On q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function II

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Abstract. A representation of a specialization of a q-deformed class one lattice $\mathfrak{gl}_{\ell+1}$-Whittaker function in terms of cohomology groups of line bundles on the space $\mathcal{QM}_d(\mathbb{P}^\ell)$ of quasi-maps $\mathbb{P}^1 \to \mathbb{P}^\ell$ of degree d is proposed. For $\ell = 1$, this provides an interpretation of non-specialized q-deformed $\mathfrak{gl}_2$-Whittaker function in terms of $\mathcal{QM}_d(\mathbb{P}^1)$. In particular the ($q$-version of) Mellin-Barnes representation of $\mathfrak{gl}_2$-Whittaker function is realized as a semi-infinite period map. The explicit form of the period map manifests an important role of q-version of Γ-function as a substitute of topological genus in semi-infinite geometry. A relation with Givental-Lee universal solution (J-function) of q-deformed $\mathfrak{gl}_2$-Toda chain is also discussed.

Introduction

In the first part [GLO1] of this series of papers we have proposed an explicit representation of a q-deformed class one lattice $\mathfrak{gl}_{\ell+1}$-Whittaker function defined as a common eigenfunction of a complete set of commuting quantum Hamiltonians of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. Here "class one" means that Whittaker function is non-zero only in the dominant domain. On q-deformed Toda chains see e.g. [Et]. The case $\ell = 1$ was discussed previously in [GLO3] (for related results in this direction see [KLS], [GiL], [GKL1], [BF], [FFJMM]). A special feature of the proposed representation is that q-deformed class one $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(\underline{p})$ with $\underline{z} = (z_1, \dots, z_{\ell+1})$ and $\underline{p} = (p_1, \dots, p_{\ell+1}) \in \mathbb{Z}^{\ell+1}$, is given by a character of a $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module $\mathcal{V}_{\underline{p}}$. The expression in terms of a character can be considered as a q-version of Shintani-Casselman-Shalika representation of class one p-adic Whittaker functions [Sh], [CS]. Indeed our representation of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function reduces, in a certain limit, to the Shintani-Casselman-Shalika representation of p-adic Whittaker function. Note that the representation q-deformed Whittaker function as a character is a q-analog of the Givental integral representation [Gi2], [GKLO] of the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function.

The main objective of this paper is a better understanding of the representation of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a character. Below we will consider a specialization of the q-deformed Whittaker function given by the trace over $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module $\mathcal{V}_{n,k}$ (in the case $\ell = 1$ there is actually no specialization). Our main result is presented in Theorem 3.1. We provide a description of $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module $\mathcal{V}_{n,k}$ as a zero degree cohomology group of a line bundle on an algebraic version $\mathcal{L}\mathbb{P}^\ell_+$ of a semi-infinite cycle $\widetilde{L\mathbb{P}^\ell}_+$ in a universal covering $\widetilde{L\mathbb{P}^\ell}$ of the space of loops in $\mathbb{P}^\ell$. We define $\mathcal{L}\mathbb{P}^\ell_+$ as an appropriate limit $d \to \infty$ of the space $\mathcal{QM}_d(\mathbb{P}^\ell)$ of degree d quasi-maps of $\mathbb{P}^1$ to $\mathbb{P}^\ell$ [Gi1], [CJS]. In particular for $\ell = 1$ this provides a description of a q-deformed $\mathfrak{gl}_2$-Whittaker function in terms of cohomology of line bundles over $\mathcal{L}\mathbb{P}^1_+$. A universal solution of the q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain [GiL] was given in terms of cohomology groups of line bundles over $\mathcal{QM}_d(X), X = G/B$ for finite d. We demonstrate how our interpretation of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function is reconciled with the results of [GiL].

Using Theorems 3.1, we interpret a q-version of the Mellin-Barnes integral representation of the specialized q-Whittaker function as a semi-infinite analog of the Riemann-Roch-Hirzebruch theorem. The corresponding Todd class is expressed in terms of a q-version of Γ-function. Analogously, the classical Γ-function appears in a description of the fundamental class of semi-infinite homology theory and enters the Mellin-Barnes integral representation of the classical Whittaker function. We briefly consider an analog of the elliptic genus arising in the $\mathbb{C}^*$-localization on $\mathcal{L}\mathbb{P}^\ell_+$. We demonstrate that proliferation of fixed points of $\mathbb{C}^*$-action obstructs identification of the result as a topological genus of an extraordinary cohomology theory. Note also that the (q-version of) Γ-function which appears in our calculations of a semi-infinite analog of the Todd class was considered as a candidate for a topological genus by Kontsevich [K] (see also [Li],[Ho]).

Let us stress that the $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module $\mathcal{V}_{n,k}$ arising in the description of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function is not irreducible. It would be natural to look for an interpretation of $\mathcal{V}_{n,k}$ as an irreducible module of a quantum affine Lie group. A relation of the geometry of semi-infinite flags to representation theory of affine Lie algebras was proposed in [FF]. The semi-infinite flag space is defined as $X^{\frac{\infty}{2}} = G(\mathcal{K})/H(\mathcal{O})N(\mathcal{K})$ where $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$, $B = HN$ is a Borel subgroup of G, N is its unipotent radical and H is the associated Cartan subgroup. The semi-infinite flag spaces are not easy to deal with. An interesting approach to the semi-infinite geometry was proposed by Drinfeld. He introduced a space of quasi-maps $\mathcal{QM}_{\underline{d}}(\mathbb{P}^1, G/B)$ that should be considered as a finite-dimensional substitute of the semi-infinite flag space $X^{\frac{\infty}{2}}$ (see e.g. [FM], [FFM], [Bra]). Thus, taking into account constructions proposed in this paper one can expect that (q-deformed) $\mathfrak{gl}_{\ell+1}$-Whittaker functions (encoding Gromov-Witten invariants and their K-theory generalizations) can be expressed in terms of representation theory of affine Lie algebras (see [GiL] for a related conjecture and [FFJMM] for a recent progress in this direction). The paper [GLO2] deals with a relation of our results with the representation theory of (quantum) affine Lie groups.

The paper is organized as follows. In Section 1, explicit solutions of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain (q-versions of Whittaker functions) are recalled. In Section 2, we derive integral expressions for the counting of holomorphic sections of line bundles in the space of quasi-maps. In Section 3 we derive a representation of specialized q-Whittaker functions in terms of cohomology of holomorphic line bundles on the space of quasi-maps of $\mathbb{P}^1$ to $\mathbb{P}^\ell$. We propose an interpretation of q-Whittaker functions as semi-infinite periods. In Section 4 the analogous interpretation of the classical Whittaker functions is discussed. In Section 5, we clarify the connection of our interpretation of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function with the results of [GiL]. Finally, in Section 6 we consider an analog of elliptic genus arising after $\mathbb{C}^*$-localization on $\mathcal{L}\mathbb{P}^\ell_+$ and its possible relation with extraordinary cohomology theories.

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1 q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function

In this section we recall a construction [GLO1] of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(\underline{p}_{\ell+1})$ defined on the lattice $\underline{p}_{\ell+1}=(p_{\ell+1,1},\ldots,p_{\ell+1,\ell+1})\in\mathbb{Z}^{\ell+1}$. We will consider only class one Whittaker functions satisfying the condition

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(\underline{p}_{\ell+1})=0$$

outside dominant domain $p_{\ell+1,1}\geq\ldots\geq p_{\ell+1,\ell+1}$.

The q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions are common eigenfunctions of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_r(\underline{p}_{\ell+1})\,=\,\sum_{I_r}\big(\widetilde{X}_{i_1}^{1-\delta_{i_2-i_1,\,1}}\cdot\ldots\cdot\widetilde{X}_{i_{r-1}}^{1-\delta_{i_r-i_{r-1},\,1}}\cdot\widetilde{X}_{i_r}^{1-\delta_{i_{r+1}-i_r,\,1}}\big)T_{i_1}\cdot\ldots\cdot T_{i_r},\tag{1.1}$$

where the sum is over ordered subsets $I_r=\{i_1<i_2<\ldots<i_r\}\subset\{1,2,\ldots,\ell+1\}$ and we assume $i_{r+1}=\ell+2$. In (1.1) we use the following notations

$$T_if(\underline{p}_{\ell+1})=f(\underline{\widetilde{p}}_{\ell+1}),\qquad \widetilde{p}_{\ell+1,k}=p_{\ell+1,k}+\delta_{k,i},\qquad i,k=1,\ldots,\ell+1,$$

$$\widetilde{X}_i=1-q^{p_{\ell+1,i}-p_{\ell+1,i+1}+1},\qquad i=1,\ldots,\ell,$$

and $\widetilde{X}_{\ell+1}=1$. We assume $q\in\mathbb{C}^*,|q|<1$. For example, the first nontrivial Hamiltonian has the following form:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_1(\underline{p}_{\ell+1})\,=\,\sum_{i=1}^{\ell}(1-q^{p_{\ell+1,i}-p_{\ell+1,i+1}+1})T_i\,+\,T_{\ell+1}.\tag{1.2}$$

The main result of [GLO1] is a construction of common eigenfunctions of quantum Hamiltonians (1.1):

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_r(\underline{p}_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})\,=\,(\sum_{I_r}\prod_{i\in I_r}z_i)\;\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}).\tag{1.3}$$

Denote by $\mathcal{P}^{(\ell+1)}\subset\mathbb{Z}^{\ell(\ell+1)/2}$ a subset of integers $p_{n,i}$, $n=1,\ldots,\ell+1$, $i=1,\ldots,n$ satisfying the Gelfand-Zetlin conditions $p_{k+1,i}\geq p_{k,i}\geq p_{k+1,i+1}$ for $k=1,\ldots,\ell$. In the following we use the standard notation $(n)_q!=(1-q)...(1-q^n)$.

Theorem 1.1 *Let $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})$ be a function given in the dominant domain $p_{\ell+1,1}\geq\ldots\geq p_{\ell+1,\ell+1}$ by*

$$\begin{aligned}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})\,=\,&\sum_{p_{k,i}\in\mathcal{P}^{(\ell+1)}}\;\prod_{k=1}^{\ell+1}z_k^{\sum_ip_{k,i}-\sum_ip_{k-1,i}}\\ &\times\frac{\prod\limits_{k=2}^{\ell}\prod\limits_{i=1}^{k-1}(p_{k,i}-p_{k,i+1})_q!}{\prod\limits_{k=1}^{\ell}\prod\limits_{i=1}^{k}(p_{k+1,i}-p_{k,i})_q!\;(p_{k,i}-p_{k+1,i+1})_q!},\end{aligned}\tag{1.4}$$

and zero otherwise. Then, $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})$ is a common solution of the eigenvalue problem (1.3).

Formula (1.4) can be written also in the recursive form.

