Yamabe Metrics and the Space of Conformal Structures

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where $x_0 = 2\log(\sqrt{\nu}+\sqrt{N})$ and

$$\beta = 2\,\frac{e^{-2x}+e^{-2\tau}-2e^{-x-\tau}\mathrm{ch}\,\xi}{(1+e^{-2x})(1+e^{-2\tau})-4e^{-x-\tau}\mathrm{ch}\,\xi} \tag{8.2}$$

with

$$e^{x} = \frac{\mathrm{ch}\,\tau+\sqrt{\mathrm{sh}^2\tau+4z(z-1)\mathrm{sh}^2\xi}}{\mathrm{ch}\,\xi+(2z-1)\mathrm{sh}\,\xi}\ . \tag{8.3}$$

Firstly we change the variable of the integration; we define $x = x(z;\xi,\tau)$ by the equation

$$\frac{\mathrm{ch}\,\tau-\mathrm{ch}(x-\xi)}{2\,\mathrm{sh}\,x\ \mathrm{sh}\,\xi} = z\ . \tag{8.4}$$

Then this function is given in the explicit form by the equality (8.3); besides

$$\frac{\partial z}{\partial x} = \frac{\mathrm{ch}x\,\mathrm{ch}\tau - \mathrm{ch}\xi}{2\,\mathrm{sh}^2x\ \mathrm{sh}\xi} \tag{8.5}$$

and we have together with (5.23) (for $\varphi(x) = \varphi_\epsilon(x-x_0)$)

$$\overset{=}{U}(\xi,\tau;\varphi_\epsilon) = \tfrac{1}{2}\,\varphi_\epsilon(\xi+\tau) + \tfrac{1}{2}\int_0^\infty W'(z)\,\frac{\mathrm{sh}x\ \mathrm{sh}\tau}{\mathrm{ch}x\,\mathrm{ch}\tau-\mathrm{ch}\xi}\,\varphi_\epsilon(x(z)-x_0)dz\ . \tag{8.6}$$

Now it follows from (8.3)

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INTRODUCTION AND MAIN THEOREMS. The aim of this article is to investigate the following two subjects relating to conformal geometry; (1) "moduli" of Yamabe metrics, metrics minimizing the Yamabe functional, whose existence has been extensively studied as Yamabe problem, (2) the space of conformal structures on a compact manifold modulo the diffeomorphism group action, especially its topology.

Two metrics are conformally equivalent when they are proportional by a positive scalar field and then they define a conformal equivalence class, called a conformal structure.

Riemannian metrics seem much richer as geometrical notion than conformal structures and one might get much more geometrical implication by dealing with Riemannian metrics than with conformal structures.

Nevertheless we can summarize significance of our study of the space of conformal structures as follows.

There exist notion and subjects which do not depend on Riemannian metric, but depend on conformal structure, namely, conformal invariants. For instance, the conformal flatness, the half conformal flatness in four dimension, and Weyl conformal curvature tensor, conformal transformations and the Hodge operator on k–forms, $k = \frac{1}{2} \dim M$ (see for further reference [3]).

The second reason, rather sophisticated than the obvious first one, is the following.

Consider on a Riemann surface conformal structure and complex structure. They are different objects, but are considered through isothermal coordinates as only distinguished forms representing the same geometrical structure. Thus the space of complex structures on a Riemann surface is interpreted as the space of conformal structures. Another example is the relation of half conformally flat structure via Penrose twistor diagram to investigate complex structure on the twistor space ([16] , for more general case see [4]). The diffeo–gauge quotient space of conformal structures on a manifold, that is, the quotient space modulo the diffeomorphism action will reflect in general dimension moduli of some geometrically interesting structures which might not come directly from Riemannian geometry.

The third reason is practical. A space of metrics satisfying some special geometrical property can be embedded in the diffeo–gauge quotient space of conformal structures so that one can study such a space of metrics as an embedded subspace.

The moduli of Einstein metrics on a compact manifold M , not diffeomorphic to S^n gives an example of such situation, because the traceless Ricci tensor Z , $\tilde{Z}$ of metrics g and $\tilde{g} = f^{-2}g$ are given by $\tilde{Z} = Z + (n-2)f^{-1}(\mathrm{Hes}(f) + \frac{\Delta f}{n} g)$ (p. 59, [6]) so that if the metrics are both Einstein, then $\mathrm{Hes}(f) + \frac{\Delta f}{n} g = 0$ and it follows that ([14]) f must be constant and hence the moduli of Einstein metrics is embedded in the space of conformal structures. In fact as is shown in [8] the moduli of Ricci flat metrics on a K3 surface or a 4–torus can be identified with the moduli of zero scalar curvature, half conformally flat conformal structures which is endowed with a finite dimensional manifold structure (probably with singularities) and carries a natural L^2–metric.

Let M be a compact, connected, oriented smooth n–manifold, $n \geq 3$ and g a smooth metric on M .

We call a conformal change $f^{\frac{4}{n-2}}g$ a <u>Yamabe metric</u> when it minimizes the Yamabe functional $Q_g(f) = (4\,\frac{n-1}{n-2}\int |df|^2 dVg + \int \rho f^2 dVg)/(\int f^N dVg)^{2/N}$

($N = \frac{2n}{n-2}$, dV_g is the volume form of g and ρ is the scalar curvature) within the conformal structure $[g]$.

A Yamabe metric, a relevant representative metric of constant scalar curvature for a conformal structure $[g]$, always exists, because the Yamabe problem was solved by Yamabe, Trudinger, Aubin, Schoen and others ([20], [18], [1], [17]).

The questions which arise next to the existence problem are the following: (1) to consider the uniqueness problem and structure problem for the "moduli" of Yamabe metrics when the uniqueness breaks down and (2) is arbitrary constant scalar curvature metric always a Yamabe metric?

