On maximizing sextics whose complements have non-abelian fundamental groups

Impossible configurations of I_n fibers on semi-stable elliptic surfaces

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Introduction

The purpose of this article is to study plane curves whose complement have non-abelian fundamental groups by using the theory of dihedral Galois coverings developed in [T1].

Let C be a reduced plane curve. The study of $\pi_1(\mathbf{P}^2 \setminus C)$ is originated from Zariski [Z] and van Kampen [Ka], and has attracted many mathematicians (see [Deg], [Del], [F], [M], [O1], [O2], [O3], [O4], etc). However, it still seems to be rather difficult to find plane curves whose complement have non-abelian fundamental groups. In particular, it is more difficult when it come to a problem to find such irreducible curves of a given degree. Hence it is worthwhile to study such curves. As Degtyarev gives a complete list in [Deg] in the case of $\deg C = 5$, in this article, we shall focus our attention on the following problem:

Question 0.1. *Find. C. with only simple singularities such that $\pi_1(\mathbf{P}^2 \setminus C)$ is non-abelian.*

In order to state our result for Question 0.1, we shall first define the index of C:

Definition 0.2. (**Persson**) *Let C be a curve with only simple singularities. We define the index of C, denoted by $i(C)$, to be the sum of all the subindices of all its simple singularities x_n ($x \in \{a, d, e\}$).*

By its definition, the index of a curve is non-negative. For a plane sextic curve C, it is known that $i(C) \leq 19$ (See [P]). Following to Persson, we shall define a maximizing sextic as follows:

Definition 0.3. *Let C be a plane sextic curve with only simple singularities. We call C a maximizing sextic if the index, $i(C)$, of C is equal to 19.*

Now we are in position to state our main result.

Theorem 0.4. *Let C be a maximizing sextic such that (i) C has at least one triple point, and (ii) C has three or more singularities each of which is of type either e_6 or a_{3k-1} ($k \geq 1$). Then*

$\pi_1(\mathbf{P}^2 \setminus C)$ *is non-abelian.*

To prove Theorem 0.4, we shall study a branched covering of $\mathbf{P}^2$ branched along C. In fact, we shall prove

Theorem 0.5. *Let C be a maximizing sextic as in Theorem 0.4. Then there exists a Galois covering $\pi : S \to \mathbf{P}^2$ branched along C having the third symmetric group as its Galois group.*

Since the Galois group, $\mathrm{Gal}(S/\mathbf{P}^2)$, of $\pi : S \to \mathbf{P}^2$ is a homomorphic image of $\pi_1(\mathbf{P}^2 \setminus C)$, Theorem 0.4 easily follows from Theorem 0.5.

This article consists of four sections. In the first section, we shall recall some results in [T1], and set up our strategy to prove Theorem 0.5. In §2, we shall consider the canonical resolution, $\mathcal{E}$, of the double covering $f : W \to \mathbf{P}^2$ branched along C. In our case, $\mathcal{E}$ is an elliptic K3 surface. Most of §2 are devoted to studying the structure of an elliptic fibration on $\mathcal{E}$. In §3, we shall prove Theorem 0.4. In §4, we shall give some examples of maximizing sextics satisfying the conditions in Theorem 0.4.

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Notations and conventions

Throughout this article, the ground field will always be the complex number field $\mathbf{C}$.

$\mathbf{C}(X) :=$ the rational function field of X.

Let X be a normal variety, and let Y be a smooth variety. Let $\pi : X \to Y$ be a finite morphism from X to Y. We define the branch locus of f, which we denote by $\Delta(X/Y)$, as follows:

$$\Delta(X/Y) = \{y \in Y \mid \sharp(\pi^{-1}(y)) < deg\pi\}.$$

For a divisor D on Y, $\pi^{-1}(D)$ denotes the set-theoretic inverse image of D, while $\pi^*(D)$ denotes the ordinary pullback. Also, $SuppD$ means the supporting set of D.

Let $\pi : X \to Y$ be an $\mathcal{S}_3$ covering of Y. Morphisms, β_1 and β_2, and the variety $D(X/Y)$ always mean those defined in §1.

Let W be a finite double covering of a smooth projective surface Σ. The "*canonical resolution*" of W always means the resolution given by Horikawa in [H].

Let S be an elliptic surface over B. We call S *minimal* if the fibration is relatively minimal. In this paper, we always assume that an elliptic surface is minimal and has a section s_0. For singular fibers of an elliptic surface, we use the notation of Kodaira [K]. Let F_v be a singular fiber over a point $v \in B$. If $\sum_i \mu_{v,i}\Theta_{v,i}$ denotes the irreducible decomposition of F_v, we always assume $\Theta_{v,0}s_0 = 1$. We call $\Theta_{v,0}$ the identity component of F_v. We denote by T a subgroup of the Néron-Severi group, $NS(S)$, generated by s_0, a fiber, and all the irreducible component of singular fibers not meeting s_0.

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.
$D_1 \approx D_2$: algebraic equivalence of divisors.
$D_1 \approx_{\mathbf{Q}} D_2$: Q-algebraic equivalence of divisors.
For singularities of a plane curve, we shall use the same notation as that in [P].

§1 Preliminaries

We shall start with the definition of an $\mathcal{S}_3$ covering.

Definition 1.1. *Let Y be a smooth projective variety. A normal variety, X, with a finite morphism $\pi : X \to Y$ is called an $\mathcal{S}_3$ covering of Y if the rational function field, $\mathbf{C}(X)$, of X is a Galois extension of $\mathbf{C}(Y)$ having the third symmetric group, $\mathcal{S}_3 = \langle \sigma, \tau | \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1\rangle$, as its Galois group.*

With the notations as above, let $\mathbf{C}(X)^\tau$ be the invariant subfield of $\mathbf{C}(X)$ by τ. As $\mathbf{C}(X)^\tau$ is a quadratic extension of $\mathbf{C}(Y)$, the $\mathbf{C}(X)^\tau$- normalization of Y is a double covering of Y. We denote it by $D(X/Y)$ and its covering morphism by β_1. X is a cyclic triple covering of $D(X/Y)$, and we denote its covering morphism by β_2. By the definition, $\pi = \beta_1 \circ \beta_2$. With these notations, we shall give the following proposition, which is fundamental in constructing an $\mathcal{S}_3$ covering.

Proposition 1.2. *Let Z be a smooth variety, and let $f : Z \to Y$ be a smooth finite double covering of a smooth projective variety Y. Let σ be the involution on Z determined by the covering transformation of f. Let D_1, D_2 and D_3 be effective divisors on Z. Suppose that*

(a) D_1 is reduced, and there is no common component between D_1 and $\sigma^* D_1$,

(b) $D_1 + 3D_2 \sim \sigma^* D_1 + 3D_3$.

Then there exists an $\mathcal{S}_3$ covering, X, of Y such that (i) $D(X/Y) = Z$, and (ii) $D_1 + \sigma^* D_1$ is the branch locus of β_2.

For a proof, see [T1].