Corollary 1.1 *Let $\mathcal{P}_{\ell+1,\ell}$ be a set of $\underline{p}_\ell = (p_{\ell,1},\dots,p_{\ell,\ell})$ satisfying the conditions $p_{\ell+1,i} \geq p_{\ell,i} \geq p_{\ell+1,i+1}$. The following recursive relation holds:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1}) \;=\; \sum_{\underline{p}_\ell\in\mathcal{P}_{\ell+1,\ell}} \Delta(\underline{p}_\ell)\; z_{\ell+1}^{\sum_i p_{\ell+1,i}-\sum_i p_{\ell,i}}\; Q_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q)\Psi^{\mathfrak{gl}_\ell}_{z_1,\dots,z_\ell}(\underline{p}_\ell),$$

where

$$\begin{aligned} Q_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q) \;&=\; \frac{1}{\prod\limits_{i=1}^{\ell}(p_{\ell+1,i}-p_{\ell,i})_q!\;(p_{\ell,i}-p_{\ell+1,i+1})_q!},\\ \Delta(\underline{p}_\ell) &= \prod_{i=1}^{\ell-1}(p_{\ell,i}-p_{\ell,i+1})_q!\;. \end{aligned} \tag{1.5}$$

Remark 1.1 *The representation* (1.4) *is a q-analog of Givental's integral representation of the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function [Gi2], [JK]:*

$$\psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(x_1,\dots,x_{\ell+1}) \;=\; \int\limits_{\mathbb{R}^{\frac{\ell(\ell+1)}{2}}} \prod_{k=1}^{\ell}\prod_{i=1}^{k} dt_{k,i}\quad e^{\frac{1}{\hbar}\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(t)}, \tag{1.6}$$

where

$$\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(t) = \imath\sum_{k=1}^{\ell+1}\lambda_k\Big(\sum_{i=1}^{k}t_{k,i}-\sum_{i=1}^{k-1}t_{k-1,i}\Big)-\sum_{k=1}^{\ell}\sum_{i=1}^{k}\Big(e^{t_{k+1,i}-t_{k,i}}+e^{t_{k,i}-t_{k+1,i+1}}\Big),$$

$\underline{\lambda} = (\lambda_1,\dots,\lambda_{\ell+1})$, $x_i := t_{\ell+1,i}$, $i = 1,\dots,\ell+1$ and $z_i = q^{\gamma_i}$, $\lambda_i = \gamma_i \log q$. For the representation theory derivation of this integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker function see [GKLO]. The representation (1.4) *of the q-Whittaker function turns into representation* (1.6) *the classical Whittaker function in appropriate limit.*

As an example consider $\mathfrak{g} = \mathfrak{gl}_2$. Let $p_1 := p_{2,1} \in \mathbb{Z}$, $p_2 := p_{2,2} \in \mathbb{Z}$ and $p := p_{1,1} \in \mathbb{Z}$. Then the function

$$\begin{aligned} \Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2) \;&=\; \sum_{p_2\leq p\leq p_1}\frac{z_1^p z_2^{p_1+p_2-p}}{(p_1-p)_q!(p-p_2)_q!}, \qquad p_1\geq p_2\,, \\ \Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2) &= 0, \qquad p_1<p_2\,, \end{aligned} \tag{1.7}$$

is a solution of the system of equations:

$$\begin{aligned} \Big\{(1-q^{p_1-p_2+1})T_1\,+\,T_2\Big\}\,\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2) \;&=\; (z_1+z_2)\,\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2),\\ T_1T_2\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2) \;&=\; z_1z_2\,\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2). \end{aligned} \tag{1.8}$$

Let us consider the following specialization of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n,k) \;:=\; \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n+k,k,\dots,k). \tag{1.9}$$

Theorem 1.2 $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n,k)$ *satisfies following difference equation:*

$$\Big\{\prod_{i=1}^{\ell+1}(1-z_iT^{-1})\Big\}\,\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n,k)\;=\;q^n\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n,k), \tag{1.10}$$

where $T\cdot f(n)=f(n+1)$.

Proof: The proof is based on the explicit expression (1.4). Let $\mathcal{P}_{n,k}$ be a Gelfand-Zetlin pattern such that $(p_{\ell+1,1},\dots,p_{\ell+1,\ell+1})=(n+k,k,\dots,k)$. Then, the relations $p_{\ell+1,i}\geq p_{\ell,i}\geq p_{\ell+1,i+1}$ for the elements of a Gelfand-Zetlin pattern imply $p_{k,i\neq 1}=0$ and we have that

$$\begin{aligned}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n,k) &= \Big(\prod_{i=1}^{\ell+1}z_i^k\Big)\sum_{\mathcal{P}_{n,k}}\frac{z_{\ell+1}^{n+k-p_{\ell,1}}}{(n+k-p_{\ell,1})_q!}\frac{z_\ell^{p_{\ell,1}-p_{\ell-1,1}}}{(p_{\ell,1}-p_{\ell-1,1})_q!}\cdots\frac{z_1^{p_{1,1}-k}}{(p_{1,1}-k)_q!}\\ &=\Big(\prod_{i=1}^{\ell+1}z_i^k\Big)\sum_{n_1+\dots+n_{\ell+1}=n}\frac{z_{\ell+1}^{n_{\ell+1}}}{(n_{\ell+1})_q!}\cdots\frac{z_1^{n_1}}{(n_1)_q!}.\end{aligned} \tag{1.11}$$

Introduce the generating function

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(t,k)=\sum_{n\in\mathbb{Z}}t^n\,\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n,k)=\prod_{j=1}^{\ell+1}\frac{z_i^k}{\prod_{m=0}^\infty(1-tz_iq^m)},$$

where we use the identity

$$\frac{1}{\prod_{m=0}^\infty(1-xq^m)}=\sum_{n=0}^\infty\frac{x^n}{(n)_q!}.$$

Due to the fact that $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n,k)=0$ for $n<0$, the generating function $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(t,k)$ is regular at $t=0$. It is easy to check now the following identity

$$\prod_{j=1}^{\ell+1}(1-tz_i)\,\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(t,k)=\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(qt,k).$$

Expanding the latter relation in powers of t, we obtain (1.10) for the coefficients of $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(t,k)$ □

Remark 1.2 *The difference equation* (1.10) *for the specialized q-Whittaker function* $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(n,k)$ *can be derived directly from the system of equations* (1.3) *for the non-specialized q-deformed Whittaker function* $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(p_1,p_2,\dots,p_{\ell+1})$ *and the condition*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(p_1,p_2,\dots,p_{\ell+1})=0$$

outside dominant domain $p_1\geq\ldots\geq p_{\ell+1}$.

Lemma 1.1 *The following integral representation for the specialized q-deformed* $\mathfrak{gl}_{\ell+1}$*-Whittaker functions holds*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k)\;=\;\Big(\prod_{i=1}^{\ell+1}z_i^k\Big)\oint_{t=0}\frac{dt}{2\pi\imath\,t}\,t^{-n}\prod_{i=1}^{\ell+1}\Gamma_q(z_it), \tag{1.12}$$

where

$$\Gamma_q(x) \,=\, \prod_{n=0}^{\infty} \frac{1}{1-q^n x}.$$

Proof: Using the identity

$$\prod_{n=0}^{\infty} \frac{1}{1-xq^n} \,=\, \sum_{m=0}^{\infty} \frac{x^m}{(m)!_q},$$

one obtains, for $n \geq 0$, that

$$\begin{aligned} \Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k) \,&=\, \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \oint\limits_{t=0} \frac{dt}{2\pi\imath\, t}\, t^{-n} \prod_{i=1}^{\ell+1}\prod_{m=0}^{\infty} \frac{1}{1-z_i t q^m} \\ &=\, \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \sum_{n_1+\ldots+n_{\ell+1}=n} \frac{z_1^{n_1}}{(n_1)_q!}\cdot\ldots\cdot\frac{z_{\ell+1}^{n_{\ell+1}}}{(n_{\ell+1})_q!} \end{aligned} \tag{1.13}$$

For $n<0$, we obviously have that $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k)=0$ $\square$

The corresponding integral representation for the classical $\mathfrak{gl}_2$-Whittaker function is given by the Mellin-Barnes representation for the $\mathfrak{gl}_2$-Whittaker function

$$\psi^{\mathfrak{gl}_2}_{\lambda_1,\lambda_2}(x_1,x_2) \,=\, e^{\frac{\imath}{\hbar}(\lambda_1+\lambda_2)x_2} \int_{\imath\sigma-\infty}^{\imath\sigma+\infty} d\lambda\; e^{\frac{\imath}{\hbar}\lambda(x_1-x_2)}\; \Gamma\Big(\frac{\lambda-\lambda_1}{\imath\hbar}\Big)\,\Gamma\Big(\frac{\lambda-\lambda_2}{\imath\hbar}\Big), \tag{1.14}$$

where $\sigma > \max\{Im\lambda_j,\, j=1,\ldots,\ell+1\}$.

Remark 1.3 *The expression*

$$\begin{aligned} \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n,k) \,&=\, \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \sum_{n_1+\ldots+n_{\ell+1}=n} \frac{z_1^{n_1}}{(n_1)_q!}\cdot\ldots\cdot\frac{z_{\ell+1}^{n_{\ell+1}}}{(n_{\ell+1})_q!}\,, \qquad n\geq 0, \\ \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(n,k) \,&=\, 0, \qquad n<0 \end{aligned} \tag{1.15}$$

is a q-analog of the Givental integral representation for the equivariant Gromov-Witten invariants of $X=\mathbb{P}^\ell$ [Gi3]

$$f_{\underline{\lambda}}(T) \,=\, \int\limits_{\mathbb{R}^\ell} \prod_{k=1}^{\ell} dt_{k,1}\;\; e^{\frac{1}{\hbar}\mathcal{F}(t)}, \tag{1.16}$$

where $\underline{\lambda}=(\lambda_1,\ldots,\lambda_{\ell+1})$, $T:=t_{\ell+1,1}$ *and*

$$\mathcal{F}(t) \,=\, \imath\lambda_1 t_{11} + \sum_{k=1}^{\ell} \imath\lambda_{k+1}(t_{k+1,1}-t_{k,1}) \,-\, e^{t_{11}} - \sum_{k=1}^{\ell} e^{t_{k+1,1}-t_{k,1}}.$$

The representation (1.15) *for specialized q-Whittaker function turns into* (1.16) *in appropriate limit.*

2 Counting holomorphic sections

In this Section we are going to provide an interpretation of the explicit expressions for q-deformed specialized class one $\mathfrak{gl}_{\ell+1}$-Whittaker functions in terms of traces of operators acting on the spaces of holomorphic sections of line bundles on infinite-dimensional manifolds. For this aim, we first consider an auxiliary problem of counting holomorphic sections on finite-dimensional manifolds approximating the infinite-dimensional ones. The relevant finite-dimensional manifolds are spaces of the quasi-maps of $\mathbb{P}^1$ to $GL_{\ell+1}(\mathbb{C})$-homogeneous spaces.