As discussed in 1–ii) these questions relate with the Yamabe invariant, the infimum of the Yamabe functional and for the structure problem we would like to raise the question whether the moduli of Yamabe metrics is compact when the volume is normalized for each conformal structure except for the standard sphere. For the standard sphere the moduli is described in an exact way as a noncompact symmetric space (Proposition 1.4).

This compactness question corresponds to the famous conjecture for conformal group $C([g])$: $C([g])$ (or its identity component) is compact if and only if $[g]$ is not conformal to the standard sphere (S^n, g_0), i.e., there does not exist any diffeomorphism $\phi : M \longrightarrow S^n$, $\phi^* g_0 = fg$, $f > 0$.

This was solved by Obata ([15]) for the identity component case by using essential conformal Killing field and by Lelong–Ferrand ([13]) for $C([g])$ from the argument of quasiconformal mappings.

For question (1), particularly for the uniqueness we observe from Theorems 1.2, 1. (see also [1]) that the moduli is unique up to positive scale factor $g \longmapsto cg$, $c > 0$ for the nonpositive Yamabe invariant case (or equivalently nonpositive scalar curvature case) and for the Einstein metric case, except for the standard sphere.

For the question (2), as shown in the remark of Theorem 1.2, any metric of nonpositive constant scalar curvature is always Yamabe. However, the situation is different for positive scalar curvature case. We exhibit indeed examples of scalar curvature positive constant Riemannian manifold, for instance the product of standard spheres $S^p \times S^q$, which are even Riemannian homogeneous but not Yamabe.

To discuss the compactness of the moduli of Yamabe metrics requires entirely analysis of the nonlinear elliptic equation describing scalar curvature and the strict inequality on the Yamabe invariant μ, $\mu < \mu(S^n, [g_0])$.

The Sobolev space analysis together with the strict inequality on μ which was keystone for solving the Yamabe problem, especially the Sobolev compactly embedding theorem, enables us to show one of our main theorem.

Theorem A. Let $[g]$ be a conformal structure not conformal to the standard sphere. Then the moduli of Yamabe metrics for $[g]$ is compact, when the volume is normalized.

Here the topology of the moduli is of course the induced topology as a topological subspace of the ambient space, the space $\mathcal{R}_M$ of all smooth Riemannian metrics with C^∞ topology. Since for any $[g]$ the conformal group $C = C([g])$ acts effectively modulo isometries on the moduli of Yamabe metrics as explained in 1–ii), the above theorem immediately gives a proof to the conformal group conjecture, a new proof from the viewpoint of Yamabe metrics.

Corollary B. The conformal group of a conformal structure $[g]$ is compact if and only if $[g]$ is not conformal to the standard sphere.

Another application of our investigation of Yamabe metric is, as discussed in 2, on the topology of the diffeo–gauge quotient space of all conformal structures, the quotient of the space of all conformal structures divided by the diffeomorphism group $\mathcal{D}iff^+(M)$, or equivalently the quotient of the space of Riemannian metrics by the semi–direct product group $C^\infty_+(M) \ltimes \mathcal{D}iff^+(M)$.

The continuity of the Yamabe invariant and the convergence property of Yamabe metrics give us the Hausdorff property. Namely,

Theorem C. Let M be a compact, connected oriented n–manifold. Then the diffeo–gauge quotient space of conformal structures on M is Hausdorff except for the point represented by the standard n–sphere $[g_0]$.

That the space is not Hausdorff at $[g_0]$ stems from that the conformal group, and hence the moduli of the Yamabe metrics is not compact.

As a direct consequence of this theorem any closed subspace of the diffeo–gauge quotient space, for instance, the moduli of Einstein metrics except for the standard sphere Einstein metric, turns out to be Hausdorff.

We have in particular

Corollary D. The following spaces are Hausdorff,

(i) the moduli of conformal flat structures on M and

(ii) the moduli of half conformally flat conformal structures on a compact, oriented 4–manifold.

Noticė that the above Hausdorff theorem only covers the corollary for the case $[g] \neq$ standard sphere $[g_0]$. For $[g_0]$ itself, however, the topology is also Hausdorff at $[g_0]$, since the moduli becomes a single point from the well known theorem of conformal geometry due to Kuiper ([11]).

Remarks (i) The diffeo–gauge quotient space of conformal structures on M , $\mathscr{C}onf_M / \mathscr{D}iff^+_{(M)}$, admits an infinite dimensional real analytic variety in which each point has a neighborhood of the conformal group quotient of a linear subspace in $C^\infty(M, End_0(TM))$ ([9]). For 2–dimensional (Riemann surface) case this space is just the moduli of Riemann surfaces and then written as the discrete group quotient of a domain in $\mathbb{C}^{3g-3}$ (g is the genus).

(ii) For M of even dimension the diffeo–gauge quotient space $\mathscr{C}onf_M / \mathscr{D}iff^+_{(M)}$ can be endowed with an L^2–metric at an even singular point by using harmonic forms of degree $\frac{1}{2}$ dim M and the generalized volume form, a volume form which preserves the conformal invariance and the diffeo–naturality ([9]).

(iii) The moduli of conformally flat structures has from the developing map a natural map into the representation space $\mathscr{R}ep(\pi_1(M), SO(n,1))$.

(iv) The moduli of half conformally flat structures has a finite dimensional real analytic variety structure admitting the induced L^2–metric ([8]).

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1. Yamabe problem and Yamabe metric

1–i) Yamabe problem. Let (M,g) be a compact, connected, oriented smooth Riemannian n–manifold, $n \geq 3$.

Metrics fg , where f runs in $C^\infty_+(M)$ = {positive smooth functions on M }, conformally equivalent to g, determine a conformal structure $[g]$.

The Yamabe problem is then the following ([20]); does there exist within $[g]$ a scalar curvature constant metric?

