NONCOMMUTATIVE GEOMETRY AND PATH INTEGRALS

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To Yuri Ivanovich Manin on his 70th birthday

Introduction

(0.1) A monomial in noncommutative variables X and Y, say, $X^iY^jX^kY^l...$, can be visualized as a lattice path in the plane, starting from 0, going i steps in the horizontal direction, j steps in the vertical one , then again k steps in the horizontal one, and so on. Usual commutative monomials are often visualized as lattice points, for example $x^a y^b$ corresponds to the point (a, b). To lift such a monomial to the noncommutative domain, is therefore the same as to choose a "history" for (a, b), i.e., a lattice path originating at 0 and ending at (a, b).

This correspondence between paths and noncommutative monomials can be extended to more general piecewise smooth paths, if we deal with exponential functions instead. Let us represent our commutative variables as $x = e^z, y = e^w$, then a monomial will be replaced by the exponential e^{az+bw} and we are free to take a and b to be any real numbers. To lift this exponential to the noncommutative domain, i.e. to a series in Z, W where $X = e^Z, Y = e^W$, one needs to choose a path γ in $\mathbb{R}^2$ joining 0 with (a, b). One can easily see this by approximating γ by lattice paths with step $1/M$, $M \to \infty$, and working with monomials in $X^{1/M} = e^{Z/M}$ and $Y^{1/M} = e^{W/M}$. Denote this exponential series $E_\gamma(Z, W)$.

This suggests the possibility of a "noncommutative Fourier transform" (NCFT) identifying appopriate spaces of functions of noncommuting variables (say, of matrices of intederminate size) with spaces of ordinary functions or measures on the space of paths. For example, to a measure μ on the space Π of paths (or some completion of it) we want to associate the function of Z, W given by

$$\mathcal{F}(\mu)(Z, W) = \int_{\gamma\in\Pi} E_\gamma(Z, W) d\mu(\gamma), \tag{0.1.1}$$

The basic phenomenon here seems to be that the two types of functional spaces (noncommutative functions of n variables vs. ordinary commutative functions but on the space paths in $\mathbb{R}^n$), have, on some fundamental level, *the same size.*

The goal of this paper and the ones to follow [K1-2] is to investigate this idea from several points of view.

(0.2) The concept of NCFT seems to implicitly underlie the very foundations of quantum mechanics such as the equivalence of the Lagrangian and Hamiltonian approaches to the theory. Indeed, the Lagrangian point of view deals with path integrals while the Hamiltonian one works with noncommuting operators. Further, it is very close to the concept of the "Wilson loop" functional (trace of the holonomy) in Yang-Mills theory [Po]. Note that the exponential E_γ, being itself the holonomy of a certain formal connection, is invariant under reparametrization of the path. Quantities invariant under reparametrization are particularly important in string theory, and the reparametrization invariance of the Wilson loop led to conjectural relations between strings and $N \to \infty$ limit of Yang-Mills theory [Po].

As the integral transform $\mathcal{F}$ should, intuitively, act between spaces of the same size, it does not lead to any loss of information and can therefore be viewed as "path integration without integration". The actual integration occurs when we restrict the function $\mathcal{F}(\mu)$ to the commutative locus, i.e., make Z and W commute. Alternatively, instead of allowing Z, W to be arbitrary matrices, we take them to be scalars. Then all paths having the same endpoint will contribute to make up a single Fourier mode of the commutativized function. We arrive at the following conclusion: *the natural homomorphism $R \to R_{ab}$ of a noncommutative ring to its maximal commutative quotient is the algebraic analog of path integration.*

(0.3) The idea that the space of paths is related to the free group and to its various versions has been clearly enunciated by K.-T. Chen [C1] in the 1950's and can be traced throughout almost all of his work [C0]. Apparently, a lot more can be said about this classical subject. Thus, the universal connection with values in the free Lie algebra (known to Chen and appearing in (2.1) below) leads to beautiful non-holonomic geometry on the free nilpotent Lie groups $G_{n,d}$, which is still far from being fully understood, see [G].

Well known examples of measures on path spaces are provided by probability theory and we spend some time in §4 below to formulate various results from probabilistic literature in terms of NCFT. Most importantly, the Fourier transform of the Wiener measure on paths in $\mathbb{R}^n$ is the noncommutative Gaussian series $\exp\left(-\sum_{i=1}^n Z_j^2\right)$ where Z_i are considered as noncommuting variables. We should mention here the recent book by Baudoin [Ba] who considered the idea of associating a noncommutative series to a stochastic process. It is clearly the same type of construction as our NCFT except in the framework of probability theory: parametrized paths, positive measures etc.

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1. Noncommutative monomials and lattice paths.

(1.1) Noncommutative polynomials and the free semigroup. Consider n noncommuting (free) variables $X_1, ..., X_n$ and form the algebra of noncommutative polynomials in these variables. This algebra will be denoted by $\mathbb{C}\langle X_1, ..., X_n\rangle$. It is the same as the tensor algebra

$$T(V) = \bigoplus_{d=1}^{\infty} V^{\otimes d}, \quad V = \mathbb{C}^n = \bigoplus \mathbb{C} \cdot X_i.$$

A noncommutative monomial in $X = (X_1, ..., X_n)$ is, as described in the Introduction, the same as a monotone lattice path in $\mathbb{R}^n$ starting at 0. We denote by F_n^+ the set of all such paths and write X^γ for the monomial corresponding to a path γ. The set F_n^+ is a semigroup with the following operation. If γ, γ' are two monotone paths as above starting at 0, then $\gamma \circ \gamma'$ is obtained by translating γ so that its beginning meets the end of γ' and then forming the composite path. It is clear that F_n^+ is the free semigroup on n generators. Thus a typical noncommutative polynomial is written as

$$f(X_1, ...X_n) = f(X) = \sum_{\gamma \in F_n^+} a_\gamma X^\gamma. \tag{1.1.1}$$

Along with $\mathbb{C}\langle X_1, ..., X_n\rangle$ we will consider the algebra $\mathbb{C}[x_1, ..., x_n]$ of usual (commutative) polynomials in the variables $x_1, ..., x_n$. A typical such polynomial will be written as

$$g(x_1, ..., x_n) = g(x) = \sum_{\alpha \in \mathbf{Z}_+^n} b_\alpha x^\alpha, \quad x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}. \tag{1.1.2}$$

The two algebras are related by the *commutativization homomorphism*

$$c : \mathbb{C}\langle X_1, ..., X_n\rangle \to \mathbb{C}[x_1, ..., x_n], \tag{1.1.3}$$

which takes X_i to x_i. For a path $\gamma \in \Gamma_n$ let $e(\gamma) \in \mathbb{Z}_+^n$ denote the end point of γ. Then we have

$$c(X^\gamma) = x^{e(\gamma)}. \tag{1.1.4}$$

This means that at the level of coefficients the commutativization homomorphism is given by the summation over paths with given endpoints: if $g(x) = c(f(X))$, then

$$b_\alpha = \sum_{e(\gamma)=\alpha} a_\gamma. \tag{1.1.5}$$

(1.2) Noncommutative power series. Let $I \subset \mathbb{C}\langle X_1, ..., X_n\rangle$ be the span of monomials of degree ≥ 1. Then clearly I is a 2-sided ideal and I^d is the span of monomials of degree $\geq d$. We define the algebra $\mathbb{C}\langle\langle X_1, ..., X_n\rangle\rangle$ as the completion of $\mathbb{C}\langle X_1, ..., X_n\rangle$ in the I-adic topology. Explicitly, elements of $\mathbb{C}\langle\langle X_1, ..., X_n\rangle\rangle$ can be seen as infinite formal linear combinations of noncommutative monomials, i.e., expressions of the form $\sum_{\gamma\in F_N^+} a_\gamma X^\gamma$. For example,

$$e^{X_1}\cdot e^{X_2} = \sum_{i,j=0}^\infty \frac{X_1^i X_2^j}{i!j!}, \quad \frac{1}{1-(X_1+X_2)} = \sum_{\gamma\in F_2^+} X^\gamma \tag{1.2.1}$$

are noncommutative power series. We will be also interested in convergence of noncommutative series. A series $f(X) = \sum_{\gamma\in F_n^+} a_\gamma X^\gamma$ will be called *entire*, if

$$\lim_{\gamma\to\infty} R^{l(\gamma)}|a_\gamma| = 0, \quad \forall R > 0. \tag{1.2.2}$$

Here $l(\gamma)$ is the length of the path γ and the limit is taken over the countable set F_n^+ (so no ordering of this set is needed). If (1.2.2) is the case, then for any N and for any square matrices $X_1^0, ..., X_n^0$ of size N the series of matrices $\sum a_\gamma (X^0)^\gamma$ obtained by specializing $X_i \to X_i^0$, converges absolutely. We denote by $\mathbb{C}\langle\langle X_1, ..., X_n\rangle\rangle^{ent}$ the set of entire series. It is clear that this set is a subring.

(1.3) Noncommutative Laurent polynomials. By a noncommutative Laurent monomial in $X_1, ..., X_n$ we will mean a monomial in positive and negative powers of the X_i such as, e.g., $X_1X_2X_1^{-1}X_2^5$. In other words, this is an element of F_n, the free noncommutative group on the generators X_i. A noncommutative Laurent polynomial is then a finite formal linear combination of such monomials i.e., an element of the group algebra of F_n. We will denote this algebra by

$$\mathbb{C}\langle X_1^{\pm 1}, ..., X_n^{\pm 1}\rangle = \mathbb{C}[F_n]. \tag{1.3.1}$$

As before, a noncommutative Laurent monomial corresponds to a lattice path in $\mathbb{R}^n$ beginning at 0 but not necessarily monotone. These paths are defined up to cancellation of pieces consisting of a sub-path and the same sub-path run in the opposite direction immediately afterwards.

We retain the notation X^γ for the monomial corresponding to a path γ. We also write $(-\gamma)$ for the path inverse to γ, so $X^{-\gamma} = (X^\gamma)^{-1}$.

(1.4) Noncommutative Fourier transform: discrete case. The usual (commutative) Fourier transform relates the spaces of functions on a locally compact abelian group G and its Pontryagin dual $\widehat{G}$. The "discrete" case $G = \mathbb{Z}^n$, $\widehat{G} = (S^1)^n$ corresponds to the theory of Fourier series.

In the algebraic formulation, the discrete Fourier transform identifies the space of finitely supported functions

$$b : \mathbb{Z}^n \to \mathbb{C}, \quad \alpha \mapsto b_\alpha, \quad |\mathrm{Supp}(b)| < \infty, \tag{1.4.1}$$

with the space $\mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ of Laurent polynomials. It is given by the well known formulas

$$(b_\alpha) \mapsto f, \quad f(x) = \sum_{\alpha\in\mathbb{Z}^n} b_\alpha x^\alpha, \tag{1.4.2}$$

$$f \mapsto (b_\alpha), \quad b_\alpha = \int_{|x_1|=...=|x_n|=1} f(x)x^{-\alpha} d^*x_1...d^*x_n, \tag{1.4.3}$$

where d^*x is the Haar measure on S^1 with volume 1. Our goal in this section is to give a generalization of these formulas for noncommutative Laurent polynomials.

Instead of (1.4.1) we consider the space of finitely supported functions

$$a : F_n \to \mathbb{C}, \quad \gamma \mapsto a_\gamma, \quad |\mathrm{Supp}(a)| < \infty. \tag{1.4.4}$$

The discrete noncommutative Fourier transform is the identification of this space with $\mathbb{C}\langle X_1^{\pm 1}, ..., X_n^{\pm 1}\rangle$ via

$$(a_\gamma) \mapsto f, \quad f(X) = \sum_{\gamma \in F_n} a_\gamma X^\gamma. \tag{1.4.5}$$

This identification ceases to look like a tautology if we regard a noncommutative Laurent polynomial as a function f which to any n invertible elements $X_1^0, ..., X_n^0$ of any associative algebra A associates an element $f(X_1^0, ..., X_n^0) \in A$. We want then to recover the coefficients a_γ in terms of the values of f on various elements of various A. Most importantly, we will consider $A = \mathrm{Mat}_N(\mathbb{C})$, the algebra of matrices of size N and let N be arbitrary. To get a generalization of (1.4.3) we replace the unit circle $|x| = 1$ by the group of unitary matrices $U(N) \subset \mathrm{Mat}_N(\mathbb{C})$. Let d^*X be the Haar measure on $U(N)$ of volume 1.

The following result is a consequence of the so-called "asymptotic freedom theorem for unitary matrices" due to Voiculescu [V], see also [HP] for a more elementary exposition.

(1.4.6) Theorem. *If $f(X) = \sum_{\gamma \in F_n} a_\gamma X^\gamma$ is a noncommutative Laurent polynomial, then we have*

$$a_\gamma = \lim_{N \to \infty} \frac{1}{N} \,\mathrm{tr} \int_{X_1, ..., X_n \in U(N)} f(X_1, ..., X_n)\, X^{-\gamma}\ d^*X_1...d^*X_n.$$

As for the commutative case, the theorem is equivalent to the following ortogonality relation. It is this relation that is usually called the "asymptotic freedom" in the literature.