2.1 Space of quasi-maps

Let us start with recalling the general construction of the quasi-map compactification of the space of holomorphic maps of $\mathbb{P}^1$ to the partial flag spaces of complex Lie group $GL_{\ell+1}$ due to Drinfeld. Let α_i, $i=1,\ldots,\ell$, be a set of simple roots of the complex Lie algebra $\mathfrak{gl}_{\ell+1}$. To any ordered subset of simple roots $\{\alpha_{i_1},\ldots,\alpha_{i_r}\}$ indexed by an ordered subset $I^P=\{i_1<\ldots<i_r\}\subset\{1,\ldots,\ell\}$ one can associate a parabolic subgroup $P\subset GL_{\ell+1}$. Namely, let $B\subset GL_{\ell+1}$ be the subgroup of upper-triangular matrices generated by Cartan torus and one-parameter unipotent subgroups corresponding to positive simple roots. Then a parabolic subgroup P is generated by B and one-parameter unipotent subgroups corresponding to negative roots $-\alpha_i$ such that $i\notin I^P$. In particular, when $r=\ell$ one gets $P=B$, and the corresponding homogeneous space $GL_{\ell+1}/B$ coincides with the full flag space. On the other hand for a parabolic subgroup $P_0\subset GL_{\ell+1}$ associated to the first simple root (i.e. $I^{P_0}=\{1\}\subset\{1,2,\ldots,\ell\}$), the corresponding homogeneous space $GL_{\ell+1}/P_0$ is isomorphic to the projective space $\mathbb{P}^\ell$. Partial flag spaces $GL_{\ell+1}/P$ possess canonical projective embeddings

$$\pi:\,GL_{\ell+1}/P\to\Pi=\prod_{j\in I^P}\mathbb{P}^{n_j-1},\qquad\qquad n_j=(\ell+1)!/j!\,(\ell+1-j)!\,. \tag{2.1}$$

The group $H^2(GL_{\ell+1}/P,\mathbb{Z})=\mathbb{Z}^r$ is naturally isomorphic to a sublattice of the weight lattice of $\mathfrak{sl}_{\ell+1}$ and is spanned by the weights ω_i indexed by I^P. Let $\mathcal{L}_j$, $j=1,\ldots,r$, be the line bundles on $GL_{\ell+1}/P$ obtained as pull backs of $\mathcal{O}(1)$ form the direct factors $\mathbb{P}^{n_j-1}$ in the right hand side (r.h.s.) of (2.1). The lattice $H^2(GL_{\ell+1}/P,\mathbb{Z})=\mathbb{Z}^r$ is generated by the first Chern classes $c_1(\mathcal{L}_i)$.

Let $\mathcal{M}_{\underline{d}}(GL_{\ell+1}/P)$ be a non-compact space of holomorphic maps of $\mathbb{P}^1$ of multi-degree $\underline{d}\in H^2(GL_{\ell+1}/P,\mathbb{Z})$ to the flag space $GL_{\ell+1}/P$. Due to (2.1), $\mathcal{M}_{\underline{d}}(GL_{\ell+1}/P)$ is a subspace of the product of space $\mathcal{M}_{d_j}(\mathbb{P}^{n_j-1})$. Explicitly, each $\mathcal{M}_{d_j}(\mathbb{P}^{n_j-1})$ can be described as a set of collections of n, *relatively prime* polynomials of degree d, up to a common constant factor. The space $\mathcal{M}_{d_j}(\mathbb{P}^{n_j-1})$ allows for a compactification by the space of quasi-maps $\mathcal{QM}_{d_j}(\mathbb{P}^{n_j-1})$ defined as a set of collections of n, polynomials of degree d, up to a common constant factor. The space of quasi-maps $\mathcal{QM}_{\underline{d}}(GL_{\ell+1}/P)$ is then constructed as a closure of $\mathcal{M}_{\underline{d}}(GL_{\ell+1}/P)$ in $\prod_j\mathcal{QM}_{\underline{d}}(\mathbb{P}^{n_j})$. Thus defined $\mathcal{QM}_{\underline{d}}(GL_{\ell+1}/P)$ is (in general singular) irreducible projective variety. A small resolution of this space is known due to [La], [Ku].

On the space of holomorphic maps $\mathcal{M}_{\underline{d}}(GL_{\ell+1}/P)$ of $\mathbb{P}^1$ to $GL_{\ell+1}/P$, there is a natural action of the group $\mathbb{C}^*\times GL_{\ell+1}$ (and, thus, of its maximal compact subgroup $S^1\times U_{\ell+1}$). Here, the action of $GL_{\ell+1}$ is induced by the standard action on flag spaces and the action of $\mathbb{C}^*$ is induced by the action of $\mathbb{C}^*$ on $\mathbb{P}^1$ given by $(y_1,y_2)\to(\xi y_1,y_2)$ in homogeneous coordinates (y_1,y_2) on $\mathbb{P}^1$. This action of $\mathbb{C}^*\times GL_{\ell+1}$ can be extended to an action on the space $\mathcal{QM}_{\underline{d}}(GL_{\ell+1}/P)$ of quasi-maps.

In the following we consider a parabolic subgroup $P_0\subset GL_{\ell+1}$ associated to the first simple root (and thus $I^{P_0} = \{1\} \subset \{1, 2, \ldots, \ell\}$. The corresponding homogeneous space $GL_{\ell+1}/P_0$ is a projective space $\mathbb{P}^\ell$. The space of quasi-maps [1] $\mathcal{QM}_d(\mathbb{P}^\ell)$ is a non-singular projective variety $\mathbb{P}^{(\ell+1)(d+1)-1}$. A quasi-map $\phi \in \mathcal{QM}_d(\mathbb{P}^\ell)$ is given by a collection

$$\big(a_0(y) : a_1(y) : \ \ldots \ a_\ell(y)\big),$$

of homogeneous polynomials $a_i(y)$ in variables $y = (y_1, y_2)$ of degree d

$$a_k(y) \ = \ \sum_{i=0}^{d} a_{k,j}\, y_1^j y_2^{d-j}, \qquad\qquad k = 0, \ldots, \ell.$$

considered up to the multiplication of all $a_i(y)$'s by a nonzero complex number. The action of $(\xi, g) \in \mathbb{C}^* \times GL_{\ell+1}$ on $\mathcal{QM}_d(\mathbb{P}^\ell)$ is given by

$$\begin{aligned} \xi \ : \quad & \big(a_0(y) : a_1(y) : \ \ldots \ a_\ell(y)\big) \longmapsto \big(a_0(y^\xi) : a_1(y^\xi) : \ \ldots : \ a_\ell(y^\xi)\big), \\ g \ : \quad & \big(a_0(y) : a_1(y) : \ \ldots \ : a_\ell(y)\big) \longmapsto \Big(\sum_{k=1}^{\ell+1} g_{1,k} a_{k-1}(y) : \ \ldots \ : \sum_{k=1}^{\ell+1} g_{\ell+1,k} a_{k-1}(y) \Big), \end{aligned} \tag{2.2}$$

where $g = \|g_{ij}\|$ and $y^\xi = (\xi y_1, y_2)$.

2.2 Generating functions of holomorphic sections

Let $\mathcal{O}(1)$ be a standard line bundle on $\mathbb{P}^{(\ell+1)(d+1)-1}$. The space of sections of the line bundle $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$ on $\mathcal{QM}_d(\mathbb{P}^\ell)$ is naturally a $\mathbb{C}^* \times GL_{\ell+1}$-module. We are interested in calculating the corresponding character.

Let $T \in GL_{\ell+1}$ be a Cartan torus, $H_1, \ldots, H_{\ell+1}$ be a basis in Lie(T), and L_0 be a generator of Lie$(\mathbb{C}^*)$. The equivariant cohomology of a point with respect to the maximal compact subgroup $G = S^1 \times U_{\ell+1}$ of $\mathbb{C}^* \times GL_{\ell+1}$ can be described as

$$H^*_G(\mathrm{pt}, \mathbb{C}) \ = \ \mathbb{C}[\lambda_1, \ldots, \lambda_{\ell+1}]^{\mathfrak{S}_{\ell+1}} \otimes \mathbb{C}[\hbar],$$

where $\lambda_1, \ldots, \lambda_{\ell+1}$ and $\hbar$ are associated with the generators $H_1, \ldots, H_{\ell+1}$ and L_0 respectively.

Let $\mathcal{L}_k$ be a one-dimensional $GL_{\ell+1}$-module such that $H_i \mathcal{L}_k = k \mathcal{L}_k$, for $i = 1, \ldots, \ell+1$. Cohomology groups $H^*(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{O}(n)) \otimes \mathcal{L}_k$ have a natural structure of $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module. We denote by $\mathcal{L}_k(n) = \mathcal{O}(n) \otimes \mathcal{L}_k$ the line bundles $\mathcal{O}(n)$ twisted by one-dimensional $GL_{\ell+1}$-module $\mathcal{L}_k$.

Let $A^{(d)}_{n,k}(\underline{z}, q)$, be the character of the $\mathbb{C}^* \times GL_{\ell+1}$-module $\mathcal{V}_{n,k,d} \ = \ H^0(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n))$, $n \geq 0$,

$$A^{(d)}_{n,k}(\underline{z}, q) \ = \ \mathrm{Tr}\,_{\mathcal{V}_{n,k,d}}\, q^{L_0}\, e^{\sum \lambda_i H_i},$$

where we assume that $q \in \mathbb{C}^*$, $|q| < 1$. This character can be straightforwardly calculated as follows. The space $\mathcal{V}_{n,k,d} = H^0(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n))$ can be identified with the space of degree n homogeneous polynomials in $(\ell+1)(d+1)$ variables $a_{k,i}$, for $k = 0, \ldots, \ell$ and $i = 0, \ldots, d$. Define

$$\mathcal{V}_{k,d} = \oplus_{n=0}^{\infty} \mathcal{V}_{n,k,d},$$

[1] The compactification of $\mathcal{M}_d(\mathbb{P}^\ell)$ by the space $\mathcal{QM}_d(\mathbb{P}^\ell)$ of quasi-maps arises naturally in the linear sigma-model description of Gromov-Witten invariants of projective spaces [W].

and the grading on $\mathcal{V}_{k,d}$ is defined by the eigenvalue decomposition with respect to the action of an operator D

$$t^D : \mathcal{V}_{n,k,d} \to t^n \; \mathcal{V}_{n,k,d}, \qquad t \in \mathbb{C}^*.$$

The action of the subgroup $(\mathbb{C}^* \times T) \subset G(\mathbb{C}) = \mathbb{C}^* \times GL_{\ell+1}$ is given by

$$e^{\sum \lambda_i H_i} : \quad \big(a_0(y) : a_1(y) : \ldots \, a_\ell(y)\big) \longmapsto \big(e^{\lambda_1} a_0(y) : e^{\lambda_2} a_1(y) : \ldots : e^{\lambda_{\ell+1}} a_\ell(y)\big), \tag{2.3}$$

where

$$a_k(y) = \sum_{j=0}^{d} a_{k,j} \, y_1^j y_2^{d-j}, \qquad\qquad k = 0, \ldots, \ell.$$

The action of the generator L_0 of $\mathbb{C}^*$ is as follows

$$q^{L_0} : \quad a_{k,j} \longmapsto q^j \; a_{k,j}. \tag{2.4}$$

Proposition 2.1 *For the $\mathbb{C}^* \times GL_{\ell+1}$-character of the module $\mathcal{V}_{n,k,d}$, the following integral representation holds*

$$A^{(d)}_{n,k}(\underline{z}, q) = \mathrm{Tr}_{\,\mathcal{V}_{n,k,d}} \, q^{L_0} \, e^{\sum \lambda_i H_i} = \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \oint_{t=0} \frac{dt}{2\pi\imath \, t^{n+1}} \prod_{m=1}^{\ell+1} \prod_{j=0}^{d} \frac{1}{(1 - tq^j z_m)}, \tag{2.5}$$

where $\underline{z} = (z_1, \ldots, z_{\ell+1})$ and $z_m = e^{\lambda_m}$.