Let C be a maximizing sextic in Theorem 0.4. Let $f : W \to \mathbf{P}^2$ be a double covering branched along C. Since W has rational double points, we can not apply Proposition 1.2 to this case. Instead, we shall consider the canonical resolution, $\mathcal{E}$, of the double covering $f : W \to \mathbf{P}^2$ which makes the following diagram commutative:

$$\begin{array}{ccc} W & \stackrel{\mu}{\leftarrow} & \mathcal{E} \\ f\downarrow & & \downarrow\tilde{f} \\ \mathbf{P}^2 & \stackrel{q}{\leftarrow} & \Sigma, \end{array}$$

where q is a succession of blowing-ups, and $\tilde{f}$ is a finite morphism of degree 2. See [H] §2 for detail for the canonical resolution.

As $\mathcal{E}$ is smooth, we can now apply Proposition 1.2 to the double covering $\tilde{f} : \mathcal{E} \to \Sigma$. Let $\bar{C}$ be the proper transform of C by q. Then, $\tilde{f}$ is branched along $\bar{C}$ and some irreducible components of the exceptional divisor of q. Therefore, by Proposition 1.2, in order to construct an $\mathcal{S}_3$ covering branched along C, by Proposition 1.2, it is enough to find three effective divisors D_1, D_2 and D_3 on $\mathcal{E}$ such that

(i) all irreducible components of D_1 are those of the exceptional divisor of μ, which are not contained in the ramification locus of $\tilde{f}$, and

(ii) these three divisors satisfy the two conditions in Proposition 1.2.

As it still seems to be intractable to find D_1, D_2 and D_3, we need one more step to reduce our problem to an easier one to deal with.

By our condition on C, it has at least one triple point. We choose one of them, and denote it by x. We call x the distinguished point. Then, by the construction of $\mathcal{E}$, it is easy to see that lines through x induce an elliptic fibration on $\mathcal{E}$. Following to Persson, we shall call this fibration "the standard fibration centered at x," and we denote it by $\varphi_x : \mathcal{E} \to \mathbf{P}^1$. φ_x has a section, s_0, which comes from an irreducible component of the exceptional divisor of the singularity $f^{-1}(x)$.

By our construction of $\mathcal{E}$, all irreducible components of the exceptional divisor of μ, except for s_0, are those of singular fibers, and s_0 is contained in the ramification locus of $\tilde{f}$. Hence every irreducible component of D_1 is that of a singular fiber not meeting s_0.

Summing up these observations, we have the following:

Proposition 1.3. *With the notations as above, suppose that there exist three effective divisors D_1, D_2 and D_3 on $\mathcal{E}$ such that*

(i) D_1 is reduced, and there is no common component between D_1 and $\sigma^* D_1$, where σ is the involution determined by $\tilde{f}$,

(ii) every irreducible component of D_1 is that of a singular fiber not meeting s_0,

(iii) every irreducible component of D_1 is that of the exceptional divisor of μ, and

(ii) $D_1 + 3D_2 \sim \sigma^* D_1 + 3D_3$.

Then there exists an $\mathcal{S}_3$ covering of $\mathbf{P}^2$ branched along C.

In the next section, we shall investigate the surface $\mathcal{E}$ in order to find the three divisors defined as above.

§2 Study of $\mathcal{E}$ and triple singularities of C

Let C be a maximizing sextic as in Theorem 0.4, and let $\tilde{f} : \mathcal{E} \to \Sigma$ be the double covering introduced in §1. As C has only simple singularities, $\mathcal{E}$ is a K3 surface. We shall choose a triple point, x, of C as the distinguished point. Let $\varphi_x : \mathcal{E} \to \mathbf{P}^1$ denote the standard fibration centered at x. Let s_0 be a section arising from x. Let $MW(\mathcal{E})$ be the Mordell-Weil group of sections of $\varphi_x : \mathcal{E} \to \mathbf{P}^1$ with s_0 being the *zero* element. Our first goal of this section is to prove

Proposition 2.1. *$MW(\mathcal{E})$ has a torsion of order 3.*

We need some preparations. For the argument as below, see §4 in [MP].

Let $H^2(\mathcal{E}, \mathbf{Z})$ be the integral second cohomology of $\mathcal{E}$. As $\mathcal{E}$ is a K3 surface, $H^2(\mathcal{E}, \mathbf{Z})$ is an even unimodular lattice. For a subgroup, J, of $H^2(\mathcal{E}, \mathbf{Z})$, $J^{\perp}$ denotes its orthogonal complement with respect to the pairing on $H^2(\mathcal{E}, \mathbf{Z})$. It is known that the Néron-Severi group is a primitive sublattice of $H^2(\mathcal{E}, \mathbf{Z})$.

Let T be a subgroup of $NS(\mathcal{E})$ generated by a fiber, s_0 and all irreducible component of singular fibers not meeting s_0. Since C is a maximizing sextic , by [P], p. 282, Corollary, $\varphi_x : \mathcal{E} \to \mathbf{P}^1$ is an extremal fibration, *i. e.* $\mathrm{rank} NS(\mathcal{E}) = \mathrm{rank} T = 20$. Hence we have

Proposition 2.2.([MP] Proposition 4.1, [S2] Theorem 1.2)

$$MW(\mathcal{E}) \cong T^{\perp\perp}/T$$

For a proof, see [MP] §4.

If J is an even sublattice of $H^2(\mathcal{E}, \mathbf{Z})$, we denote its dual lattice by $J^{\vee}$. By using the pairing on $H^2(\mathcal{E}, \mathbf{Z})$, J is canonically embedded in $J^{\vee}$. The group $J^{\vee}/J$ is called the discriminant-form group of J, and denoted by G_J. There is a bilinear form on $J^{\vee}$ induced by the bilinear form on J, and we also denote it by $(\, , \,)$. Thus, we can define a $\mathbf{Q}/\mathbf{Z}$-valued quadratic form q_J on G_J in the following way:

$$q_J(x \bmod J) = \frac{1}{2}(x, x) \bmod \mathbf{Z} \quad \text{for } x \in J^{\vee}.$$

Note that q_J defines non-degenerate bilinear form on J. Let J_1, J_2 be sublattices of an even unimodular lattice $\tilde{J}$ such that $J_1^{\perp} = J_2$ and $J_2^{\perp} = J_1$. Then we have $G_{J_1} \cong G_{J_2}$.

Proposition 2.3. **([MP], Proposition 4.2)** *Let T be the subgroup $NS(\mathcal{E})$ as before. Then we have the following:*

(i) There exists a subgroup H of G_T isomorphic to $MW(\mathcal{E})$.

(ii) $G_{T^{\perp\perp}} \cong G_{T^{\perp}}$; $H^{\perp} \cong (T^{\perp\perp})^{\vee}/T$, where $H^{\perp}$ denotes the orthogonal complement of H with respect to the pairing induced by q_{G_T}.

(iii) $G_{T^{\perp\perp}} \cong H^{\perp}/H$; $\sharp(G_{T^{\perp\perp}}) = \sharp(G_J)/(\sharp H)^2$.

For a proof, see [MP], Proposition 4.2.

