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THE GROWTH RATE OF FLOER HOMOLOGY AND SYMPLECTIC ZETA FUNCTION

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Abstract. The main theme of this paper is to compute for a symplectomorphism $\phi : M \to M$ of a compact surface, the asymptotic invariant $F_\infty(\phi)$ which is defined to be the growth rate of the sequence $\dim HF_*(\phi^n)$ of the total dimensions of symplectic Floer homologies of the iterates of ϕ. We prove that the asymptotic invariant coincides with the largest dilatation of the pseudo-Anosov components of ϕ and its logarithm coincides with the topological entropy . This implies that the symplectic zeta function of ϕ has a positive radius of convergence.

Contents

1. Introduction

The main theme of this paper is to compute for a symplectomorphism $\phi : M \to M$ of a compact surface, the asymptotic invariant $F_\infty(\phi)$, introduced in [8], which is defined to be the growth rate of the sequence $\dim HF_*(\phi^n)$ of the total dimensions of symplectic Floer homologies of the iterates of ϕ. We prove a conjecture from [8] which suggests that the asymptotic invariant coincides with the largest dilatation of the pseudo-Anosov components of ϕ and its logarithm coincides with the topological entropy. The asymptotic invariant also provides the radius of convergence of the symplectic zeta function.We show that the symplectic zeta function of ϕ has a positive radius of convergence which admits exact algebraic estimation via Reidemeister trace formula.

Our main result is following

Theorem 1.1. *Let ϕ be a perturbed standard form map ψ in a reducible mapping class g of compact surface of genus ≥ 2 and λ is the largest dilatation of the pseudo-Anosov* *components($\lambda = 1$ if there is no pseudo-Anosov components). Then*

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = \lambda = \exp(h(\psi)) = L^\infty(\psi) = N^\infty(\psi),$$

where $h(\psi)$ is the topological entropy and $L^\infty(\psi)$ and $N^\infty(\psi)$ are asymptotic(absolute) Lefshetz number and asymptotic Nielsen number.

Remark 1.2. The genus one case follows from [8] and Pozniak's thesis[23].

Although the exact evaluation of the asymptotic invariant would be desirable, in general, its estimation is a more realistic goal and as we shall show, one that is sufficient for some applications.

We suggested in [8] that the asymptotic invariant potentially may be important for the applications. Recent paper of Ivan Smith [26] gives application of the asymptotic invariant to the important question of faithfulness of a representation of extended mapping class group via considerations motivated by Homological Mirror Symmetry.

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2. Preliminaries

2.1. Symplectic Floer homology.

2.1.1. *Review of monotonicity and weak monotonicity.* In this section we discuss the notion of monotonicity and weak monotonicity as defined in [24, 14, 2]. Monotonicity plays important role for Floer homology in two dimensions. Throughout this article, M denotes a compact connected and oriented 2-manifold of genus ≥ 2. Pick an everywhere positive two-form ω on M.

Let $\phi \in \mathrm{Symp}(M, \omega)$, the group of symplectic automorphisms of the two-dimensional symplectic manifold (M, ω) (when M has boundary we consider the group of orientation-preserving diffeomorphisms of M with no fixed points on the boundary). The mapping torus of ϕ, $T_\phi = \mathbb{R} \times M/(t+1, x) \sim (t, \phi(x))$, is a 3-manifold fibered over $S^1 = \mathbb{R}/\mathbb{Z}$. There are two natural second cohomology classes on T_ϕ, denoted by $[\omega_\phi]$ and c_ϕ. The first one is represented by the closed two-form ω_ϕ which is induced from the pullback of ω to $\mathbb{R} \times M$. The second is the Euler class of the vector bundle $V_\phi = \mathbb{R} \times TM/(t+1, \xi_x) \sim (t, \mathrm{d}\phi_x \xi_x)$, which is of rank 2 and inherits an orientation from TM.

$\phi \in \mathrm{Symp}(M, \omega)$ is called **monotone**, if $[\omega_\phi] = (\mathrm{area}_\omega(M)/\chi(M)) \cdot c_\phi$ in $H^2(T_\phi; \mathbb{R})$; throughout this article $\mathrm{Symp}^m(M, \omega)$ denotes the set of monotone symplectomorphisms.

Now $H^2(T_\phi; \mathbb{R})$ fits into the following short exact sequence [24, 14]

$$0 \longrightarrow \frac{H^1(M; \mathbb{R})}{\mathrm{im}(\mathrm{id} - \phi^*)} \xrightarrow{d} H^2(T_\phi; \mathbb{R}) \xrightarrow{r^*} H^2(M; \mathbb{R}), \longrightarrow 0. \tag{1}$$

where the map r^* is restriction to the fiber. The map d is defined as follows. Let $\rho : I \to \mathbb{R}$ be a smooth function which vanishes near 0 and 1 and satisfies $\int_0^1 \rho \, \mathrm{d}t = 1$. If θ is a closed 1-form on M, then $\rho \cdot \theta \wedge \mathrm{d}t$ defines a closed 2-form on T_ϕ; indeed $d[\theta] = [\rho \cdot \theta \wedge \mathrm{d}t]$. The map $r : M \hookrightarrow T_\phi$ assigns to each $x \in M$ the equivalence class of $(1/2, x)$. Note, that

$r^*\omega_\phi = \omega$ and r^*c_ϕ is the Euler class of TM. Hence, by (1), there exists a unique class $m(\phi) \in H^1(M;\mathbb{R})/\operatorname{im}(\mathrm{id} - \phi^*)$ satisfying $d\,m(\phi) = [\omega_\phi] - (\mathrm{area}_\omega(M)/\chi(M)) \cdot c_\phi$, where $\chi(M)$ denotes the Euler characteristic of M. Therefore, ϕ is monotone if and only if $m(\phi) = 0$.

Because c_ϕ controls the index, or expected dimension, of moduli spaces of holomorphic curves under change of homology class and ω_ϕ controls their energy under change of homology class, the monotonicity condition ensures that the energy is constant on the index one components of the moduli space, which implies compactness and, as a corollary, finite count in a differential of the Floer complex.

We recall the fundamental properties of $\mathrm{Symp}^m(M,\omega)$ from [24, 14]. Let $\mathrm{Diff}^+(M)$ denotes the group of orientation preserving diffeomorphisms of M.

(Identity) $\mathrm{id}_M \in \mathrm{Symp}^m(M,\omega)$.

(Naturality) If $\phi \in \mathrm{Symp}^m(M,\omega), \psi \in \mathrm{Diff}^+(M)$, then $\psi^{-1}\phi\psi \in \mathrm{Symp}^m(M,\psi^*\omega)$.

(Isotopy) Let $(\psi_t)_{t\in I}$ be an isotopy in $\mathrm{Symp}(M,\omega)$, i.e. a smooth path with $\psi_0 = \mathrm{id}$. Then $m(\phi \circ \psi_1) = m(\phi) + [\mathrm{Flux}(\psi_t)_{t\in I}]$ in $H^1(M;\mathbb{R})/\operatorname{im}(\mathrm{id} - \phi^*)$; see [24, Lemma 6]. For the definition of the flux homomorphism see [21].

(Inclusion) The inclusion $\mathrm{Symp}^m(M,\omega) \hookrightarrow \mathrm{Diff}^+(M)$ is a homotopy equivalence.