Proof: A simple calculation gives us that

$$A^{(d)}_{k}(\underline{z}, t, q) = \mathrm{Tr}_{\,\mathcal{V}_{k,d}} \, t^D \, q^{L_0} \, e^{\sum \lambda_i H_i} = \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \prod_{m=1}^{\ell+1} \prod_{j=0}^{d} \frac{1}{(1 - tq^j z_m)}. \tag{2.6}$$

The projection on the subspace of $\mathcal{V}_{k,d}$ of the grading n with respect to D can be realized by taking a residue,

$$\mathrm{Tr}_{\,\mathcal{V}_{n,k,d}} \, q^{L_0} \, e^{\sum \lambda_i H_i} = \oint_{t=0} \frac{dt}{2\pi\imath \, t^{n+1}} \; \mathrm{Tr}_{\,\mathcal{V}_{k,d}} \, t^D \, q^{L_0} \, e^{\sum \lambda_i H_i}. \tag{2.7}$$

This gives us the integral expression (2.5) $\Box$

2.3 Equivariant Euler characteristic of line bundles on $\mathcal{QM}_d(\mathbb{P}^\ell)$

Characters (2.5) of the space of holomorphic sections can be related to equivariant holomorphic Euler characteristics of line bundles on $\mathcal{QM}_d(\mathbb{P}^\ell)$. First we recall the standard facts about line bundles on projective spaces. Line bundles $\mathcal{O}(n)$ on projective spaces $\mathbb{P}^N$ are equivariant with respect to the standard action of U_{N+1} on $\mathbb{P}^N$. The U_{N+1}-equivariant Euler characteristic of $\mathcal{O}(n)$ is given by the character

$$\chi_{U_{N+1}}(\mathbb{P}^N, \mathcal{O}(n)) = \sum_{m=0}^{N} (-1)^m \, \mathrm{Tr}_{\,H^m(\mathbb{P}^N, \mathcal{O}(n))} \, e^{\sum \lambda_i H_i}, \qquad e^{\sum \lambda_i H_i} \in U_{N+1}. \tag{2.8}$$

Cohomology groups of $\mathcal{O}(n)$ on projective space $\mathbb{P}^N$ have the following properties (see e.g. [OSS])

$$\begin{aligned} \dim H^m(\mathbb{P}^N, \mathcal{O}(n)) &= 0, \qquad m \neq 0, N, \\ \dim H^N(\mathbb{P}^N, \mathcal{O}(n)) &= 0, \qquad n \geq 0, \\ \dim H^0(\mathbb{P}^N, \mathcal{O}(n)) &= 0, \qquad n < 0. \end{aligned} \tag{2.9}$$

Taking into account (2.9) the expression (2.8) reduces to

$$\chi_{U_{N+1}}(\mathbb{P}^N, \mathcal{O}(n)) = \mathrm{Tr}_{\,H^0(\mathbb{P}^N, \mathcal{O}(n))}\, e^{\sum \lambda_i H_i} + (-1)^N \,\mathrm{Tr}_{\,H^N(\mathbb{P}^N, \mathcal{O}(n))}\, e^{\sum \lambda_i H_i}\,. \tag{2.10}$$

We have $\mathcal{QM}_d(\mathbb{P}^\ell) = \mathbb{P}^{(\ell+1)(d+1)-1}$ and, thus, for $n \geq 0$, we can identify $A^{(d)}_{n,k}(\underline{z}, q)$ with equivariant holomorphic Euler characteristic of $\mathcal{L}_k(n)$

$$A^{(d)}_{n,k}(\underline{z}, q) = \mathrm{Tr}_{\,H^0(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n))}\, e^{\hbar L_0 + \sum \lambda_i H_i} = \chi_G(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n)), \qquad n \geq 0,$$

where $G = S^1 \times U_{\ell+1}$. The equivariant Euler characteristic of a holomorphic vector bundle on the projective space possesses a canonical holomorphic integral representation. According to the Riemann-Roch-Hirzebruch (RRH) theorem, one can express the U_{N+1}-equivariant holomorphic Euler characteristic of a U_{N+1}-equivariant vector bundle $\mathcal{E}$ on $\mathbb{P}^N$ as follows

$$\begin{aligned} \chi_{U_{N+1}}(\mathbb{P}^N, \mathcal{E}) &= \sum_{m=0}^{N} (-1)^m \mathrm{Tr}_{\,H^m(\mathbb{P}^N, \mathcal{E})}\, e^{\sum \lambda_i H_i} = \\ &= \langle \mathrm{Ch}_{U_{N+1}}(\mathcal{E})\ \mathrm{Td}_{U_{N+1}}(\mathcal{T}\mathbb{P}^N)\,, [\mathbb{P}^N] \rangle, \end{aligned} \tag{2.11}$$

where H_i are generators of the Lie algebra $\mathfrak{gl}_{N+1}$, $\mathcal{T}\mathbb{P}^N$ is the tangent bundle to $\mathbb{P}^N$, $\mathrm{Ch}_{U_{N+1}}(\mathcal{E})$ is a U_{N+1}-equivariant Chern character of $\mathcal{E}$ and $\mathrm{Td}_{U_{N+1}}(\mathcal{E})$ is a U_{N+1}-equivariant Todd genus of $\mathcal{E}$ [A].

The tangent bundle $\mathcal{T}\mathbb{P}^N$ to the projective space $\mathbb{P}^N$ is U_{N+1}-equivariantly stable-equivalent to $\mathcal{O}(1)^{\oplus(N+1)}$ as the following lemma shows.

Lemma 2.1 *The following relation holds in U_{N+1}-equivariant topological K-theory $K_{U_{N+1}}(\mathbb{P}^N)$*

$$[\mathcal{T}\mathbb{P}^N] \oplus [\mathcal{O}] = [\mathcal{O}(1)]^{\oplus(N+1)}, \tag{2.12}$$

where $[\mathcal{E}]$ is a class of a vector bundle $\mathcal{E}$ in $K_{U_{N+1}}(\mathbb{P}^N)$.

Proof: For a tangent sheaf to the complex projective space $\mathbb{P}^N$ we have the Euler exact sequence (see e.g. [GH])

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus(N+1)} \longrightarrow \mathcal{T}\mathbb{P}^N \longrightarrow 0 \tag{2.13}$$

The maps (2.13) are explicitly U_{N+1}-equivariant and, thus, we obtain the relation (2.12) in U_{N+1}-equivariant K-groups of $\mathbb{P}^N$ $\Box$

Lemma 2.1 and the fact that the Todd class depends only on stable equivalence class of a vector bundle allows us to rewrite RRH-theory on projective spaces as follows

$$\chi_{U_{N+1}}(\mathbb{P}^N, \mathcal{E}) = \langle \mathrm{Ch}_{U_{N+1}}(\mathcal{E})\ \mathrm{Td}_{U_{N+1}}(\mathcal{O}(1)^{\oplus(N+1)}\,, [\mathbb{P}^N] \rangle. \tag{2.14}$$

In the following we will consider only the case of line bundles and thus we take $\mathcal{E} = \mathcal{O}(n)$, $n \in \mathbb{Z}$. The pairing of the cohomology classes with the fundamental class entering the formulation of RRH-theorem can be expressed explicitly using a particular model for the cohomology ring $H^*(\mathbb{P}^N, \mathbb{C})$. The cohomology ring $H^*(\mathbb{P}^N, \mathbb{C})$ is generated by an element $x \in H^2(\mathbb{P}^N, \mathbb{C})$ with a single relation $x^{N+1} = 0$

$$H^*(\mathbb{P}^N, \mathbb{C}) = \mathbb{C}[x]/x^{N+1}. \tag{2.15}$$

The U_{N+1}-equivariant analog of (2.15) is given by

$$H^*_{U_{N+1}}(\mathbb{P}^\ell, \mathbb{C}) = \mathbb{C}[x] \otimes \mathbb{C}[\lambda_1, \cdots, \lambda_{N+1}]^{\mathfrak{S}_{N+1}} \Big/ \Big(\prod_{j=1}^{N+1} (x - \lambda_j) \Big),$$

which is naturally a module over $H^*_{U_{N+1}}(\mathrm{pt}, \mathbb{C}) = \mathbb{C}[\lambda_1, \cdots, \lambda_{N+1}]^{\mathfrak{S}_{N+1}}$ where $\mathfrak{S}_{N+1}$ is the permutation group of a set of $N+1$ elements. The pairing of an element of $H^*_{U_{N+1}}(\mathbb{P}^N, \mathbb{C})$ represented by $P(x, \lambda)$ with a U_{N+1}-equivariant fundamental cycle $[\mathbb{P}^N]$ can be expressed in terms of the integral

$$\langle P(\lambda), [\mathbb{P}^N] \rangle = \frac{1}{2\pi\imath} \oint_{C_0} dx \, \frac{P(x, \lambda)}{\prod_{j=1}^{N+1} (x - \lambda_j)},$$

where the integration contour C_0 encircles the poles $x = \lambda_j$, $j = 1, \ldots, (N+1)$. The pairing for $H^*(\mathbb{P}^N, \mathbb{C})$ is obtained by a specialization $\lambda_j = 0$, $j = 1, \cdots, (N+1)$. The equivariant Chern character and Todd class can be written in terms of this model of $H^*_{U_{N+1}}(\mathbb{P}^N, \mathbb{C})$ as follows (see e.g. [H])

$$\mathrm{Ch}_{U_{N+1}}(\mathcal{O}(n)) \, = \, e^{nx}, \qquad \mathrm{Td}_{U_{N+1}}\big(\mathcal{O}(1)^{\oplus (N+1)}\big) \, = \, \prod_{j=1}^{N+1} \frac{(x - \lambda_j)}{1 - e^{-(x - \lambda_j)}}.$$

Therefore we have the following integral representation of the equivariant holomorphic Euler characteristic ($t = e^{-x}$, $z_i = e^{\lambda_i}$, $i = 1, \ldots, N+1$):

$$\begin{aligned} \chi_{U_{N+1}}(\underline{z}) \, &= \, \langle \mathrm{Ch}_{U_{N+1}}(\mathcal{O}(n)) \, \mathrm{Td}_{U_{N+1}}\big(\mathcal{O}(1)^{\oplus(N+1)}\big), [\mathbb{P}^N] \rangle \\ &= \, \frac{1}{2\pi\imath} \oint_{C_0} \frac{dx}{\prod_{i=1}^{N+1} (x - \lambda_i)} \, e^{nx} \prod_{i=1}^{N+1} \frac{(x - \lambda_i)}{(1 - e^{-(x - \lambda_i)})} \end{aligned} \tag{2.16}$$

$$= \, -\frac{1}{2\pi\imath} \oint_{C_0} \frac{dt}{t^{n+1}} \prod_{i=1}^{N+1} \frac{1}{1 - t z_i}\,, \tag{2.17}$$

where in the last expression the integration contour C_0 encircles the poles $t = z_j^{-1}, j = 1, \ldots, \ell + 1$. The integral representation (2.17) can be obtained directly using a particular realization of (U_{N+1}-equivariant) K-theory on $\mathbb{P}^N$ (see e.g. [A]). The K-group $K(\mathbb{P}^N)$ is generated by a class t of the line bundle $\mathcal{O}(1)$ satisfying the relation $(1 - t)^{N+1} = 0$. We have the following isomorphisms for (U_{N+1}-equivariant) K-groups of projective spaces

$$K(\mathbb{P}^N) = \mathbb{C}[t, t^{-1}]/(1 - t)^{N+1}, \qquad K_{U_{N+1}}(\mathbb{P}^N) = \mathbb{C}[t, t^{-1}, z, z^{-1}] \Big/ \prod_{j=1}^{N+1} (1 - t z_j)\,. \tag{2.18}$$

The equivariant analog of the pairing with the fundamental class of $\mathbb{P}^N$ in K-theory is given by

$$\langle R\,,[\mathbb{P}^N]\rangle_K \;=\; -\frac{1}{2\pi\imath}\oint\limits_{C_0}\frac{dt}{t}\;\frac{R(t)}{\prod_{j=1}^{N+1}(1-tz_j)}, \tag{2.19}$$

where $R(t)$ is a rational function representing an element of $K_{U_{N+1}}(\mathbb{P}^N)$ and the integration contour C_0 encircles the poles $t=z_j^{-1}$, $j=1,\ldots,(N+1)$.