Let $\psi : S \to B$ be an elliptic surface. Let $R = \{v \in \mathbf{P}^1 | \psi^{-1}(v) \text{ is reducible}\}$. Let T_v be a subgroup of T generated by all irreducible components of $\varphi_x^{-1}(v)$ not meeting s_0. Then we can rewrite T in such a way as $T = \mathbf{Z}s_0 \oplus \mathbf{Z}F \oplus \oplus_{v \in R} T_v$, where F is a general fiber. With this expression, we have:

Lemma 2.4.

$$G_T \cong \oplus_{v \in R} G_{T_v},$$

where

G_{T_v}	The type of $\varphi_r^{-1}(v)$
$\{0\}$	II, II^*
$\mathbf{Z}/2\mathbf{Z}$	III, III^*
$\mathbf{Z}/3\mathbf{Z}$	IV, IV^*
$\mathbf{Z}/n\mathbf{Z}$	I_n, $n \geq 2$
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	I_n^*, n is even
$\mathbf{Z}/4\mathbf{Z}$	I_n^*, n is odd

For a proof, see [M], p. 70.

Now we shall slightly modify Proposition 4.4 in [MP] for our purpose.

Proposition 2.5. *Let $\psi : S \to \mathbf{P}^1$ be an extremal elliptic K3 surface with a section s_0. Suppose that there exist $v_i \in \mathbf{P}^1$ $(i = 1, 2, 3)$ such that, for every i, $\psi^{-1}(v_i)$ is of type either IV, IV^* or I_{3k} $(k \geq 1)$. Then there exists a non-trivial torsion element of order 3.*

Proof. Suppose that there exists no torsion element of order 3. Then, by Proposition 2.2, there is no element of order 3 in $T^{\perp\perp}/T \cong H$, where H is a group in Proposition 2.3.

Claim 2.6. *Let $S_3((T^{\perp\perp})^\vee/T)$, $S_3(G_T)$ and $S_3(G_{T^{\perp\perp}})$ be the 3-Sylow subgroups of $T^{\perp\perp}/T$, G_T and $G_{T^{\perp\perp}}$, respectively. Then, we have*

$$S_3(T^{\perp\perp}/T) \cong S_3(G_T) \cong S_3(G_{T^{\perp\perp}}).$$

Proof of Claim 2.6. There exists a natural surjective homomorphism is $(T^{\perp\perp})^\vee/T \to G_{T^{\perp\perp}}$. As the kernel of this homomorphism is $T^{\perp\perp}/T$, we have $S_3(T^{\perp\perp}/T) \cong S_3(G_{T^{\perp\perp}})$.
Since $(T^{\perp\perp})^\vee/T \subset G_T$, we have

$$S_3(G_{T^{\perp\perp}}) \cong S_3((T^{\perp\perp})^\vee/T) \hookrightarrow S_3(G_T).$$

By Proposition 2.3 (iii) and $3 \nmid \sharp(T^{\perp\perp}/T) = \sharp(H)$. Therefore, we have

$$S_3(G_{T^{\perp\perp}}) \cong S_3(G_T).$$

Now we shall go back to prove Proposition 2.5.
Since $\psi : S \to \mathbf{P}^1$ is extremal, we have rank$T = 20$. As rank$H^2(S, \mathbf{Z}) = 22$, rank$T^\perp = 2$. This implies that $G_{T^\perp}$ is isomorphic to $\mathbf{Z}/n_1\mathbf{Z} \oplus \mathbf{Z}/n_2\mathbf{Z}$ as an abelian group. Hence the number of generators of $S_3(S_{T^\perp})$ is less than 2. On the other hand, by Claim 2.6, $S_3(G_{T^{\perp\perp}})$ is generated by three or more elements. But, by Proposition 2.3 (i), this is impossible.

By Proposition 2.5, in order to show Proposition 2.1, it is enough to show the following:

Proposition 2.7. *Let $\varphi_r : \mathcal{E} \to \mathbf{P}^1$ be as before. Then φ_r has three singular fibers $F_i = \varphi_r^{-1}(v_i)$ $(v_i \in \mathbf{P}^1, i = 1, 2, 3)$ each of which is of type either IV, IV^* or I_{3b} $(b \geq 1)$.*

Before we go on to prove Proposition 2.7, we shall prove the following lemma.

Lemma 2.8. *Let $\varphi_x : \mathcal{E} \to \mathbf{P}^1$ be as before. Let $l_{x,1}, \ldots, l_{x,s}$ be lines which meet C at x with multiplicities ≥ 4. Let $\bar{l}_{x,i}$ $(i = 1, \ldots, s)$ be the proper transforms of $l_{x,i}$ $(i = 1, \ldots, s)$ by $q : \Sigma \to \mathbf{P}^2$, respectively. Then $\tilde{f}^*\bar{l}_{x,i}$ is irreducible for every i.*

Proof. It is easy to see that all irreducible component of $\tilde{f}^*\bar{l}_{x,i}$'s are those of singular fibers not meeting s_0. On the other hand, by the construction of $\mathcal{E}$, $18 - s$ of the 19 irreducible components of the exceptional divisor of $\mu : \mathcal{E} \to W$ are also those of singular fibers not meeting s_0. As $\mathcal{E}$ is an elliptic K3 surface, the number of irreducible component of singular fibers not meeting s_0 is at most 18. Hence $\tilde{f}^*\bar{l}_{x,i}$ is irreducible for every i.

Proof of Proposition 2.7. By the construction of $\varphi_x : \mathcal{E} \to \mathbf{P}^1$, the singular fibers of φ_x are determined by the singularities of C and the position of lines connecting x and singularities of C (see [M] pp. 38-39).

Let x_1, x_2 and x_3 be the three singular points described in Theorem 0.4. There are two cases:

i) the triple point x is in $\{x_1, x_2, x_3\}$, or

ii) the triple point x is not in $\{x_1, x_2, x_3\}$.

In the case i), we can apply Proposition IV 2.2 in [M] to obtain the desired result.

In the case ii), we may assume that $x = x_1 = e_6$. The singular fibers corresponding to x_2 and x_3 are of type either IV, IV^* or I_{3b} $(b \geq 1)$. We shall now look into the singular fiber arising from x_1. Let l_{x_1} be the line meeting C at x_1 with multiplicity 4, and let $\bar{l}_{x_1}$ be the proper transform of l_{x_1} by $q : \Sigma \to \mathbf{P}^2$. Then we have

Claim 2.9. *l_{x_1} meets C at two distinct points other than x_1.*

Proof of Claim 2.9. If l_{x_1} meets C at one point other than x_1, by looking into the canonical resolution, we can easily see that $\tilde{f}^*\bar{l}_{x_1}$ consists of two irreducible components. This contradicts to Lemma 2.8.

By Claim 2.9, the following claim is straightforward.

Claim 2.10. *The singular fiber arising from x_1 is of type I_6.*

By Claim 2.10, the singular fiber arising from x_1 is of type I_6. Hence we have the desired result for the case ii) as above.

Now we know that $MW(\mathcal{E})$ has a torsion of order 3.