(Floer homology) To every $\phi \in \mathrm{Symp}^m(M,\omega)$ symplectic Floer homology theory assigns a $\mathbb{Z}_2$-graded vector space $HF_*(\phi)$ over $\mathbb{Z}_2$, with an additional multiplicative structure, called the quantum cap product, $H^*(M;\mathbb{Z}_2) \otimes HF_*(\phi) \longrightarrow HF_*(\phi)$. For $\phi = \mathrm{id}_M$ the symplectic Floer homology $HF_*(\mathrm{id}_M)$ are canonically isomorphic to ordinary homology $H_*(M;\mathbb{Z}_2)$ and quantum cap product agrees with the ordinary cap product. Each $\psi \in \mathrm{Diff}^+(M)$ induces an isomorphism $HF_*(\phi) \cong HF_*(\psi^{-1}\phi\psi)$ of $H^*(M;\mathbb{Z}_2)$-modules.

(Invariance) If $\phi, \phi' \in \mathrm{Symp}^m(M,\omega)$ are isotopic, then $HF_*(\phi)$ and $HF_*(\phi')$ are naturally isomorphic as $H^*(M;\mathbb{Z}_2)$-modules. This is proven in [24, Page 7]. Note that every Hamiltonian perturbation of ϕ (see [4]) is also in $\mathrm{Symp}^m(M,\omega)$.

Now let g be a mapping class of M, i.e. an isotopy class of $\mathrm{Diff}^+(M)$. Pick an area form ω and a representative $\phi \in \mathrm{Symp}^m(M,\omega)$ of g. Then $HF_*(\phi)$ is an invariant of g, which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form ω by Moser's isotopy theorem [20] and naturality of Floer homology.

2.1.2. *Weak monotonicity.* We give now, following A. Cotton-Clay [2], a notion of *weak monotonicity* such that $HF_*(\phi)$ is well-defined for and invariant among weakly monotone maps. Monotonicity implies weak monotonicity, and so $HF_*(g) = HF_*(\phi)$ for any weakly monotone ϕ in mapping class g. The properties of weak monotone symplectomorphism of surface play a crucial role in the computation of Floer homology for pseudo-Anosov and reducible mapping classes(see [2]).

Definition 2.1. A map $\phi : M \to M$ is **weakly monotone** if $[\omega_\phi]$ vanishes on $\ker(c_\phi|_{T(T_\phi)})$, where $T(T_\phi) \subset H_2(M_\phi;\mathbb{R})$ is generated by tori T such that $\pi|_T : T \to S^1$ is a fibration with fiber S^1, where the map $\pi : T_\phi \to S^1$ is the projection.

Throughout this article $\mathrm{Symp}^{wm}(M,\omega)$ denotes the set of weakly monotone symplectomorphisms.

2.1.3. *Floer homology.* Let $\phi \in \mathrm{Symp}(M, \omega)$.There are two ways of constructing Floer homology detecting its fixed points, $\mathrm{Fix}(\phi)$. Firstly, the graph of ϕ is a Lagrangian submanifold of $M \times M, (-\omega) \times \omega)$ and its fixed points correspond to the intersection points of graph(ϕ) with the diagonal $\Delta = \{(x, x) \in M \times M\}$. Thus we have the Floer homology of the Lagrangian intersection $HF_*(M \times M, \Delta, graph(\phi))$. This intersection is transversal if the fixed points of ϕ are nondegenerate, i.e. if 1 is not an eigenvalue of $d\phi(x)$, for $x \in \mathrm{Fix}(\phi)$. The second approach was mentioned by Floer in [12] and presented with details by Dostoglou and Salamon in [4].We follow here Seidel's approach [24] which, comparable with [4], uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed. As a consequence, the usual invariance of Floer homology under Hamiltonian isotopies is extended to the stronger property stated above. Let now ϕ is monotone or weakly monotone. Firstly, we give the definition of $HF_*(\phi)$ in the special case where all the fixed points of ϕ are non-degenerate, i.e. for all $y \in \mathrm{Fix}(\phi)$, $\det(\mathrm{id} - \mathrm{d}\phi_y) \neq 0$, and then following Seidels approach [24] we consider general case when ϕ has degenerate fixed points. Let $\Omega_\phi = \{y \in C^\infty(\mathbb{R}, M) \,|\, y(t) = \phi(y(t+1))\}$ be the twisted free loop space, which is also the space of sections of $T_\phi \to S^1$. The action form is the closed one-form α_ϕ on Ω_ϕ defined by

$$\alpha_\phi(y)Y = \int_0^1 \omega(dy/dt, Y(t))\, dt.$$

where $y \in \Omega_\phi$ and $Y \in T_y\Omega_\phi$, i.e. $Y(t) \in T_{y(t)}M$ and $Y(t) = \mathrm{d}\phi_{y(t+1)}Y(t+1)$ for all $t \in \mathbb{R}$.

The tangent bundle of any symplectic manifold admits an almost complex structure $J : TM \longrightarrow TM$ which is compatible with ω in sense that $(v, w) = \omega(v, Jw)$ defines a Riemannian metric. Let $J = (J_t)_{t\in\mathbb{R}}$ be a smooth path of ω-compatible almost complex structures on M such that $J_{t+1} = \phi^* J_t$. If $Y, Y' \in T_y\Omega_\phi$, then $\int_0^1 \omega(Y'(t), J_tY(t))\mathrm{d}t$ defines a metric on the loop space Ω_ϕ. So the critical points of α_ω are the constant paths in Ω_ϕ and hence the fixed points of ϕ. The negative gradient lines of α_ω with respect to the metric above are solutions of the partial differential equations with boundary conditions

$$\begin{cases} u(s, t) = \phi(u(s, t+1)), \\ \partial_s u + J_t(u)\partial_t u = 0, \\ \lim_{s\to\pm\infty} u(s, t) \in \mathrm{Fix}(\phi) \end{cases} \tag{2}$$

These are exactly Gromov's pseudoholomorphic curves [15].

For $y^\pm \in \mathrm{Fix}(\phi)$, let $\mathcal{M}(y^-, y^+; J, \phi)$ denote the space of smooth maps $u : \mathbb{R}^2 \to M$ which satisfy the equations (2). Now to every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ we associate a Fredholm operator D_u which linearizes (2) in suitable Sobolev spaces. The index of this operator is given by the so called Maslov index $\mu(u)$, which satisfies $\mu(u) = \deg(y^+) - \deg(y^-) \bmod 2$, where $(-1)^{\deg y} = \mathrm{sign}(\det(\mathrm{id} - \mathrm{d}\phi_y))$. We have no bubbling, since for surface $\pi_2(M) = 0$. For a generic J, every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is regular, meaning that D_u is onto. Hence, by the implicit function theorem, $\mathcal{M}_k(y^-, y^+; J, \phi)$ is a smooth k-dimensional manifold and is the subset of those $u \in \mathcal{M}(y^-, y^+; J, \phi)$ with $\mu(u) = k \in \mathbb{Z}$. Translation of the s-variable defines a free $\mathbb{R}$-action on 1-dimensional manifold $\mathcal{M}_1(y^-, y^+; J, \phi)$ and hence the quotient is a discrete set of points. The energy of a map $u : \mathbb{R}^2 \to M$ is given by $E(u) = \int_\mathbb{R} \int_0^1 \omega\big(\partial_t u(s, t), J_t\partial_t u(s, t)\big)\, \mathrm{d}t\mathrm{d}s$ for all $y \in \mathrm{Fix}(\phi)$. P.Seidel and A. Cotton-Clay have proved in [24] and [2] that if ϕ is monotone or weakly monotone, then the energy is constant on each $\mathcal{M}_k(y^-, y^+; J, \phi)$. Since all fixed points of ϕ are nondegenerate the set $\mathrm{Fix}(\phi)$ is a finite set and the $\mathbb{Z}_2$-vector space $CF_*(\phi) := \mathbb{Z}_2^{\#\,\mathrm{Fix}(\phi)}$ admits a $\mathbb{Z}_2$-grading with $(-1)^{\deg y} = \mathrm{sign}(\det(\mathrm{id} - \mathrm{d}\phi_y))$, for all $y \in \mathrm{Fix}(\phi)$. The boundedness of the energy $E(u)$ for monotone or weakly monotone ϕ implies that the 0-dimensional quotients $\mathcal{M}_1(y_-, y_+, J, \phi)/\mathbb{R}$ are actually finite sets. Denoting by $n(y_-, y_+)$ the number of points mod 2 in each of them, one defines a differential $\partial_J : CF_*(\phi) \to CF_{*+1}(\phi)$ by $\partial_J y_- = \sum_{y_+} n(y_-, y_+) y_+$. Due to gluing theorem this Floer boundary operator satisfies $\partial_J \circ \partial_J = 0$. For gluing theorem to hold one needs again the boundedness of the energy $E(u)$. It follows that $(CF_*(\phi), \partial_J)$ is a chain complex and its homology is by definition the Floer homology of ϕ denoted $HF_*(\phi)$. It is independent of J and is an invariant of ϕ.