Using the representation of the pairing (2.19) one can represent RRH expression for the Euler characteristic as

$$\chi_{U_{N+1}}(\underset{\sim}{z}) \;=\; \langle[\mathcal{O}(n)],[\mathbb{P}^N]\rangle_K = -\frac{1}{2\pi\imath}\oint_{C_0}\frac{dt}{t^{n+1}}\prod_{i=1}^{N+1}\frac{1}{1-tz_i}\,. \tag{2.20}$$

This reproduces the representation (2.17).

Now we would like to apply the integral representation for equivariant Euler characteristic to the $S^1\times U_{\ell+1}$-equivariant line bundle $\mathcal{L}_k(n)$ on $\mathcal{QM}_d(\mathbb{P}^\ell)$.

Consider the $S^1\times U_{\ell+1}$-equivariant cohomology of the projective space $\mathbb{P}(V_{(\ell+1)(d+1)})$ where the vector space $V_{(\ell+1)(d+1)}=\oplus_{j=1}^{\ell+1}\oplus_{m=0}^{d}V_{j,m}$ has the structure of an S^1-module with an action given by

$$e^{\imath\theta}\,:\,V_{j,m}\to e^{\imath m\theta}\,V_{j,m},\qquad \dim V_{j,m}=1,\quad \theta\in S^1,$$

and each $V_m=V_{1,m}\oplus V_{2,m}\oplus\ldots V_{\ell+1,m}$ is standard $U_{\ell+1}$-module. Then for the $G=S^1\times U_{\ell+1}$-equivariant cohomology of $\mathbb{P}(V_{(\ell+1)(d+1)})$ we have an isomorphism

$$H^*_{S^1\times U_{\ell+1}}(\mathbb{P}(V_{(\ell+1)(d+1)})) \;=\; \mathbb{C}[x,\lambda,\hbar]\Big/\prod_{j=1}^{\ell+1}\prod_{m=0}^{d}(x-\lambda_j-\hbar m), \tag{2.21}$$

where x is a generator of $H^*(\mathbb{P}(V_{(\ell+1)(d+1)}),\mathbb{C})$. The pairing with the $S^1\times U_{\ell+1}$-equivariant fundamental cycle $[\mathbb{P}(V_{(\ell+1)(d+1)})]$ can be expressed in the form of the contour integral

$$\langle P(\lambda,\hbar),[\mathbb{P}(V_{(\ell+1)(d+1)})]\rangle \;=\; \frac{1}{2\pi\imath}\oint_C\frac{P(x,\lambda,\hbar)\;dx}{\prod_{j=1}^{\ell+1}\prod_{m=0}^{d}(x-\lambda_j-\hbar m)}, \tag{2.22}$$

where the integration contour C encircles the poles $x=\lambda_j+m\hbar$, $j=1,\ldots,\ell+1$, $m=0,1,\ldots,d$.

Specializing to the action of $G=S^1\times U_{\ell+1}$ on $\mathcal{QM}_d(\mathbb{P}^\ell)\simeq\mathbb{P}^{(\ell+1)(d+1)-1}$ described in Section 2.1 we obtain

$$\mathrm{Ch}_G(\mathcal{L}_k(n)) \;=\; e^{nx+k(\lambda_1+\ldots+\lambda_{\ell+1})},\qquad\qquad \mathrm{Td}_G(\mathcal{T}\mathbb{P}^{(\ell+1)(d+1)-1}) \;=\; \prod_{i=1}^{\ell+1}\prod_{m=0}^{d}\frac{x-m\hbar-\lambda_i}{1-e^{\lambda_i+m\hbar-x}}.$$

Let $q=e^{\hbar}$, $t=e^{-x}$, and $z_i=e^{\lambda_i}$, $1\le i\le\ell+1$, then

$$\begin{aligned}\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{L}_k(n)) \;&=\; \big\langle\mathrm{Ch}_G(\mathcal{L}_k(n))\,\mathrm{Td}_G(\mathcal{T}\mathbb{P}^{(\ell+1)(d+1)-1}),[\mathbb{P}^\ell]\big\rangle\\ =\;\frac{1}{2\pi\imath}\oint_C dx\,\prod_{i=1}^{\ell+1}\prod_{m=0}^{d}&\frac{1}{(x-\lambda_i-m\hbar)}\;e^{nx+k(\lambda_1+\ldots+\lambda_{\ell+1})}\;\prod_{i=1}^{\ell+1}\prod_{n=0}^{d}\frac{(x-\lambda_i-m\hbar)}{(1-e^{-(x-\lambda_i-m\hbar)})}\\ &=\;-\Big(\prod_{i=1}^{\ell+1}z_i^k\Big)\oint\limits_C\frac{dt}{2\pi\imath\,t^{n+1}}\prod_{i=1}^{\ell+1}\prod_{m=0}^{d}\frac{1}{(1-tz_iq^m)}.\end{aligned} \tag{2.23}$$

For $n \geq 0$ one has the identity

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n)) = \mathrm{Tr}\,_{\mathcal{V}_{n,k,d}} \, q^{L_0} \, e^{\sum \lambda_i H_i}.$$

Deforming the contour for $n \geq 0$ we obtain the following integral representation for the character

$$\mathrm{Tr}\,_{\mathcal{V}_{n,k,d}} \, q^{L_0} \, e^{\sum \lambda_i H_i} \; = \; \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \oint\limits_{t=0} \frac{dt}{2\pi\imath \, t^{n+1}} \prod_{i=1}^{\ell+1} \prod_{m=0}^{d} \frac{1}{(1 - t z_i q^m)}$$

which coincides with (2.5).

Remark 2.1 *Without a restriction $n \geq 0$, the integral representation for the equivariant Euler characteristic can be represented as a difference of two terms*

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n)) \; =$$

$$= \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \oint\limits_{t=0} \frac{dt}{2\pi\imath \, t^{n+1}} \prod_{i=1}^{\ell+1} \prod_{m=0}^{d} \frac{1}{(1 - t z_i q^m)} + \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \oint\limits_{t=\infty} \frac{dt}{2\pi\imath \, t^{n+1}} \prod_{i=1}^{\ell+1} \prod_{m=0}^{d} \frac{1}{(1 - t z_i q^m)}.$$

This decomposition corresponds to the decomposition (2.10)

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n)) \; =$$

$$= \mathrm{Tr}\,_{H^0(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n))} \, q^{L_0} \, e^{\sum \lambda_i H_i} + (-1)^{(\ell+1)(d+1)-1} \mathrm{Tr}\,_{H^{(\ell+1)(d+1)-1}(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n))} \, q^{L_0} \, e^{\sum \lambda_i H_i} \, .$$

Remark 2.2 *In the limit $q \to 0$ one has an integral representation for a character $\chi^{(0)}_{(n,k)}$ of an irreducible finite-dimensional representation $V_{n,k,0} = \mathrm{Sym}^n \mathbb{C}^{\ell+1} \otimes \mathcal{L}_k$ of $GL_{\ell+1}$:*

$$\chi^{(0)}_{(n,k)}(\underline{z}) \; = \; \mathrm{tr}\,_{V_{n,k,0}} e^{\lambda_1 H_1 + \ldots + \lambda_{\ell+1} H_{\ell+1}} \; = \; \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \oint\limits_{t=0} \frac{dt}{2\pi\imath \, t^{n+1}} \prod_{i=1}^{\ell+1} \frac{1}{1 - t z_i}. \qquad (2.24)$$

where $z_i = \exp \lambda_i$ and the $GL(\ell+1)$-module $V_{n,k,0}$, $n \geq 0$ is realized as a zero cohomology space $H^0(\mathbb{P}^\ell, \mathcal{L}_k(n))$.

3 K-theory of $\mathcal{L}\mathbb{P}^\ell_+$ and q-Whittaker functions

In this section we establish a direct connection between q-deformed class one specialized $\mathfrak{gl}_{\ell+1}$-Whittaker functions and geometry of the space $\mathcal{L}\mathbb{P}^\ell_+$ defined as an appropriate limit of $\mathcal{QM}_d(\mathbb{P}^\ell)$ when $d \to +\infty$. Geometrically the $\mathcal{L}\mathbb{P}^\ell_+$ should be considered as a space of algebraic disks in $\mathbb{P}^\ell$ (see [Gi1] for details). In general, let $LX = \mathrm{Map}(S^1, X)$ be the space of free contractible loops in a compact Kähler manifold X. There is a natural action of S^1 on LX by loop rotations. The universal covering $\widetilde{LX}$ can be defined as a space of maps $D \to X$ of the disk D considered up to a homotopy of the map preserving the image of the boundary loop $S^1 \subset D$. The group of covering transformations of the universal cover $\widetilde{LX} \to LX$ is isomorphic to $\pi_2(X)$. Let $\widetilde{LX}_+ \subset \widetilde{LX}$ be a semi-infinite cycle of loops that are boundaries of holomorphic maps $D \to X$. For $X = \mathbb{P}^\ell$ define an algebraic version $\mathcal{L}\mathbb{P}^\ell_+$ of $\widetilde{L\mathbb{P}^\ell}_+$ as a set of collections of regular series

$$a_i(z) = a_{i,0} + a_{i,1} z + a_{i,2} z^2 + \cdots \, , \qquad\qquad 0 \leq i \leq \ell,$$

modulo the action of $\mathbb{C}^*$. The topology on this space should be defined by considering $\mathcal{L}\mathbb{P}^\ell_+$ as a limit of $\mathcal{QM}_d(\mathbb{P}^\ell)$ when $d \to \infty$. This space inherits the action of $G = S^1 \times U(\ell+1)$ defined previously on $\mathcal{QM}_d(\mathbb{P}^\ell)$. In the following we do not define appropriate topology rigorously leaving this for another occasion. Instead we define the limit $d \to +\infty$ on the level of cohomology algebra $H^*(\mathcal{QM}_d(\mathbb{P}^\ell), \mathbb{C})$ and the space of holomorphic sections of line bundles on $\mathcal{QM}_d(\mathbb{P}^\ell)$. Let us take the limit $d \to +\infty$ of the character $A^{(d)}_{n,k}(\underline{z}, q)$ given by the integral expression (2.5). The limit of $A^{(d)}_{n,k}(\underline{z}, q)$ can be interpreted as a character of a $\mathbb{C}^* \times GL_{\ell+1}$-module $\mathcal{V}_{n,k,\infty}$ defined as follows. Let $\mathcal{V}_{k,\infty}$ be a linear space of polynomials of infinite number of variables $a_{i,m}$, $i = 0, \dots, \ell$, $m \in \mathbb{Z}_{\geq 0}$. Let L_0 be a generator of $\mathrm{Lie}(\mathbb{C}^*)$, $T \in GL_{\ell+1}$ be a Cartan torus and $H_1, \dots, H_{\ell+1}$ be a basis in $\mathrm{Lie}(T)$. Define the action of L_0 and H_j on the generators $a_{i,m}$ as follows

$$L_0 : \; a_{i,m} \longrightarrow m \, a_{i,m};$$

$$e^{\sum_j \lambda_j H_j} : \; a_{i,m} \longrightarrow e^{\lambda_i} \, a_{i,m}$$

This supplies $\mathcal{V}_{k,\infty}$ with the structure of a $\mathbb{C}^* \times GL_{\ell+1}$-module. Now the linear subspace $\mathcal{V}_{n,k,\infty} \subset \mathcal{V}_{k,\infty}$ is defined as a subspace of polynomials of the variables $a_{i,m}$, $i = 0, \dots, \ell$, $m \in \mathbb{Z}_{\geq 0}$ of the total degree n.