Proposition 2.11. *Every singular fiber of $\mathcal{E}$ is of type either IV, IV^* or I_n $(n \geq 1)$.*

This is immediate by [S1] Remark 1.10 or [M] Chapter VII, §3.

By Proposition 2.11, we can determine types of triple points on C.

Proposition 2.12. *Let C be a maximizing sextic as in Theorem 0.4. Then every triple point of C is either d_4, d_5 or e_6.*

Proof. Choose x, arbitrary triple point of C, as the distinguished point, and let $\varphi_x : \mathcal{E} \to \mathbf{P}^1$ be the standard fibration centered at x. Let $f^{-1}(x)$ be the rational double point on W lying over x. Then $f^{-1}(x)$ is of type either D_n $(n \geq 4)$ or E_n $(n = 6, 7, 8)$. Let E be the exceptional divisor arising from $f^{-1}(x)$. From the construction of $\mathcal{E}$, the section s_0 of $\varphi_x : \mathcal{E} \to \mathbf{P}^1$ is an irreducible component of E. For each type of $f^{-1}(x)$, the location of the irreducible component corresponding to s_0 is illustrated as follows (The vertex, $\otimes$, corresponds to the section):

(Figure 1)

All irreducible component of E other than the section component are those of singular fibers of $\varphi_x : \mathcal{E} \to \mathbf{P}^1$. Hence if x is of type either d_n $(n \neq 4, 5, 7)$, e_7 or e_8, then φ_x has a singular fiber of type either I_b^* or III^* by Figure 1. This contradicts to Proposition 2.11. We shall next look into the case of $x = d_7$. Let l_x be a line which meets the singular branch of d_7 with multiplicity either 4 or 5 (Note that the case of multiplicity 3 does not occur). Let $\bar{l}_x$ be the proper transform of l_x by $q : \Sigma \to \mathbf{P}^2$. By Lemma 2.8, $\tilde{f}^*\bar{l}_x$ is an irreducible component of a singular fiber, F, arising from d_7. The type of F depends on the intersection multiplicity between l_x and the singular branch of d_7 as follows:

Multiplicity	4	5
Type of F	I_1^*	IV^*

By Proposition 2.11, F is of type IV^*. In this case, however, by looking into the canonical resolution, we can show that $\tilde{f}^*\bar{l}_x$ consists of two irreducible components. This contradicts to Lemma 2.8.

In the rest of this section, we shall look into singular fibers arising from the distinguished triple point.

Proposition 2.13.

x	Singular fibers
d_4	I_2, I_2, I_2
d_5	I_2, I_4
e_6	I_6

Proof. In the case of $x = e_6$, this is nothing but Claim 2.10. We shall go on to the cases of $x = d_4$, d_5.

$x = d_4$. There exist three lines, $l_{x,i}$ ($i = 1, 2, 3$), which meet C at x with multiplicities ≥ 4. Let $\bar{l}_{x,i}$ be the proper transform of $l_{x,i}$ by $q : \Sigma \to \mathbf{P}^2$ for each i. Then, by Lemma 2.8, $\tilde{f}^*\bar{l}_{x,i}$ is irreducible for every i. Suppose that $\tilde{f}^*\bar{l}_{x,i}$ is an irreducible component of a singular fiber, F_i, for each i.

Claim 2.14. *For every i, F_i is a singular fiber of type I_2.*

Proof of Claim 2.14. For each i, $l_{x,i}$ is a tangent line to one of three irreducible branches of d_4 of order 2, 3 or 4. By performing the canonical resolution, we have the following singular fibers for each case.

The order of tangency	2	3	4
Singular fiber	I_2	II	IV

By Proposition 2.11, the middle case does not occur. For the right case, $\tilde{f}^*\bar{l}_{x,i}$ has two irreducible component. This contradicts to Lemma 2.8.

Our statement for the case $x = d_4$ follows from Claim 2.14. We shall go on to the remaining case.

$x = d_5$. There are two lines $l_{x,1}$ and $l_{x,2}$ which meet C at x with multiplicities ≥ 4; $l_{x,1}$ is a tangent line at x of the smooth branch of d_5 and $l_{x,2}$ is the cuspidal tangent line of the singular branch of d_5. Let F_i ($i = 1, 2$) denote singular fibers which contain $\tilde{f}^*\bar{l}_{x,i}$ ($i = 1, 2$), respectively. By the same argument as in the case $x = d_4$, F_1 is of type I_2. For F_2, by performing the canonical resolution, we can show that it is of type I_n ($n \geq 4$). In the case of $n \geq 5$, $\tilde{f}^*\bar{l}_{x,2}$ consists of two irreducible components. This contradicts to Lemma 2.8. Hence F_2 is of type I_4.

§3 Proof of Theorem 0.5.

The goal of this section is to find three effective divisors D_1, D_2 and D_3 on $\mathcal{E}$ satisfying the conditions in Proposition 1.3. In this section, we shall show that the existence of a 3-torsion in $MW(\mathcal{E})$ implies the existence of D_1, D_2 and D_3 on $\mathcal{E}$.

Let s denote a section corresponding to a 3-torsion in $MW(\mathcal{E})$. Then, by [S2], (8.2), we have

$$s \sim_{\mathbf{Q}} s_0 + 2F - \text{the contribution terms arising from singular fibers.} \quad (*)$$

By Proposition 2.11, every singular fiber of $\varphi_x : \mathcal{E} \to \mathbf{P}^1$ is either IV, IV^* or I_n ($n \geq 1$). For each case, the contribution term is as follows:

$$IV: \quad \tfrac{2}{3}\Theta_1 + \tfrac{1}{3}\sigma^*\Theta_1 \qquad (3.1)$$

$$IV^*: \quad \tfrac{4}{3}\Theta_1 + \tfrac{2}{3}\Theta_2 + 2\Theta_3 + \Theta_4 + \tfrac{4}{3}\sigma^*\Theta_2 + \tfrac{2}{3}\sigma^*\Theta_1 \qquad (3.2)$$

$$I_n: \quad \tfrac{(n-k)}{n}\Theta_1 + \tfrac{2(n-k)}{n}\Theta_2 + \cdots + \tfrac{k(n-k)}{n}\Theta_k + \tfrac{k(n-k-1)}{n}\Theta_{k+1} + \cdots \tfrac{k}{n}\Theta_{n-1}, \ \Theta_{n-i} = \sigma^*\Theta_i \ (1 \leq i \leq [\tfrac{n}{2}]) \qquad (3.3)$$

where σ is the covering transformation of $\tilde{f} : \mathcal{E} \to \Sigma$, and we label irreducible components as below. Also, we assume that s hits Θ_1 at IV and IV^*, and Θ_k at I_n.