If ϕ has degenerate fixed points one needs to perturb equations (2) in order to define the Floer homology. Equivalently, one could say that the action form needs to be perturbed. The necessary analysis is given in [24], it is essentially the same as in the slightly different situations considered in [4]. But Seidel's approach also differs from the usual one in [4]. He uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed.

2.2. **Nielsen classes and Reidemeister trace.** Before discussing the results of the paper, we briefly describe the few basic notions of Nielsen fixed point theory which will be used. We assume X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, i.e. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(Fix(\tilde{f})) \subset Fix(f)$ is called *the fixed point class of f determined by the lifting class* $[\tilde{f}]$.Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X. A fixed point class is called *essential* if its index is nonzero. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$.The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy invariants. In the category of compact, connected polyhedra, the Nielsen number of a map is, apart from certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as f.

Let $f : X \to X$ be given, and let a specific lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ be chosen as reference. Let Γ be the group of covering translations of $\tilde{X}$ over X. Then every lifting of f can be written uniquely as $\alpha \circ \tilde{f}$, with $\alpha \in \Gamma$. So elements of Γ serve as coordinates of liftings with respect to the reference $\tilde{f}$. Now for every $\alpha \in \Gamma$ the composition $\tilde{f} \circ \alpha$ is a lifting of f so there is a unique $\alpha' \in \Gamma$ such that $\alpha' \circ \tilde{f} = \tilde{f} \circ \alpha$. This correspondence $\alpha \to \alpha'$ is determined by the reference $\tilde{f}$, and is obviously a homomorphism. The endomorphism $\tilde{f}_* : \Gamma \to \Gamma$ determined by the lifting $\tilde{f}$ of f is defined by $\tilde{f}_*(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha$. It is well known that $\Gamma \cong \pi_1(X)$. We shall identify $\pi = \pi_1(X, x_0)$ and Γ in the usual way.

We have seen that $\alpha \in \pi$ can be considered as the coordinate of the lifting $\alpha \circ \tilde{f}$. We can tell the conjugacy of two liftings from their coordinates: $[\alpha \circ \tilde{f}] = [\alpha' \circ \tilde{f}]$ iff there is $\gamma \in \pi$ such that $\alpha' = \gamma \alpha \tilde{f}_*(\gamma^{-1})$.

So we have the Reidemeister bijection: Lifting classes of f are in 1-1 correspondence with $\tilde{f}_*$-conjugacy classes in group π, the lifting class $[\alpha\circ\tilde{f}]$ corresponds to the $\tilde{f}_*$-cojugacy class of α.

By an abuse of language, we say that the fixed point class $p(\mathrm{Fix}\,\alpha\circ\tilde{f})$, which is labeled with the lifting class $[\alpha\circ\tilde{f}]$,corresponds to the $\tilde{f}_*$-conjugacy class of α. Thus the $\tilde{f}_*$-conjugacy classes in π serve as coordinates for the fixed point classes of f, once a reference lifting $\tilde{f}$ is chosen.

2.2.1. *Reidemeister trace.* The results of this section are well known(see [18],[7, 11]).We shall use this results later in section to estimate the radius of convergence of the symplectic zeta function. The fundamental group $\pi=\pi_1(X,x_0)$ splits into $\tilde{f}_*$-conjugacy classes.Let π_f denote the set of $\tilde{f}_*$-conjugacy classes,and $\mathbb{Z}\pi_f$ denote the Abelian group freely generated by π_f .We will use the bracket notation $a\rightarrow[a]$ for both projections $\pi\rightarrow\pi_f$ and $\mathbb{Z}\pi\rightarrow\mathbb{Z}\pi_f$. Let x be a fixed point of f.Take a path c from x_0 to x.The $\tilde{f}_*$-conjugacy class in π of the loop $c\cdot(f\circ c)^{-1}$,which is evidently independent of the choice of c, is called the coordinate of x.Two fixed points are in the same fixed point class F iff they have the same coordinates.This $\tilde{f}_*$-conjugacy class is thus called the coordinate of the fixed point class F and denoted $cd_\pi(F,f)$ (compare with description in section 2). The generalized Lefschetz number or the Reidemeister trace [18] is defined as

$$L_\pi(f):=\sum_F \mathrm{ind}(F,f)\cdot cd_\pi(F,f)\in\mathbb{Z}\pi_f, \tag{3}$$

the summation being over all essential fixed point classes $F$of f.The Nielsen number $N(f)$ is the number of non-zero terms in $L_\pi(f)$,and the indices of the essential fixed point classes appear as the coefficients in $L_\pi(f)$.This invariant used to be called the Reidemeister trace because it can be computed as an alternating sum of traces on the chain level as follows [18] . Assume that X is a finite cell complex and $f:X\rightarrow X$ is a cellular map. A cellular decomposition e_j^d of X lifts to a π-invariant cellular structure on the universal covering $\tilde{X}$.Choose an arbitrary lift $\tilde{e}_j^d$ for each e_j^d . They constitute a free $\mathbb{Z}\pi$-basis for the cellular chain complex of $\tilde{X}$. The lift $\tilde{f}$ of f is also a cellular map.In every dimension d, the cellular chain map $\tilde{f}$ gives rise to a $\mathbb{Z}\pi$-matrix $\tilde{F}_d$ with respect to the above basis,i.e $\tilde{F}_d=(a_{ij})$ if $\tilde{f}(\tilde{e}_i^d)=\sum_j a_{ij}\tilde{e}_j^d$,where $a_{ij}\in\mathbb{Z}\pi$.Then we have the Reidemeister trace formula

$$L_\pi(f)=\sum_d(-1)^d[\mathrm{Tr}\,\tilde{F}_d]\in\mathbb{Z}\pi_f. \tag{4}$$

Now we describe alternative approach to the Reidemeister trace formula proposed by Jiang [18]. This approach is useful when we study the periodic points of f,i.e. the fixed points of the iterates of f.