The scalar curvatures $\rho, \tilde{\rho}$ of g and $\tilde{g} = f^{\frac{4}{n-2}} g$ satisfy the elliptic equation

$$(1.1) \qquad \tilde{\rho}\, f^{\frac{n+2}{n-2}} = 4\,\frac{n-1}{n-2}\,\Delta f + \rho f \ ,$$

where $\Delta = -\nabla^i \nabla_i$ is the g–Laplacian.

<u>Theorem</u> ([20], [18], [1], [17]). For any metric g on M there exist $f \in C_+^\infty(M)$ and constant $\tilde{\rho}$ satisfying (1.1). In other words the Yamabe problem is solvable for any (M,g).

We will briefly summarize how this theorem was solved (see [12] for the survey of the Yamabe problem).

Yamabe considered the functional

$$(1.2) \qquad Q_g(f) = \left[4\,\frac{n-1}{n-2} \int |df|^2 dV_g + \int \rho f^2 dV_g\right] \Big/ \left[\int f^N dV_g\right]^{2/N} \ ,$$

$f \in C_+^\infty(M)$, and observed that the Euler–Lagrange equation of (1.2) coincides with the equation (1.1).

In fact f is a critical point of (1.2) if and only if f satisfies (1.1) with $\tilde{\rho} = Q_g(f) \Big/ \left(\int f^N dV_g\right)^{1-2/N}$.

As is easily obtained (p. 288, [1]), for a conformal change $\tilde{g} = f^{\frac{4}{n-2}} g$, $Q_g(\varphi f) = Q_{\tilde{g}}(\varphi)$, $\varphi \in C_+^\infty(M)$. Then $Q_g(f) = Q_{\tilde{g}}(1) = \int \tilde{\rho}\, dV_{\tilde{g}} \big/ \left(\int dV_{\tilde{g}}\right)^{2/N}$, from which we derive the value of constant scalar curvature $\tilde{\rho}$ for the solution f of (1.1) as above.

Obviously $Q_{\psi^* g}(\psi^* f) = Q_g(f)$ for any orientation preserving diffeomorphism $\psi : M \longrightarrow M$ so that

$$\mu(M,[g]) = \mu([g]) = \inf\{Q_g(f); f \in C_+^\infty(M)\} \tag{1.3}$$

is conformal and diffeomorphism invariant. We call this Yamabe invariant.

Theorem ([20], [18], [1]). The Yamabe problem can be solved for any (M,g) of $\mu([g]) < \mu(S^n,[g_0])$ ((S^n,g_0) is the standard n–sphere).

The difficulty of solving the Yamabe problem is caused by that the value $N = \frac{2n}{n-2}$ is the border exponent for Sobolev embedding, that is, the continuous Sobolev embedding $L_1^2(M) \subset L^N(M)$ is not compact. So to verify this theorem by passing away by this difficulty one perturbs the Yamabe functional as $Q_g^s(f) = \left[4\frac{n-1}{n-2}\int |df|^2 + \int \rho f^2\right] \Big/ \left[\int f^s\right]^{\frac{2}{s}}$, $s < N$ and shows that Q_g^s has a positive smooth function f_s which minimizes Q_g^s and then that the sequence $\{f_s\}$ has a subsequence uniformly converging to a positive smooth function f as $s \longrightarrow N$, $\lim_{s\to N} Q_g^s(f_s) = Q_g(f)$.

Theorem. i) ([1]) If (M,g) has dimension ≥ 6 and is not locally conformally flat, then $\mu([g]) < \mu(S^n,[g_0])$.

ii) ([17]) If (M,g) has dimension 3, 4, 5 or if (M,g) is locally conformally flat, then $\mu([g]) < \mu(S^n, [g_0])$, provided (M,g) is not conformal to the standard sphere.

The Yamabe problem was solved by these theorems (see for reference [12], which is an excellent survey of the Yamabe problem and references cited in [12]).

1–ii) Yamabe metric.

Definition 1.1. For a smooth metric g a conformal change $\tilde{g} = f^{\frac{4}{n-2}} g$ is called Yamabe metric for g or conformal structure [g] if $\tilde{g}$ minimizes $Q_g(f)$, namely, $Q_g(f) = \mu([g])$.

We see easily that if $\tilde{g}$ is Yamabe, so is $c\tilde{g}$, $c > 0$ and remark that the scalar curvature $\tilde{\rho}$ of a Yamabe metric $\tilde{g}$ is

$$\tilde{\rho} = \mu([g]) \cdot \mathrm{Vol}(\tilde{g})^{\frac{2}{N}-1} \quad . \tag{1.4}$$

We consider first the uniqueness of Yamabe metric.

Theorem 1.2 ([1], [8]). Let [g] be a conformal structure of $\mu([g]) \leq 0$. Then, up to scale factor Yamabe metrics are the same.

Proof. Suppose $g_1, g_2 = f^{\frac{4}{n-2}} g_1$ are two Yamabe metrics of [g].

i) The case $\mu = \mu([g]) = 0$. The scalar curvatures ρ_1, ρ_2 are zero and then from (1.1) $\Delta f = 0$ and hence f is constant.

ii) The case $\mu < 0$. Suppose more generally that g_1, g_2 are negative constant scalar curvature metrics within $[g]$. By rescaling we may assume $Vol(g_1) = Vol(g_2)$ and $\rho_2 \leq \rho_1 < 0$. At a point $x_o \in 0$ where f is maximal $\Delta f = -\nabla^i\nabla_i f \geq 0$. On the other hand from (1.1)

$$4\,\frac{n-1}{n-2}\,\Delta f = (\rho_2-\rho_1)f^{N-1} + \rho_1(f^{N-2}-1)f \ .$$

So at x_0 $\rho_1(f^{N-2}-1)f \geq (\rho_1-\rho_2)f^{N-1} \geq 0$. Since $\rho_1-\rho_2 \geq 0$ and $\rho_1 < 0$ $(f^{N-2}-1)f \leq 0$ at x_o and hence $f(x_o) \leq 1$. Therefore $f = 1$ on M since $Vol(g_2) = \int f^N dV_{g_1} = \int dV_{g_1} = Vol(g_1)$.