(Figure 2)

We shall rewrite these explicit formulas for the contribution terms in the following way:
For a singular fiber of type IV,

$$\frac{1}{3}\sigma^*\Theta_1 - \frac{1}{3}\Theta_1 + \Theta_1. \qquad (3.4)$$

For a singular fiber of type IV^*,

$$\begin{aligned}\frac{1}{3}(\Theta_1 + \sigma^*\Theta_2) \quad &- \quad \frac{1}{3}\sigma^*(\Theta_1 + \sigma^*\Theta_2) \\ &+ \quad \Theta_1 + \sigma^*\Theta_1 + 2\Theta_2 + 2\Theta_3 + \Theta_4. \qquad (3.5)\end{aligned}$$

For a singular fiber of type I_n, we shall first look into at which component s hits. Let T be the subgroup of $NS(\mathcal{E})$ as before. By Proposition 2.2, $3s \in T$. Since the denominator of the coefficient of Θ_1 is $\frac{(n-k)}{n}$ by (3.3), $\frac{3k}{n} \in \mathbf{Z}$. This implies $3|n$, as $0 \leq k \leq n-1$. Put $n = 3l$. Then, as $\frac{k}{l} \in \mathbf{Z}$, we may assume that $k = b$. (If $k = 2b$, we shall label the irreducible components in another direction.) As $\sigma^*\Theta_k = \Theta_{n-k}$, $(1 \leq k \leq [\frac{n}{2}])$, we can now rewrite (3.3) in the following way:

For a singular fiber of type I_{3b} with b even,

$$\begin{aligned}&\textstyle\sum_{k\equiv 1 \pmod 3} && \left\{\tfrac{1}{3}\tilde{\sigma}^*\Theta_k - \tfrac{1}{3}\Theta_k + \left(2[\tfrac{k}{3}] + 1\right)\Theta_k + [\tfrac{k}{3}]\tilde{\sigma}^*\Theta_k\right\} \\ +&\textstyle\sum_{k\equiv 2 \pmod 3} && \left\{\tfrac{1}{3}\Theta_k - \tfrac{1}{3}\tilde{\sigma}^*\Theta_k + \left(2[\tfrac{k}{3}] + 1\right)\Theta_k + \left([\tfrac{k}{3}] + 1\right)\tilde{\sigma}^*\Theta_k\right\} \\ +&\textstyle\sum_{k\equiv 0 \pmod 3} && \left(\tfrac{2k}{3}\Theta_k + \tfrac{k}{3}\tilde{\sigma}^*\Theta_k) + \tfrac{b}{2}\Theta_{\frac{3}{2}b}\right). \quad (3.6.1)\end{aligned}$$

For a singular fiber of type I_{3b} with b odd,

$$\begin{aligned}&\textstyle\sum_{k\equiv 1 \pmod 3} && \left\{\tfrac{1}{3}\tilde{\sigma}^*\Theta_k - \tfrac{1}{3}\Theta_k + \left(2[\tfrac{k}{3}] + 1\right)\Theta_k + [\tfrac{k}{3}]\tilde{\sigma}^*\Theta_k\right\} \\ +&\textstyle\sum_{k\equiv 2 \pmod 3} && \left\{\tfrac{1}{3}\Theta_k - \tfrac{1}{3}\tilde{\sigma}^*\Theta_k + \left(2[\tfrac{k}{3}] + 1\right)\Theta_k + \left([\tfrac{k}{3}] + 1\right)\tilde{\sigma}^*\Theta_k\right\} \\ +&\textstyle\sum_{k\equiv 0 \pmod 3} && \left(\tfrac{2k}{3}\Theta_k + \tfrac{k}{3}\tilde{\sigma}^*\Theta_k\right). \quad (3.6.2)\end{aligned}$$

Now by these formulas, it is easy to see that we can rewrite $(*)$ in the form of

$$D_1 - \sigma^* D_1 \approx 3(D_3 - D_2)$$

such that

(i) D_1 is reduced; every irreducible component is that of singular fibers not meeting s_0,

(ii) D_1 and $\sigma^* D_1$ have no common component, and

(iii) both D_2 and D_3 are effective.

As $\mathcal{E}$ is simply connected, we can replace $\approx$ by $\sim$ in the above equivalence.

Now it only remains to show that every irreducible component of D_1 is that the exceptional divisor of μ.

Let Θ be an arbitrary irreducible component of D_1. Since Θ is an irreducible component of a singular fiber not being the identity component, $f \circ \mu(\Theta)$ is either a point or a line meeting C at

x with multiplicity ≥ 4. We shall show that the latter does not occur. Suppose that $f \circ \mu(\Theta)$ is such a line. If $x = e_6$ then by Lemma 2.8, Θ is an irreducible component, Θ_3, of a singular fiber of type I_6. By (3.6), Θ_3 does not appear in D_1. Hence, this case does not occur. If $x = d_4, d_5$, by Proposition 2.11, Θ is an irreducible component of a singular fiber of type either I_2 or I_4. As the contribution terms arise from only singular fibers of type IV, IV^* or I_{3b} ($b \geq 1$), this case also does not occur.

Thus, the three effective divisors D_1, D_2 and D_3 satisfy the conditions in Proposition 1.3.

§4 Examples

In this section, we shall give several examples of maximizing sextics which satisfy the conditions in Theorem 0.4. For this purpose, we shall use the same method as that in §2, [T2]. Namely, let $\psi : S \to \mathbf{P}^1$ be an elliptic K3 surface with a section s_0, and let σ denote the involution on S determined by the inversion morphism of the group law on S. Then the quotient surface $S/\langle\sigma\rangle$ is a smooth rational surface. $S/\langle\sigma\rangle$ is not minimal in general. We shall consider when $S/\langle\sigma\rangle$ is blown down to $\mathbf{P}^2$. Namely, we shall consider the "inverse" process of the canonical resolution (See [B] and [N] for detail).

Let $\psi : S \to \mathbf{P}^1$ be an extremal elliptic K3 surface. Then, by the proof of Proposition 2.11, we have the following:

Proposition 4.1. *Let $\psi : S \to \mathbf{P}^1$ be an extremal elliptic K3 surface with a section s_0. Suppose that $\psi : S \to \mathbf{P}^1$ satisfies one of the three conditions as follows:*

(i) There exists a singular fiber of type I_6.

(ii) There exist two singular fibers: one is of type I_2 and the other is of type I_4.

(iii) There exist three singular fibers of type I_2.

Then there exists a maximizing sextic, C, with a triple point x such that $\psi : S \to \mathbf{P}^1$ is the standard fibration centered at x.

By Proposition 4.1, we can give several examples. We shall summarize them as follows:

	Singularities of C	Singular fibers of $\mathcal{E}$
1	e_6, e_6, e_6, a_1	I_6, IV^*, IV^*, I_2
2	$e_6, a_5, a_2, a_2, a_2, a_2$	$I_6, I_6, I_3, I_3, I_3, I_3$
3	e_6, a_{11}, a_2	$I_6, I_{12}, I_3, I_1, I_1, I_1$
4	e_6, a_8, a_3, a_2	$I_6, I_9, I_4, I_3, I_1, I_1$
5	e_6, a_8, a_2, a_2, a_1	$I_6, I_9, I_3, I_3, I_2, I_1$
6	e_6, a_5, a_4, a_2, a_2	$I_6, I_6, I_5, I_3, I_3, I_1$
7	d_5, a_8, a_2, a_2, a_2	$I_4, I_2, I_9, I_3, I_3, I_3$
8	$e_6, a_5, a_3, a_2, a_2, a_1$	$I_6, I_6, I_4, I_3, I_3, I_2$
9	e_6, a_5, a_5, a_3	$I_6, I_6, I_6, I_4, I_1, I_1$
10	d_5, a_5, a_5, a_2, a_2	$I_2, I_4, I_6, I_6, I_3, I_3$
11	d_4, a_5, a_5, a_5	$I_2, I_2, I_2, I_6, I_6, I_6$
12	d_4, a_{11}, a_2, a_2	$I_2, I_2, I_2, I_{12}, I_3, I_3$

For the existence of elliptic K3 surfaces as above, see [P] for the first one and [MP] for the rest. We can easily show that C is irreducible for the first seven cases in the table.

