Modular space associated to a graded one-dimensional complete intersection singularity

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Abstract: Modular families and modular spaces for quasi-homogeneous complete intersection singularities of curves with modularity one are computed. In particular, it is proved that the modular space of any unimodular singularity has a covering which is biholomorphically equivalent to a punctured or twice punctured Riemann sphere. Some related questions concerning singularities of higher modularity are also discussed.

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1. Introduction

The problem of classification of singularities is a significant part of general deformation theory. In principal, one may consider different equivalence relations between singularities such as the right equivalence (change of coordinates in the source of defining mapping), contact equivalence (change of coordinates and multiplication with a unit, that is, preserving the isomorphism class of the corresponding germ), and others (see e.g. [20]). In the framework of the corresponding deformation theory, the first step in solving the classification problem is the description of simple singularities. As a rule, one can write down a finite or at least a perceptible list of normal forms or similar data. However a scheme of classification of more complicated singularities seems to be a rather nontrivial and hard problem since such phenomenon as moduli or parametric families may occur. Among different approaches to further classification, the most fruitful and useful are probably those which are based on the analytical and geometrical methods. We shall discuss here some aspects of the approach originated in the work of V.P.Palamodov [13].

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Given a singularity X_0, the base space of its minimal versal deformation contains a remarkable subspace M consisting of points that satisfy a certain condition of (semi)universality. It is called the maximal modular germ, and its dimension is called the modularity of X_0.

Some fibers of the family induced over M can be isomorphic, though nearby fibers are non-isomorphic in virtue of the universality. In many interesting cases this allows to consider a natural action of a discrete group G on M. More precisely, the group G is the quotient of the automorphism group of the distinguished fiber X_0 of the modular family by the connected component of a unit. Thus for any two points M close enough to the distinguished one the corresponding fibers are isomorphic, if and only if these points belong to the same orbit of the action of the discrete group G on M. It does mean that the quotient M/G may be regarded as the germ of a modular space, i.e. as a local prototype of the corresponding moduli space. In fact, for certain singularities all fibers X_t over M, as well as the distinguished one, have the same (the greatest possible) dimensions of their first tangent cohomology groups. It is called the Tjurina number of X_0 and denoted by $\tau = \tau(X_0)$. For example, one can prove (see [3]) that for quasi-homogeneous isolated complete intersection singularities the germ M is reduced and coincides with the stratum $\{\tau_{\max} = \text{const}\}$. It consists of points $s \in S$ for which $\tau(X_s) = \tau(X_0)$.

The notions of the modular deformation, maximal modular germ, and modular space in the category of complex spaces have been introduced and investigated by V.P.Palamodov [13] and somewhat later, in a more general context, by O.Laudal [9]. In the special case of an isolated complete intersection singularity a closely related notion of the stratum $\{\tau_{\max} = \text{const}\}$ has been introduced and studied in the earlier works of B.Teissier (see [20], (4.8)) and M.Lindner [12].

The further development of the theory enables one to consider a special stratification of the base space of the versal deformation. Its strata M_τ correspond to singular fibers with fixed Tjurina numbers $\tau_{\min} \leq \tau \leq \tau_{\max}$. This leads directly to the analysis of the structure of the coarse moduli space of a given algebraic or analytical variety. It turns out that, in general, the coarse moduli space does not lie in the categories of algebraic or analytical varieties. In particular, it may be non-Hausdorff and can possesses many other unexpected properties. The reason is that the kernel of Kodaira-Spencer map has a highly complicated and complex structure. A series of works of O.Laudal, G.-M.Greuel and their followers (see [11], [7], [8], and others) are devoted to the study of different aspects of this problematic. Thus they construct a (coarse) moduli space for isolated hypersurface singularities with certain fixed invariants.

The present paper deals with the problem of explicit calculation of modular families, modular and moduli spaces. In the first section we briefly discuss the notion of modular germ, due to V.P.Palamodov [13]. The second section is devoted to the study of the properties of modular germs for graded singularities. In the next two sections we compute the modular spaces for unimodular quasi-homogeneous germs of curves, which are complete intersection with Milnor numbers less than seventeen and modularity one. In fact, our computational results give a partial answer on a question posed in [16], Problem 4.4. In the final section singularities of higher modularity are considered. We also discuss some questions, relating to the notion of adjacency and to computing modular spaces for non-complete intersection singularities.

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2. Modular germ and modularity

First, we recall the notion of modular germ and modular deformation, following [13], [14]. Let $X_0 = (X_0, 0)$ be the germ of an isolated singularity, and let $\phi : (X, X_0) \to (Y, *)$ be a flat deformation of X_0.

Definition 2.1. A subgerm $M \subset Y$ is called *modular* if for any two morphisms of germs $g : Z \to M$ and $h : Z \to Y$ the following condition holds: If there is an isomorphism $g^*(f) \cong h^*(f)$ of the induced deformations, then the germs $g(Z) = h(Z)$ coincide. The restriction of the deformation ϕ to the modular germ M is called the *modular deformation* of X_0.

For simplicity, in what follows, we only consider the case when ϕ is the *minimal versal deformation* of X_0; it is denoted by $\varphi : (X, X_0) \to (S, *)$. Let $T^0(X_0)$ be the module of holomorphic vector fields on X_0, and let $T^0(X/S)$ be the module of "relative" vector fields on X/S. The latter consists of the so-called *vertical* vector fields on X, i.e. of those vector fields that are tangent to any fiber of the minimal versal deformation at each of its points (see [15]). In the case when X_0 is a complete intersection, they are usually called hamiltonian fields (see [4]). Then the following natural mapping

$$u : T^0(X/S) \to T^0(X_0)$$

is defined. The quotient module

$$T^0_\bullet(X_0) = T^0(X_0)/T^0(X/S)$$

is called the *truncated* zero-dimensional tangent cohomology of X_0. One then gets the following conditions of modularity.