(1.4.7) Reformulation. *Let $\gamma \in F_n$ be a nontrivial lattice path. Then*

$$\lim_{N \to \infty} \frac{1}{N} \,\mathrm{tr} \int_{X_1, ..., X_n \in U(N)} X^\gamma\ d^*X_1...d^*X_n = 0.$$

Note that for $\gamma = 0$ the integral is equal to 1 for any N.

Passing to the $N \to \infty$ limit is unavoidable here since for any given N there exist noncommutative polynomials which vanish identically on $\mathrm{Mat}_N(\mathbb{C})$. An example is provided by the famous Amitsur-Levitsky polynomial

$$f(X_1, ..., X_{2N}) = \sum_{\sigma \in S_{2N}} \mathrm{sgn}(\sigma) X_{\sigma(1)} \cdot ... \cdot X_{\sigma(2N)}.$$

2. Noncommutative exponential functions.

(2.1) The universal connection and noncommutative exponentials. Let us introduce the "logarithmic" variables $Z_1, ..., Z_n$, so that we have the embedding

$$\mathbb{C}\langle X_1, ..., X_n\rangle \subset \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle, \quad X_i \mapsto e^{Z_i}. \tag{2.1.1}$$

The algebra $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ is a projective limit of finite-dimensional algebras, namely

$$\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle \quad = \quad \underleftarrow{\lim}_d \ \mathbb{C}\langle Z_1, ..., Z_n\rangle / I^d, \tag{2.1.2}$$

where the ideal I is as in (1.2).

Consider the space $\mathbb{R}^n$ with coordinates $y_1, ..., y_n$. On this space we have the following 1-form with values in $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$:

$$\Omega \quad = \quad \sum_i Z_i dy_i \quad \in \quad \Omega^1(\mathbb{R}^n) \otimes \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle. \tag{2.1.3}$$

We consider the form as a connection on $\mathbb{R}^n$. One can see it as the universal translation invariant connection on $\mathbb{R}^n$, an algebraic version of the connection of Kobayashi on the path space, see [Si], §3.

Let γ be any piecewise smooth path in $\mathbb{R}^n$. We define the noncommutative exponential function corresponding to γ to be the holonomy of the above connection along γ:

$$E_\gamma(Z) = E_\gamma(Z_1, ..., Z_n) = P \exp \int_\gamma \Omega \quad \in \quad \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle. \tag{2.1.4}$$

The holonomy can be understood by passing to finite-dimensional quotients as in (2.1.2) and solving an ordinary differential equation with values in each such quotient. It is clear that $E_\gamma(Z)$ becomes unchanged under parallel translations of γ, since the form Ω is translation invariant. So in the following we will always assume that γ begins at 0.

Further, $E_\gamma(Z)$ is invariant under reparametrizations of γ: this is a general property of the holonomy of any connection. So let us give the following definition.

(2.1.5) Definition. *Let M be a C^∞-manifold. An (oriented) unparametrized path in M is an equivalence class of pairs $(I, \gamma : I \to M)$ where I is a smooth manifold with boundary diffeomorhic to $[0,1]$ and γ is a piecewise smooth map $I \to M$. Two such pairs (I, γ) and (I', γ') are equivalent if there is an orientation preserving piecewise smooth homeomorphism $\phi : I \to I'$ such that $\gamma = \gamma' \circ \phi$.*

We will denote an unparametrized path simply by γ.

(2.1.6) Example. Let γ be a straight segment in $\mathbb{R}^2$ joining (0,0) and (1,1). Let also δ be the path consisting of the horizontal segment $[(0,0),(0,1)]$ and the vertical segment $[(0,1),(1,1)]$. Let σ be the path consisting of the vertical segment $[(0,0),(1,0)]$ and the horizontal segment $[(1,0),(1.1)]$. Then

$$E_\gamma(Z_1, Z_2) = e^{Z_1+Z_2}, \quad E_\delta(Z_1, Z_2) = e^{Z_1}e^{Z_2}, \quad E_\sigma(Z) = e^{Z_2}e^{Z_1}.$$

More generally, if γ is a lattice path corresponding to the integer lattice $\mathbb{Z}^n$, then $E_\gamma(Z) = X^\gamma$ is the noncommutative monomial in $X_i = e^{Z_i}$ associated to γ as in §1.

Let γ, γ' be two unparametrized paths in $\mathbb{R}^n$ starting at 0. Their product $\gamma \circ \gamma'$ is the path obtained by translating γ so that its beginning meets the end of γ' and then forming the composite path. The set of γ's with this operation forms a semigroup. For a path γ we denote by γ^{-1} the path obtained by translating γ so that its end meets 0 and then taking it with the opposite orientation. Finally, we denote by Π_n the set of paths as above modulo cancellations, i.e., forgetting sub-paths of a given path consisting of a segment and then immediately of the same segment run in the opposite direction. Clearly the set Π_n forms a group which we will call *the group of paths* in $\mathbb{R}^n$.

The standard properties of the holonomy of connections imply the following:

(2.1.7) Proposition. *(a) We have*

$$E_{\gamma\circ\gamma'}(Z) = E_\gamma(Z) \cdot E_{\gamma'}(Z), \quad E_{\gamma^{-1}}(Z) = E_\gamma(Z)^{-1}$$

(equalities in $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$).
(b) The series $E_\gamma(Z)$ is entire, i.e., it converges for any given N by N matrices $Z_1^0, ..., Z_n^0$.
(c) If $Z_1^0, ..., Z_n^0$ are Hermitian, then $E_\gamma(iZ_1^0, ..., iZ_n^0)$ is unitary.

The property (a) implies that $E_\gamma(Z)$ depends only on the image of γ in the group Π_n. Further, let us consider the commutativisation homomorphism

$$c : \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle \to \mathbb{C}[[z_1, ..., z_n]]. \tag{2.1.8}$$

The following is also obvious.

(2.1.9) Proposition. *If $a = (a_1, ..., a_n)$ is the endpoint of γ, then*

$$c(E_\gamma(Z)) = e^{(a,z)}$$

is the usual exponential function.

Thus there are as many ways to lift $e^{(a,z)}$ into the noncommutative domain as there are paths in $\mathbb{R}^n$ joining 0 and a.

(2.2) Idea of a noncommutative Fourier transform. The above observations suggest that there should be a version of Fourier transform which would identify an appropriate space of measures on Π_n with an appropriate space of functions of n noncommutative variables $Z_1, ..., Z_n$, via the formula

$$\mu \mapsto f(Z_1, ..., Z_n) = \int_{\gamma\in\Pi_n} E_\gamma(iZ_1, ..., iZ_n)\mathcal{D}\mu(\gamma). \tag{2.2.1}$$

The integral in (2.2.1) is thus a path integral. The concept of a "function of noncommuting variables" is of course open to interpretation. Several such interpretations are currently being considered in Noncommutative Geometry.

In the present paper we adopt a loose point of view that a function of n noncommutative variables is an element of an algebra R equipped with a homomorphism $\mathbb{C}\langle Z_1, ..., Z_n\rangle \to$

R. We will assume that this homomorphism realizes R as some kind of completion, or localization (or both) of $\mathbb{C}\langle Z_1, ..., Z_n\rangle$. In other words, that R does not have "superfluous" elements, independent of the images of the Z_i. See [Ta] for an early attempt to define noncommutative functions in the analytic context.

(2.2.2) Examples. We can take $R = \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle^{ent}$, the algebra of entire power series. Alternatively we can take R to be the skew field of "noncommutative rational functions" in $Z_1, ..., Z_n$ constructed by P. Cohn [Coh]. Thus expressions such as

$$\exp(Z_1^2 + Z_2^2), \quad \left(Z_1^2 + Z_2^2\right)^{-1}, \quad (Z_1Z_2 - Z_2Z_1)^{-1} + Z_3^{-2}Z_1$$

are considered as noncommutative functions.

It will be important for us to be able to view a "function" $f(Z_1, ..., Z_n)$ as above as an actual function defined on appropriate subsets of n-tuples of N by N matrices for each N and taking values in matrices of the same size.

Similarly, the group Π_n can also possibly be replaced by various related objects (completions). In this paper we will consider several approaches such as completion by a pro-algebraic group or completion by continuous paths.

Alternatively, functions on Π_n should correspond to "noncommutative measures", or distributions on the space of noncommutative functions. Examples of such "measures" are being studied in the Free Probability Theory [HP] [NS] [VDN]. See §6 below.

Note that we have a surjective homomorphism of groups

$$e : \Pi_n \to \mathbb{R}^n, \quad \gamma \mapsto e(\gamma). \tag{2.2.3}$$

Here $e(\gamma)$ is the endpoint of γ. One important property of the NCFT is the following principle which is just a consequence of Proposition 2.1.9: under the Fourier transform the integration over paths with given beginning and end, i.e., the pushdown of measures on Π_n to measures on $\mathbb{R}^n$ corresponds to a simple algebraic operation: the commutativization homomorphism

$$c : R \to R/([R, R]), \tag{2.2.4}$$

where R is a noncommutative algebra and the RHS is the maximal commutative quotient of R.

(2.3) Relation to Chen's iterated integrals. Let us recall the main points of Chen's theory. Let M be a smooth manifold, γ an unparametrized path and ω a 1-form on M.

Along with the "definite integral" $\int_\gamma \omega$, we can consider the "indefinite integral" which is a function "on γ", or, more precisely, on the abstract interval I such that γ is a map $I \to M$. For any $t \in I$ we have the sub-path $\gamma_{\leq t}$ going from the beginning of I until t and we have the function

$$\int_{(\gamma)} \omega : I \to \mathbb{C}, \quad t \mapsto \int_{\gamma_{\leq t}} \omega.$$

If now ω_1 and ω_2 are two 1-forms on M, we can form a new 1-form on γ by multiplying (the restriction of) ω_2 and the function $\int_{(\gamma)} \omega_1$. Then this form can be integrated along γ. The result is called the *iterated integral*

$$\int_\gamma \left(\omega_2 \cdot \int_{(\gamma)} \omega_1\right).$$

Note that if we think of γ as a map $\gamma : I \to M$, then the iterated integral is equal to

$$\int_{t_1 \le t_2 \in I} \gamma^*(\omega_1)(t_1)\gamma^*(\omega_1)(t_2).$$

Note that integration over all $t_1, t_2 \in I$ would give the product $\left(\int_\gamma \omega_1\right) \cdot \left(\int_\gamma \omega_2\right)$.

Similarly, one defines the d-fold iterated integral of d 1-forms $\omega_1, ..., \omega_d$ on M by induction:

$$\int_\gamma^{\to} \omega_1 \cdot ... \cdot \omega_d = \int_\gamma \left(\omega_d \cdot \int_{(\gamma)}^{\to} \omega_1 \cdot ... \cdot \omega_{d-1}\right),$$

where the $(d-1)$-fold indefinite iterated integral is defined as the function on I of the form

$$t \to \int_{\gamma_{\le t}}^{\to} \omega_1 \cdot ... \cdot \omega_{d-1}.$$

As before the iterated integral is equal to the integral over the d-simplex:

$$\int_\gamma^{\to} \omega_1 \cdot ... \cdot \omega_d = \int_{t_1 \le ... \le t_d \in I} \gamma^*\omega_1(t_1)...\gamma^*\omega_d(t_d).$$

The concept of iterated integrals extends in an obvious way to 1-forms with values in any associative (pro-)finite-dimensional $\mathbb{C}$-algebra R. The well known Picard series for the holonomy of a connection consists exactly of such iterated integrals. We state this as follows.

(2.3.1) Proposition. *Let R be any (pro-)finite-dimensional associative $\mathbb{C}$-algebra, and Ω be a 1-form on M with values in R considered as a connection form. Then the parallel transport along an unparametrized path γ has the form*

$$P \exp \int_\gamma \Omega = \sum_{d=0}^{\infty} \int_\gamma^{\to} \Omega \cdot ... \cdot \Omega.$$

Let us specialize this to $M = \mathbb{R}^n$, $R = \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ and $\Omega = \sum Z_i dy_i$. We obtain:

(2.3.2) Corollary. *The coefficient of the series $E_\gamma(Z_1, ..., Z_n)$ at any noncommutative monomial $Z_{i_1}...Z_{i_d}$ is equal to the iterated integral*

$$\int_\gamma^{\to} dy_{i_1} \cdot ... \cdot dy_{i_d}.$$

Thus E_γ is the generating function for all the iterated integrals involving constant 1-forms on $\mathbb{R}^n$.