The mapping torus T_f of $f:X\rightarrow X$ is the space obtained from $X\times[o,\infty)$ by identifying $(x,s+1)$ with $(f(x),s)$ for all $x\in X, s\in[0,\infty)$.On T_f there is a natural semi-flow $\phi:T_f\times[0,\infty)\rightarrow T_f, \phi_t(x,s)=(x,s+t)$ for all $t\geq 0$.Then the map $f:X\rightarrow X$ is the return map of the semi-flow ϕ.A point $x\in X$ and a positive number $\tau>0$ determine the orbit curve $\phi_{(x,\tau)}:=\phi_t(x)_{0\leq t\leq\tau}$ in T_f. Take the base point x_0 of X as the base point of T_f.It is known that the fundamental group $H:=\pi_1(T_f,x_0)$ is obtained from π by adding a new generator z and adding the relations $z^{-1}gz=\tilde{f}_*(g)$ for all $g\in\pi=\pi_1(X,x_0)$.Let

H_c denote the set of conjugacy classes in H. Let $\mathbb{Z}H$ be the integral group ring of H, and let $\mathbb{Z}H_c$ be the free Abelian group with basis H_c.We again use the bracket notation $a \to [a]$ for both projections $H \to H_c$ and $\mathbb{Z}H \to \mathbb{Z}H_c$. If F^n is a fixed point class of f^n, then $f(F^n)$ is also fixed point class of f^n and $\mathrm{ind}(f(F^n), f^n) = \mathrm{ind}(F^n, f^n)$. Thus f acts as an index-preserving permutation among fixed point classes of f^n.By definition, an n-orbit class O^n of f to be the union of elements of an orbit of this action.In other words, two points $x, x' \in \mathrm{Fix}(f^n)$ are said to be in the same n-orbit class of f if and only if some $f^i(x)$ and some $f^j(x')$ are in the same fixed point class of f^n.The set $\mathrm{Fix}(f^n)$ splits into a disjoint union of n-orbits classes.Point x is a fixed point of f^n or a periodic point of period n if and only if orbit curve $\phi_{(x,n)}$ is a closed curve. The free homotopy class of the closed curve $\phi_{(x,n)}$ will be called the H -coordinate of point x,written $cd_H(x, n) = [\phi_{(x,n)}] \in H_c$.It follows that periodic points x of period n and x' of period n' have the same H-coordinate if and only if $n = n'$ and x,x' belong to the same n-orbits class of f. Thus it is possible equivalently define $x, x' \in \mathrm{Fix}(f^n)$ to be in the same n-orbit class if and only if they have the same $H-$coordinate. Jiang [18] has considered generalized Lefschetz number with respect to H

$$L_H(f^n) := \sum_{O^n} \mathrm{ind}(O^n, f^n) \cdot cd_H(O^n) \in \mathbb{Z}H_c, \tag{5}$$

and proved following trace formula:

$$L_H(f^n) = \sum_d (-1)^d [\mathrm{Tr}(z\tilde{F}_d)^n] \in \mathbb{Z}H_c, \tag{6}$$

where $\tilde{F}_d$ be $\mathbb{Z}\pi$-matrices defined in (16) and $z\tilde{F}_d$ is regarded as a $\mathbb{Z}H$-matrix.

2.3. **Computation of symplectic Floer homology.** In this section we describe known results from [2], [14], [8, 9] about computation of symplectic Floer homology for different mapping classes.

We recall firstly Thurston classification theorem for homeomorphisms of surfase M of genus ≥ 2.

Theorem 2.2. [27] *Every homeomorphism $\phi : M \to M$ is isotopic to a homeomorphism f such that either*
(1) f is a periodic map; or
(2) f is a pseudo-Anosov map, i.e. there is a number $\lambda > 1$, the dilation of f, and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that $f(F^s, \mu^s) = (F^s, \frac{1}{\lambda}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$; or
(3)f is reducible map, i.e. there is a system of disjoint simple closed curves $\gamma = \{\gamma_1, ..., \gamma_k\}$ in $intM$ such that γ is invariant by f (but γ_i may be permuted) and γ has a f-invariant tubular neighborhood U such that each component of $M \setminus U$ has negative Euler characteristic and on each(not necessarily connected) f-component of $M \setminus U$, f satisfies (1) or (2).

The map f above is called a Thurston standard form of ϕ. In (3) it can be chosen so that some iterate f^m is a generalised Dehn twist on U. Such a f , as well as the f in (1) or (2), will be called standard. A key observation is that if f is standard, so are all iterates of f.

Thurston classification theorem for homeomorphisms of surfase implies that every mapping class of M is precisely one of the following: periodic, pseudo-Anosov or reducible.

2.3.1. *Periodic mapping classes.*

Theorem 2.3. [14], [9] *If ϕ is a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) \leq 0$, then ϕ is monotone symplectomorphism with respect to some ϕ-invariant area form and*

$$\dim HF_*(\phi) = L(\phi) = N(\phi)$$

where $L(\phi), N(\phi)$ denote the Lefschetz and the Nielsen number of ϕ correspondingly.

2.3.2. *Algebraically finite mapping classes.* A mapping class of M is called algebraically finite if it does not have any pseudo-Anosov components in the sense of Thurston's theory of surface diffeomorphism.The term algebraically finite goes back to J. Nielsen
In [14] the diffeomorphisms of finite type were defined . These are special representatives of algebraically finite mapping classes adopted to the symplectic geometry.

Definition 2.4. We call $\phi \in \mathrm{Diff}_+(M)$ of **finite type** if the following holds. There is a ϕ-invariant finite union $N \subset M$ of disjoint non-contractible annuli such that:
(1) $\phi|M \setminus N$ is periodic, i.e. there exists $\ell > 0$ such that $\phi^\ell|M \setminus N = \mathrm{id}$.
(2) Let N' be a connected component of N and $\ell' > 0$ be the smallest integer such that $\phi^{\ell'}$ maps N' to itself. Then $\phi^{\ell'}|N'$ is given by one of the following two models with respect to some coordinates $(q,p) \in I \times S^1$:

(twist map) $\qquad (q,p) \longmapsto (q, p - f(q))$

(flip-twist map) $\qquad (q,p) \longmapsto (1-q, -p - f(q)),$

where $f : I \to \mathbb{R}$ is smooth and strictly monotone. A twist map is called positive or negative, if f is increasing or decreasing.
(3) Let N' and ℓ' be as in (2). If $\ell' = 1$ and $\phi|N'$ is a twist map, then $\mathrm{im}(f) \subset [0,1]$, i.e. $\phi|\mathrm{int}(N')$ has no fixed points.
(4) If two connected components of N are homotopic, then the corresponding local models of ϕ are either both positive or both negative twists.

By M_{id} we denote the union of the components of $M \setminus \mathrm{int}(N)$, where ϕ restricts to the identity.

The monotonicity of diffeomorphisms of finite type was investigated in details in [14]. Let ϕ be a diffeomorphism of finite type and ℓ be as in (1). Then ϕ^ℓ is the product of (multiple) Dehn twists along N. Moreover, two parallel Dehn twists have the same sign, by (4). We say that ϕ has uniform twists, if ϕ^ℓ is the product of only positive, or only negative Dehn twists.
Furthermore, we denote by ℓ the smallest positive integer such that ϕ^ℓ restricts to the identity on $M \setminus N$.

If ω' is an area form on M which is the standard form $\mathrm{d}q \wedge \mathrm{d}p$ with respect to the (q,p)-coordinates on N, then $\omega := \sum_{i=1}^{\ell} (\phi^i)^*\omega'$ is standard on N and ϕ-invariant, i.e. $\phi \in \mathrm{Symp}(M,\omega)$. To prove that ω can be chosen such that $\phi \in \mathrm{Symp}^m(M,\omega)$, Gautschi distinguishes two cases: uniform and non-uniform twists. In the first case he proves the following stronger statement.

Lemma 2.5. [14] *If ϕ has uniform twists and ω is a ϕ-invariant area form, then $\phi \in \mathrm{Symp}^m(M,\omega)$.*

In the non-uniform case, monotonicity does not hold for arbitrary ϕ-invariant area forms.