We observe from the proof that any constant scalar curvature metric g is a Yamabe metric provided $\mu([g]) \leq 0$.

Another case for which Yamabe metric is unique up to constant scale is the Einstein metric case. Namely we have

<u>Theorem</u> 1.3. Let g be an Einstein metric on M , a compact connected, oriented n–manifold, not conformal to the standard sphere. Then any Yamabe metric for the metric g is constantly proportional to g .

<u>Proof</u>. Let $\tilde{g}$ be a Yamabe metric for g , $g = f^2\tilde{g}$, $f > 0$. Then the formula for the traceless Ricci tensors Z , $\tilde{Z}$ $(Z_{ij} = R_{ij} - \frac{\rho}{n} g_{ij})$ is

(1.5) $$\tilde{Z} + (n-2)f^{-1}(\widetilde{Hes}(f) + \frac{1}{n}\tilde{\Delta}f\tilde{g}) = Z = 0$$

(p. 59 [6] or (3.3), [12]).

Apply the same method in the proof of Proposition 3.1, [12] to integrate

$$\int_M f|\tilde{Z}|^2 dV_{\tilde{g}} = -(n-2)\int_M (\tilde{Z}, \widetilde{Hes}(f) + \frac{1}{n}\tilde{\Delta}f\tilde{g})dV_{\tilde{g}}$$

$$= -(n-2)\int (\tilde{Z}, \widetilde{Hes}(f))dV_{\tilde{g}} = (n-2)\int \tilde{\nabla}_k \tilde{Z}^{jk} f_j = 0 \ ,$$

where we used (1.5) and $\tilde{\nabla}_n \tilde{Z}^{jk} = 0$ since the scalar curvature of $\tilde{g}$ is constant. So $\tilde{Z} = 0$, or $\tilde{g}$ is Einstein and moreover the positive scalar field f satisfies the Hessian equations $\widetilde{Hes}(f) + \frac{1}{n}\tilde{\Delta}f \cdot \tilde{g} = 0$. But f must be constant since the equations have nontrivial solutions only when (M,g) is isometric to (S^n,g_0) (see [14]) . So $g = c^2\tilde{g}$, $c > 0$ constant, that is, g is a Yamabe metric and there are no Yamabe metrics other than g up to constant factor.

Relative to the uniqueness problem for Yamabe metric of $\mu > 0$ one can exhibit a conformal structure possessing plenty of Yamabe metrics. The standard n–sphere (S^n,g_0) , the exceptional case of Theorem 1.3, has indeed many Yamabe metrics. The Yamabe invariant is $\mu([g_0]) = n(n-1)\omega_n^{2/n}$ (ω_n is the volume of (S^n,g_0)) and Yamabe metrics for $[g_0]$ are characterized as

<u>Proposition</u> 1.4 (Th. 3.2, [12]). Yamabe metrics of $(S^n,[g_0])$ are exactly constant multiple of g_0 and pull back metrics of g_0 by conformal transformations.

So, from this proposition the "moduli" of Yamabe metrics, more precisely, the g_0–component of the moduli is parametrized as $\mathbb{R}^+ \times SO(n+1,1)/_{SO(n+1)}$, since the identity component of the group of orientation preserving conformal transformations of $(S^n, [g_0])$ is $SO(n+1,1)$ and the identity component of the isometry group is $SO(n+1)$. The right factor $SO(n+1,1)/_{SO(n+1)}$ is a noncompact symmetric space, H^{n+1}, a hyperbolic space.

Given a conformal structure $[g]$ we define the moduli of volume normalized Yamabe metrics;

$$\mathcal{YM}([g]) = \{\text{Yamabe metrics } \tilde{g} \text{ in } [g]\ ,\ \mathrm{Vol}(\tilde{g}) = \mathrm{Vol}(g)\ \}\ .$$

Pull back metric of $\tilde{g}$ under an orientation preserving diffeomorphism $\psi : M \longrightarrow M$ yields a Yamabe metric in $[\psi^* g]$, since the Yamabe functional is diffeomorphism invariant so that $\psi^* : \mathcal{YM}([g]) \longrightarrow \mathcal{YM}([g_1])$, $g_1 = \psi^* g$, gives rise to an isomorphsim.

When $\psi : M \longrightarrow M$ preserves a conformal structure $[g]$, i.e., $\psi^* g = fg$, $f \in C^\infty_+(M)$, ψ induces the action ψ^* on $\mathcal{YM}([g])$.

If we let $I(\tilde{g})$ be the group of orientation preserving $\tilde{g}$–isometries $\{\psi : M \longrightarrow M;\ \psi^*\tilde{g} = \tilde{g}\}$, then $I(\tilde{g})$ is the isotropy subgroup at $\tilde{g}$ in $\mathcal{YM}([g])$ for the action of the conformal group $C([g]) = \{\psi : M \longrightarrow M;\ \psi^*\tilde{g} = f\tilde{g}\ ,\ \psi$ is orientation preserving$\}$ so that $C([g])/_{I(\tilde{g})}$ is embedded in $\mathcal{YM}([g])$.

The following is easily obtained.

<u>Proposition</u> 1.5. If a conformal structure $[g]$ has the unique Yamabe metric $\tilde{g}$ up to constant scale factor, then $C([g]) = I(\tilde{g})$.