(2.3.3) Example. By the above

$$E_\gamma(Z) = 1 + \sum a_i Z_i + \sum b_{ij} Z_i Z_j + ...$$

where $a_i = \int_\gamma dy_i$ is the ith coordinate of the endpoint of γ and

$$b_{ij} = \int_\gamma \left(dy_i \cdot \int_{(\gamma)} dy_j \right) = \int_\gamma y_j dy_i.$$

Suppose that γ is closed, so $a_i = 0$. Then $b_{ii} = 0$ and for $i \neq j$ we have that b_{ij} is the oriented area encirlced by γ after the projection to the (i, j)-plane.

The following was proved by Chen [C2].

(2.3.4) Theorem. *The homomorphism $\Pi_n \to \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle^*$ sending γ to E_γ is injective. In other words, if a path γ has all iterated integrals as above equal to 0, then γ is (equivalent modulo cancellations to) a constant path (situated at 0).*

(2.4) Group-like and primitive elements. Let $FL(Z_1, ..., Z_n)$ be the free Lie algebra generated by $Z_1, ..., Z_n$. It is characterized by the obvious universal property, see [R] for background. This property implies that we have a Lie algebra homomorphism

$$h : FL(Z_1, ..., Z_n) \to \mathbb{C}\langle Z_1, ..., Z_n\rangle, \tag{2.4.1}$$

and this homomorphism identifies $\mathbb{C}\langle Z_1, ..., Z_n\rangle$ with the universal enveloping algebra of $FL(Z_1, ..., Z_n)$. Further, let us consider the Hopf algebra structure on $\mathbb{C}\langle Z_1, ..., Z_n\rangle$ given on the generators by

$$\Delta(Z_i) = Z_i \otimes 1 + 1 \otimes Z_i. \tag{2.4.2}$$

The following result, originally due to K. Friedrichs, is a particular case of a general property of enveloping algebras.

(2.4.3) Theorem. *The image of h consists precisely of all primitive elements, i.e., of elements f such that $\Delta(f) = f \otimes 1 + 1 \otimes f$.*

We will also use the term *Lie elements* for primitive elements of $\mathbb{C}\langle Z_1, ..., Z_n\rangle$.

Further, consider the noncommutative power series algebra $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$. It is naturally a topological Hopf algebra with respect to the comultiplication given by (2.4.2) on generators and extended by additivity, multiplicativity and continuity.

The free Lie algebra is graded:

$$FL(Z_1, ..., Z_n) = \bigoplus_{d \geq 1} FL(Z_1, ..., Z_n)_d, \tag{2.4.4}$$

where $FL(Z_1, ..., Z_n)_d$ is the span of Lie monomials containing exactly d letters. We denote by

$$\mathfrak{g}_n = \prod_{d \geq 1} FL(Z_1, ..., Z_n)_d \tag{2.4.4}$$

its completion, i.e., the set of formal *Lie series.* This is a complete topological Lie algebra. We clearly have an embedding of $\mathfrak{g}_n$ into $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ induced by the embedding of the graded components as above. Further, degree-by-degree considerations and Theorem 2.4.3 imply the following:

(2.4.5) Corollary. *A noncommutative power series $f \in \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ lies in $\mathfrak{g}_n$ if and only if it is primitive, i.e., $\Delta(f) = f \otimes 1 + 1 \otimes f$ with respect to the topological Hopf algebra structure defined above.*

Along with primitive (or Lie) series in $Z_1, ..., Z_n$ we will consider group-like elements of $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$, i.e., series Φ satisfying

$$\Delta(\Phi) = \Phi \otimes \Phi. \tag{2.4.6}$$

The completed tensor product $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle\widehat{\otimes}\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ consists of series in $2n$ variables $Z_i' = Z_i \otimes 1$ and $Z_i'' = 1 \otimes Z_i$ which satisfy $[Z_i', Z_j''] = 0$ and no other relations. Thus a series $\Phi(Z_1, ..., Z_n)$ is group-like if it satisfies the *exponential property*:

(2.4.7)
$$F(Z_1' + Z_1'', ..., Z_n' + Z_n'') = F(Z_1', ..., Z_n') \cdot F(Z_1'', ..., Z_n''), \quad \text{provided} \quad [Z_i', Z_j''] = 0, \forall i, j.$$

We denote by G_n the set of primitive elements in $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$. Elementary properties of cocommutative Hopf algebras and elementary convergence arguments in the adic topology imply the following:

(2.4.8) Proposition. *(a) G_n is a group with respect to the multiplication.*
(b) The exponential series defines a bijection

$$\exp : \mathfrak{g}_n \to G_n,$$

with the inverse given by the logarithmic series.
(c) The image of any series $\Phi \in G_n$ under the commutativization homomorphism (2.1.8) is a formal series of the form $e^{(a,z)}$ for some $a \in \mathbb{C}^n$.
(d) If $\Phi \in G_n$, then

$$\Phi(-Z_1, ..., -Z_n) = \Phi(Z_1, ..., Z_n)^{-1}$$

(equality of power series).

(2.4.9) Example. The above proposition implies that the series

$$\log(e^{Z_1} \cdot e^{Z_2}) \in \mathbb{C}\langle\langle Z_1, Z_2\rangle\rangle$$

is in fact a Lie series. It is known as the Campbell-Hausdorff series and its initial part has the form

$$\log(e^{Z_1} \cdot e^{Z_2}) = Z_1 + Z_2 + \frac{1}{2}[Z_1, Z_2] + ...$$

Let $G_n(\mathbb{R}) \subset G_n$ be the set of group-like series with real coefficients. Further, the Lie algebra $FL(Z_1, ..., Z_n)$ is in fact defined over rational numbers. In particular, it makes sense to speak about its real part. By taking the completion as above, we define the real part of the completed free algebra $\mathfrak{g}_n(\mathbb{R})$. It is clear that the exponential series establishes a bijection between $\mathfrak{g}_n(\mathbb{R})$ and $G_n(\mathbb{R})$.

The following fact was also pointed out by Chen [C2].

(2.4.9) Theorem. *If $\gamma \in \Pi_n$ is a path in $\mathbb{R}^n$ as above, then $E_\gamma(Z)$ is group-like. Moreover, it lies in the real part $G_n(\mathbb{R})$.*

Note that a typical element $\Phi = \Phi(Z_1, ..., Z_n) \in G_n$ is a priori just a formal power series and does not have to converge for any given matrix values of the Z_i (unless they are all 0). At the same time, series of the form $\Phi = E_\gamma$, $\gamma \in \Pi_n$, converge for all values of the Z_i. This leads to the proposal, formulated by Chen [C3] to view series from G_n with good covergence properties as corresponding to "generalized paths", i.e., paths perhaps more general than piecewise C^∞ ones. Theory of stochastic integrals, see below, provides a step in a similar direction.

(2.5) Finite-dimensional approximations to G_n and $\mathfrak{g}_n$. Let us recall a version of the Malcev theory for nilpotent Lie algebras. Let k be a field of characteristic 0. A Lie algebra $\mathfrak{g}$ over k is called nilpotent of degree d if all d-fold iterated commutators in $\mathfrak{g}$ vanish. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. It is a Hopf algebra with the comultiplication given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$. The subspace I in $U(\mathfrak{g})$ generated by all nontrivial Lie monomials in elements of $\mathfrak{g}$, is an ideal, with $U(\mathfrak{g})/I = \mathbb{C}$.

(2.5.1) Lemma. *If $\mathfrak{g}$ is nilpotent of some degree, then $\bigcap I^n = 0$.*

Thus the I-adic completion

$$\widehat{U}(\mathfrak{g}) = \varprojlim U(\mathfrak{g})/I^n \tag{2.5.2}$$

is a complete topological algebra containing $U(\mathfrak{g})$. As before, the standard Hopf algebra structure on $U(\mathfrak{g})$ gives rise to a topological Hopf algebra structure on $\widehat{U}(\mathfrak{g})$. We then have the following fact.

(2.5.3) Theorem. *(a) $\mathfrak{g}$ is the set of primitive elements of $\widehat{U}(\mathfrak{g})$*
(b) The set G of group-like elements in $\widehat{U}(\mathfrak{g})$ is the nilpotent group associated, via the Malcev theory, to the Lie algebra $\mathfrak{g}$.
(c) If $k = \mathbb{R}$ or $\mathbb{C}$, then G is the simply connected real or complex Lie group with Lie algebra G.
(d) The exponential map establishes a bijection between $\mathfrak{g}$ and G.

Let now $k = \mathbb{C}$ and

$$\mathfrak{g}_{n,d} = FL(X_1, ..., X_n)/FL(X_1, ..., X_n)_{\geq d+1}. \tag{2.5.4}$$

This is a finite-dimensional Lie algebra known as the free nilpotent Lie algebra of degree d generated by n elements. It satisfies the obvious universal property. Then

$$\mathfrak{g}_n = \varprojlim{}_n\, \mathfrak{g}_{n,d}.$$

So $\mathfrak{g}_n$ is the free pro-nilpotent Lie algebra on n generators.

Let $R_{n,d}$ be the quotient of $R_n = \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ by the closed ideal generated by all the $(d+1)$-fold commutators of the Z_i. For example, $R_{n,1} = \mathbb{C}[[Z_1, ..., Z_n]]$ is the usual (commutative) power series algebra.

The topological Hopf algebra structure on R_n descends to $R_{n,d}$, and we easily see the following:

(2.5.5) Proposition. *$R_{n,d}$ is isomorphic to $\widehat{U}(\mathfrak{g}_{n,d})$ as a topological Hopf algebra.*

We denote by $G_{n,d} \subset R^*_{n,d}$ the group of group-like elements of $R_{n,d}$. Then the above facts imply:

(2.5.6) Proposition. *(a) $G_{n,d}$ is the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{n,d}$.*
(b) G_n is the projective limit of $G_{n,d}$.

Thus $G_{n,d}$ is the "free unipotent complex algebraic group of degree d with n generators" while G_n is the free prounipotent group.

As above, taking $k = \mathbb{R}$, we get the real parts $G_{n,d}(\mathbb{R})$ and $\mathfrak{g}_{n,d}(\mathbb{R})$. The homomorphism $E : \Pi_n \to G_n(\mathbb{R})$ gives rise, for any $d \geq 1$ to the homomorphism

$$\epsilon_{n,d} : \Pi_n \to G_{n,d}(\mathbb{R}) \tag{2.5.7}$$

whose target is a finite-dimensional Lie group.

(2.5.8) Proposition. *For any $d \geq 1$ the homomorphism $\epsilon_{n,d}$ is surjective.*

In other words, the group G_n can be seen as a (pro-)algebraic completion of the path group Π_n.

Proof: Let $\Pi_n^{rect} \subset \Pi_n$ be the subgroup of rectangular paths, i.e., paths consisting of segments each going in the direction of some particular coordinate. As a group, Π_n^{rect} is the free product of n copies of $\mathbb{R}$. Let $Z_{i,d} \in \mathfrak{g}_{n,d}$ be the image of Z_i. Then the image of Π_n^{rect} in $G_{n,d}(\mathbb{R})$ is the subgroup generated by the 1-parameter subgroups $\exp(t \cdot Z_{i,d})$, $t \in \mathbb{R}$, $i = 1, ..., n$. As the $Z_{i,d}$ generate $\mathfrak{g}_{n,d}$ as a Lie algebra, the corresponding 1-parameter subgroups generate $G_{n,d}(\mathbb{R})$ as a group. Therefore $\epsilon_{n,d}(\Pi_n^{rect}) = G_{n,d}(\mathbb{R})$.

(2.6) Complex exponentials. Consider the complexification $\mathbb{C}^n$ of the space $\mathbb{R}^n$ from (2.1). The form Ω from (2.1.3) is then a holomorphic form on $\mathbb{C}^n$ with values in $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$. In particular, we have the noncommutative exponential function

$$E_\gamma(Z) \in G_n \subset \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$$

for any unparametrized path γ in $\mathbb{C}^n$ starting at 0. Because Ω is holomorphic, $E_\gamma(Z)$ is, in addition to invariance under cancellations, also invariant under holomorphic deformations of sub-paths of γ. Let $\Pi_n^{\mathbb{C}}$ be the quotient of Π_{2n}, the group of paths in $\mathbb{C}^n = \mathbb{R}^{2n}$ by the equivalence relation generated by such deformations. Obviously, $\Pi_n^{\mathbb{C}}$ is a group, and the correspondence $\gamma \mapsto E_\gamma$ gives rise to a homomorphism

$$E : \Pi_n^{\mathbb{C}} \to G_n. \tag{2.6.1}$$

Unlike the real case, it seems to be unknown whether (2.6.1) is injective. As before, we see that the composite homomorphism

$$\epsilon^{\mathbb{C}}_{n,d} : \Pi^{\mathbb{C}}_n \to G_{n,d} \tag{2.6.2}$$

is surjective.