Lemma 2.6. [14] *If ϕ does not have uniform twists, there exists a ϕ-invariant area form ω such that $\phi \in \mathrm{Symp}^m(M,\omega)$. Moreover, ω can be chosen such that it is the standard form $\mathrm{d}q \wedge \mathrm{d}p$ on N.*

Theorem 2.7. [14] *Let ϕ be a diffeomorphism of finite type, then ϕ is monotone with respect to some ϕ-invariant area form and*

$$HF_*(\phi) = H_*(M_{\mathrm{id}}, \partial_{M_{\mathrm{id}}}; \mathbb{Z}_2) \oplus \mathbb{Z}_2^{L(\phi|M\setminus M_{\mathrm{id}})}.$$

Here, L denotes the Lefschetz number.

2.3.3. *Pseudo-Anosov mapping classes.*

Theorem 2.8. *([2], see also [9]) If ϕ is any symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class g, then ϕ is weakly monotone, $HF_*(\phi)$ is well defined and*

$$\dim HF_*(\phi) = \dim HF_*(g) = \sum_{x \in \mathrm{Fix}(\psi)} |\operatorname{Ind}(x)|,$$

where ψ is the standard pseudo-Anosov representative of g.

2.3.4. *Reducible mapping classes.* Recently, A. Cotton-Clay [2] calculated Seidel's symplectic Floer homology for reducible mapping classes. This result completing all previous computations.

In the case of reducible mapping classes a energy estimate forbids holomorphic discs from crossing reducing curves except when a pseudo-Anosov component meets an identity component (with no twisting). Let us introduce some notation following [2]. Recall the notation of M_{id} for the collection of fixed components as well as the tree types of boundary: 1) $\partial_+ M_{id}$, $\partial_- M_{id}$ denote the collection of components of ∂M_{id} on which we've joined up with a positive(resp. negative) twist; 2) the collection of components of ∂M_{id} which meet a pseudo-Anosov component will be denoted $\partial_p M_{\mathrm{id}}$. Additionally let M_1 be the collection of periodic components and let M_2 be the collection of pseudo-Anosov components with punctures(i.e. before any perturbation) instead of boundary components wherever there is a boundary component that meets a fixed component. We further subdivide M_{id}. Let M_a be the collection of fixed components which don't meet any pseudo-Anosov components. Let $M_{b,p}$ be the collection of fixed components which meet one pseudo-Anosov component at a boundary with p prongs. In this case, we assign the boundary components to $\partial_+ M_{\mathrm{id}}$ (this is an arbitrary choice). Let $M^o_{b,p}$ be the collection of the $M_{b,p}$ with each component punctured once. Let $M_{c,q}$ be the collection of fixed components which meets at least two pseudo-Anosov components such that the total number of prongs over all the boundaries is q. In this case, we assign at least one boundary component to $\partial_+ M_{\mathrm{id}}$ and at least one to $\partial_- M_{\mathrm{id}}$ (and beyond that, it does not matter).

Theorem 2.9. [2] *Let ϕ be a perturbed standard form map* [2] *in a reducible mapping class g with choices of the signs of components of $\partial_p M_{\mathrm{id}}$. Then $HF_*(\phi)$ is well-defined and*

$$\dim HF_*(g) = \dim HF_*(\phi) = \dim H_*(M_a, \partial_+ M_{\mathrm{id}}; \mathbb{Z}_2) +$$

$$+\sum_p(\dim H_*(M^0_{b,p},\partial_+M_{b,p};\mathbb{Z}_2)+(p-1)|\pi_0(M_{b,p})|)+$$
$$+\sum_q(\dim H_*(M_{c,q},\partial_+M_{c,q};\mathbb{Z}_2)+q|\pi_0(M_{c,q)})|)+$$
$$+L(\phi|M_1)+\dim HF_*(\phi|M_2),$$

where $L(\phi|M_1)$ is the Lefschetz number of $\phi|M_1$, the $L(\phi|M_1)$ summand is all in even degree, the other two summands(with $p-1$ and q are all in odd degree, and $HF_*(\phi|M_2)$ denotes the Floer homology for ϕ on the pseudo-Anosov components M_2

Remark 2.10. The first summand and the $L(\phi|M_1)$ are as in R. Gautschi's Theorem 2.7 [14]. The last summand comes from the pseudo-Anosov components and is calculated via the Theorem 2.8. The sums over p and q arise in the same manner as the first summand.

Corollary 2.11. *As an application, A. Cotton-Clay gave recently* [3] *a sharp lower bound on the number of fixed points of area-preserving map in any prescribed mapping class(rel boundary), generalising the Poincare-Birkhoff fixed point theorem.*

3. The growth rate and symplectic zeta function

3.1. **Topological entropy and Nielsen numbers.** The most widely used measure for the complexity of a dynamical system is the topological entropy. For the convenience of the reader, we include its definition. Let $f : X \to X$ be a self-map of a compact metric space. For given $\epsilon > 0$ and $n \in \mathbb{N}$, a subset $E \subset X$ is said to be (n,ϵ)-separated under f if for each pair $x \neq y$ in E there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) > \epsilon$. Let $s_n(\epsilon, f)$ denote the largest cardinality of any (n,ϵ)-separated subset E under f. Thus $s_n(\epsilon, f)$ is the greatest number of orbit segments $x, f(x), ..., f^{n-1}(x)$ of length n that can be distinguished one from another provided we can only distinguish between points of X that are at least ϵ apart. Now let

$$h(f,\epsilon) := \limsup_n \frac{1}{n}\cdot \log\, s_n(\epsilon, f)$$
$$h(f) := \limsup_{\epsilon\to 0} h(f,\epsilon).$$

The number $0 \leq h(f) \leq \infty$, which to be independent of the metric d used, is called the topological entropy of f. If $h(f,\epsilon) > 0$ then, up to resolution $\epsilon > 0$, the number $s_n(\epsilon, f)$ of distinguishable orbit segments of length n grows exponentially with n. So $h(f)$ measures the growth rate in n of the number of orbit segments of length n with arbitrarily fine resolution.

A basic relation between topological entropy $h(f)$ and Nielsen numbers was found by N. Ivanov [17]. We present here a very short proof by Boju Jiang of the Ivanov's inequality.

Lemma 3.1. [17]

$$h(f) \geq \limsup_n \frac{1}{n}\cdot \log N(f^n)$$

Proof. Let δ be such that every loop in X of diameter $< 2\delta$ is contractible. Let $\epsilon > 0$ be a smaller number such that $d(f(x), f(y)) < \delta$ whenever $d(x, y) < 2\epsilon$. Let $E_n \subset X$ be a set consisting of one point from each essential fixed point class of f^n. Thus $\mid E_n \mid= N(f^n)$. By the definition of $h(f)$, it suffices to show that E_n is (n,ϵ)-separated. Suppose it is not so. Then there would be two points $x \neq y \in E_n$ such that $d(f^i(x), f^i(y)) \leq \epsilon$ for $o \leq i < n$ hence for all $i \geq 0$. Pick a path c_i from $f^i(x)$ to $f^i(y)$ of diameter $< 2\epsilon$ for $o \leq i < n$ and let $c_n = c_0$. By the choice of δ and ϵ , $f \circ c_i \simeq c_{i+1}$ for all i, so $f^n \circ c_0 \simeq c_n = c_0$. such that This means x, y in the same fixed point class of f^n, contradicting the construction of E_n. □

This inequality is remarkable in that it does not require smoothness of the map and provides a common lower bound for the topological entropy of all maps in a homotopy class.