We will now prove the compactness theorem, Theorem A by showing the following convergence theorem

Theorem 1.6. Let g be a smooth metric with $\mu = \mu([g]) < \mu_0 = \mu(S^n, [g_0])$. Assume $\{g_i\}$ is a sequence of volume normalized Yamabe metrics of $[g]$. Then $\{g_i\}$ has a subsequence converging uniformly to a Yamabe metric of $[g]$.

Proof. From $\{g_i\}$ we have a sequence $\{f_i\}$ of positive smooth functions by $g_i = f_i^{N-2} g$. Each f_i is a solution of the equation

$$a\Delta f_i + \rho f_i = \mu f_i^{N-1}, \quad a = 4\,\frac{n-1}{n-2}, \tag{1.6}$$

and L^N–norm is $\|f_i\|_N = 1$ because of normalized volume.

It suffices to show that $\{f_i\}$ has a subsequence which uniformly converges to a positive smooth function satisfying (1.6).

We first claim that L^r–norm of f_i is uniformly bounded, $\|f_i\|_r \leq C$ for some $r > N$. We apply word by word the argument in the proof of Proposition 4.4, [12].

Choose $\delta > 0$. Multiply (1.6) by $f_i^{1+2\delta}$ and integrate over M. Then

$$\int_M \{a < df_i, (1+2\delta) f_i^{2\delta} df_i > + \rho f_i^{2+2\delta}\}\, dV_g$$

$$= \mu \int_M f_i^{N+2\delta} dV_g \ .$$

If we set $w_i = f_i^{1+\delta}$, then this can be written as

$$(1.7)\qquad \frac{1+2\delta}{(1+\delta)^2}\int_M a\,|dw_i|^2 = \int_M (\mu\, w_i^2\, f_i^{N-2} - \rho\, w_i^2)\ .$$

Apply the Sobolev inequality due to Aubin which holds universally for all compact manifolds (see Theorem 2.3 in [12]) and have that for any $\epsilon > 0$ there is $C_\epsilon > 0$, depending only on M, g and ϵ

$$\|w_i\|_N^2 \le (1+\epsilon)\frac{a}{\mu_0}\int |dw_i|^2 + C_\epsilon \int w_i^2$$

$$\le (1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\int \frac{\mu}{\mu_0} w_i^2 f_i^{N-2}$$

$$+ C'_\epsilon \|w_i\|_2^2$$

$$\le (1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\frac{\mu}{\mu_0}\|w_i\|_N^2\,\|f_i\|_N^{N-2}$$

$$+ C'_\epsilon \|w_i\|_2^2$$

$$= (1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\frac{\mu}{\mu_0}\|w_i\|_N^2 + C'_\epsilon\|w_i\|_2^2\ .$$

In the last inequality we used the Hölder inequality for dual indices $(\frac{N}{2}, \frac{N}{N-2})$. So,

$$(1.8)\qquad \left(1-(1+\epsilon)\frac{(1+\delta)^2}{1+2\delta}\frac{\mu}{\mu_0}\right)\|w_i\|_N^2 \le C'_\epsilon\|w_i\|_2^2\ ,$$

from which $\|w_i\|_N^2 \le C\|w_i\|_2^2$ for a constant $C > 0$, independent of i, because of the assumption $\frac{\mu}{\mu_0} < 1$.

Since $\|w_i\|_2^2 = \int w_i^2 = \int f_i^{2+2\delta} = \|f_i\|_{2+2\delta}^{2+2\delta}$, by applying again the Hölder inequality for dual indices $(N/_{2+2\delta}, R)$, $R = N/_{(N-2-2\delta)}$ for sufficiently small $\delta > 0$ we have the L^2–norm $\|w_i\|_2^2 \leq \|f_i\|_N^{2+2\delta} = 1$. Therefore $\|w_i\|_N = \left[\int f_i^{N(1+\delta)}\right]^{1/N} = \|f_i\|_{N(1+\delta)}^{1+\delta}$ is uniformly bounded from above.

Now $\{f_i\}$ is uniformly bounded in $L^r(M)$, $r = N(1+\delta)$, $\delta > 0$ so that from the regularity theorem, Theorem 4.1 in [12] they are bounded uniformly also in the Hölder space $C^{2,\alpha}(M)$ and then they have a subsequence converging in C^2–norm to a function $f \in C^2(M)$ which satisfies (1.6).

By applying the regularity theorem again the limit f is smooth, $f \in C^\infty(M)$, and it is strictly positive since $\|f\|_N = \lim\|f_i\|_N = 1$ (this strictly positivity is proved also from Lemma 6 in [1]).

By applying the argument in the proof of Proposition 2.2 in **2** without difficulty one can prove that the quotient space $C([g])/_{I(g)}$ is for any Yamabe metric g of $\mu < \mu_0$ closed in the moduli $\mathcal{YM}([g])$ so that $C([g])/_{I(g)}$ and hence $C([g])$ is compact. So Corollary B is verified.

With respect to the structure problem of the moduli of Yamabe metrics we would like to present the following

<u>Orbit Conjecture</u>. Every connected component of the moduli of volume normalized Yamabe metrics can be written as a quotient space of the conformal group $C([g])$.

Evidently the moduli $\mathcal{YM}$ of the standard sphere is from Proposition 1.4 a quotient space of $C([g_0])$, and both $\mathcal{YM}([g])$ and $C([g])$ are compact from

Theorem A and Corollary B when $[g]$ is not the standard sphere.

Moreover from Theorems 1.2, 1.3 the moduli consists only of a single point for any conformal structure either of $\mu \leq 0$ or represented by an Einstein metric.

Remark that a similar discussion appears in [2] on the moduli of Einstein Kähler metrics.

We are able to ask a condition relative to eigenvalue of the Laplacian for the moduli to admit a continuous parameter.