(2.6.3) Example. Let C be a complex analytic curve, $c_0 \in C$ be a point, and $\phi : C \to \mathbb{C}^n$ a holomorphic map such that $\phi(c_0) = 0$. Denote by $p : \widetilde{C} \to C$ the universal covering of C corresponding to the base point c_0. In other words, $\widetilde{C}$ is ths space of pairs (c, γ), where $c \in C$ and γ is a homotopy class of paths joinig c_0 and c. Then, by the above, ϕ induces a map $\widetilde{\phi} : \widetilde{C} \to \Pi^{\mathbb{C}}_n$. The composition

$$\varpi_d = \epsilon^{\mathbb{C}}_{n,d} \circ \widetilde{\phi} : \widetilde{C} \to G_{n,d}$$

can be called the period map of degree d. The restriction of ϖ_d to $p^{-1}(c_0) = \pi_1(C, c_0)$ is a homomorphism

$$m_d : \pi_1(C, c_0) \to G_{n,d}$$

called the monodromy homomorphism of degree d. We get then the "Albanese map"

$$\alpha_d : C \to G_{n,d}/\mathrm{Im}(m_d).$$

The particular case when C is the maximal Abelian covering of a smooth projective curve of genus n, and ϕ is the Abel-Jacobi map, corresponds to the setting of Parshin [Pa]. Iterated integrals of modular forms were studied by Manin [Ma].

In the subsequent paper [K1] we will use complex noncommutative exponentials to construct invariants of degenerations of families of curves in an algebraic variety.

3. Generalities on NCFT

(3.0) Formal FT on nilpotent groups. Let us start with the general situation of (2.5) with $k = \mathbb{R}$. Thus $\mathfrak{g}$ is a finite dimensional nilpotent real Lie algebra and G is the corresponding simply connected Lie group. Then G is realized inside $\widehat{U}(\mathfrak{g})$ as the set of group-like elements. In general, we can think of elements of $\widehat{U}(\mathfrak{g})$ as some kind of formal series (infinite formal linear combinations of elements of a Poincare-Birkhoff-Witt basis of $U(\mathfrak{g})$).

To keep the notation straight, we denote by $E_g \in \widehat{U}(\mathfrak{g})$ the element corresponding to $g \in G$.

(3.0.1) Example. Let $G = \mathbb{R}^n$ with coordinates $y_1, ..., y_n$, then $\widehat{U}(\mathfrak{g})$ is the ring $\mathbb{C}[[z_1, ..., z_n]]$ of formal Taylor series. If $g = (y_1, ..., y_n) \in G$, then $E_g = E_g(z) = \exp\left(\sum_i y_i z_i\right)$ is the exponential series with the vector of exponents $(y_1, ..., y_n)$.

The above example motivates the following definition. Let μ be a measure on G, or, more generally, a distribution (understood as a generalized measure, i.e., as a functional on the space of C^∞-functions). Its formal Fourier transform is the element (formal series) given by

$$\widehat{\mathcal{F}}(\mu) = \int_{g\in G} E_g d\mu \quad \in \quad \widehat{U}(\mathfrak{g}), \tag{3.0.2}$$

whenever the integral is defined.

Recall that for two distributions μ, ν on a Lie group G their convolution is defined by

$$\mu * \nu = m_*(\mu \boxtimes \nu), \tag{3.0.4}$$

where $m : G \times G \to G$ is the multiplication, and $\mu \boxtimes \nu$ is the Cartesian product of μ and ν. Here we assume that the pushdown under m is defined. The following is then straightforward.

(3.0.5) Proposition. *For two (generalized) measures μ, ν on G we have*

$$\widehat{\mathcal{F}}(\mu * \nu) = \widehat{\mathcal{F}}(\mu) \cdot \widehat{\mathcal{F}}(\nu),$$

(product in $\widehat{U}(\mathfrak{g})$).

(3.1) Pro-measures and formal NCFT. We now specialize the above to the case when $G = G_{n,d}(\mathbb{R})$. In other words, we consider the projective system of Lie groups

$$\cdots \to G_{n,3}(\mathbb{R}) \to G_{n,2}(\mathbb{R}) \to G_{n,1}(\mathbb{R}) = \mathbb{R}^n \tag{3.1.1}$$

with projective limit $G_n(\mathbb{R})$. For $d \geq d'$ let

$$p_{dd'} : G_{n,d}(\mathbb{R}) \to G_{n,d'}(\mathbb{R}) \tag{3.1.2}$$

be the projection. By a *pro-measure* on $G_n(\mathbb{R})$ we will mean a compatible system of measures on the $G_{n,d}(\mathbb{R})$. In other words, a pro-measure is a system $\mu_\bullet = (\mu_d)$ such that each μ_d is a measure on $G_{n,d}(\mathbb{R})$ such that for any $d \geq d'$ the pushdown $(p_{dd'})_*(\mu_d)$ is defined as a measure on $G_{n,d'}(\mathbb{R})$ and is equal to $\mu_{d'}$. Equivalently, this means that for any continuous function f on $G_{n,d'}(\mathbb{R})$ we have

$$\int_{G_{n,d'}(\mathbb{R})} f \cdot d\mu_{d'} \quad = \quad \int_{G_{n,d}(\mathbb{R})} (f \circ p_{dd'}) \cdot d\mu_d, \tag{3.1.3}$$

whenever the LHS is defined.

More generally, by a pro-distribution we mean a system of distributions on the $G_{n,d}(\mathbb{R})$ (understood as generalized measures, i.e., as functionals on C^∞-functions) compatible in the similar sense, i.e., satisfying (3.1.3) for C^∞-functions f.

For $\Phi = \Phi(Z_1, ..., Z_n) \in G_n$ we denote by $\Phi_{i_1,...,i_p}$ the coefficient of Φ at $Z_{i_1} \cdots Z_{i_p}$. It is clear that $\Phi_{i_1,...,i_p}$ depends only on the image of Φ in $G_{n,p}$, so it makes sense to speak about $\Psi_{i_1,...,i_p}$ for $\Phi \in G_{n,d}$, $d \geq p$.

Let $\mu_\bullet$ be a pro-distribution on $G_n(\mathbb{R})$. Its formal Fourier transform is the formal series $\widehat{\mathcal{F}}(\mu_\bullet) \in \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ defined as follows:

$$\widehat{\mathcal{F}}(\mu_\bullet) = \sum_{p=0}^{\infty} \sum_{i_1,...,i_p} \left(\int_{\Psi \in G_{n,d}(\mathbb{R})} \Psi_{i_1...i_p} \cdot d\mu_d \right) Z_{i_1} \cdots Z_{i_p}. \tag{3.1.4}$$

Here for each p the number d is any integer greater or equal to p, and we assume that all the integrals converge.

The convolution operation extends, in an obvious way, to pro-distributions on $G_m(\mathbb{R})$ and we get:

(3.1.6) Proposition. *If $\mu_\bullet, \nu_\bullet$ are two pro-distributions, then*

$$\widehat{\mathcal{F}}(\mu_\bullet * \nu_\bullet) = \widehat{\mathcal{F}}(\mu_\bullet) \cdot \widehat{\mathcal{F}}(\nu_\bullet)$$

(product in $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$).

(3.2) Delta-functions. In classical analysis the Fourier transform of $\delta^{(m)}$, the mth derivative of the delta function, is the monomial z^m. We now give a noncommutative analog of this fact.

First of all, let δ_d be the delta function on $G_{n,d}(\mathbb{R})$ supported at 1. Then $\delta_\bullet = (\delta_d)$ is a pro-distribution, and

$$\widehat{\mathcal{F}}(\delta_\bullet) = 1 \in \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle. \tag{3.2.1}$$

Next, first derivatives of the delta function at a point on a C^∞ manifold correspond to elements of the complexified tangent space to the manifold at this point. This, if $\xi \in FL(Z_1, ..., Z_n)$, and ξ_d is the image of ξ in $\mathfrak{g}_{n,d} = T_1 G_{n,d}(\mathbb{R}) \otimes \mathbb{C}$, then we have the distribution $\partial_{\xi_d}(\delta_d)$ on $G_{n,d}(\mathbb{R})$, and these distributions form a pro-distribution $\partial_\xi(\delta_\bullet)$.

Further, for any Lie group G with Lie algebra $\mathfrak{g}$ the iterated derivatives of the delta function at 1 correspond to elements of $U(\mathfrak{g} \otimes \mathbb{C})$, the universal enveloping algebra. Thus for any $\psi \in U(\mathfrak{g}_{n,d})$ we have a punctual distribution $D_\psi(\delta_d)$ on $G_{n,d}(\mathbb{R})$.

Let now $f \in \mathbb{C}\langle Z_1, ..., Z_n\rangle$ be a noncommutative polynomial. Recall that $\mathbb{C}\langle Z_1, ..., Z_n\rangle$ is the enveloping algebra of $FL(Z_1, ..., Z_d)$. This for any d we have the image of f in $U(\mathfrak{g}_{n,d})$, which we denote by f_d. As before, the distributions $D_{f_d}(\delta_d)$ form a pro-distribution which we denote $D_f(\delta_\bullet)$.

(3.2.2) Theorem. *We have $\widehat{\mathcal{F}}(D_f(\delta_\bullet)) = f$. In other words, $\widehat{\mathcal{F}}$ takes iterated derivatives of the delta function into (noncommutative) polynomials.*

Let $L_{f,d}$ be the left invariant differential operator on $G_{n,d}(\mathbb{R})$ corresponding to $f_d \in U(\mathfrak{g}_{n,d})$). Similarly, let $R_{f,d}$ be the right invariant differential operator corresponding to f_d. Recall that distributions (volume forms) form a right module over the ring of differential operators. In other words, if P is a differential operator acting on functions by $\phi \mapsto P\phi$, then we write the action of the adjoint operator on volume forms by $\omega \mapsto \omega P$. Thus, if $\mu_\bullet = (\mu_d)$ is a pro-distribution, and $f \in \mathbb{C}\langle Z_1, ..., Z_n\rangle$, then we have pro-distributions $\mu_\bullet L_f = (\mu_d L_{f,d})$ and $\mu_\bullet R_f = (\mu_d R_{f,d})$. Since applying $R_{f,d}$ or $L_{f,d}$ to a distribution is the same as the right or left convolution with $D_{f_d}(\delta_d)$, Proposition 3.1.6 implies the following.

(3.2.3) Proposition. *If $\phi \in \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ is the Fourier transform of $\mu_\bullet$, then for any $f \in \mathbb{C}\langle Z_1, ..., Z_n\rangle$ the product $f \cdot \phi$ is the Fourier transform of $\mu_\bullet L_f$, and $\phi \cdot f$ is the Fourier transform of $\mu_\bullet R_f$.*

(3.3) Measures and convergent NCFT. Let $p_d : G_n(\mathbb{R}) \to G_{n,d}(\mathbb{R})$ be the projection. By a cylindrical open set in $G_n(\mathbb{R})$ we mean a set of the form $p_d^{-1}(U)$, where $d \geq 1$ and $U \subset G_{n,d}(\mathbb{R})$ is an open set. These sets form thus a basis of the projective limit topology on $G_n(\mathbb{R})$. We denote by $\mathfrak{S}$ the σ-algebra of sets in $G_n(\mathbb{R})$ generated by cylindrical open sets. Its elements will be simply called Borel subsets in $G_n(\mathbb{R})$.

(3.3.1) Example. Let

$$G_n(\mathbb{R})^{ent} = G_n(\mathbb{R}) \cap \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle^{ent}$$

be the subgroup formed by entire series, see (1.2.2). Since for $\Phi \in G_n(\mathbb{R})$ each given coefficient of f depends on the image of Φ in some $G_{n,d}(\mathbb{R})$, the condition (1.2.2) implies that $G_n(\mathbb{R})^{ent}$ is a Borel subset. Note further that for $\Phi \in G_n(\mathbb{R})^{ent}$ and any Hermitian matrices $Z_1, ..., Z_n$ (of any size N) the matrix $\Phi(iZ_1, ..., iZ_n)$ is unitary. This follows from the reality of the coefficients in Φ and from Proposition 2.4.8(d).

By a measure on $G_n(\mathbb{R})$ we mean a complex valued, countably additive measure on the σ-algebra $\mathfrak{S}$. If μ is such a measure, we define its Fourier transform to be the function of indeterminate Hermitian N by N matrices $Z_1, ..., Z_n$ (with indeterminate N) given by

$$\mathcal{F}(\mu)(Z_1, ..., Z_n) = \int_{\Phi \in G_n(\mathbb{R})^{ent}} \Phi(iZ_1, ..., iZ_n) d\mu(\Phi). \tag{3.3.2}$$

As usual, by a probability measure on $G_n(\mathbb{R})$ we mean a real, nonnegative-valued measure on $\mathfrak{S}$ of total volume 1.

Given a pro-measure $\mu_\bullet = (\mu_d)$ on $G_n(\mathbb{R})$, the correspondence

$$p_d^{-1}(U) \mapsto \mu_d(U), \quad U \in G_{n,d}(\mathbb{R}), \tag{3.3.3}$$

defines a finite-additive function on cylindrical open sets in $G_n(\mathbb{R})$. The following fact is a version of the basic theorem of Kolmogoroff ([SW], Thm. 1.1.10) that a stochastic process is uniquely determined by its finite-dimensional distributions. More precisely, Kolmogoroff's theorem is about probability measures on an infinite product of measure spaces. The modification to the case of a projective limit is immediate.