3.2. **Asymptotic invariant.** Let $\Gamma = \pi_0(Diff^+(M))$ be the mapping class group of a closed connected oriented surface M of genus ≥ 2. Pick an everywhere positive two-form ω on M. A isotopy theorem of Moser [20] says that each mapping class of $g \in \Gamma$, i.e. an isotopy class of $Diff^+(M)$, admits representatives which preserve ω. Due to Seidel[24] and Cotton-Clay [2] we can pick a monotone(weakly monotone) representative $\phi \in \mathrm{Symp}^m(M, \omega)$(or $\phi \in \mathrm{Symp}^{wm}(M, \omega)$ of g. Then $HF_*(\phi)$ is an invariant of g, which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form ω by Moser's theorem and naturality of Floer homology.

Taking a dynamical point of view, we consider now the iterates of monotone(weakly monotone) symplectomorphism ϕ. Symplectomorphisms ϕ^n are also monotone(weakly monotone) for all $n > 0$ [14, 2].

The growth rate of a sequence a_n of complex numbers is defined by

$$\mathrm{Growth}(a_n) := max\{1, \limsup_{n \to \infty} |a_n|^{1/n}\}$$

which could be infinity. Note that $\mathrm{Growth}(a_n) \geq 1$ even if all $a_n = 0$. When $\mathrm{Growth}(a_n) > 1$, we say that the sequence a_n grows exponentially.

In [8] we have introduced the asymptotic invariant $F^\infty(g)$ of mapping class $g \in Mod_M = \pi_0(Diff^+(M))$ as the growth rate of the sequence $\{a_n = \dim HF_*(\phi^n)\}$ for a monotone(or weakly monotone) representative $\phi \in \mathrm{Symp}^m(M, \omega)$ of g:

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n))$$

Example 3.2. If ϕ is a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) < 0$, then the periodicity of the sequence $\dim HF_*(\phi^n)$ implies that for the corresponding mapping class g the asymptotic invariant

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = 1$$

Example 3.3. Let ϕ be a monotone diffeomorphism of finite type of a compact connected surface M of Euler characteristic $\chi(M) < 0$ and g a corresponding algebraically finite mapping class. Then the total dimension of $HF_*(\phi^n)$ grows at most linearly (see [2, 26, 9]. Taking the growth rate in n, we get that the asymptotic invariant

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = 1$$

.

For any set S let $\mathbb{Z}S$ denote the free Abelian group with the specified basis S.The norm in $\mathbb{Z}S$ is defined by

$$\| \sum_i k_i s_i \| := \sum_i \mid k_i \mid \in \mathbb{Z}, \tag{7}$$

when the s_i in S are all different.

For a $\mathbb{Z}H$-matrix $A = (a_{ij})$,define its norm by $\|A\| := \sum_{i,j} \|a_{ij}\|$.Then we have inequalities $\|AB\| \leq \|A \mid \cdot \|B\|$ when A, B can be multiplied, and $\| \operatorname{tr} A\| \leq \|A\|$ when A is a square matrix.For a matrix $A = (a_{ij})$ in $\mathbb{Z}S$, its matrix of norms is defined to be the matrix $A^{norm} := (\|a_{ij}\|)$ which is a matrix of non-negative integers.In what follows, the set S wiill be π, H or H_c.We denote by $s(A)$ the spectral radius of A, $s(A) = \lim_n \sqrt[n]{\|A^n\|}$ which coincide with the largest modul of an eigenvalue of A.

Remark 3.4. The norm $\|L_H(f^n)\|$ is the sum of absolute values of the indices of all the n-orbits classes O^n . It equals $\|L_\pi(f^n)\|$, the sum of absolute values of the indices of all the fixed point classes of f^n, because any two fixed point classes of f^n contained in the same n-orbit class O^n must have the same index. The norm $\|L_\pi(f^n)\|$ is homotopy type invariant.

We define the asymptotic absolute Lefschetz number [18] to be the growth rate

$$L^\infty(f) = \operatorname{Growth}(\|L_\pi(f^n)\|)$$

We also define the asymptotic Nielsen number[17] to be the growth rate

$$N^\infty(f) = \operatorname{Growth}(N(f^n))$$

All these asymptotic numbers are homotopy type invariants.

Lemma 3.5. *If ϕ is any symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class g, then*

$$\dim HF_*(\phi) = \dim HF_*(g) = \|L_\pi(\psi)\|,$$

where ψ is a standard pseudo-Anosov representative of g.

Proof. It is known that for pseudo-Anosov map ψ fixed points are topologically separated, i.e. each essential fixed point class of ψ consists of a single fixed point(see [27, 17, 7]). Then the generalized Lefschetz number or the Reidemeister trace [18] is

$$L_\pi(\psi) := \sum_F \operatorname{ind}(F, \psi) \cdot cd_\pi(F, \psi) = \sum_{x \in \operatorname{Fix}(\psi)} \operatorname{ind}(x) \cdot cd_\pi(x, \psi) \in \mathbb{Z}\pi_\psi, \tag{8}$$

where the summation being over all essential fixed point classes F of ψ i.e over all fixed points of ψ. So, the result follows from the theorem 2.8 and the definition of the norm $\|L_\pi(\psi)\|$.

Remark 3.6. Lemma 3.5 provides via Reidemeister trace formula a new combinatorial formula to compute $\dim HF_*(g)$ comparable to the train-track combinatorial formula of A. Cotton-Clay in [2].

Theorem 3.7. [5, 17, 18] *Let f be a pseudo-Anosov homeomorphism with dilatation $\lambda > 1$ of surfase M of genus ≥ 2. Then*

$$h(f) = log(\lambda) = \log N^\infty(f) = \log L^\infty(f)$$

Theorem 3.8. [18] *Suppose f is a standard homeomorphism of surfase M of genus ≥ 2 and λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components). Then*

$$h(f) = log(\lambda) = \log N^\infty(f) = \log L^\infty(f)$$

Theorem 3.9. *If ϕ is any symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class g with dilatation $\lambda > 1$ of surfase M of genus ≥ 2. Then*

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = \lambda = \exp(h(\psi)) = L^\infty(\psi) = N^\infty(\psi)$$

where ψ is a standard pseudo-Anosov representative of g.

Proof. By lemma 3.5 we have that $\dim HF_*(\phi^n) = \|L_\pi(\phi^n)\|$ for every n. So, the result follows from theorem 3.7.

Theorem 3.10. *Let ϕ be a perturbed standard form map ψ in a reducible mapping class g of compact surface of genus ≥ 2 and λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components). Then*

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = \lambda = \exp(h(\psi)) = L^\infty(\psi) = N^\infty(\psi)$$

Proof. It follows from the theorem 2.9 that for every n

$$\dim HF_*(g^n) = \dim HF_*(\phi^n) = \dim H_*(M_a, \partial_+ M_{\mathrm{id}}; \mathbb{Z}_2) +$$
$$+ \sum_p (\dim H_*(M^0_{b,p}, \partial_+ M_{b,p}; \mathbb{Z}_2) + (p-1)|\pi_0(M_{b,p})|) +$$
$$+ \sum_q (\dim H_*(M_{c,q}, \partial_+ M_{c,q}; \mathbb{Z}_2) + q|\pi_0(M_{c,q)})|) +$$
$$+ L(\phi^n|M_1) + \dim HF_*(\phi^n|M_2).$$

We need to investigate only the growth of the last summand in this formula because the rest part in the formula grows at most linearly [2, 26]. We have $\dim HF_*(\phi^n|M_2) = \sum_j \dim HF_*(\phi_j^n|M_2)$, where the sum is taken over different pseudo-Anosov components of $\phi^n|M_2$. It follows from the theorem 3.9 that $\dim HF_*(\phi_j^n|M_2)$ grows as λ_j^n, where λ_j is the dilatation of the pseudo-Anosov component $\phi_j|M_2$.

Taking growth rate in n, we get $F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = \max_j \lambda_j = \lambda = \exp(h(\psi))$.