Proposition 1.7. Let g be a Yamabe metric of $\mu([g]) > 0$. Assume there exist Yamabe metrics g_t in $[g]$ with an effective parameter t, $|t| < \epsilon$, $g_0 = g$, $\mathrm{Vol}(g_t) = \mathrm{Vol}(g)$. Then $\rho/n-1$ is the eigenvalue of the Laplacian Δ_g.

Proof. For the metrics $g_t = f_t^{\frac{4}{n-2}} g$ the scalar curvature ρ_t is constant, given by $\rho_t = \mu([g]) \cdot \mathrm{Vol}(g_t)^{\frac{2}{N}-1}$ which is equal to ρ, the scalar curvature of g. Differentiate the equation at $t = 0$ with respect to t, $\rho f_t^{N-1} = a\,\Delta_g f_g + \rho f_t$, from which $\Delta_g \left[\frac{df}{dt}\right]_{t=0} = \frac{\rho}{n-1}\left[\frac{df}{dt}\right]_{t=0}$ for $\left[\frac{df}{dt}\right]_{t=0} \not\equiv 0$.

This proposition shows us that the "tangent" space of the moduli of Yamabe metrics is represented in the eigenspace of the Laplacian for the eigenvalue $\frac{\rho}{n-1}$.

The following is a necessary, but useful condition for a metric g to be Yamabe, whose proof appeared in Th. 11, [1].

Proposition 1.8 ([1]). The first eigenvalue λ_1 of the Yamabe metric Laplacian satisfies $\lambda_1 \geq \frac{\rho}{n-1}$.

So, if $\lambda_1 > \frac{\rho}{n-1}$, then from these propositions the moduli of Yamabe metrics is discrete near g.

Remarks (i) From Proposition 1.8 the upper bound of the Yamabe invariant is $\mu([g]) \leq (n-1)\lambda_1 \cdot \mathrm{Vol}(g)^{1-2/N}$ (for a lower bound of $\mu([g])$ refer to [10]).

(ii) Moreover the standard sphere is characterized in terms of Einstein Yamabe metric of $\mu([g]) > 0$ as follows. Let (M,g) be a Yamabe metric of $\mu([g]) > 0$ which is Einstein and satisfies $\lambda_1 = \rho/n-1$. Then (M,g) is, when rescaled, isometric to the standard sphere ([14], and also p. 180, [5]).

(iii) It is concluded by using Proposition 1.8 that the product of the standard metrics on the product of spheres $S^p \times S^q$, $p,q \geq 2$, $p > q+1$, is not Yamabe even it is a homogeneous metric of positive scalar curvature. In fact from G, [5] $\lambda_1(S^p \times S^q) = \min\{\lambda_1(S^p), \lambda_1(S^q)\} = q$. On the other hand the scalar curvature ρ of the product metric is $\rho = \sum_{i=1}^{p} R^{(1)}_{ii} + \sum_{j=1}^{q} R^{(2)}_{jj} = p(p-1)+q(q-1)$ ($R^{(1)}_{ii}$ and $R^{(2)}_{jj}$ are the Ricci curvature tensors of S^p, S^q, respectively) and hence $\frac{\rho}{p+q-1} - q = p(p-q-1)/(p+q-1) > 0$ so that the product metric is not a Yamabe metric.

The product metric g on $S^p \times S^p$ is an Einstein metric and hence from Theorem 1.3 a Yamabe metric unique for the conformal structure $[g]$.

1–iii) The continuity of Yamabe metrics.

Finally we state the continuity property of Yamabe metric which will be used for the topology of the diffeo–gauge quotient space of conformal structures.

Theorem 1.9 Let $\{\gamma_i\}$ be a sequence of smooth conformal structures and $\{g_i\}$ a sequence of volume normalized Yamabe metrics g_i representing γ_i. Assume $\{g_i\}$ converges to a smooth metric g in C^2–norm. Then g is a Yamabe metric for a conformal structure γ, the limit of γ_i, of Yamabe invariant $\mu = \lim_i \mu(\gamma_i)$.

We need for the proof the continuity of the Yamabe invariant ([10]): Let $\{h_i\}$ be a sequence of smooth metrics of scalar curvature ρ_i. If h_i converges to h in C^0–norm and ρ_i to ρ, the scalar curvature of h in C^0–norm, then $\lim \mu([h_i]) = \mu([h])$. We omit the proof which is obviously shown.

For each Yamabe metric g_i $Q_{g_i}(1) = \int \rho_i dV_{g_i} / (\int dV_{g_i})^{2/N}$ which goes to $Q_g(1) = \int \rho dV_g / (\int dV_g)^{2/N}$ as $i \longrightarrow \infty$. On the other hand, since the scalar curvature of g_i converges to the scalar curvature of g, from the continuity of Yamabe invariant $Q_{g_i}(1) = \mu(\gamma_i) \longrightarrow \mu([g])$. So $\mu([g]) = Q_g(1)$ which means that g is a Yamabe metric for $[g]$ of Yamabe invariant $\lim_i \mu(\gamma_i)$.

The following is a convergence theorem of Yamabe metrics, a slightly general than Theorem 1.6.

Theorem 1.10. Let $\{h_i\}$ be a sequence of smooth metrics of Yamabe invariant $\mu_i < \mu_0 = \mu(S^n, [g_0])$. Assume $\{h_i\}$ has a subsequence converging to a smooth metric h of $\mu < \mu_0$ in C^2–norm. Then any sequence $\{g_i\}$ consisting of volume normalized Yamabe metrics, $g_i = f_i^{N-2} h_i$, for $[h_i]$ has a subsequence which converges to a Yamabe metric $g = f^{N-2}h$ for $[h]$.