(3.3.4) Theorem. *If $\mu_\bullet$ is a probability pro-measure (i.e., each μ_d is a probability measure), then the correspondence (3.3.4) extends to a unique probability measure $\mu = \varprojlim \mu_d$ on $G_n(\mathbb{R})$, so that $\mu_d = p_{d*}(\mu)$.*

Thus, probability measures and pro-measures are in bijection.

4. Noncommutative Gaussian and the Wiener measure

(4.1) Informal overview. By the noncommutative Gaussian we mean the following noncommutative power series

$$\Xi(Z) = \exp\left(-\frac{1}{2}\sum_{i=1}^{n} Z_i^2\right) \quad \in \quad \mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle^{ent}. \tag{4.1.1}$$

As the series is entire, we will denote by the same symbol $\Xi(Z_1, ..., Z_n)$ its value on any given square matrices $Z_1, ..., Z_n$. In the classical (commutative) analysis the Fourier transform of a Gaussian is another Gaussian. In this section we present a noncommutative extension of this fact. Informally the answer can be formulated as follows.

(4.1.2) Informal theorem. *"The" measure on the space of paths whose Fourier transform gives $\Xi(Z)$, is the Wiener measure.*

We write "the" in quotes because so far there is no uniqueness result for NCFT, so (4.1.2) can be read in one direction: that the NCFT of the Wiener measure is $\Xi(Z)$. Still, there are two more issues one has to address in order to make (4.1.2) into a theorem. First, the Wiener measure (see below for a summary) is defined on the space of parametrized paths, while NCFT is defined for measures on the space of unparametrized paths. This can be addressed by considering the pushdown of the Wiener measure (i.e., by performing the integration over the space of parametrized paths).

Second, and more importantly, the Wiener measure is defined on the space of continuous paths, and piecewise smooth paths form a subset of measure 0. On the other hand, the series $E_\gamma(Z)$ is a solution of a differential equation involving the time derivatives of γ and so is a priori not defined if γ is just a continuous path. This difficulty is resolved by using the theory of stochastic integrals and stochastic differential equations which indeed provides a way of associating $E_\gamma(Z)$ to all continuous γ except those forming a set of Wiener measure 0.

Once these two modifications are implemented, (4.1.2) becomes an instance of the familiar principle in the theory of stochastic differential equations: that the direct image of the Wiener measure under the map given by the solution of a stochastic differential equation, is the heat measure for the corresponding (hypo)elliptic operator, see [Bel] [Ok] [Bi1].

(4.2) The hypo-Laplacians and their heat kernels. Let $Z_{i,d}$ be the image of Z_i in $\mathfrak{g}_{n,d}$, and $L_{i,d}$ be the left invariant vector field on $G_{n,d}(\mathbb{R})$ corresponding to $Z_{i,d}$. We consider $L_{i,d}$ as a first order differential operator on functions. The dth hypo-Laplacian is the operator

$$\Delta_d = \sum_{i=1}^{n} L_{i,d}^2 \tag{4.2.1}$$

in functions on $G_{n,d}(\mathbb{R})$. For $d \geq d'$ the operators Δ_d and $\Delta_{d'}$ are compatible:

$$\Delta_d(p_{dd'}^* f) = p_{dd'}^*(\Delta_{d'} f), \quad \forall f \in C^2(G_{d'}(\mathbb{R})), \tag{4.2.2}$$

where the projection $p_{dd'}$ is as in (3.1.2). This follows because a similar compatibility holds for each $L_{i,d}$ and $L_{i,d'}$.

For $d > 1$ the number of summands in (4.2.1) is less than the dimension of $G_{n,d}(\mathbb{R})$, so Δ_d is not elliptic. However, Δ_d is hypoelliptic [Ho], i.e., every distribution solution of $\Delta_d u = 0$ is real analytic. This follows from Theorem 1.1 of Hörmander [Ho], since the $Z_{i,d}$ generate $\mathfrak{g}_{n,d}$ as a Lie algebra. Further, it is obvious that Δ_d is positive:

$$(\Delta_d u, u) \geq 0, \quad u \in C_0^\infty(G_{n,d}(\mathbb{R})). \tag{4.2.3}$$

General properties of positive hypoelliptic operators [Ho] imply that the heat operator $\exp(-t\Delta_d)$, $t > 0$, is given by a positive C^∞ kernel. Because this operator is left invariant, we get part (a) of the following theorem:

(4.2.4) Theorem. *(a) The operator* $\exp(\Delta_d/2)$ *is given by convolution with a uniquely defined probability measure* θ_d *on* $G_{n,d}(\mathbb{R})$*. This measure is real analytic with respect to the Haar measure.*
(b) For $d \geq d'$ *the measures* θ_d *and* $\theta_{d'}$ *are compatible:* $(p_{dd'})_*(\theta_d) = \theta_{d'}$.

Part (b) above follows from (4.2.2).

Thus we obtain a probability pro-measure $\theta_\bullet = (\theta_d)$ on $G_n(\mathbb{R})$ and hence a probability measure $\theta = \varprojlim \theta_d$.

(4.2.5) Examples. (a) the group $G_{n,1}$ is identified with the space $\mathbb{R}^n$ from (2.1) with coordinates $y_1, ..., y_n$, and $Z_{i,1} = \partial/\partial y_i$. Therefore Δ_1 is the standard Laplacian on $\mathbb{R}^n$, and

$$\theta_1 = \frac{dy_1 ... dy_n}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^n y_i^2\right)$$

is the usual Gaussian measure on $\mathbb{R}^n$. Each θ_d, $d > 1$ is thus a lift of this measure to $G_{n,d}$.

(b) For $d = 2$ an explicit formula for θ_2 was obtained by Gaveau in [G]. Here we consider the case $n = 2$ where the formula was also obtained by Hulanicki [Hu]. In this case $\mathfrak{g}_{2,2}$ is the Heisenberg Lie algebra with basis consisting of $Z_{1,2}, Z_{2,2}$ and the central element $h = [Z_{1,2}, Z_{2,2}]$. Denoting y_1, y_2, v the corresponding exponential coordinates on $G_{2,2}$, we have

$$\theta_2 = \frac{dy_1 dy_2 dv}{(2\pi)^2} \int_{\tau=-\infty}^{\infty} \frac{2\tau}{\sinh(2\tau)} \cdot \exp\left(i\tau v - (y_1^2 + y_2^2)\frac{2\tau}{\tanh(2\tau)}\right) d\tau.$$

In fact, all known formulas in the literature (see [BGG] for a survey) involve integration over auxuliary parameters.

(4.2.6) Theorem. *The formal Fourier transform of the pro-measure* $\theta_\bullet$ *is equal to the noncommutative Gaussian* $\Xi(Z)$.

Proof: This follows from the fact that the delta-pro-distribution $D_{Z_i}(\delta_\bullet)$ corresponding to the generator $Z_i \in \mathfrak{g}_n$ is taken by $\mathcal{F}$ into the monomial Z_i. For each d the corresponding distribution takes a function f on $G_{n,d}(\mathbb{R})$ into the value of $L_i(f)$ at the unit element of $G_{n,d}(\mathbb{R})$. Further, convolution of such pro-distributions corresponds to composition of left invariant differential operators in the spaces of functions of the $G_{n,d}(\mathbb{R})$. So the system of the heat kernel operators on the $G_{n,d}(\mathbb{R})$, $d \geq 1$, given by $\exp\left(-\frac{1}{2}\sum L_i^2\right)$ has, as a pro-distribution, the Fourier transform equal to $\exp\left(-\frac{1}{2}\sum Z_i^2\right)$. □

(4.3) The Wiener measure. Let P_n be the space of continuous parametrized paths $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = 0$. The Wiener measure w on P_n is first defined on cylindrical open sets $C(t_1, ..., t_m, U_1, ..., U_m)$, where $0 < t_1 < ..., < t_m < 1$ and $U_i \subset \mathbb{R}^n$ is open. By definition,

$$C(t_1, ..., t_m, U_1, ..., U_m) = \big\{\gamma :\ \gamma(t_i) \in U_i,\ i = 1, ..., m\big\},$$

and

$$w\big(C(t_1, ..., t_m, U_1, ..., U_m)\big) = \tag{4.3.1}$$

$$= \int_{(y^{(1)},...,y^{(m)}) \in U_1 \times ... \times U_m} \prod_{i=0}^{m} \frac{\exp\big(-\|y^{(i+1)} - y^{(i)}\|^2 / 2(t_{i+1} - t_i)\big)}{\big(2\pi(t_{i+1} - t_i)\big)^{1/2}} dy^{(1)} ... dy^{(m)}.$$

Here we put $t_0 = 0, t_{m+1} = 1$ and $y^{(0)} = 0$. Further, it is proved that w extends to a probability measure on the σ-algebra generated by the above subsets.

The Brownian motion is the family of $\mathbb{R}^n$-valued functions (random variables) on P_n parametrized by $t \in [0,1]$:

$$b(t) = (b_1(t), ..., b_n(t)), \quad b(t) : P_n \to \mathbb{R}^n,\ b(t)(\gamma) = \gamma(t). \tag{4.3.2}$$

let $P_n^{sm} \subset P_n$ be the subset of piecewise smooth paths. Then it is well known that $w(P_n^{sm}) = 0$.

As well known, the Wiener measure has the following intuitive interpretation

$$dw(\gamma) = \exp\left(-\int_0^1 \|\gamma'(t)\|^2 dt\right) \mathcal{D}\gamma, \quad \mathcal{D}\gamma = \prod_{t=0}^{1} d\gamma(t). \tag{4.3.3}$$

In other words, $\mathcal{D}\gamma$ is the (nonexistent) Lebesgue measure on the infinite-dimensional vector space of all paths, while the integral in the exponential is the action of a free particle.

(4.5) Reminder on stochastic integrals. Let $\omega = \sum_{i=1}^n \phi_i(y) dy_i$ be a 1-form on $\mathbb{R}^n$ with (complex valued) C^∞ coefficients. If $\gamma : [0,1] \to \mathbb{R}^n$ is a piecewise smooth path, then we can integrate ω along γ, getting a number

$$\int_\gamma \omega = \int_0^1 \gamma^*(\omega) = \int_0^1 \sum_i \phi_i(\gamma(t)) \gamma_i'(t) dt. \tag{4.5.1}$$

This gives a map (function)

$$\int(\omega) : P_n^{sm} \longrightarrow \mathbb{C}. \tag{4.5.2}$$

If $\gamma(t)$ is just a continuous path without any differentiability assumptions, then (4.5.1) is not defined, so there is no immediate extension of the map (4.5.2) to the space P_n. The theory of stochastic integrals provides several (a priori different) ways to construct such an extension. The two best known approaches are the Ito and Stratonovich integrals over the Brownian motion, see [SW] [KW]. They are functions

$$\int^{\rm Ito}(\omega), \quad \int^{\rm Str}(\omega) : \quad P_n \to \mathbb{C}, \tag{4.5.3}$$

defined everywhere outside some subset of Wiener measure 0, and measurable with respect to this measure. To construct them, see [Ok], p. 14-16, one has to consider Riemann sum approximations to the integral but restrict to Riemann sums of some particular type. For a piecewise smooth path γ the integral is the limit of sums

$$\sum_{i=1}^n \sum_{\nu=1}^m \phi_i(\gamma(\xi_\nu))\big((\gamma_i(t_\nu) - \gamma_i(t_{\nu-1})\big), \tag{4.5.4}$$

where $0 = t_0 < t_1 < ... < t_m = 1$ is a decomposition of $[0, 1]$ into intervals, and $\xi_\nu \in [t_{\nu-1}, t_\nu]$ are some chosen points. In the smooth case the limit exists provided $\max(t_\nu - t_{\nu-1})$ goes to 0 (in particular, the choice of ξ_ν is inessential).

Now, to obtain $\int^{\rm Ito}(\omega)$, one chooses the class of Riemann sums with

$$t_\nu = \nu/m, \quad \xi_\nu = t_{\nu-1}, \quad m = 2^q, q \to \infty. \tag{4.5.5}$$

In other words, for each q the above sum defines a function $\mathcal{S}_q^{\rm Ito}(\omega) : P_n \to \mathbb{C}$, and

$$\int^{\rm Ito}(\omega) = \lim_{q\to\infty} \mathcal{S}_q^{\rm Ito}(\omega). \tag{4.5.6}$$

To obtain $\int^{\rm Str}(\omega)$, one chooses the class of Riemann sums with

$$t_\nu = \nu/m, \quad \xi_\nu = (t_{\nu-1} + t_\nu)/2, \quad m = 2^q, q \to \infty. \tag{4.5.7}$$

Each such sum gives a function $\mathcal{S}_q^{\rm Str}(\omega) : P_n \to \mathbb{C}$, and

$$\int^{\rm Str}(\omega) = \lim_{q\to\infty} \mathcal{S}_q^{\rm Str}(\omega). \tag{4.5.8}$$

It follows that $\int^{\rm Str}(\omega)$ is invariant under smooth reparametrizations of the path considered as transformations acting on P_n.