Theorem 3.1 *Let $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k)$ be a specialization (1.9) of the solution of q-deformed Toda lattice defined in the Theorem 1.1. Then the following holds.*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k) \; = \; \chi_{n,k}(\underline{z}), \tag{3.1}$$

where

$$\chi_{n,k}(\underline{z}) = \lim_{d \to \infty} A^{(d)}_{n,k}(\underline{z}, q) = \mathrm{Tr}\,_{\mathcal{V}_{n,k,\infty}} \, q^{L_0} \, e^{\sum \lambda_i H_i}, \qquad z_i = e^{\lambda_i}.$$

Proof: For the function $\chi_{n,k}(\underline{z}) = \lim_{d\to\infty} A^{(d)}_{n,k}(\underline{z}, q)$ the following integral representation holds:

$$\chi_{n,k}(\underline{z}) \; = \; \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \oint_{t=0} \frac{dt}{2\pi\imath \, t^{n+1}} \prod_{m=1}^{\ell+1} \prod_{j=0}^{\infty} \frac{1}{(1 - tq^j z_m)} \; = \tag{3.2}$$

$$= \Big(\prod_{i=1}^{\ell+1} z_i^k\Big) \oint_{t=0} \frac{dt}{2\pi\imath \, t^{n+1}} \prod_{m=1}^{\ell+1} \Gamma_q(tz_m).$$

The relations (3.1) follows directly from the explicit integral expression (3.2) and Lemma 1.1. The representation in terms of the trace over $\mathcal{V}_{n,k,\infty}$ follows from the statement of Proposition 2.1 with obvious modifications for $d \to +\infty$ $\Box$

For $n \geq 0$ we can identify the character $A^{(d)}_{n,k}$ with the equivariant Euler characteristic expressed through Riemann-Roch-Hirzebruch formula

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n)) \; = \; \int_{\mathcal{QM}_d(\mathbb{P}^\ell)} \mathrm{Ch}_G(\mathcal{L}_k(n)) \, \mathrm{Td}_G(\mathcal{T}\mathcal{QM}_d(\mathbb{P}^\ell)). \tag{3.3}$$

Taking the limit $d \to +\infty$ we obtain formal Riemann-Roch-Hirzebruch formula for $\chi_G(\mathcal{L}\mathbb{P}^\ell_+, \mathcal{L}_k(n))$. Using the description (2.18), (2.19) of the equivariant K-groups of projective spaces and taking the limit $d \to +\infty$ in the integral representation of the Euler characteristic (2.23) one obtains the following integral representation for the equivariant Euler characteristic of line bundles on $\mathcal{L}\mathbb{P}^{\ell}_{+}$

$$\begin{aligned}\chi_G(\mathcal{L}\mathbb{P}^{\ell}_{+},\mathcal{L}_k(n)) \,=\, -\Big(\prod_{j=1}^{\ell+1} z_j^k\Big)\oint\limits_C \frac{dt}{2\pi\imath\, t^{n+1}}\prod_{i=1}^{\ell+1}\prod_{j=0}^{\infty}\frac{1}{(1-tq^jz_i)} \,=\\ =\, -\Big(\prod_{j=1}^{\ell+1} z_j^k\Big)\oint\limits_C \frac{dt}{2\pi\imath\, t^{n+1}}\prod_{i=1}^{\ell+1}\Gamma_q(tz_i),\end{aligned} \tag{3.4}$$

where the integration contour C encircles all poles except $t=0$ and

$$\Gamma_q(y) \,=\, \prod_{n=0}^{\infty}\frac{1}{1-yq^n}.$$

Porblem 3.1 *Define an equivariant (co)homology theory for $\mathcal{L}\mathbb{P}^{\ell}_{+}$ in such a way that Chern and Todd classes $\mathrm{Ch}_G(\mathcal{L}_k(n))$, $\mathrm{Td}_G(\mathcal{T}\mathcal{L}\mathbb{P}^{\ell}_{+})$ make sense and the expression*

$$\int_{\mathcal{L}\mathbb{P}^{\ell}_{+}}\mathrm{Ch}_G(\mathcal{L}_k(n))\,\mathrm{Td}_G(\mathcal{T}\mathcal{L}\mathbb{P}^{\ell}_{+})$$

is well-defined and is equal to

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k) \,=\, -\Big(\prod_{j=1}^{\ell+1} z_j^k\Big)\oint\limits_C \frac{dt}{2\pi\imath\, t^{n+1}}\prod_{i=1}^{\ell+1}\Gamma_q(tz_i). \tag{3.5}$$

Remark 3.1 *The conjectural relation above provides a description of the specialized q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a semi-infinite period*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k) = \int_{\mathcal{L}\mathbb{P}^{\ell}_{+}}\mathrm{Ch}_G(\mathcal{L}_k(n))\,\mathrm{Td}_G(\mathcal{T}\mathcal{L}\mathbb{P}^{\ell}_{+}),\qquad n\geq 0. \tag{3.6}$$

The K-theory of the semi-infinite spaces $\mathcal{L}\mathbb{P}^{\ell}_{+}$ is closely connected with a quantum version of K-theory of projective spaces proposed in [GiL]. The generating function $F(n,\underline{z},q)$ of the correlation functions in K-theory version of Gromov-Witten theory with the target space $\mathbb{P}^{\ell}$ obeys the following difference equation [GiL]

$$\Big\{\prod_{i=1}^{\ell+1}(1-z_iT^{-1})\Big\}\cdot F(n,\underline{z},q) \,=\, q^nF(n,\underline{z},q), \tag{3.7}$$

where $T\cdot f(n)=f(n+1)$. The specialized q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function satisfies the same equation (3.7) (see Lemma 1.1 and relation (1.10)). Therefore the Whittaker function can be considered as a correlation function of some special operator singled out by the class one condition (i.e. the condition $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(n,k)=0$ for $n<0$). We provide some information on this operator in the Section 5.

4 Quantum cohomology and Whittaker function

In the previous Section we proposed a description of q-deformed class one specialized $\mathfrak{gl}_{\ell+1}$-Whittaker function in terms of a semi-infinite version of Riemann-Roch-Hirzebruch theorem. This expresses the q-Whittaker function as a semi-infinite period. Its classical (i.e. non-deformed) counterpart can be also expressed in terms of a semi-infinite period. In this Section we provide this conjectural representation [2].

We start from recalling the notion of quantum cohomology. The quantum cohomology $QH^*(X)$ of a compact symplectic manifold X can be defined in terms of semi-infinite geometry of a universal cover $\widetilde{LX}$ of the loop space LX. One of the descriptions is given by a Morse-Smale-Bott-Novikov-Floer complex constructed in terms of critical points of an area functional on $\widetilde{LX}$. Its cohomology groups (interpreted as Floer cohomology groups $FH^*(\widetilde{LX})$ of $\widetilde{LX}$) are isomorphic to the semi-infinite cohomology $H^{\infty/2+*}(LX)$ arising naturally in the Hamiltonian formalism of a topological two-dimensional sigma model with the target space X. In the following we will use an equivariant version of quantum cohomology $QH^*(\mathbb{P}^\ell)$ of projective spaces considered in [Gi1] (see also [CJS] for a non-equivariant version).

We have defined the universal covering $\widetilde{LX}$ of the loop space LX as a space of maps $D \to X$ of the disk D considered up to a homotopy preserving the image of the boundary loop $S^1 \subset D$. The group of covering transformations of the universal cover $\widetilde{LX} \to LX$ is isomorphic to the image $\Gamma \subset H_2(X)$ of the Hurewicz homomorphism $\pi_2(X) \to H_2(X)$ where $H_2(X)$ denotes integral homology modulo torsion. Let r be the rank of Γ and $\mathbb{C}\big[\Gamma\big] \simeq \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$ be its group algebra.

As a vector space the quantum cohomology $QH^*(X)$ of X as a vector space is isomorphic to the ordinary cohomology $H^*(X, \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}])$, over the group algebra $\mathbb{C}\big[\Gamma\big] \simeq \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$. Let S^1 act on the loop space LX by loop rotations. For the corresponding S^1-equivariant quantum cohomology we have the following isomorphism of vector spaces:

$$QH^*_{S^1}(X) \,=\, H^*\big(X, \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}](\hbar)\,\big),$$

where we use the identification

$$H^*_{S^1}(\mathrm{pt}, \mathbb{C}) = H^*(BS^1, \mathbb{C}) = \mathbb{C}[\hbar],$$

and the standard localization of the equivariant cohomology $H^*_{S^1}(\mathrm{pt})$ with respect to the maximal ideal generated by $\hbar$ is implied.

The quantum cohomology space $QH^*_{S^1}(X)$ has a natural structure of a module over an algebra $\mathcal{D}$ generated by $u_i = \exp \tau_i$, $v_i = -\hbar\partial/\partial\tau_i$, $i = 1, \ldots, r$. More precisely, $QH^*_{S^1}(X)$ as a linear space over $\mathbb{C}(\hbar)$ is generated by solutions of the system of linear differential equations

$$\nabla_i\, f(\underline{\tau}) = 0, \qquad f(\underline{\tau}) = (f_1(\underline{\tau}), f_2(\underline{\tau}), \ldots, f_n(\underline{\tau})), \qquad n = \dim H^*(X), \tag{4.1}$$

where the flat connection $\nabla = \sum_{i=1}^r d\tau_i \nabla_i$ provides an action of v_i on $QH^*_{S^1}(X)$.

The $\mathcal{D}$-module $QH^*_{S^1}(\mathbb{P}^\ell)$ is of rank one. It is generated over $\mathcal{D}$ by an element f_* satisfying the relation $(v^{\ell+1} - u)f_* = 0$ i.e. the quantum cohomology can be represented as $QH^*(\mathbb{P}^\ell) \simeq \mathcal{D}/(v^{\ell+1} - u)$. Explicitly we have the differential equation for the generator $f_*(\tau)$

$$\Big\{ \Big(-\hbar\frac{\partial}{\partial\tau}\Big)^{\ell+1} \,-\, e^\tau \Big\}\, f_*(\tau, \hbar) \,=\, 0. \tag{4.2}$$

[2]Whittaker functions naturally arise in the description of Gromov-Witten invariants of flag spaces. In the mirror dual description they expressed in terms of periods of top-dimensional holomorphic forms on non-compact Calabi-Yau spaces [Gi2]. Thus, the possibility to express Whittaker functions as semi-infinite periods leads to a formulation of the mirror symmetry as an identification of two period maps - semi-infinite and finite ones.

The representation (4.1) arises after transformation of (4.2) to the matrix differential equation of the first order.