Corollary 3.11. *The asymptotic invariant $F^\infty(g) > 1$ if and only if ψ has a pseudo-Anosov component.*

3.3. **Radius of convergence of the symplectic zeta function.** In [8] we have introduced a symplectic zeta function

$$F_\phi(z) \quad = \quad \exp\left(\sum_{n=1}^\infty \frac{\dim HF_*(\phi^n)}{n} z^n\right).$$

$F_\phi(z)$ is an invariant of mapping class g, which we denote by $F_g(z)$. A motivation for this definition was a connection [8, 14] between Nielsen numbers and Floer homology and nice analytic properties of Nielsen zeta function [22, 8, 6, 7, 10, 11]

We denote by R the radius of convergence of the symplectic zeta function $F_g(z) = F_\phi(z)$.

In this section we give exact algebraic lower estimation for the radius R using Reidemeister trace formula for generalized Lefschetz numbers.

Theorem 3.12. *For any symplectomorphism ϕ of compact surface M of genus ≥ 2 the symplectic zeta function has positive radius of convergence $R = \frac{1}{\lambda}$, where λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components). Radius of convergence R admits following estimations*

$$R \geq \frac{1}{\max_d \|z\tilde{F}_d\|} > 0 \tag{9}$$

and

$$R \geq \frac{1}{\max_d s(\tilde{F}_d^{norm})} > 0 \tag{10}$$

Proof. By the homotopy type invariance of the invariants we can suppose that ϕ is a cell map of a finite cell complex. It follows from lemma 3.5 that $\dim HF_*(\phi^n) = \|L_\pi(\phi^n)\|$.The norm $\|L_H(\phi^n)\|$ is the sum of absolute values of the indices of all the n-orbits classes O^n . It equals $\|L_\pi(\phi^n)\|$, the sum of absolute values of the indices of all the fixed point classes of ϕ^n, because any two fixed point classes of ϕ^n contained in the same n-orbit class O^n must have the same index. From this we have $\dim HF_*(\phi^n) = \|L_\pi(\phi^n)\| = \|L_H(\phi^n)\| = \|\sum_d (-1)^d [\mathrm{tr}(z\tilde{F}_d)^n]\| \leq \sum_d \|[\mathrm{tr}(z\tilde{F}_d)^n]\| \leq \sum_d \|\mathrm{tr}(z\tilde{F}_d)^n\| \leq \sum_d \|(z\tilde{F}_d)^n\| \leq \sum_d \|(z\tilde{F}_d)\|^n$. The radius of convergence R is given by Caushy-Adamar formula:

$$\frac{1}{R} = \limsup_n \sqrt[n]{\frac{\dim HF_*(\phi^n)}{n}} = \limsup_n \sqrt[n]{\dim HF_*(\phi^n)} = \lambda.$$

Therefore we have:

$$R = \frac{1}{\limsup_n \sqrt[n]{\dim HF_*(\phi^n)}} \geq \frac{1}{\max_d \|z\tilde{F}_d\|} > 0.$$

Inequalities:

$$\begin{aligned} \dim HF_*(\phi^n) \leq \|L_\pi(f^n)\| = \|L_H(f^n)\| = \|\sum_d (-1)^d [\mathrm{tr}(z\tilde{F}_d)^n]\| \leq \sum_d \|[\mathrm{tr}(z\tilde{F}_d)^n]\| \\ \leq \sum_d || \mathrm{tr}(z\tilde{F}_d)^n|| \leq \sum_d \mathrm{tr}((z\tilde{F}_d)^n)^{norm} \leq \sum_d \mathrm{tr}((z\tilde{F}_d)^{norm})^n \\ \leq \sum_d \mathrm{tr}((\tilde{F}_d)^{norm})^n \end{aligned}$$

and the definition of spectral radius give estimation:

$$R = \frac{1}{\limsup_n \sqrt[n]{\dim HF_*(\phi^n)}} \geq \frac{1}{\max_d s(\tilde{F}_d^{norm})} > 0.$$

Example 3.13. Let X be surface with boundary, and $f : X \to X$ be a map.Fadell and Husseini(see [18]) devised a method of computing the matrices of the lifted chain map for surface maps.Suppose $\{a_1, ..., a_r\}$ is a free basis for $\pi_1(X)$. Then X has the homotopy type of a bouquet B of r circles which can be decomposed into one 0-cell and r 1-cells corresponding to the a_i,and f has the homotopy type of a cellular map $g : B \to B$. By the homotopy type invariance of the invariants,we can replace f with g in computations.The homomorphism $\tilde{f}_*:\pi_1(X)\to\pi_1(X)$ induced by f and g is determined by the images $b_i=\tilde{f}_*(a_i), i=1,..,r$.The fundamental group $\pi_1(T_f)$ has a presentation $\pi_1(T_f)=<a_1,...,a_r,z|a_iz=zb_i, i=1,..,r>$.Let

$$D=(\frac{\partial b_i}{\partial a_j})$$

be the Jacobian in Fox calculus(see [18]).Then,as pointed out in [18], the matrices of the lifted chain map $\tilde{g}$ are

$$\tilde{F}_0=(1), \tilde{F}_1=D=(\frac{\partial b_i}{\partial a_j}).$$

Now, we can find estimations for the radius R as above.

Let $\mu(d), d\in\mathbb{N}$, be the Möbius function.

Theorem 3.14. [8] *Let ϕ be a non-trivial orientation preserving periodic diffeomorphism of least period m of a compact connected surface M of Euler characteristic $\chi(M)<0$. Then the zeta function $F_\phi(z)$ is a radical of a rational function and*

$$F_\phi(z)=\prod_{d|m}\sqrt[d]{(1-z^d)^{-P(d)}},$$

where the product is taken over all divisors d of the period m, and $P(d)$ is the integer $P(d)=\sum_{d_1|d}\mu(d_1)\dim HF_*(\phi^{d/d_1})$.

We denote by $L_\phi(z)$ the Weil zeta function

$$L_\phi(z):=\exp\left(\sum_{n=1}^{\infty}\frac{L(\phi^n)}{n}z^n\right),$$

where $L(\phi^n)$ is the Lefschetz number of of ϕ^n.

Theorem 3.15. [8] *If ϕ is a hyperbolic diffeomorphism of a 2-dimensional torus T^2, then the symplectic zeta function $F_\phi(z)$ is a rational function and $F_\phi(z)=(L_\phi(\sigma\cdot z))^{(-1)^r}$, where r is equal to the number of $\lambda_i\in Spec(\tilde{\phi})$ such that $|\lambda_i|>1$, p is equal to the number of $\mu_i\in Spec(\tilde{\phi})$ such that $\mu_i<-1$ and $\sigma=(-1)^p$.*

3.4. Concluding remarks and questions.

Question 3.16. *(Entropy conjecture for symplectomorphisms)*

Is it always true that for symplectomorphisms of compact symplectic manifolds

$$h(\phi)\geq\log F^\infty(\phi)=\log\text{Growth}(\dim HF_*(\phi^n))?$$

Question 3.17. *Is it true that for a symplectomorphism ϕ of an aspherical compact symplectic manifold*

$$F^\infty(\phi):=\text{Growth}(\dim HF_*(\phi^n))=L^\infty(\phi)=N^\infty(\phi)?$$