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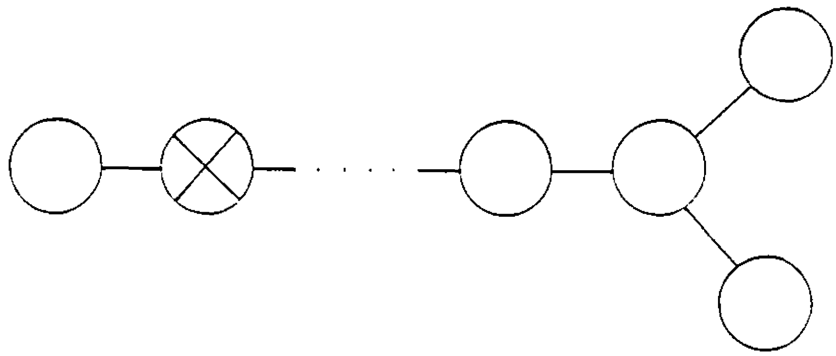
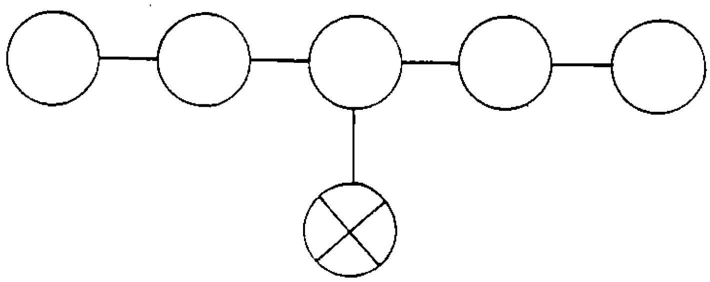
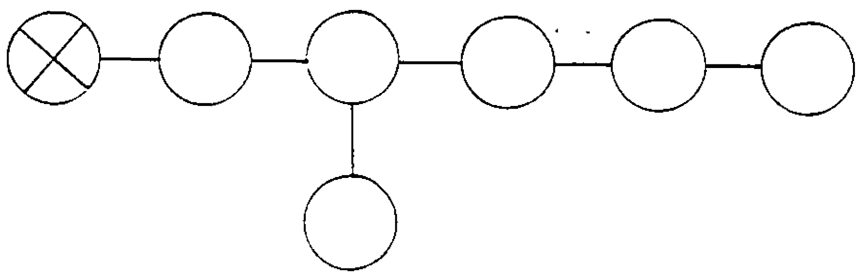
$x = d_n \ (n \geq 4)$