The $(S^1\times U_{\ell+1})$-equivariant analog of quantum cohomology $QH^*(\mathbb{P}^\ell)$ allows for a similar representation with the differential equation (4.2) replaced by

$$\Big\{\prod_{k=1}^{\ell+1}\Big(-\hbar\frac{\partial}{\partial\tau}+\lambda_k\Big)\,-\,e^{\tau}\Big\}\,f_*(\tau,\underline{\lambda},\hbar)\;=\;0. \tag{4.3}$$

Lemma 4.1 *The general solution of* (4.3) *is given by a linear combination of the integrals*

$$f^{(a)}(\tau,\underline{\lambda},\hbar)=\int_{\gamma_a}d\lambda\; e^{\frac{\lambda\tau}{\hbar}}\prod_{k=1}^{\ell+1}\hbar^{\frac{\lambda_k-\lambda}{\hbar}}\,\Gamma\Big(\frac{\lambda_k-\lambda}{\hbar}\Big),\qquad a=1,\cdots,n \tag{4.4}$$

with a suitable choice of integration contours γ_a.

Proof: Note that the function

$$Q(\lambda,\underline{\lambda})\;=\;\prod_{k=1}^{\ell+1}\hbar^{\frac{\lambda_k-\lambda}{\hbar}}\,\Gamma\Big(\frac{\lambda_k-\lambda}{\hbar}\Big). \tag{4.5}$$

obeys the difference equation

$$\prod_{k=1}^{\ell+1}(\lambda-\lambda_k)\,Q(\lambda,\underline{\lambda})\;=\;(-1)^{\ell+1}Q(\lambda-\hbar,\underline{\lambda}). \tag{4.6}$$

Therefore, the function

$$f(\tau,\underline{\lambda})=\int_{\gamma}d\lambda\; e^{\frac{\lambda\tau}{\hbar}}\;Q(\lambda,\underline{\lambda},\hbar) \tag{4.7}$$

satisfies (4.3) provided the choice of the contour γ allows for an integration by parts. The contours can be chosen is such a way that the total derivatives do not give a contribution into the integral (4.4) □

A particular choice of γ in (4.7) gives us a special solution of the equation (4.3)

$$f_*(\tau,\underline{\lambda},\hbar)\;=\int\limits_{\sigma-\imath\infty}^{\sigma+\imath\infty}d\lambda\;\,e^{\frac{\lambda\tau}{\hbar}}\;\prod_{k=1}^{\ell+1}\hbar^{\frac{\lambda_k-\lambda}{\hbar}}\,\Gamma\Big(\frac{\lambda_k-\lambda}{\hbar}\Big), \tag{4.8}$$

where σ is such that $\sigma<min\{\,\mathrm{Re}\,\lambda_j,j=1,\ldots,\ell+1\}$. This is a unique solution of (4.3) exponentially decaying when $\tau\to+\infty$.

Remark 4.1 *In the case of* $\ell=1$, *the differential equation* (4.2) *is equivalent to an eigenvalue problem for the quadratic Hamiltonian of the* $\mathfrak{sl}_2$*-Toda chain. The solution given in the integral form* (4.8) *coincides in this case with the Mellin-Barnes representation of the* $\mathfrak{gl}_2$*-Whittaker function (1.14) for* $x_2=0$ *and* $x_1=\tau$.

Replacing formally Γ-functions by infinite products over their poles one has for $f_*(\tau,\underline{\lambda},\hbar)$ the following expression

$$\int dx\, e^{\tau x/\hbar}\prod_{j=1}^{\ell+1}\prod_{n=0}^{\infty}\frac{1}{x-\lambda_j-\hbar n}. \tag{4.9}$$

This formal representation can be interpreted using the model for cohomology of $\mathcal{QM}_d(\mathbb{P}^\ell)$ discussed n Section 2.3. Naively (4.9) can be considered as an integral over $\mathcal{L}\mathbb{P}^\ell_+$ of $\exp(\tau\omega/\hbar)$ where ω is an element of the second $S^1\times U_{\ell+1}$-equivariant cohomology of $\mathcal{L}\mathbb{P}^\ell_+$. Recall that we define $\mathcal{L}\mathbb{P}^\ell_+$ on the level of cohomology as a limit of $\mathcal{QM}_d(\mathbb{P}^\ell)$ when $d\to+\infty$. However a correct regularization for (4.9) is given by (4.8) and, thus, a geometric interpretation of (4.8) implies some modification of $\mathcal{L}\mathbb{P}^\ell_+$. We attribute the difference between (4.9) and (4.8) to the fact that the proper interpretation of $\mathcal{L}\mathbb{P}^\ell_+$ as $d\to+\infty$ limit deserves more care in this case and does not coincide with a straightforward limit $d\to+\infty$ on the level of cohomology. Let us denote the corresponding hypothetically modified limit by $\mathbf{L}\mathbb{P}^\ell_+$.

Porblem 4.1 *Find the space* $\mathbf{L}\mathbb{P}^\ell_+$ *and construct equivariant (co)homology for* $\mathbf{L}\mathbb{P}^\ell_+$ *in such a way that the integral*

$$\int_{\mathbf{L}\mathbb{P}^\ell_+} e^{\tau\,\omega/\hbar},\qquad \omega\in H^2_{S^1\times U_{\ell+1}}(\mathbf{L}\mathbb{P}^\ell_+,\mathbb{C})$$

is well-defined and is equal to $f_*(\tau,\underline{\lambda},\hbar)$ *given by* (4.8).

5 S^1-localization

In this Section we calculate the equivariant Euler characteristic $\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{L}_k(n))$ for $G=S^1\times U_{\ell+1}$ using Borel localization for S^1-action. This yields a direct relation between our construction of q-Whittaker functions and the results of [GiL].

The character (2.23) can be calculated using an equivariant localization as follows. We have a compact Lie group $G=S^1\times U_{\ell+1}$ acting on a projective space $X=\mathcal{QM}_d(\mathbb{P}^\ell)\equiv\mathbb{P}^{(\ell+1)(d+1)-1}$. Recall that $\mathcal{QM}_d(\mathbb{P}^\ell)$ is defined as a set of $(\ell+1)$ polynomials each of degree d considered up to common constant factor

$$\big(a_0(y),a_1(y),\dots a_\ell(y)\big)\sim\big(\rho\, a_0(y),\rho\, a_1(y),\dots\rho\, a_\ell(y)\big),\qquad \rho\in\mathbb{C}^*,$$

where

$$a_k(y)=\sum_{i=0}^d a_{k,i}\,y_1^i\,y_2^{d-i},\qquad k=0,\dots,\ell.$$

The action of an element $q=e^{\imath\theta}$ of S^1 on $a_{k,j}$ is given by

$$q:\ a_{k,j}\to q^j a_{k,j}.$$

The line bundles $\mathcal{L}_k(n)$ are equivariant with respect to the action of S^1. Let $X^{S^1}\subset X$ be a set of S^1-fixed points. It is a union of smooth components Y_i. The Bott localization formula gives the following expression for the equivariant Euler characteristic (2.23) (see e.g. [BGV])

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell),\mathcal{L}_k(n))=\sum_{Y_i\in X^{S^1}}\int_{Y_i}\frac{\mathrm{Ch}_G(\mathcal{L}_k(n)|_{Y_i})\mathrm{Td}_G(TY_i)}{\mathrm{E}_G(\mathcal{N}_{Y_i})}, \tag{5.1}$$

where the sum runs over all components Y_i in X^{S^1}, $\mathcal{N}_{Y_i}$ is the normal bundle of Y_i in X, $\mathrm{Ch}_G(\mathcal{L}_k(n))$, $\mathrm{Td}_G(Y_i)$ are equivariant Chern character and Todd class, and $\mathrm{E}_G(\mathcal{N}_{Y_i})$ is the equivariant Euler class of $\mathcal{N}_{Y_i}$.

It is easy to infer that the subvarieties Y_i, $i = 0, \dots, \ell$ in $\mathcal{QM}_d(\mathbb{P}^\ell)$ are isomorphic to the projective spaces $\mathbb{P}^\ell$ and are defined by the equations

$$Y_i = \{a_{k,j} = 0,\, j \neq i\}, \qquad i = 0, \dots, \ell.$$

To calculate the action of S^1 on the normal bundle to Y_i we consider the intersection of Y_i with open subsets $U_{a_{k,i}} = \{a_{k,i} \neq 0\}$ of $\mathcal{QM}_d(\mathbb{P}^\ell)$. Natural coordinates on $U_{a_{k,i}}$ are

$$\xi_{r,j} = a_{r,j}/a_{k,i}, \qquad (r, j) \neq (k, i),$$

and the intersections $Y_i \cap U_{a_{k,i}} = \{a_{k,i} \neq 0\}$ are defined by the equations

$$\xi_{r,i} = 0, \qquad r \neq k.$$

Thus one can take a collection of coordinates $\xi_{r,j}$, $j \neq i$ as a local section of the dual to the normal bundle $\mathcal{N}_{Y_i}$. The action of S^1 on $\mathcal{N}_{Y_i}$ can then be found by considering the action on section $\xi_{r,j}$:

$$\xi_{r,j} \to q^{j-i}\xi_{r,j}.$$

Similarly, one can show that $q \in S^1$ acts on the restriction of the line bundle $\mathcal{O}()$ on Y_i by multiplication on $q^{n\,i}$. The fixed point formula (5.1) reduces to the following explicit expression

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^\ell), \mathcal{L}_k(n)) =$$

$$= -\Big(\prod_{j=1}^{\ell+1} z_j^k\Big) \sum_{i=0}^{d} \int_{C_0} \frac{dt}{2\pi\imath t^{n+1}} \frac{q^{ni}}{\prod_{j=1}^{\ell+1}\prod_{m=0,m\neq i}^{d}(1 - t z_j q^{m-i})} \frac{1}{\prod_{j=1}^{\ell+1}(1 - t z_j)},$$

where the integration contour C_0 encircles the $(\ell + 1)$ poles defined by the equations $t = z_j^{-1}$, $j = 1, \dots, \ell + 1$.

Lemma 5.1 *The following identity holds for $n \geq 0$*

$$\oint_C \frac{dt}{2\pi\imath t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1}\prod_{m=0}^{d}(1 - t z_j q^m)} =$$

$$= \sum_{i=0}^{d} \int_{C_0} \frac{dt}{2\pi\imath t^{n+1}} \frac{q^{ni}}{\prod_{j=1}^{\ell+1}\prod_{m=0,m\neq i}^{d}(1 - t z_j q^{m-i})} \frac{1}{\prod_{j=1}^{\ell+1}(1 - t z_j)}, \tag{5.2}$$

where the integration contour C encircles poles defined by the equations $t = z_j^{-1}q^{-i}$, $j = 1, \dots, \ell+1$, $i = 0, \dots, d$ and the integration contour C_0 encircles $(\ell + 1)$ poles defined by the equations $t = z_j^{-1}$, $j = 1, \dots, \ell + 1$.

Proof: We have that

$$\oint_C \frac{dt}{2\pi\imath t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1}\prod_{m=0}^{d}(1 - t z_j q^m)} = \sum_{i=0}^{d} \int_{C_i} \frac{dt}{2\pi\imath t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1}\prod_{m=0}^{d}(1 - t z_j q^m)},$$

where the integration contour C_i encircles $(\ell+1)$ poles defined by the equations $t = z_j^{-1}q^{-i}$, $j=1,\dots,\ell+1$. Making the change of variables $t \to tq^{-i}$ in the r.h.s., we obtain that

$$\oint_C \frac{dt}{2\pi\imath t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1}\prod_{m=0}^{d}(1-tz_jq^m)} =$$

$$\sum_{i=0}^{d}\int_{C_i} \frac{dt}{2\pi\imath t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1}\prod_{m=0}^{d}(1-tz_jq^m)} = \sum_{i=0}^{d} q^{ni}\int_{C_0} \frac{dt}{2\pi\imath t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1}\prod_{m=0}^{d}(1-tz_jq^{m-i})}$$

□

We are going to consider a continuation of the expression (5.2) to $q \in \mathbb{C}^*$, $|q|<1$ and the limit $d \to \infty$.