Inspired by the Hasse-Weil zeta function of an algebraic variety over a finite field, Artin and Mazur [1] defined the zeta function for an arbitrary map $f:X\to X$ of a topological space X:

$$AM_f(z):=\exp\left(\sum_{n=1}^{\infty}\frac{\#\,\text{Fix}(f^n)}{n}z^n\right),$$

where $\#\operatorname{Fix}(f^n)$ is the number of isolated fixed points of f^n. Artin and Mazur showed that for a dense set of the space of smooth maps of a compact smooth manifold into itself the number of periodic points $\#\operatorname{Fix}(f^n)$ grows at most exponentially and the Artin-Mazur zeta function $AM_f(z)$ has a positive radius of convergence [1]. Later Manning [19] proved the rationality of the Artin - Mazur zeta function for diffeomorphisms of a smooth compact manifold satisfying Smale axiom A. On the other hand there exist maps for which Artin-Mazur zeta function is transcendental . The symplectic zeta function $F_\phi(z)$ can be considered as some analog of the Artin-Mazur zeta function $AM_f(z)$ because periodic points of ϕ^n provide the generators of symplectic Floer homologies $HF_*(\phi^n)$. This motivate following

Conjecture 3.18. *For any compact symplectic manifold M and symplectomorphism $\phi: M \to M$ with well defined Floer homology groups $HF_*(\phi^n)$, $n \in \mathbb{N}$ the symplectic zeta function $F_\phi(z)$ has a positive radius of convergence.*

Question 3.19. *Is the symplectic zeta function $F_\phi(z)$ an algebraic function of z?*

Remark 3.20. Given a symplectomorphism ϕ of surface M, one can form the symplectic mapping torus $M^4_\phi = T^3_\phi \rtimes S^1$, where T^3_ϕ is usual mapping torus . Ionel and Parker [16] have computed the degree zero Gromov invariants [16](these are built from the invariants of Ruan and Tian) of M^4_ϕ and of fiber sums of the M^4_ϕ with other symplectic manifolds. This is done by expressing the Gromov invariants in terms of the Lefschetz zeta function $L_\phi(z)$ [16]. The result is a large set of interesting non-Kahler symplectic manifolds with computational ways of distinguishing them. In dimension four this gives a symplectic construction of the exotic elliptic surfaces of Fintushel and Stern [13]. This construction arises from knots. Associated to each fibered knot K in S^3 is a Riemann surface M and a monodromy diffeomorphism f_K of M. Taking $\phi = f_K$ gives symplectic 4-manifolds ${M^4}_\phi(K)$ with Gromov invariant $Gr({M^4}_\phi(K)) = A_K(t)/(1-t)^2 = L_\phi(t)$, where $A_K(t)$ is the Alexander polynomial of knot K. Next, let $E^4(n)$ be the simply-connected minimal elliptic surface with fiber F and canonicla divisor $k = (n-2)F$. Forming the fiber sum $E^4(n,K) = E^4(n)\#_{(F=T^2)}{M^4}_\phi(K)$ we obtain a symplectic manifold homeomorphic to $E^4(n)$. Then for $n \geq 2$ the Gromov and Seiberg-Witten invariants of $E^4(K)$ are $Gr(E^4(n,K)) = SW(E^4(n,K)) = A_K(t)(1-t)^{n-2}$ [13, 16]. Thus fibered knots with distinct Alexander polynomials give rise to symplectic manifolds $E^4(n,K)$ which are homeomorphic but not diffeomorphic. In particular, there are infinitely many distinct symplectic 4-manifolds homeomorphic to $E^4(n)$ [13] .

In higher dimensions it gives many examples of manifolds which are diffeomorphic but not equivalent as symplectic manifolds. Theorem 13 in [8] implies that the Gromov invariants of M^4_ϕ are related to symplectic Floer homology of ϕ via Lefschetz zeta function $L_\phi(z)$. We hope that the symplectic zeta function $F_\phi(z)$ give rise to a new invariant of symplectic 4-manifolds.

References

[1] M. Artin and B. Mazur, On periodic points, Annals of Math., 81 (1965), 82-99.

[2] A. Cotton-Clay, Symplectic Floer homology of area-preserving surface diffeomorphisms, Geometry and Topology 13(2009), 2619-2674.

[3] A. Cotton-Clay, A sharp bound on fixed points of surface symplectomorphisms in each mapping class, arXiv: 1009.0760[math.SG], 2010.

[4] S. Dostoglou and D. Salamon, Self dual instantons and holomorphic curves, Annals of Math., 139 (1994), 581–640.
[5] A. Fathi, F. Laudenbach, and V. Poénaru, Travaux de Thurston sur les surfaces, Astérisque, vol. 66–67, Soc. Math. France, 1979.
[6] A.L. Fel'shtyn, New zeta functions for dynamical systems and Nielsen fixed point theory. in : Lecture Notes in Math. 1346, Springer, 1988, 33-55.
[7] A. Fel'shtyn, *Dynamical zeta functions, Nielsen theory and Reidemeister torsion*, Mem. Amer. Math. Soc. **147** (2000), no. 699, xii+146. MR2001a:37031
[8] Fel'shtyn A.L. Floer homology, Nielsen theory and symplectic zeta functions. Proceedings of the Steklov Institute of Mathematics, Moscow, vol. 246, 2004, pp. 270-282.
[9] Fel'shtyn A.L. Nielsen theory, Floer homology and a generalisation of the Poincare - Birkhoff theorem. Journal of Fixed Point Theory and Applications, no.2, v.3(2008), 191-214.
[10] A.L.Fel'shtyn,R.Hill, The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion. K-theory,v.8,n.4,1994,p.367-393
[11] A.L.Fel'shtyn,R.Hill, Trace formulae, zeta functions, congruences and Reidemeister torsion in Nielsen theory. Forum Mathematicum, v.10, n.6, 1998, 641-663.
[12] A. Floer. Morse theory for Lagrangian intersections. J. Differential Geom., 28(1988), 513-547.
[13] R. Fintushel, R. Stern, Knots, links and 4-manifolds, Invention. Math. 134(1998), 363-400.
[14] R. Gautschi. Floer homology of algebraically finite mapping classes J. Symplectic Geom. 1(2003), no.4, 715-765.
[15] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. Invent. Math.82(1985), 307-347.
[16] E.-N. Ionel, Th. Parker, Gromov invariants and symplectic maps, Math. Ann. 314(1999), 127-158.
[17] N. V. Ivanov, Entropy and the Nielsen Numbers. Dokl. Akad. Nauk SSSR 265 (2) (1982), 284-287 (in Russian); English transl.: Soviet Math. Dokl. 26 (1982), 63-66.
[18] B. Jiang, Estimation of the number of periodic orbits. Pacific Jour. Math., 172(1996), 151-185.
[19] A. Manning, Axiom A diffeomorphisms have rational zeta function. Bull. London Math. Soc. 3 (1971), 215-220.
[20] J. Moser. On the volume elements on a manifold. Trans. Amer. Math. Soc., 120:286–294, 1965.
[21] D. McDuff and D. A. Salamon. J-holomorphic Curves and Symplectic Topology. AMS Colloquium Publications, Vol. 52, 2004.
[22] V. B. Pilyugina and A. L. Fel'shtyn, The Nielsen zeta function. Funktsional. Anal. i Prilozhen. 19 (4) (1985), 61-67 (in Russian); English transl.: Functional Anal. Appl. 19 (1985), 300-305.
[23] M. Pozniak, Floer homology, Novikov rings and clean intersections, PhD thesis,University of Warwick (1994).
[24] P. Seidel. Symplectic Floer homology and the mapping class group. Pacific J. Math. 206(2002), no. 1, 219-229.
[25] P. Seidel. Braids and symplectic four-manifolds with abelian fundamental group. Turkish J. Math. 26(2002), no.1, 93-100.
[26] Ivan Smith. Floer cohomology and pencils of quadrics. arXiv:1006.1099v1[math.SG], 2010.
[27] W. P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. Bull. Amer. Math. Soc., 19(2):417–431, 1988.

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