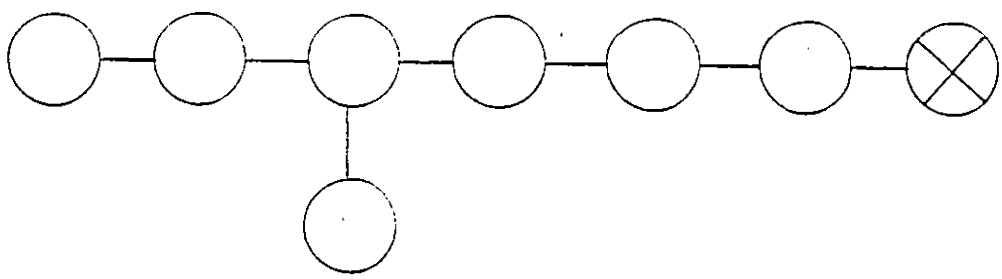
$x = e_6$



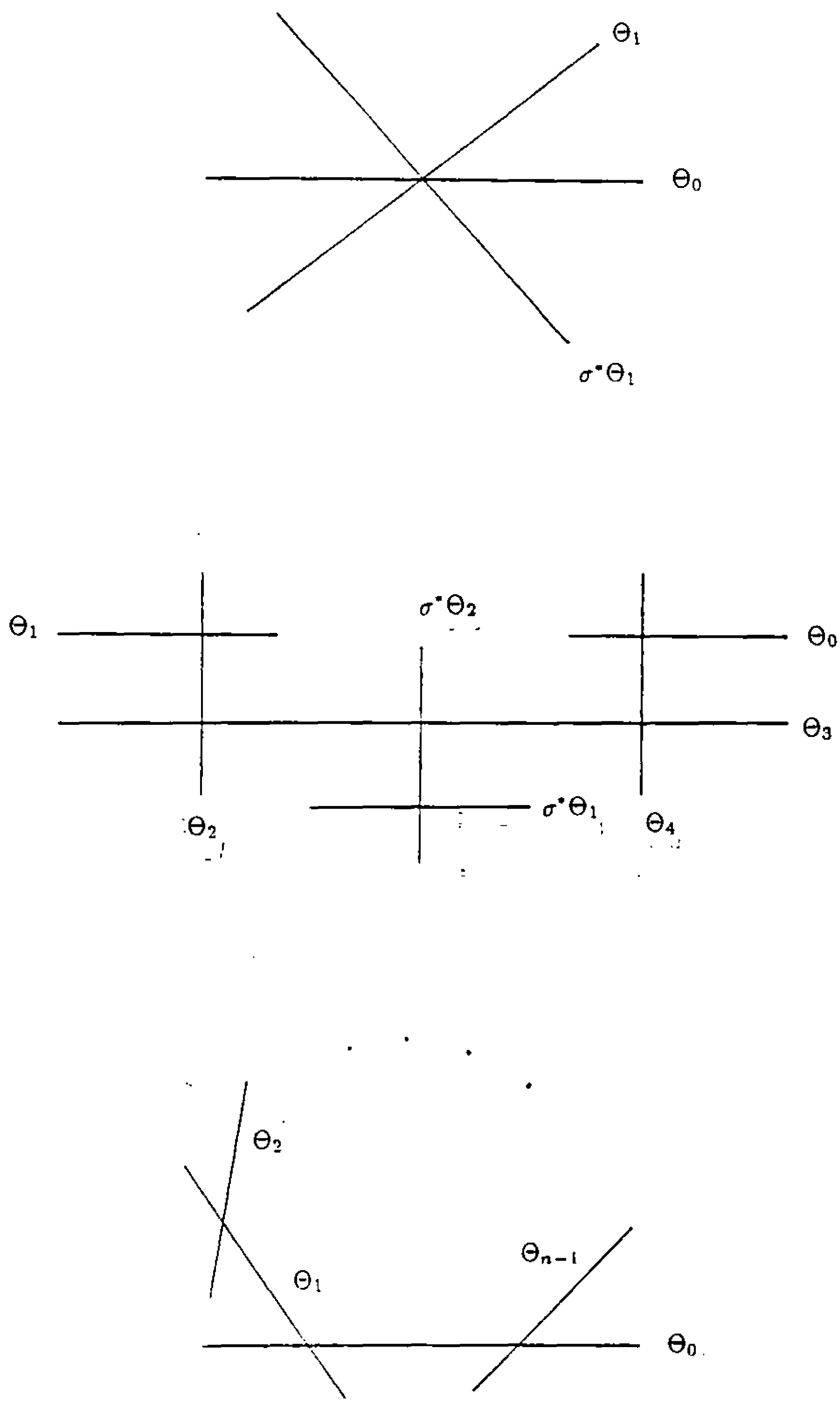
$x = e_7$



$x = e_8$



(Figure 1)


Θ_1
Θ_0
$\sigma^*\Theta_1$
Θ_1
$\sigma^*\Theta_2$
Θ_0
Θ_3
Θ_2
$\sigma^*\Theta_1$
Θ_4
Θ_2
Θ_1
Θ_{n-1}
Θ_0

(Figure 2)

Impossible configurations of I_n fibers on semi-stable elliptic surfaces

Hiro-o TOKUNAGA [1]

Abstract

Let $\varphi : S \to \mathbf{P}^1$ be a semi-stable elliptic surface with a section. In this note, we shall consider configurations of I_n fibers of φ, and give a criterion for impossible configurations.

Introduction

Let $\varphi : S \to C$ be a semi-stable elliptic surface over a curve C with a section s_0, *i.e.*, all singular fibers are of I_n type (see [K] for notations for singular fibers). In this note, we shall consider a question as follows:

Question. *Let $n_1, ..., n_r$ be given positive integers. Does there exist any semi-stable elliptic surface $\varphi : S \to C$ with singular fibers $I_{n_1}, ..., I_{n_r}$?*

It is known that $\sum_{i=1}^r n_i$ is divisible by 12 if such a semi-stable elliptic surface exists. In the cases that $\sum_{i=1}^r n_i = 12$ or 24 and $C = \mathbf{P}^1$, this question is solved completely by Miranda and Persson in [MP], [P].

Our result on this question is as follows:

Theorem 0.1. *Let $n_1, ..., n_r$ be positive integers with $24 | \sum_{i=1}^r n_i$. Suppose that there exists a prime, p, satisfying properties as follows:*

(i) p divides $r - 3$ or more of n_i's.

(ii) If we rearrange n_i' s in such a way that $p \not| n_i$ $(1 \leq i \leq t)$ and $p \mid n_i$ $(t+1 \leq i \leq r)$, then $2\sum_{i=1}^t n_i + \frac{3}{p^2}\sum_{i=t+1}^r n_i > \sum_{i=t+1}^r n_i$ (resp. $2\sum_{i=1}^t n_i > \sum_{i=t+1}^r n_i$) for $p \geq 3$ (resp. $p = 2$).

Then there exists no semi-stable elliptic surface $\varphi : S \to \mathbf{P}^1$ with singular fibers $I_{n_1}, ..., I_{n_r}$.

By applying Theorem 0.1 to the case that $\varphi : S \to C$ is an elliptic K3 surface, we obtain many impossible configurations I_n fibers on elliptic K3 surfaces. In fact, by Theorem 0.1, we can easily check that 87 of the 135 cases listed in Corollary 3.3, Propositions 3.4, 3.5 and 3.6 in [MP] are impossible.

Let $MW(S)$ denote the Mordell-Weil group of sections of $\varphi : S \to C$. To prove Theorem 0.1, we shall look into existence of p-torsions in $MW(S)$. We shall first prove

Proposition 0.2. *Let p be a fixed prime and let $\varphi : S \to C$ be a semi-stable elliptic surface with a section s_0. We shall put singular fibers, $I_{n_1}, ..., I_{n_r}$ $(n_i \geq 1)$, of φ in such a way that*

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$p \not| n_i$ $(i = 1,...,t)$ *and* $p|n_i$ $(i = t+1,...,r)$. *If* $2\sum_{i=1}^{t} n_i + \frac{3}{p^2}\sum_{i=t+1}^{r} n_i > \sum_{i=t+1}^{r} n_i$ *(resp.* $2\sum_{i=1}^{t} n_i > \sum_{i=t+1}^{r} n_i$*), then* $MW(S)$ *has no torsion of order* $p\,(p \geq 3)$ *(resp. order 2).*

Remark (i) In the case of $p = 3$, the inequality in the conditions of Proposition 0.2 is $3\sum_{i=1}^{t} n_i > \sum_{i=t+1}^{r} n_i$.

(ii) The inequality in the conditions of Proposition 0.2 is sharp for $p = 2, 3$. In fact, there exist rational elliptic surfaces, S_1 and S_2, as below (*cf.* [P]):

	Singular fibers	Torsion
S_1	I_4, I_3, I_2, I_2, I_1	$\mathbf{Z}/2\mathbf{Z}$
S_2	I_3, I_3, I_3, I_2, I_1	$\mathbf{Z}/3\mathbf{Z}$.

We shall next show that a criterion for existence of p-torsion, which is a generalization of the Length Criterion (Proposition 4.4, [MP]).

Proposition 0.3. *Let* p *be a prime. Let* $\varphi : S \to \mathbf{P}^1$ *be a semi-stable elliptic surface with a section* s_0 *having singular fibers* $I_{n_1},...,I_{n_r}$. *If* $24|\sum_{i=1}^{r} n_i$ *and* p *divides* $r - 3$ *or more of* n_i*'s, then there exists a non-trivial* p*-torsion element in* $MW(S)$.

By Propositions 0.2 and 0.3, we easily obtain Theorem 0.1.

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§1 Proof of Proposition 0.2

We shall start with a basic fact as below:

Lemma 1.1. *Let* $\varphi : S \to C$ *be a semi-stable elliptic surface as in Proposition 0.2. Let* $\chi(\mathcal{O}_S)$ *be the holomorphic Euler characteristic of* S. *Then,* $12\chi(\mathcal{O}_S) = \sum_{i=1}^{r} n_i$.

This is an easy corollary from Theorem 12.2 in [K].