The more common notation for the stochastic integrals (considered as random variables on P_n) is:

$$\int^{\rm Ito}(\omega) = \int_0^1 \omega(b(t))db(t)), \quad \int^{\rm Str}(\omega) = \int_0^1 \omega(b(t)) \circ db(t)), \tag{4.5.9}$$

where $b(t)$ is the Brownian motion (4.3.2). Thus $db(t)$ and $\circ db(t)$ stand for the two ways (due to Ito and Stratonovich) of regularizing the (a priori divergent) differential of the Brownian path $b(t)$. See [Ok] for the relation between the two regularization schemes. By restricting to the truncated path $[0, s]$, $s \leq 1$, one defines the stochastic integrals $\int_0^s$ in each of the above setting.

(4.6) Stochastic holonomy. Let G be a Lie group which we suppose to be embedded as a closed subgroup of $GL_N(\mathbb{C})$ for some N and let $\mathfrak{g} \subset \mathrm{Mat}_N(\mathbb{C})$ be the Lie algebra of G. Let $A = \sum A_i(y)dy_i$ be a smooth $\mathfrak{g}$-valued 1-form on $\mathbb{R}^n$, which we consider as a connection in the trivial G-bundle over $\mathbb{R}^n$. If $\gamma : [0,1] \to \mathbb{R}^n$ is a piecewise smooth path, then we have the holonomy of A along γ:

$$\mathrm{Hol}_\gamma(A) = P \exp \int_\gamma A \quad \in \quad G. \tag{4.6.1}$$

It is the value at $t = 1$ of the solution $U(t) \in GL_N(\mathbb{C})$ of the differential equation

$$\frac{dU}{dt} = U(t)\biggl(\sum_i A_i(\gamma(t)) \cdot \gamma_i'(t)\biggr), \quad U(0) = 1. \tag{4.6.2}$$

The holonomy defines thus a map

$$\mathrm{Hol}(A) : P_n^{sm} \to G. \tag{4.6.3}$$

As before, (4.6.2) and thus $\mathrm{Hol}_\gamma(A)$ have no immediate sense without some differentiability assumptions on A. The theory of stochastic differential equations [Ok] [KW] resolves this difficulty by replacing the above differential equation by an integral equation and understanding the integral in a regularized sense as in (4.5). Thus, one defines the Ito and Stratonovich stochastic holonomies which are measurable maps

$$\mathrm{Hol}^{\mathrm{Ito}}(A),\ \mathrm{Hol}^{\mathrm{Str}}(A) : P_n \to G, \tag{4.6.3}$$

defined outside a subset of Wiener measure 0. For example, $\mathrm{Hol}^{\mathrm{Str}}(A)$ is defined as the value at $t = 1$ of the G-valued stochastic process $U(t)$ satisfying the Stratonovich integral equation

$$U(t) = 1 + \int_0^t U(s)\biggl(\sum_i A_i(b(s)) \circ db_i(s)\biggr). \tag{4.6.4}$$

We will be particularly interested in the case when A_i are constant, i.e., our connection is translation invariant. In this case $B(t) = \sum A_i b_i(t)$ is a (possibly degenerate) Brownian motion on $\mathfrak{g}$ and $U(t)$ is the corresponding left invariant Brownian motion on G as studied by McKean, see [McK], §4.7, and also [HL]. In particular, the Stratonovich holonomy can be represented as a "product integral" in the sense of McKean

$$\mathrm{Hol}^{\mathrm{Str}}(A) = \prod_{t\in[0,1]} \exp(dB(t)) \quad := \quad \lim_{q\to\infty} \prod_{\nu=1}^{2^q} \exp\biggl(B\left(\frac{\nu}{2^q}\right) - B\left(\frac{\nu-1}{2^q}\right)\biggr), \tag{4.6.5}$$

see [HL], Thm.2. Here the product is taken in the order of increasing ν.

Again, the product integral representation implies invariance of $\mathrm{Hol}^{\mathrm{Str}}(A)$ under smooth reparametrizations of the path.

(4.7) The Malliavin calculus and the Feynman-Kac-Bismut formula. We now specialize (4.6) to the case when $G = G_{n,d}(\mathbb{R})$, $\mathfrak{g} = \mathfrak{g}_{n,d}(\mathbb{R})$ and $A = A^{(d)}$ is the constant 1-form $A^{(d)} = \sum_{i=1}^n Z_{i,d} dy_i$. We get the stochastic holonomy map

$$\mathrm{Hol}^{\mathrm{Str}}(A^{(d)}) : P_n \to G_{n,d}(\mathbb{R}). \tag{4.7.1}$$

(4.7.2) Theorem. *The probability measure θ_d on $G_{n,d}(\mathbb{R})$ is equal to $\mathrm{Hol}^{\mathrm{Str}}(A^{(d)})_*(w)$, the push-down of the Wiener measure under the holonomy map.*

Proof: This is a fundamental property of (hypo)elliptic diffusions holding for any vector fields $\xi_1, ..., \xi_n$ on a manifold M such that iterated commutators of the ξ_i span the tangent space at every point. In this case the operator $\Delta = \sum \mathrm{Lie}^2_{\xi_i}$ is hypoelliptic and has a uniquely defined, smooth heat kernel $\Theta(x, y), x, y \in M$ which is a function in x and a volume form in y and represents the operator $\exp(-\Delta/2)$. Further, the heat equation

$$\partial u/\partial t = -\Delta(u)/2 \tag{4.7.3}$$

is the "Kolmogoroff backward equation" for the M-valued stochastic process $U(t)$ satisfying the Stratonovich differential equation

$$dU = \sum L_{\xi_i}(U) \circ db_i \tag{4.7.4}$$

with the $b_i(t)$ being as before. This means that the fundamental solution of (4.7.3) is the pushforward of the Wiener measure under the process $U(t)$. See [Ok], Th. 8.1. Our case is obtained by specializing to $M = G_{n,d}(\mathbb{R})$, $\xi_i = Z_{i,d}$. □

Further, let θ be the probability measure on $G_n(\mathbb{R}) = \varprojlim_d G_{n,d}(\mathbb{R})$ corresponding to the pro-measure (θ_d) by Theorem 3.3.5. Note that the maps $\mathrm{Hol}^{\mathrm{Str}}(A^{(d)})$ for various d unite into a map

$$\mathrm{Hol}^{\mathrm{Str}}(A) : P_n \to G_n(\mathbb{R}), \quad A = \sum Z_i dy_i. \tag{4.7.4}$$

We get the following corollary.

(4.7.5) Corollary. *The measure θ is the pushdown of the Wiener measure under* $\mathrm{Hol}^{\mathrm{Str}}(A)$.

(4.7.6) Theorem. *(a) The support of the measure θ is contained in $G_n(\mathbb{R})^{ent}$, the set of entire group-like power series.*
(b) The convergent Fourier transform of θ is equal to $\Xi(Z)$. In other words (taking into account part (a) and (4.7.5)), for any given Hermitian matrices $Z_1, ..., Z_n$ of any given size N we have

$$\exp\left(-\frac{1}{2}\sum_{j=1}^n Z_j^2\right) = \int_{\gamma \in P_n} \mathrm{Hol}^{\mathrm{Str}}_\gamma(A)(iZ_1, ..., iZ_n) dw(\gamma).$$

Proof: (a) First of all, one can write $\mathrm{Hol}^{\mathrm{Str}}(A)$, similarly to (2.3.2), as the generating function of stochastic iterated integrals understood in the sense of Stratonovich:

$$\mathrm{Hol}^{\mathrm{Str}}(A) = \sum_m \sum_{J=(j_1,\dots,i_m)} Z_{j_1}\dots Z_{j_m} \cdot \int^{\rightarrow} \circ db_{j_1}\dots \circ db_{j_m}. \tag{4.7.7}$$

See [FN] for the definition of stochastic iterated integrals as well as for the proof. So the question is whether the series (4.7.7) (with coefficients being random variables on P_n) is entire almost surely (i.e., outside a set of Wiener measure 0). Questions of this nature ("convergence of stochastic Taylor series") were studied by Ben Arous [Be]. To get a link to his notation, denote the stochastic integral in the RHS of (4.7.7) by B_J and the monomial $Z_{j_1}\dots Z_{j_m}$ by simply Z^J. The stochastic Taylor series of [Be] are expressions of the form

$$\sum_m \sum_{j=(j_1,\dots,j_m)} x_J \cdot B_J, \tag{4.7.8}$$

where (x_J) is a family of (nonrandom) complex numbers labelled by the multi-indices J. Let us write $m = l(J)$ for the total degree of the monomial corresponding to J. For the noncommutative formal series $\mathrm{Hol}^{\mathrm{Str}}(A) = \sum_J B_J Z^J$ to be entire (1.2.2), it should satisfy $|B_J| \cdot R^{l(J)} \to 0$ for any $R > 0$ which is equivalent to saying that $\sum_J |B_J| \cdot R^{l(J)} < \infty$ for any $R > 0$. The last sum is an example of a Ben Arous series, and Corollary 1 of [Be] gives its almost sure convergence, whence the claim.

(b) It follows from (a) and Theorem 4.2.6 about the formal Fourier transform. $\square$

5. Futher examples of NCFT

(5.1) Near-Gaussians. In classical analysis, a near-Gaussian is a function of the form $f(z) \cdot e^{-\|z\|^2/2}$ where $f(z)$, $z = (z_1, ..., z_n)$, is a polynomial. In that setting, the Fourier transform of a near-Gaussian is another near-Gaussian.

A natural noncommutative analog of a near-Gaussian is a function of the form

$$F(Z) \cdot \Xi(Z) \cdot G(Z), \quad F, G \in \mathbb{C}\langle Z_1, ..., Z_n\rangle. \tag{5.1.1}$$

It can be represented as a (formal) Fourier transform using Proposition 3.2.3:

$$F(Z) \cdot \Xi(Z) \cdot G(Z) = \widehat{\mathcal{F}}(\theta_\bullet L_F R_G), \tag{5.1.2}$$

where L_F is the system of left invariant differential operators on the $G_{n,d}(\mathbb{R}), d \geq 1$, corresponding to F, while R_G is the system of right invariant differential operators corresponding to G.

It seems difficult to realize the measures $\theta_d L_F R_G, \quad d \geq 1$, in terms of some transparent measures on the space P_n, as it requires using group translations on Π_n^{cont}, the group of continuous paths obtained by quotienting P_n by reparametrizations and cancellations.

(5.2) The Green pro-measure. Let g_d be the fundamental solution of the dth hypo-Laplacian on $G_{n,d}(\mathbb{R})$ centered at 1, the unit element, i.e.,

$$\Delta_d(g_d) = \delta_1. \tag{5.2.1}$$

By the general properties of hypoelliptic operators, g_d is a measure (volume form) on $G_{n,d}(\mathbb{R})$ smooth away from 1. In fact, if we denote by $\theta_{d,t}$ the kernel of $\exp(-t\Delta_d/2)$, $t > 0$, i.e., the heat kernel measure at time t, then

$$g_d = \int_{t=0}^{\infty} \theta_{d,t} dt. \tag{5.2.2}$$

This expresses the fact that the Green measure of a domain is equal to the amount of time a diffusion path spends in the domain. It is clear therefore that $g_\bullet = (g_d)$ is a pro-measure on $G_n(\mathbb{R})$.

(5.2.3) Examples. (a) For $d = 1$ we have the Green function of the usual Euclidean Laplacian in $\mathbb{R}^n$ which has the form

$$g_1(y) = \frac{1}{4\pi} \ln(y_1^2 + y_2^2) dy_1 dy_2, \quad n = 2,$$

$$g_1(y) = -\frac{((n/2) - 2)!}{4\pi^{n/2}} \left(\sum y_i^2\right)^{1-n/2} dy_1 ... dy_n.$$

(b) Consider the case $n = 2, d = 2$ corresponding to the Heisenberg group, and let us use the exponential coordinates y_1, y_2, v as in Example 4.2.5(b). Then

$$g_2(y_1, y_2, v) = \frac{1}{\pi} \frac{1}{\sqrt{(y_1^2 + y_2^2) + v^2}} dy_1 dy_2 dv,$$

as was found by Folland [Fo], see also [G], p. 101.

(5.3) The method of kernels. More generally, if $F(Z_1, ..., Z_n)$ is a "noncommutative function" such that the operator $F(L_{1,d}, ..., L_{n,d})$ in functions on $G_{n,d}(\mathbb{R})$ makes sense and posesses a distribution kernel $K_d(x, y)dy$, then the distribution $\mu_d = K_d(1, y)dy$ is precisely the dth component of the pro-distribution whose Fourier transform is F.

