, hence compact. The validity of theorem 6.2.3 follows readily from this.

6.5. **A category of spectral triples.** Let A and B be smooth C^*-algebras. The results from 5.4 suggest that triples $(\mathcal{E}, D, \nabla)$ consisting of a smooth (A,B)-bimodule equipped with a smooth regular operator D and a smooth connection ∇ form a category, in which the composition law is

$$(\mathcal{E}, D, \nabla)\circ(\mathcal{E}', D', \nabla') := (\mathcal{E}\tilde{\otimes}\mathcal{F}, D\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla D', \nabla\tilde{\otimes}_B\nabla').$$

An essential piece for this statement to hold is missing, and we will prove it now.

Proposition 6.5.1. *Let A, B, C be C^*-algebras, $(\mathcal{E}, S, \nabla)$ and $(\mathcal{F}, T, \nabla')$ be (A,B)- and (B,C)-bimodules equipped with selfadjoint regular operators S and T, and C^1-connections ∇ and ∇', respectively. Suppose that $[T,b]\in \mathrm{End}^*_B(\mathcal{E})$ for all $b\in \mathcal{B}_1\subset B$, a dense subalgebra of B. Then the product connection $\nabla\tilde{\otimes}_B\nabla'$ is an $S\tilde{\otimes}1+1\tilde{\otimes}_\nabla T$-connection.*

Proof. Since

$$[\nabla\tilde{\otimes}_B\nabla', S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T] = [\nabla\tilde{\otimes}_B\nabla', S\tilde{\otimes}1] + [\nabla\tilde{\otimes}_B\nabla', 1\tilde{\otimes}_\nabla T],$$

and $[\nabla\tilde{\otimes}_B\nabla', S\tilde{\otimes}1] = [\nabla, S]\tilde{\otimes}1$, which is completely bounded, we compute

$$(-1)^{\partial e}[\nabla\tilde{\otimes}_B\nabla', 1\tilde{\otimes}_\nabla T](e\otimes f)$$

to find

$$e\otimes[\nabla', T]f + \nabla_{\nabla'}(e)Tf + \nabla\tilde{\otimes}_B\nabla'(\nabla_T(e)f) - \nabla_T(e)\nabla'(f) - 1\tilde{\otimes}_\nabla T(\nabla_{\nabla'}(e)f).$$

The first term is completely bounded, and in working out the last four terms write $\nabla(e) = \sum e_i\otimes db_i$. Then

$$\begin{aligned}
\nabla_{\nabla'}(e)Tf &= \sum e_i\otimes[\nabla', b_i]Tf, && (6.13)\\
\nabla_T(e)\nabla'(f) &= \sum e_i\otimes[T, b_i]\nabla'(f), && (6.14)\\
\nabla\tilde{\otimes}_B\nabla'(\nabla_T(e)f) &= \sum e_i\otimes\nabla'[T,b_i]f + \nabla_{\nabla'}(e_i)[T,b_i]f, && (6.15)\\
1\tilde{\otimes}_\nabla T(\nabla_{\nabla'}(e)f) &= \sum_i e_i\otimes T[\nabla', b_i]f + \nabla_T(e_i)[\nabla', b_i]f. && (6.16)
\end{aligned}$$

Combining 6.13,6.14 and the first terms on the right hand sides of 6.15 and 6.16 give a term

$$\sum_i e_i\otimes[[\nabla', T], b_i]f = \nabla_{[\nabla', T]}(e)f,$$

and the terms remaining from 6.15 and 6.16 give a term

$$(\nabla_{\nabla'}\nabla_T - \nabla_T\nabla_{\nabla'})(e\otimes f).$$

Thus, we have shown that

$$[\nabla\tilde{\otimes}_B\nabla', 1\tilde{\otimes}_\nabla T] = 1\tilde{\otimes}_\nabla[\nabla', T] + [\nabla_{\nabla'}, \nabla_T],$$

which is a completely bounded map $\mathcal{E}\tilde{\otimes}_B\mathcal{F}\to\mathcal{E}\tilde{\otimes}_B\mathcal{F}\tilde{\otimes}_C\Omega^1(C)$. $\square$

Definition 6.5.2. Let A and B be C^*-algebras, and $(\mathcal{H}, D)$ and $(\mathcal{H}', D')$ be smooth spectral triples for A and B respectively. A C^k-*correspondence* $(\mathcal{E}, S, \nabla)$ between $(\mathcal{H}, D)$ and $(\mathcal{H}', D')$ is an unbounded C^k-(A, B)-bimodule with C^k-connection , such that $[S, \mathcal{A}_{i+1}] \subset \mathrm{End}^*_{\mathcal{B}_i}(E^i)$ and $\mathcal{H} \cong \mathcal{E}\tilde{\otimes}_B\mathcal{H}'$ and $D_i = (S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla D')_i$ for $i = 0, ..., k$ under this isomorphism. The correspondence is *smooth* if it is C^k for all k. Two correspondences are said to be equivalent if they are C^k- or smoothly unitarily isomorphic such that the unitary intertwines the operators. The set of isomorphism classes of such correspondences is denoted by $\mathfrak{Cor}_k(D, D')$ or $\mathfrak{Cor}(D, D')$ in the smooth case.