To prove this theorem we slightly modify the compactness argument in the proof of Theorem 1.6 by dealing the equation $(a\Delta_i + \rho_i)f_i = \mu_i f_i^{N-1}$ simultaneously. Since metrics h_i are uniformly convergent in C^2–sense, there exists an $\epsilon > 0$ from Theorem 1.9 such that $\mu_i < \mu + \epsilon < \mu_o$ for sufficiently large i . Moreover, since the constant C_ϵ in the inequality (Th. 2.3, [12]) depends on a metric in C^0–sense. C'_ϵ in (1.8) is uniformly bounded from above. So, together with the uniform boundedness of L^N–norm of f_i every part of the argument of the proof of Theorem 1.6 works and the theorem is verified.

2. The topology of diffeo–gauge quotient space.

2–i) For a compact, connected, oriented smooth n–manifold M , $n \geq 3$, we denote by $\mathcal{D}iff^+(M)$ the group of orientation preserving diffeomorphisms $\psi : M \longrightarrow M$ and by $\mathcal{D}iff^0(M)$ $\{\psi \in \mathcal{D}iff^+(M); \psi$ is isotopic to $id_M\}$.

Herr $\psi : M \longrightarrow M$ is isotopic to the identity transformation id_M when there is a path $\{\psi_t\}$ in $\mathcal{D}iff^+(M)$, $0 \leq t \leq 1$, $\psi_0 = \psi$ and $\psi_1 = id_M$.

$\mathcal{D}iff^0(M)$ is as a group finitely generated by diffeomorphisms which are generated as transformations of M by smooth vector fields. $\mathcal{D}iff^0(M)$ is the connected component group of $\mathcal{D}iff^+(M)$ and the quotient group $\mathcal{D}iff^+(M)/_{\mathcal{D}iff^0(M)}$ is the group of isotopic equivalence classes, called the mapping class group and also called the Teichmüller modular group for Riemann surface case.

For a conformal structure $[g]$ every $\psi \in \mathcal{D}iff^+(M)$ defines by pull back another conformal structure $[\psi^* g]$ which we write as $\psi^*[g]$.

The conformal groups $C([g])$ and $C^0([g])$ for $[g]$ are defined in

$\mathcal{D}iff^+(M)$, $\mathcal{D}iff^0(M)$ as subgroups consisting of ϕ fixing $[g]$.

Since $\mathcal{D}iff^+(M)$ acts on the space $\mathcal{C}onf_M$ of all smooth conformal structures $[g]$ on M , we obtain as the quotient the space $\mathcal{C}onf_M / \mathcal{D}iff^+(M)$ of diffeomorphism equivalence classes $\overline{\gamma}$, represented by $\gamma = [g]$. We call this quotient space the diffeo–gauge quotient space of conformal structures, since diffeomorphisms behave like gauge transformations in the Yang–Mills gauge theory.

$\mathcal{C}onf_M / \mathcal{D}iff^+(M)$ is a fibred space over another diffeo–gauge quotient space $\mathcal{C}onf_M / \mathcal{D}iff^0(M)$ with fibre $\mathcal{D}iff^+_M / \mathcal{D}iff^0(M)$.

Similarly we define the diffeo–gauge quotient space of smooth Riemannian metrics on M , $\mathcal{R}_M / \mathcal{D}iff^+(M)$ and $\mathcal{R}_M / \mathcal{D}iff^0(M)$ with projections

$\pi : \mathcal{R}_M / \mathcal{D}iff^+(M) \longrightarrow \mathcal{C}onf_M / \mathcal{D}iff^+(M)$ and

$\pi : \mathcal{R}_M / \mathcal{D}iff^0(M) \longrightarrow \mathcal{C}onf_M / \mathcal{D}iff^0(M)$.

Notice that the projections have a "canonical section" over subspaces where the moduli of Yamabe metrics consists of a single point.

2–ii) The Hausdorff property

The space $\mathcal{C}onf_M / \mathcal{D}iff^+(M)$ has the quotient topology induced from the projection π . Since the following diagram commutes

$$\begin{array}{ccc} \mathcal{R}_M & \longrightarrow & \mathcal{C}onf_M \\ \downarrow & & \downarrow \\ \mathcal{R}_M / \mathcal{D}iff^+(M) & \xrightarrow{\ \pi\ } & \mathcal{C}onf_M / \mathcal{D}iff^+(M) \end{array} ,$$

the topology of $\mathcal{C}onf_M / \mathcal{D}iff^+(M)$ comes originally from the naturally defined topology of $\mathcal{R}_M$.

<u>Theorem</u> 2.1 (Theorem C in the introduction). The subspace $\{\overline{\gamma};\ \mu(\overline{\gamma}) < \mu(S^n, [g_0])\}$ of the space $\mathcal{C}onf_M/\mathcal{D}iff^+(M)$ (or of $\mathcal{C}onf_M/\mathcal{D}iff^0(M)$) has the Hausdorff property.

For the proof we need the Sobolev space completion of spaces of metrics and of diffeomorphisms.

Consider the product space $F = M \times M$. We follow the argument of Ebin ([7]). Sections of $F \longrightarrow M$ are considered as maps of M into M so that $C^1(M,M) = \{C^1\text{–maps: } M \longrightarrow M\}$ is identified with $C^1(F) = \{C^1\text{–sections of } F \longrightarrow M\}$ where the topology of C^1–maps (sections) are the uniform convergence topology up to the first derivative.

Define C^1–diffeomorphisms as $C^1\mathcal{D}^+ = \{\psi \in C^1(M,M);\ \psi^{-1} \in C^1(M,M)$, ψ is orientation preserving$\}$. $C^1\mathcal{D}^+$ is open in $C^1(M,M)$.

Pick $s > n+2 (\geq \frac{n}{2} + 1)$ and define $\mathcal{D}iff_s^+ = \mathcal{D}iff_s^+(M) = C^1\mathcal{D}^+ \cap H^s(F)$ where $H^s(F) = L_s^2(F)$ is the space of L_s^2–sections of F, {sections ψ of F; $\|\psi\|_{2,s} < \infty$ }. $\mathcal{D}iff_s^+$ is a topological group under the mapping composition.