For example, hypoelliptic calculus allows us to consider $F(Z) = \phi\left(\sum_{i=1}^n Z_i^2\right)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is any C^∞ function decaying at infinity such as $\phi(u) = e^{-u^2/2}$ or $\phi(u) = 1/u$, or $1/(u^2 + 1)$. This leads to a considerable supply of pro-distributions.

(5.4) Probabilistic meaning. An idea in probability theory very similar to our NCFT, viz. the idea of associating a noncommutative power series to a stochastic process, was proposed by Baudoin [Ba], who called this series "expectation of the signature" and emphasized its importance. From the general viewpoint of probability theory one can look at this series (the Fourier transform of a probability measure on the space of paths) as being rather an analog of the characteristic function of n random variables. Indeed, if $x_1, ..., x_n$ are random variables, then their joint distribution is a probability measure on $\mathbb{R}^n$, and the characteristic function is the (usual) Fourier transform of this measure:

$$f(z_1, ..., z_n) = \mathbb{E}\left[e^{i(z,x)}\right] \tag{5.4.1}$$

which is an entire function of n variables. Each time when we have a natural lifting of the characteristic function to the noncommutative domain, we can therefore expect some n-dimensional stochastic process lurking in the background.

6. Fourier transform of noncommutative measures

(6.1) Nomcommutative measures. Following the general approach of Noncommutative Geometry [Con], we consider a possibly noncommutative $\mathbb{C}$-algebra R (with unit) as a replacement of a "space" ($\mathrm{Spec}(A)$). A measure on R is then simply a linear functional ("integration map") $\tau : I \to \mathbb{C}$ defined on an appropriate subspace $I \subset R$ whose elements have the meaning of integrable functions. We will call a measure τ *finite*, if $I = R$, and *normalized*, if it is finite and $\tau(1) = 1$. If R has a structure of a $*$-algebra, then a finite measure τ is called positive if $\tau(aa^*) \geq 0$ for any $a \in R$. A (noncommutative) *probability measure* on a $*$-algebra A is a normalized, positive measure.

(6.1.1) Examples. (a) Let $R = \mathrm{Mat}_N(\mathbb{C})$ with the $*$-algebra structure given by the Hermitian conjugation. Then $\tau(a) = \frac{1}{N}\operatorname{Tr}(a)$ is a probability measure.

(b) Let $R = \mathbb{C}\langle Z_1, ..., Z_n\rangle$ with the $*$-algebra structure given by $Z_i^* = Z_i$. Let Herm_N be the space of Hermitian N by N matrices. We denote by $dZ = \prod_{i,j=1}^N dZ_{ij}$ the standard volume form on Herm_N. Let $\mu = \mu_N$ be a volume form on $(\mathrm{Herm}_N)^n$ of exponential decay at infinity. Then we have a finite measure on R given by

$$\tau(f) = \frac{1}{N}\operatorname{Tr}\int_{Z_1,...,Z_n \in \mathrm{Herm}_N} f(Z_1, ..., Z_n) d\mu(Z_1, ..., Z_n).$$

If μ_N is a normalized (resp. probability) measure in the usual sense, then τ is a normalized (resp. probability) measure in the noncommutative sense. An important example is

$$\mu_N = \exp\left(-S(Z_1, ..., Z_n)\right) dZ_1...dZ_n,$$

where the "action" $S(Z_1, ..., Z_n) \in \mathbb{C}\langle Z_1, ..., Z_n\rangle$ is a noncommutative polynomial with appropriate growth conditions at infinity.

(c) Let $R = \mathbb{C}\langle X_1^{\pm 1}, ..., X_n^{\pm 1}\rangle$ with the $*$-algebra structure given by $X_i^* = X_i^{-1}$. If $\mu = \mu_N$ is a finite measure on $U(N)^n$, then we have a finite measure τ on R given by

$$\tau(f) = \frac{1}{N}\operatorname{Tr}\int_{X_1,...,X_n \in U(N)} f(X_1, ..., X_n) d\mu(X_1, ..., X_n),$$

which is normalized (resp. probability) if μ_N is so in the usual sense.

(6.2) Free products. Let $R_1, ..., R_n$ be algebras with unit. Then we have their free product $R_1 \star .. \star R_n$. This is an algebra containing all the R_i and characterized by the following universal property: for any algebra B and any homomorphisms $f_i : R_i \to B$ there is a unique homomorphism $f : R_1 \star ... \star R_n \to B$ restricting to f_i on R_i for each i. Explicitly, $R_1 \star ... \star R_n$ is obtained as the quotient of the free (tensor) algebra generated by the vector space $R_1 \oplus ... \oplus R_n$ by the relations saying the products of elements from each R_i are given by the existing multiplication in R_i. We will also use the notation $\bigstar_{i=1}^n R_i$.

(6.2.1) Example. If each $R_i = \mathbb{C}[Z_i]$ is the polynomial algebra in one variable, then $R_1 \star ... \star R_n = \mathbb{C}\langle Z_1, ..., Z_n\rangle$ is the algebra of noncommutative polynomials. If each $R_i = \mathbb{C}[X_i, X_i^{-1}]$ us the algebra of Laurent polynomials, then $R_1 \star ... \star R_n = \mathbb{C}\langle X_1^{\pm 1}, ..., X_n^{\pm 1}\rangle$ is the algebra of noncommutative Laurent polynomials.

The following description of the free product follows easily from definition (see [VDN]).

(6.2.2) Proposition. *Suppose that for each i we chose a subspace $R_i^\circ \subset R_i$ which is a complement to $\mathbb{C}\cdot 1$. Then as a vector space*

$$R_1 \star ... \star R_n = \mathbb{C}\cdot 1 \oplus \bigoplus_{k>0} \bigoplus_{i_1 \neq i_2 \neq ... \neq i_k} R_{i_1}^\circ \otimes ... \otimes R_{i_k}^\circ .$$

The following definition of the free product of (noncommutative) measures is due to Voiculescu, see [VDN].

(6.2.3) Proposition-Definition. *Let R_i, $i = 1, ..., n$ be associative algebras with 1, and $\tau_i : R_i \to \mathbb{C}$ be finite normalized measures. Then there exists a unique finite normalized measure $\tau = \bigstar \tau_i$ on $\bigstar_{i=1}^n R_i$ with the following properties:*
(1) $\tau|_{R_i} = \tau_i$.
(2) If $i_1 \neq ... \neq i_k$ and $a_\nu \in R_\nu$ are such that $\tau_{i_\nu}(a_\nu) = 0$, then $\tau(a_{i_1} ... a_{i_k}) = 0$.
If the R_i are $*$-algebras, and each τ_i is a probability measure, then so is τ.

Both the existence and the uniqueness of τ follow at once from (6.2.2), if we take $R_i^\circ = \mathrm{Ker}(\tau_i)$. The problem of finding $\tau(a_1...a_k)$ for arbitrary elements $a_\nu \in R_{i_\nu}$ is clearly equivalent to that of writing $a_1 ... a_k$ in the normal form (6.2.2). To do this, one writes

$$a_\nu = \tau_{i_\nu}(a_\nu)\cdot 1 + a_\nu^\circ, \tag{6.2.4}$$

with a_ν° defined so as to satisfy (6.2.4) and we have $\phi_{i_\nu}(a_\nu^\circ) = 0$. Then one uses the conditions (1) and (2) to distribute.

(6.2.5) Examples. Suppose we have two algebras A and B and normalized measures $\phi : A \to \mathbb{C}$ and $\psi : B \to \mathbb{C}$. Let $\chi : A \star B \to \mathbb{C}$ be the free product of ϕ and ψ. Then for $a, a' \in A$ and $b, b' \in B$ we have, after some calculations:

$$\chi(ab) = \phi(a)\psi(b), \quad \chi(aba') = \phi(aa')\psi(b),$$

$$\chi(aba'b') = \phi(aa')\psi(b)\psi(b') + \phi(a)\phi(a')\psi(bb') - \phi(a)\phi(a')\psi(b)\psi(b').$$

See [NS], Thm. 14.4, for a general formula for $\chi(a_1 b_1 ... a_m b_m)$, $a_i \in A$, $b_i \in B$.

(6.2.6) Examples. (a) Let $R_i = \mathbb{C}[x^{\pm 1}]$, $i = 1, ..., n$ and let τ_i be given by the integration over the normalized Haar measure d^*x on the unit circle. Thus

$$\tau_i\left(f(x) = \sum_m a_m x^m\right) = \int_{|x|=1} f(x) d^*x = a_0.$$

The free product of these measures is the functional on $\mathbb{C}\langle X_1^{\pm 1}, ..., X_n^{\pm 1}\rangle$ given by

$$\tau\left(f(X) = \sum_{\gamma\in F_n} a_\gamma X^\gamma\right) = a_0,$$

the constant term of a noncommutative Laurent polynomial. The asymptotic freedom theorem for unitary matrices (1.4.7) says that this functional is the limit, as $N \to \infty$, of the functionals from Example 6.1.1(c) with μ_N, for each N, being the normalized Haar measure on $U(N)^n$.

(b) Let $R_i = \mathbb{C}[z]$, $i = 1, ..., n$, and let $\tau_i = \delta(z - a_i)$ be the Dirac delta-function situated at a point $a_i \in \mathbb{C}$, i.e., $\tau_i(f) = f(a_i)$. Then the free product $\tau = \tau_1 \star ... \star \tau_n$ is given by:

$$\tau(f(Z_1, ..., Z_n)) = f(a_1 \cdot 1, ..., a_n \cdot 1),$$

in other words, it depends only on the image of f in the ring of commutative polynomials $\mathbb{C}[z_1, ..., z_n]$. This can be seen from the procedure (6.2.4) using the fact that each $\tau_i : \mathbb{C}[z] \to \mathbb{C}$ is a ring homomorphism.

(c) Let $R_i = \mathbb{C}[z]$, $i = 1, ..., n$, and let τ_i be the integration over the standard Gaussian probability measure

$$\tau_i(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{-z^2/2} dz.$$

Their free product is a probability measure on $\mathbb{C}\langle Z_1, ..., Z_n\rangle$ denoted by ξ_n and called the *free Gaussian measure*. The asymptotic freedom for Hermitian Gaussian ensembles [V] can be formulated as follows.

(6.2.7) Theorem. *The measure ξ_n is the limit, as $N \to \infty$, of the measures from Example 6.1.1(b) where, for each N, we take for μ_N the Gaussian probability measure on the vector space $(\mathrm{Herm}_N)^n$ corresponding to the scalar product $\sum \mathrm{Tr}(A_i B_i)$ on this vector space:*

$$\mu = \mu_N = \frac{1}{(2\pi)^{nN^2/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^n Z_i^2\right) dZ_1 ... dZ_n.$$

(6.3) The Fourier transform of noncommutative measures. In classical analysis, Fourier transform is defined for measures on $\mathbb{R}^n$, not on an arbitrary curved manifold. We will call a measure on $\mathbb{R}^n_{\mathrm{NC}}$ ("noncommutative $\mathbb{R}^n$") a datum consisting of a $*$-algebra R, a $*$-homomorphism $\mathbb{C}\langle Z_1, ..., Z_n\rangle \to R$ (i.e., a choice of self-adjoint elements in R which we will still denote Z_i), and a measure τ on R. Elements of R for which τ is defined, will be thought of as functions integrable with respect to the measure. This concept is thus very similar to that of n noncommutative random variables in noncommutative probability theory, except we do not require any positivity or normalization.

Let τ be a measure on $\mathbb{R}^n_{\mathrm{NC}}$. Its Fourier transform is the complex valued function $\mathfrak{F}(\tau)$ on the group Π_n of piecewise smooth paths in $\mathbb{R}^n$ defined as follows:

$$\mathfrak{F}(\tau)(\gamma) = \tau(E_\gamma(iZ_1, ..., iZ_n)), \quad \gamma \in \Pi_n. \tag{6.3.1}$$

Here we assume that the "entire function" $E_\gamma(iZ_1, ..., iZ_n)$ lies in the domain of definition of τ. In physical terminology, $\mathfrak{F}(\tau)(\gamma)$ is the "Wilson loop functional" (defined here for non-closed paths as well).

(6.3.2) Example: delta-functions. (a) For every $J = (j_1, ..., j_m)$ we have the measure $\delta^{(J)}$ on $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ given by

$$\delta^{(J)} \left(\sum_m \sum_{I=(i_1,...,i_m)} a_I Z_{i_1} ... Z_{i_m} \right) = a_J.$$

The Fourier transform of $\delta^{(J)}$ is the function $W_J : \Pi_n \to \mathbb{C}$ which associates to a path γ the iterated integral along γ labelled by J:

$$W_J(\gamma) = \int_\gamma^{\to} dy_{j_1} ... dy_{j_m}.$$

We will call these functions *monomial* functions on Π_n

(b) If we take for τ the free product of (underived) delta-functions $\delta_{a_1} \star ... \star \delta_{a_n}$, as in Example 6.2.6(b), then

$$\mathfrak{F}(\tau)(\gamma) = \exp\bigl(i(e(\gamma), a)\bigr),$$

where $e(\gamma) \in \mathbb{R}^n$ is the endpoint of γ. This follows from the fact that τ is supported on the commutative locus, i.e., $\tau(E_\gamma(iZ))$ depends only on the image of $E_\gamma(iZ)$ in the commutative power series ring, which is $\exp\bigl(ie(\gamma), z\bigr)$).