The requirement $[S, \mathcal{A}_{i+1}] \subset \mathrm{End}^*_{\mathcal{B}_i}(E^i)$ can be viewed as a transversality condition.

Theorem 6.5.3. *There is a category whose objects are C^k-spectral triples and whose morphisms are the sets $\mathfrak{Cor}_k(D, D')$. The bounded transform $\mathfrak{b}(\mathcal{E}, D, \nabla) = (\mathcal{E}, \mathfrak{b}(D))$ defines a functor $\mathfrak{Cor}_k \to KK$.*

Proof. Composition of correspondences $(\mathcal{E}, S, \nabla)$ and $(\mathcal{F}, T, \nabla')$ is defined by

$$(\mathcal{E}\tilde{\otimes}_B\mathcal{F}, S\tilde{\otimes}1 + 1\tilde{\otimes}_\nabla T, \nabla\tilde{\otimes}_B\nabla').$$

This is associative by theorem 5.3.2 and proposition 5.4.1, and defines a correspondence again by by propositions 5.4.3 and 6.2.2. That the composite of two C^k-correspondences is again a C^k-correspondence, follows by examining the diagram the diagram after theorem 5.4.5 and using the transversality condition. □

As mentioned in the introduction, a category with unbounded cycles as objects can be constructed in a similar way. A morphism of unbounded cycles $A \to (\mathcal{E}, D) \leftrightarrows B$ and $A' \to (\mathcal{E}', D') \leftrightarrows B'$ is given by a correspondence $A \to (\mathcal{F}, S, \nabla) \leftrightarrows A'$ and a bimodule $B \to \mathcal{F}' \leftrightarrows B'$, where B is represnted by compact operators. The bounded transform functor then takes values in the *morphim category* KK^2.

Furthermore, we would like to note that the category of spectral triples constructed is a 2-category. A morphism of morphisms $f : (\mathcal{E}, D, \nabla) \to (\mathcal{E}', D'\nabla')$ is given by an element $F \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$, commuting with the left A-module structures and making the diagrams

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "D"'] & \mathcal{E}' \arrow[d, "D'"'] \\ \mathcal{E} \arrow[r, "F"] & \mathcal{E}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "\nabla"'] & \mathcal{E}' \arrow[d, "\nabla'"'] \\ \mathcal{E}\tilde{\otimes}_B\Omega^1(B) \arrow[r, "F"] & \mathcal{E}'\tilde{\otimes}_B\Omega^1(B), \end{tikzcd}$$

commutative.

The external product of correspondences is defined in the expected way:

$$(\mathcal{E}, D, \nabla) \otimes (\mathcal{E}', D'\nabla') := (\mathcal{E}\overline{\otimes}\mathcal{E}', D\overline{\otimes}1 + 1\overline{\otimes}D', \nabla\overline{\otimes}1 + 1\overline{\otimes}\nabla).$$

In this way, $\mathfrak{Cor}$ becomes a symmetric monoidal category.

Definition 6.5.4. Let $(\mathcal{E}, S)$ and $(\mathcal{F}, T)$ be unbounded (A, B)-bimodules. They are said to be *weakly equivalent* if there exists a unitary in $u \in \mathrm{Hom}^*_B(\mathcal{E}, \mathcal{F})$ with the property that $u^*Tu - S$ is densely defined and extends to a bounded operator in $\mathrm{End}^*_B(\mathcal{E})$.

Weak equivalence is an equivalence relation. On the level of correspondences, all compatible connections (if they exist) become equivalent. The resulting category $\mathfrak{WCor}$, is the category of *weak correspondences* of spectral triples (or unbounded bimodules). The functor $\mathfrak{Cor} \to KK$ factors through the quotient map $\mathfrak{Cor} \to \mathfrak{WCor}$. When one is merely interested in obtaining K-theoretic information of some sort, working in $\mathfrak{WCor}$ can be much easier than working in $\mathfrak{Cor}$.

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