Proposition 5.1 *The specialization* (1.9), (1.12) *of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function can be written in the following form*

$$\Psi_{\underline{z}}^{\mathfrak{gl}_{\ell+1}}(n,k) = \langle I_{n,k}(\underline{z})\,\tilde{L}(\underline{z}),[\mathbb{P}^\ell]\rangle_K, \tag{5.3}$$

where

$$\tilde{L}(\underline{z},t) = \frac{1}{\prod_{j=1}^{\ell+1}\prod_{k=1}^{\infty}(1-tz_jq^k)} = \prod_{j=1}^{\ell+1}\Gamma_q(qtz_j),$$

$$I_{n,k}(\underline{z},t) = \Big(\prod_{j=1}^{\ell+1} z_j^k\Big)\, t^{-n}\sum_{i=0}^{\infty} q^{ni}\frac{1}{\prod_{j=1}^{\ell+1}\prod_{m=1}^{i}(1-tz_jq^{-m})}, \tag{5.4}$$

and the pairing $\langle\,,\,\rangle_K$ is the standard pairing (2.19) *on $K_{U_{\ell+1}}(\mathbb{P}^\ell)$ taking values in $K_{U_{\ell+1}}(pt)$.*

The representation of the Whittaker function given in Proposition 5.1 establishes a direct connection with the results of Givental-Lie [GiL]. In [GiL] the function (5.4) was interpreted as a universal solution of the reduction (1.10) of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. Indeed, the function $I_{n,k}(\underline{z},t)$ satisfies the eigenvalue problem

$$\prod_{j=1}^{\ell+1}(1-z_iT^{-1})\,I_{n,k}(\underline{z},t) = q^n\,I_{n,k}(\underline{z},t), \tag{5.5}$$

modulo the relation $\prod_{j=1}^{\ell+1}(1-tz_j)=0$ holding in $K_{U_{\ell+1}}(\mathbb{P}^\ell)$ and is uniquely determined by the normalization condition

$$I_{n,k}(\underline{z},t)|_{q=0} = \Big(\prod_{j=1}^{\ell+1} z_j^k\Big)\,t^{-n}, \qquad n \geq 0.$$

The solution $I_{n,k}(\underline{z},t)$ is universal in the sense that taking the pairing

$$\langle I_{n,k}(\underline{z}),f\rangle_K = -\frac{1}{2\pi\imath}\oint_{C_0}\frac{dt}{t}\,\frac{I_{n,k}(\underline{z},t)\,f(t)}{\prod_{j=1}^{\ell+1}(1-z_jt)}, \tag{5.6}$$

with arbitrary $f \in K_{U_{\ell+1}}(\mathbb{P}^\ell)$ one obtains a solution (5.6) of the q-deformed reduced $\mathfrak{gl}_{\ell+1}$-Toda chain (1.10).

6 Semi-infinite Todd genus and q-Gamma function

In the explicit expression for the cohomological pairing on $\mathcal{L}\mathbb{P}^{\ell}_{+}$ conjectured in Problem 3.1 the q-Gamma function Γ_q plays the role similar to the Todd genus in the analogous pairing for underlying finite-dimensional space $\mathbb{P}^{\ell}$. The S^1-localization discussed in the previous section reduces the pairing of the Chern and Todd classes on $\mathcal{L}\mathbb{P}^{\ell}_{+}$ to the paring of some cohomology classes on $\mathbb{P}^{\ell}$. It is an interesting problem to interpret the resulting cohomology classes on $\mathbb{P}^{\ell}$ in terms of some geometric objects on $\mathbb{P}^{\ell}$. For example in an analogous case of S^1-localization of K-theory on the loop space LX, the elliptic genus of X arises. The corresponding elliptic cohomology is an instance of an extraordinary cohomology theory. In this Section we discuss the result of S^1-localization on $\mathcal{L}\mathbb{P}^{\ell}_{+}$ from the extraordinary cohomology perspective. We demonstrate that an intrinsic non-local nature of the $\mathbb{C}^*$-localization on $\mathcal{L}\mathbb{P}^{\ell}_{+}$ obstructs a straightforward relation with the formalism of extraordinary cohomology theories and corresponding multiplicative genera. Let us remark that classical Γ-function was considered as a candidate for a topological genus by Kontsevich in [K]. Also a kind of Γ-genus also appeared in obviously related context in [Li], [Ho] (see also [CGi1], [CGi2] for a discussion of formal groups in a quantum version of cobordism theory).

We first recall standard facts on multiplicative topological genera and formal group laws corresponding to complex oriented cohomology theories (see e.g. [BMN]). The Hirzerbruch multiplicative genus is a homomorphism $\varphi : \Omega^* \to \mathcal{R}$ of the ring of complex cobordisms $\Omega^* = \Omega^*(\mathrm{pt})$ to a ring of coefficients $\mathcal{R}$. One has a Thom isomorphisms $\Omega^* \otimes \mathbb{Q} = \mathbb{Q}[x_1, x_2, \cdots]$, $\deg(x_i) = -2i$ and the topological genus φ is characterized by its values on generators x_i that can be represented by Pontryagin-Thom duals to complex projective spaces $\mathbb{P}^i$. Equivalently φ defined over $\mathbb{Q}$ can be described in terms of a one-dimensional commutative formal group law

$$f_{\varphi}(z, w) = e_{\varphi}(\log_{\varphi}(z) + \log_{\varphi}(w)), \tag{6.1}$$

expressed through the logarithm function

$$\log_{\varphi}(z) = z + \sum_{n=1}^{\infty} \frac{\varphi([\mathbb{P}^n])}{n+1}\, z^{n+1},$$

and its inverse $e_{\varphi}(u)$. For instance rational cohomology and K-theory correspond to additive and multiplicative group laws

$$f_H(z, w) = z + w, \qquad f_K(z, w) = z + w - zw.$$

To a genus φ one associates a multiplicative sequence $\{\Phi_n(c_i)\}$, $\deg(\Phi_n) = n$ of cohomology classes

$$P_{\varphi} = \sum_{n=0}^{\infty} \Phi_n(c_i) = \prod_{j=1}^{N} \frac{x_j}{e_{\varphi}(x_j)},$$

and a map

$$X \to \varphi(X) = \langle \Phi(\mathcal{T}X), [X] \rangle, \qquad \dim_{\mathbb{C}} X = N.$$

Here $\mathcal{T}X$ is the tangent bundle to a manifold X, $[X]$ is the fundamental class in the homology of X and $\langle\, ,\, \rangle$ is a standard pairing. The classes x_i are defined in terms of Chern classes c_i of $\mathcal{T}X$ using a splitting of $\mathcal{T}X$

$$c(X) = 1 + \sum_{i=1}^{N} c_i(X) = \prod_{j=1}^{N} (1 + x_j).$$

In the case of additive and multiplicative group laws we have respectively that

$$P_{\varphi_H}(x)=1, \qquad \log_{\varphi_H}(z)=z, \qquad e_{\varphi}(u)=u;$$

$$P_{\varphi_K}(x) \,=\, \prod_{j=1}^{n}\frac{x_j}{1-e^{-x_j}}, \qquad \log_{\varphi_K}(z)=-\ln(1-z), \qquad e_{\varphi_K}(u)\,=\,1-e^{-u}\,.$$

Note that $P_{\varphi_K}(x)$ defines the Todd class of $\mathcal{T}X$. For example, the equivariant Riemann-Roch-Hirzebruch theorem for a line bundle $\mathcal{L}_k(n)$ on $\mathbb{P}^\ell$ can be represented in the following form

$$\begin{aligned}\chi_{U_{\ell+1}}(\mathbb{P}^\ell,\mathcal{L}_k(n)) \,&=\, \big\langle \mathrm{Ch}_{U(\ell+1)}\big(\mathcal{L}_k(n)\big)\,\mathrm{Td}_{U(\ell+1)}(\mathcal{T}\mathbb{P}^\ell),\,[\mathbb{P}^\ell]\big\rangle \qquad (6.2)\\ &= \frac{1}{2\pi\imath}\oint_{C_0} dx\; e^{nx+k(\lambda_1+\ldots+\lambda_{\ell+1})}\prod_{i=1}^{\ell+1}\frac{1}{1-e^{\lambda_i-x}}\\ &= \frac{1}{2\pi\imath}\oint_{C_0} dx\; e^{nx+k(\lambda_1+\ldots+\lambda_{\ell+1})}\prod_{i=1}^{\ell+1}\frac{1}{e_{\varphi_K}(x-\lambda_i)}.\end{aligned}$$

Here $e_{\varphi_K}(x)$ is the exponent corresponding to K-theory (see the Remark 2.2).

Using the conjectural relation in Problem 3.1, the $S^1\times U_{\ell+1}$-equivariant Riemann-Roch-Hirzebruch theorem for a trivial line bundle on $\mathcal{L}\mathbb{P}^\ell_+$ can be represented in the form similar to (6.2) for $k=n=0$

$$\chi_{S^1\times U_{\ell+1}}(\mathcal{L}\mathbb{P}^\ell_+,\mathcal{L}_0(0)) \,=\, \frac{1}{2\pi\imath}\oint_C dx\,\prod_{j=1}^{\ell+1}\prod_{m=0}^{\infty}\frac{1}{1-e^{\lambda_j+m\hbar-x}} = \qquad (6.3)$$

$$= \frac{1}{2\pi\imath}\oint_C dx\,\prod_{i=1}^{\ell+1}\frac{1}{e_{\varphi_q}(x-\lambda_i)},$$

where C encircles all the poles $x=\lambda_j+m\hbar$, $j=1,\ldots,(\ell+1)$, $m\in\mathbb{Z}_{\geq 0}$ and we used a notation

$$e_{\varphi_q}(u;\hbar)=\frac{1}{\Gamma_q(e^{-u})} \,=\, \prod_{n=0}^{\infty}\left(1-e^{n\hbar-u}\right). \qquad (6.4)$$

However, despite a similarity of (6.2) and (6.3) the difference in integration contours does not allow directly to interpret $e_{\varphi_q}(u;\hbar)$ as a topological genus corresponding to a extraordinary cohomology theory on $\mathbb{P}^\ell$. The way to transform (6.3) into an integral over the contour C_0 was discussed in Section 5. As a result the integration contour in (6.3) can be replaces by C_0 at the expense of multiplying the integrand by the correction factor $I_{0,0}(\underline{z},e^{-x})$ (see (5.4) for explicit expression for $I_{n,k}(\underline{z},e^{-x})$). Now we have an expression for the equivariant Euler characteristic on $\mathcal{L}\mathbb{P}^\ell_+$ in terms of the pairing of cohomology classes on $\mathbb{P}^\ell$. However this correction factor appears to spoil the multiplicative property of (6.4). The underlying reason for this is the appearance of an infinite number of copies of $\mathbb{P}^\ell$ as components of fixed point set in S^1-localization. Thus, the situation is very much different from, for example, the elliptic genus (see e.g. [Se]) where the fixed point set of S^1 acting on LX is simply X itself. It is conceivable that the failure to interpret S^1-localization on $\mathcal{L}\mathbb{P}^\ell_+$ in terms of an extraordinary topological genus implies actually the existence of a meaningful quantum version of the an extraordinary cohomology theory formalism.

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