Let $\varphi : S \to C$ be a semi-stable elliptic surface as in Proposition 0.2. Let T be a subgroup of the Néron-Severi group of S generated by s_0 and the irreducible components of fibers of $\varphi : S \to C$ by T. Also, we shall always assume that irreducible components, $\Theta_j^{(i)}$ $(0 \leq j \leq n_i - 1)$, of the I_{n_i} fiber are labeled in such a way that $\Theta_0^{(i)}\Theta_1^{(i)} = \Theta_1^{(i)}\Theta_2^{(i)} \cdots = \Theta_{n-1}^{(i)}\Theta_0^{(i)} = 1$, and that $\Theta_0^{(i)}$ meets the section s_0. Under these circumstances, we have the following:

Lemma 1.2. *Let* $\varphi : S \to C$ *be an elliptic surface as above. Suppose that there exists a torsion element of order* p *in* $MW(S)$, *and let* s *denote the corresponding section. If* s *meets* $\Theta_j^{(i)}$ *at the* I_{n_i} *singular fiber, then* $j = 0$ *for* $(1 \leq j \leq t)$, *and* $j \equiv 0 \bmod \frac{n_i}{p}$ *for* $(t+1 \leq j \leq r)$.

Proof. By the formula (8.2) in [S], we have

$$s \approx_{\mathbf{Q}} s_0 + (ss_0 + \chi(\mathcal{O}_S))F - \textit{the contribution terms form the singular fibers.}$$

where F denotes a class of a fiber of $\varphi : S \to C$ and $\approx_{\mathbf{Q}}$ denotes $\mathbf{Q}$-linear equivalence of divisors.

Since s is a p-torsion, by Theorem 1.3 in [S], $ps \in T$. Hence, in the above equivalence, the denominators of the coefficients of the irreducible components appearing in the contribution terms are either 1 or p. Suppose that s meets $\Theta_j^{(i)}$ at the I_{n_i} singular fiber. Then, by (8.16) in [S], the coefficient of $\Theta_1^{(i)}$ in the contribution terms is $\frac{n_i - j}{n_i}$. Since $ps \in T$, we have $p\left(\frac{n_i-j}{n_i}\right) = p - \frac{p}{n_i}j \in \mathbf{Z}$. This implies $j = 0$ for $i = 1, ..., t$ and $j \equiv 0 \bmod \frac{n_i}{p}$ for $i = t+1, ..., r$.

Now we shall prove Proposition 0.2 for p : odd prime. By Lemma 1.2, we may assume that s meets $\Theta_0^{(i)}$ at the I_{n_i} fibers $(i = 1, ...t)$ and $\Theta_{n_ik_i/p}^{(i)}$ at the I_{n_i} fibers $(i = t+1, ..., r,\ 0 \le k_i \le p$. Let $\langle\ ,\ \rangle$ denote Shioda's pairing defined in [S]. Then, by Theorem 8.6 in [S], we have

$$\begin{aligned}
\langle s, s\rangle &= 2\chi(\mathcal{O}_s) + 2s_0s - \sum_{i=t+1}^{r} \frac{1}{n_i}\frac{n_ik_i}{p}\left(n_i - \frac{n_ik_i}{p}\right) \\
&= 2\chi(\mathcal{O}_S) + 2s_0s - \frac{1}{p^2}\sum_{i=t+1}^{r} n_ik_i(p-k_i) \\
&\ge 2\chi(\mathcal{O}_\mathcal{E}) - \frac{p^2-1}{4p^2}\sum_{i=t+1}^{r} n_i.
\end{aligned}$$

Hence, by Lemma 1.1, we have

$$\langle s, s\rangle \ge \frac{1}{12}\left(2\sum_{i=1}^{t} n_i + \frac{3}{p^2}\sum_{i=t+1}^{r} n_i - \sum_{i=t+1}^{r} n_i\right).$$

Therefore, our assumption implies $\langle s, s\rangle > 0$. On the other hand, as s is a torsion element, we have $\langle s, s\rangle = 0$ by Theorem 8.4 in [S]. This is a contradiction. In the case of $p = 2$, in the same way as above, we have

$$\langle s, s\rangle \ge \frac{1}{6}\sum_{i=1}^{t} n_i - \frac{1}{12}\sum_{i=t+1}^{r} n_i.$$

With the same argument as in the cases of $p \ge 3$, we have the desired result.

§2 Proof of Proposition 0.3

We need settings to prove Proposition 0.3.

Lemma 2.1. Let $\varphi : S \to \mathbf{P}^1$ be an elliptic surface with a section s_0. If $\varphi : S \to \mathbf{P}^1$ has at least one singular fiber, then $H^1(S, \mathbf{Z}) = 0$

Proof. By our assumption, $b_1(S) = 0$ and $\chi(\mathcal{O}_S) > 0$. Hence, $H_1(S, \mathbf{Z})$ is a finite abelian group. Suppose that $H^1(S, \mathbf{Z})$ has a non-trivial element of order $m > 1$. Then, there exists an étale covering $\pi : \hat{S} \to S$. Since $s_0 \cong \mathbf{P}^1$, π^*s_0 has m irreducible components each of which is isomorphic to $\mathbf{P}^1$. Hence, $\hat{S}$ also has an elliptic fibration $\hat{\varphi} : \hat{S} \to \mathbf{P}^1$. Let F and F_1 denote fibers of φ and $\hat{\varphi}$, respectively. By the canonical bundle formula, we have $K_S \approx (\chi(\mathcal{O}_S) - 2)F$ and $K_{\hat{S}} \approx (\chi(\mathcal{O}_{\hat{S}}) - 2)F_1$. Since π is étale, $K_{\hat{S}} \approx \pi^* K_S$ and $\chi(\mathcal{O}_{\hat{S}}) = m\chi(\mathcal{O}_S)$. Hence, as $\pi^* F \approx dF_1$, where d is a divisor of m, we have $d(\chi(\mathcal{O}_S) - 2))F_1 \approx (m\chi(\mathcal{O}_S) - 2)F_1$. Thus, we have $((m-d)\chi(\mathcal{O}_S) + 2d - 2)F_1 \approx 0$. This holds if $m = d = 1$, or $\chi(\mathcal{O}_S) = 0$ and $d = 1$. Both of two cases, however, are impossible.

Lemma 2.2. *Let $\varphi : S \to \mathbf{P}^1$ be an elliptic surface as in Proposition 0.3. Then, $H^2(S, \mathbf{Z})$ is an even unimodular integral lattice.*

Proof. By Lemma 2.1, $H^2(S, \mathbf{Z})$ is torsion-free. Hence, by Poincaré duality, $H^2(S, \mathbf{Z})$ is a unimodular lattice. We shall prove that it is even. Let $w_2 \equiv -K_S \bmod 2$ and u_2 be the second Stiefel-Whitney class of S and the second Wu class of S, respectively. Then, by [HFK], p. 43, we have

$$\alpha^2 \equiv u_2\alpha \equiv -K_S\alpha \bmod 2$$

for arbitrary $\alpha \in H^2(S, \mathbf{Z})$. By our assumption and the canonical bundle formula, $K_S \equiv 0 \bmod 2$. Therefore, $H^2(S, \mathbf{Z})$ is an even unimodular integral lattice.

With Lemma 2.1 and Theorem 3.1 in [S], we just repeat the argument in §4 in [MP] by replacing the assumption $\sum_{i=1}^r n_i = 24$ by $\sum_{i=1}^r n_i = 12\chi(\mathcal{O}_S)$. Then we can easily check that Proposition 4.4 in [MP] is generalized to Proposition 0.3.

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