(6.4) Convolution and product. Let τ, σ be two measures on $\mathbb{R}^n_{\mathrm{NC}}$, so we have homomorphisms

$$\alpha : \mathbb{C}\langle Z_1, ..., Z_n\rangle \to R, \quad \beta : \mathbb{C}\langle Z_1, ..., Z_n\rangle \to S$$

and τ is a linear functional on R while σ is a linear functional on S. Their (tensor) convolution is the measure $\tau * \sigma$ which corresponds to the homomorphism

$$\mathbb{C}\langle Z_1, ..., Z_n\rangle \to R \otimes S, \quad Z_i \mapsto \alpha(Z_i) \otimes 1 + 1 \otimes \beta(Z_i), \tag{6.4.1}$$

and the linear functional

$$\tau * \sigma : R \otimes S \to \mathbb{C}, \quad r \otimes s \mapsto \tau(r) \otimes \sigma(s).$$

For commutative algebras this corresponds to the usual convolution of measures with respect to the group structure on $\mathbb{R}^n$.

(6.4.2) Proposition. *The Fourier transform of the convolution of measures is the product of their Fourier transforms:*

$$\mathfrak{F}(\tau * \sigma) = \mathfrak{F}(\tau) \cdot \mathfrak{F}(\sigma).$$

Proof: This is a consequence of the fact that the elements $E_\gamma(iZ_1, ..., iZ_n)$ of $\mathbb{C}\langle\langle Z_1, ..., Z_n\rangle\rangle$ are group-like, see the exponential property (2.4.7). $\square$

(6.5) Formal Fourier transform of noncommutative measures. The product of two monomial functions on Π_n is a linear combination of monomial functions. This expresses Chen's shuffle relations among iterated integrals:

$$W_{j_1,...,j_m} W_{j_{m+1},...,j_{m+p}} = \sum_s W_{j_{s(1)},...,j_{s(m+p)}}, \tag{6.5.1}$$

the sum being over the set of (m,p)-shuffles. An indentical formula holds for the convolution of the measures $\delta^{(j_1,...,j_m)}$ and $\delta^{(j_{m+1},...,j_{m+p})}$, as both formulas describe the Hopf algebra structure on $\mathbb{C}\langle\langle Z_1,...,Z_n\rangle\rangle$.

The $\mathbb{C}$-algebra with basis $W_J = W_{j_1,...,j_m}$ and multiplication law (6.5.1) is nothing but the algebra

$$\mathbb{C}[G_n] = \varinjlim \mathbb{C}[G_{n,d}] \tag{6.5.2}$$

of regular functions on the group scheme $G_n = \varprojlim G_{n,d}$. The multiplication in G_n corresponds to the Hopf algebra structure given by

$$\Delta(W_{j_1,...,j_m}) = \sum_{\nu=0}^{m+1} W_{j_1...j_\nu} \otimes W_{j_{\nu+1},...,j_m}. \tag{6.5.3}$$

Elements of $\mathbb{C}[G_n]$ can be called polynomial functions on Π_n.

Note that formal infinite linear combinations (series) $\sum_J c_J W_J$ still form a well defined algebra via (6.5.1), which we denote $\mathbb{C}[[G_n]]$. This is the algebra of functions on the formal completion of G_n at 1. The rule (6.5.3) makes $\mathbb{C}[[G_n]]$ into a topological Hopf algebra.

Let τ be a measure on $\mathbb{R}^n_{\mathrm{NC}}$. We will call the formal Fourier transform of τ the series

$$\widehat{\mathfrak{F}}(\tau) = \sum_{J=(j_1,...,j_m)} \tau(Z_{i_1}...Z_{j_m}) \cdot W_J \quad \in \quad \mathbb{C}[[G_n]]. \tag{6.5.4}$$

As before, we see that convolution of measures is taken into the product in $\mathbb{C}[[G_n]]$.

7. Towards the inverse NCFT

(7.0) In this section we sketch a possible approach to the problem of finding the inverse to the NCFT $\mathcal{F}$ from (2.2). In other words, given a "noncommutative function" $f = f(Z_1, ..., Z_n)$, how to find a measure μ on (possibly some completion of) Π_n such that $\mathcal{F}(\mu) = f$? Note that unlike in classical analysis, the dual Fourier transform $\mathfrak{F}$ (from noncommutative measures to functions on Π_n) does not provide even a conjectural answer, since there is no natural identification of functions and measures.

So we take as our starting point the case of discrete NCFT (1.4.5) where Theorem 1.4.6 provides a neat inversion formula.

(7.1) Fourier series and Fourier integrals. We recall the classical procedure expressing Fourier integrals as scaling limits of Fourier series, see [W], §5. Let $f(x)$ be a piecewise continuous, $\mathbb{C}$-valued function on $\mathbb{R}$ of sufficiently rapid decay. We can restrict f to the interval $[-\pi, \pi]$ which is a fundamental domain for the exponential map $z \mapsto \exp(iz)$, $\mathbb{R} \to S^1$, and then represent f on this interval as a Fourier series in e^{inz}, $n \in \mathbb{Z}$.

Next, let us scale the interval to $[-A, A]$ instead. Then the orthonormal basis of functions is formed by

$$\frac{1}{\sqrt{2A}} \exp\left(\frac{n\pi i z}{A}\right), \quad n \in \mathbb{Z}, \tag{7.1.1}$$

so on the new interval we have

$$f(z) = \frac{1}{2A} \sum_{n\in\mathbb{Z}} \exp\left(\frac{n\pi i z}{A}\right) \int_{-A}^{A} f(w) \exp\left(\frac{-n\pi i w}{A}\right) dy. \tag{7.1.2}$$

If we associate the Fourier coefficients to the scaled lattice points, putting

$$g\left(\frac{n\pi}{A}\right) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(z) \exp\left(\frac{-n\pi i z}{A}\right) dz, \tag{7.1.3}$$

then

$$f(z) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} g\left(\frac{n\pi}{A}\right) \exp\left(\frac{n\pi i z}{A}\right) \Delta\left(\frac{n\pi}{A}\right), \quad z \in [-A, A], \tag{7.1.4}$$

where $\Delta\left(\frac{n\pi}{A}\right) = \frac{\pi}{A}$ is the step of the dual lattice. So when $A \to \infty$, the formulas (7.1.3-4) "tend to" the formulas for two mutually inverse Fourier transforms for functions on $\mathbb{R}$. In other words, the measures on $\mathbb{R}$ (with coordinate y) given by infinite combinations of shifted Dirac delta functions:

$$\frac{1}{\sqrt{2\pi}} \frac{\pi}{A} \sum_{n\in\mathbb{Z}} g\left(\frac{n\pi}{A}\right) \delta\left(y - \frac{n\pi}{A}\right), \tag{7.1.5}$$

converge, as $A \to \infty$, to a measure whose Fourier transform is f.

(7.2) Matrix fundamental domains. We now consider the analog of the above formalism for Hermitian matrices instead of elements of $\mathbb{R}$ and unitary matrices instead of those of S^1. Let $\mathrm{Herm}_N^{\leq A}$ be the set of Hermitian N by N matrices whose eigenvalues all lie in $[-A, A]$. Then $\mathrm{Herm}_N^{\leq\pi}$ is a fundamental domain for the exponential map

$$Z \mapsto X = \exp(iZ), \quad \mathrm{Herm}_N \to U(N). \tag{7.2.1}$$

Note that the Jacobian of the map (7.2.1) is given by

$$J(Z) = \det{}_{N^2\times N^2} \frac{e^{\mathrm{ad}(Z)} - 1}{Z} = \prod_{j,k} \frac{e^{i(\lambda_j - \lambda_k)} - 1}{\lambda_j - \lambda_k} = \prod_{j<k} 2\frac{1 - \cos(\lambda_j - \lambda_k)}{(\lambda_k - \lambda_k)^2}. \tag{7.2.2}$$

Here $\lambda_1, ..., \lambda_N$ are the eigenvalues of Z, see [Hel], p. 255. Using the formula for the volume of $U(N)$, see, e.g., [Mac], we can write the normalized Haar measure on $U(N)$ transferred into $\mathrm{Herm}_N^{\leq\pi}$, as

$$d^*X = \frac{J(Z)dZ}{V_N}, \quad V_N = \prod_{m=0}^{N-1} \frac{2\pi^{m+1}}{m!}. \tag{7.2.3}$$

Let $f(Z_1, ..., Z_n)$ be a "good" noncommutative function (for example an entire function or a rational function defined for all Hermitian $Z_1, ..., Z_n$ and having good decay at infinity). Then we can restrict f to $(\mathrm{Herm}_N^{\leq\pi})^n$ and transfer it, via the map (7.2.1), to a matrix function on $U(N)^n$. This matrix function is clearly nothing but

$$f(-i\log(X_1), ..., -i\log(X_n)), \tag{7.2.4}$$

where $-i\log : U(N) \to \mathrm{Herm}_N^{\leq\pi}$ is the branch of the logarithm defined using our choice of the fundamental domain. Although (7.2.4) is far from being a noncommutative Laurent polynomial (indeed, it is typically discontinuous), one can hope to use the procedure of Theorem 1.4.6 to expand it into a noncommutative *Fourier series.* In other words, assuming that for each $\gamma \in F_n$ the limit

$$a_\gamma = \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr} \int_{X_1,...,X_n \in U(N)} f(-i\log(X_1), ..., -i\log(X_n))X^{-\gamma} \prod_{j=1}^n d^*X_j = \tag{7.2.5}$$

$$= \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr} \int_{Z_1,...,Z_n \in \mathrm{Herm}_N^{\leq\pi}} f(Z_1, ..., Z_n)E_{\gamma^{-1}}(iZ_1, ..., iZ_n)) \prod_{j=1}^n \frac{J(Z_j)dZ_j}{V_N}$$

exists, we can form the series

$$\sum_\gamma a_\gamma X^\gamma \quad = \quad \sum_\gamma a_\gamma \; E_\gamma(iZ_1, ..., iZ_n), \quad Z_j \in \mathrm{Herm}_N^{\leq\pi}. \tag{7.2.6}$$

By analogy with the classical case one can expect that this series converges to $f|_{(\mathrm{Herm}_N^{\leq\pi})^n}$ away from the boundary.

(7.3) Scaling the period. In the situation of (7.2) let us choose $A > 0$ and restrict f to $(\mathrm{Herm}_N^{\leq A})^n$. The same procedure would then expand the restriction into a series in

$$X_j^{\pi/A} = \exp\left(i\frac{\pi}{A}Z_j\right), \quad j = 1, ..., n. \tag{7.3.1}$$

Let $F_n^{\pi/A} \subset G_n(\mathbb{R})$ be the group generated by the $X_j^{\pi/A}$. We can think of elements of $F_n^{\pi/A}$ as rectangular paths in $\mathbb{R}^n$ with increments being integer multiples of π/A. The coefficients of the series for the restriction given then a function

$$g_A : F_n^{\pi/A} \to \mathbb{C},$$

so the series will have the form

$$f(Z) = \sum_{\gamma \in F_n^{\pi/A}} g_A(\gamma) E_\gamma(iZ), \quad Z = (Z_1, ..., Z_n), \ Z_j \in \mathrm{Herm}_N^{\leq A}.$$

Now, as $A \to \infty$, we would like to say that the g_A, considered as linear combinations of Dirac measures on Π_n (or some completion) tend to a limit measure. Although Π_n is not a manifold, we can pass to finite dimensional approximations

$$\Pi_n \subset G_n(\mathbb{R}) \xrightarrow{p_d} G_{n,d}(\mathbb{R}).$$

Let $F_{n,d}^{\pi/A} = p_d(F_n^{\pi/A})$. This is a free nilpotent group of degree d on generators $p_d(X_j^{\pi/A})$, and is a discrete subgroup ("lattice") in $G_{n,d}(\mathbb{R})$. As $A \to \infty$, these lattices are getting dence in $G_{n,d}(\mathbb{R})$. Supposing that the direct image (summation over the fibers) $p_{d*}(g_A)$ exists as a function on $F_{n,d}^{\pi/A}$ or, what is the same, a measure on $G_{n,d}(\mathbb{R})$ supported on the discrete subgroup $F_{n,d}^{\pi/A}$, we can then ask for the existence of the limit

$$\mu_d = \lim_{A\to\infty} p_{d*}(g_A) \quad \in \quad \mathrm{Meas}(G_{n,d}(\mathbb{R})).$$

These measures, if they exist, would then form a pro-measure $\mu_\bullet$ which is the natural candidate for the inverse Fourier transform of f. The author hopes to address these issues in a future paper.

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