Notice that from a Sobolev lemma there is $c > 0$ for L_k^2–norm such taht $|D^\alpha f(x)| \leq c\|f\|_{2,k+|\alpha|}$ for $k > n/2$ (so $H^s(F) \subset C^1(F)$) and $\|f_1 f_2\|_{2,k} \leq c\|f_1\|_{2,k}\|f_2\|_{2,k}$ for $k \geq n+2$.

The action $H^{s+1}(F) \times H^s(S^2(T^*M)) \longrightarrow H^s(S^2(T^*M))$; $(\psi,h) \longmapsto \psi^* h$ is continuous.

The following proposition is not directly needed, but is useful for the proof of Theorem 2.1.

<u>Proposition</u> 2.2. $\mathcal{R}_{M,s}/\mathcal{D}iff_{s+1}^+$ is Hausdorff. Here $\mathcal{R}_{M,s}$ is the space of metrics on

M of finite L_s^2–norm.

It suffices to show from the following Hausdorff criterion that the map $u : \mathcal{D}iff^+_{s+1} \times \mathcal{R}_{M,s} \longrightarrow \mathcal{R}_{M,s} \times \mathcal{R}_{M,s}$; $(\psi,g) \longmapsto (g,\psi^* g)$ has the closed image.

Lemma ([19]). Let G be a topological group acting continuously on a topological space X and Z the set of all orbits: $Z = X/G$. The quotient topology on Z is Hausdorff if and only if the image of the map: $G \times X \longrightarrow X \times X$; $(a,x) \longmapsto (x,a\cdot x)$ is closed.

Proof of Proposition 2.2 Let $\{g_i\}$, $\{g_i'\}$ be sequences of metrics in $\mathcal{R}_{M,s}$.

Suppose $\psi_i^* g_i = g_i'$ for an $\psi_i \in \mathcal{D}iff^+_{s+1}$ and the sequences have limit g , g′ in $\mathcal{R}_{M,s}$. It is not hard to show that ψ_i is uniformly bounded in $H^{s+1}(F)$ since the second derivatives $\frac{\partial^2 \psi_i}{\partial x^a \partial x^b}$ of ψ_i are represented by the Christoffel symbols of g_i , g_i' and the first derivatives $\frac{\partial x_i}{\partial x^a}$. So $\{\psi_i\}$ has a subsequence converging to ψ in $H^{s+1}(F)$ and hence from the Sobolev lemma $\psi \in C^1(M,M)$.

Since $H^{s+1}(F) \times H^s(S^2(T^*M)) \longrightarrow H^s(S^2(T^*M))$ is continuous, $g' = \lim g_i' = (\lim \psi_i)^*(\lim g_i) = \psi^* g$. So ψ has maximal rank at everypoint and the sign of the determinant $\det \psi$ is positive since $\det g'(\psi(x)) = (\det \psi)^2(x) \cdot \det g(x)$ and ψ is a limit of orientation preserving diffeomorphisms.

Since the degree of ψ is one and the mapping degree theorem applies, ψ is bijective and $\psi^{-1} \in C^1\mathcal{D}^+$ and hence $\psi \in C^1\mathcal{D}^+ \cap H^{s+1}(F) = \mathcal{D}iff^+_{s+1}$. So Im(u) is closed.

Remark. The group $\mathscr{D}iff^{o}_{s+1}$ is the connected component group of $\mathscr{D}iff^{+}_{s+1}$. So the space $\mathscr{R}_{M,s}/\mathscr{D}iff^{o}_{s+1}$ is also Hausdorff.

To verify theorem 2.1 we suppose that $\{\gamma_i\}$ and $\{\gamma_i'\}$ are sequences in $\mathscr{C}onf^{Y}_{M,s} = \{\gamma \in \mathscr{C}onf_{M,s};\ \mu(\gamma) < \mu_0\}$ satisfying $\gamma_i' = \overset{*}{\psi}_i \gamma_i$, $\psi_i \in \mathscr{D}iff^{+}_{s+1}$ and having limits γ, γ' in $\mathscr{C}onf^{Y}_{M,s}$.

For each i γ_i and γ_i' have representatives in $\mathscr{R}_{M,s}$ the Yamabe metrics of unit volume g_i and g_i'.

Since $\overset{*}{\psi}_i g_i$ is Yamabe in γ_i', we have $\overset{*}{\psi}_i g_i = f_i^{N-2} g_i'$ for a positive L^2_s–function f_i.

We now show that sequences $\{g_i\}$, $\{g_i'\}$ have subsequences converging to Yamabe metrics g and g' which project onto γ and γ' and whose Yamabe invariant is less than μ_0.

In fact the sequence $\{\gamma_i\}$ admits a sequence $\{h_i\}$ in $\mathscr{R}_{M,s}$ converging to h in L^2_s–norm and projecting onto $\{\gamma_i\}$ so that from Theorem 1.10 the sequence $\{g_i\}$ of Yamabe metrics has a subsequence which converges to a Yamabe metric g. The Yamabe invariant $\mu < \mu_0$ from Theorem 1.9. $\{g_i'\}$ has also a subsequence converging to a Yamabe metric g' of $\mu < \mu_0$.

The intermediate sequence $\{f_i^{N-2} g_i' = \overset{*}{\psi}_i g_i\}$ has also a subsequence which converges to $f^{N-2} g'$, same as the previous case, so that $\{\psi_i\}$ has a converging subsequence and hence the image of the map

$\mathscr{D}iff^{+}_{s+1} \times \mathscr{C}onf_{M,s} \longrightarrow \mathscr{C}onf_{M,s} \times \mathscr{C}onf_{M,s}$ is closed.

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