Theorem 2.2 ([14], (6.2)). *Let $\varphi : X \to S$ be the versal deformation of X_0. A subgerm $M \subset S$ is modular if for any artinian subgerm $Z \subset M$ the restriction*

$$T^0(X/S)|_Z \longrightarrow T^0(X_0)$$

is surjective. In other words, any element of $T^0(X_0)$ or, equivalently, of $T^0_\bullet(X_0)$ can be extended to a vertical vector field on X/S.

Theorem 2.3 ([13], [14]). *Let $\varphi : (X, X_0) \to (S, *)$ be the minimal versal deformation of an isolated singularity X_0. Then there exists a subgerm $M \subset S$ with the following properties:*

1) *The germ M is modular and contains all modular subgerms of $(S, *)$.*

2) *The support M_{red} of the germ M is the set of points $s \in S$ such that φ is the minimal versal deformation of the fiber germ X_s.*

3) *The Zariski tangent space $\mathcal{T}_*(M) \subset \mathcal{T}_*(S)$ may be defined as follows:*

$$\mathcal{T}_*(M) = \{t \in \mathcal{T}_*(S) : [D_f(t), v] = 0 \text{ for all } v \in T^0(X_0)\},$$

where the bracket denotes the Lie operation in the tangent cohomology and

$$D_f : \mathcal{T}_*(S) \to T^1(X_0)$$

is the Kodaira-Spencer map of the deformation φ.

4) *The set of those $s \in S$ such that $\dim_{\mathbf{C}} T^1(X_s) = \dim_{\mathbf{C}} T^1(X_0)$ is contained in the support M_{red}. Furthermore, if $T^2(X_0) = 0$ then both sets are identical.*

Definition 2.4. The conditions of the Theorem (2.3) characterize the germ M which is called the *maximal modular germ.* The restriction of the minimal versal deformation $\varphi|_M$ is called the *maximal modular deformation* of X_0.

Remark 2.5. It is not difficult to see that any element $v \in u(T^0(X/S))$ is contained in the kernel of the canonical action of $T^0(X_0)$ on $T^1(X_0)$. Thus, in the third condition of the Theorem one can replace the space $T^0(X_0)$ by $T^0_{\bullet}(X_0)$.

Remark 2.6. It is well known that the minimal versal deformation of an isolated singularity is defined uniquely up to a *non-canonical* isomorphism only. However, the maximal modular germ it is defined uniquely up to a *canonical* isomorphism in the sense of [14], Proposition 7.4.

In fact, it follows from these theorems that the structure sheaf of a modular germ M may contain nilpotent elements. Thus, in general, a modular germ may be non-reduced, that is, $M \neq M_{\mathrm{red}}$. The corresponding examples have been considered by V.P. Palamodov in 1976 (cf. [15], [16], Ex. 3.2). Here is an example of a non-reduced modular germ which is a *multiple* point.

Example 2.7. On the maximal modular germ of an isolated hypersurface singularity given by the semi-quasi-homogeneous function $f = x^3 + y^7 + xy^5$.

The Tjurina number of the singularity X_0 is equal to 11. The tangent space $T_*(S)$ at the distinguished point of the base of the versal deformation is isomorphic to the space of infinitesimal deformations of the first order:

$$T^1(X_0) = \{1, x, y, x^2, xy, y^2, x^2y, xy^2, y^3, xy^3, y^4\}.$$

The truncated zero-dimensional tangent cohomology of X_0 is generated by two elements. To be precise,

$$T^0_\bullet(X_0) = T^0(X_0)/T^0(X/S) = \{\delta_x, \delta_y\}\mathcal{O}_{X_0},$$

where

$$\delta_x = (7x^2 - 49y^5c)\partial_x + (3xy - 5xy^2c + 7y^4c)\partial_y,$$
$$\delta_y = (7xy + 35y^4c)\partial_x + (3y^2 - 21xc - 5y^3c)\partial_y,$$

and $c = 1/(147 + 25y)$.

Since $\delta_x(f) = 21fx$ and $\delta_y(f) = 21fy$ the natural action of $T^0(X_0)$ on $T^1(X_0)$ (see [15]) gives us the following:

$$[\delta_x, e] = \delta_x(e) - 21xe, \quad [\delta_y, e] = \delta_y(e) - 21ye,$$

where $e \in T^1(X_0) \cong T_*(S)$. Hence

$$\delta_x : \mathbf{m}_0^k \to \mathbf{m}_0^{k+1},$$

where $\mathbf{m}_0 \subset \mathcal{O}_{X_0}$ is the maximal ideal. That is, the action of δ_x on $T^1(X_0)$ is *nilpotent.* The same is true also for the operator δ_y. In virtue of Engel's theorem, there is a nilpotent element, namely $x^2y \in \mathrm{Ker}\delta_x \cap \mathrm{Ker}\delta_y$. Clearly, any vertical vector field can be extended in the direction along the tangent vector $e = x^2y \in T_*(S)$ in the first neighbourhood $\mathbf{C}[t]/(t^2)$. In other words, the element e is annihilated by elements of $T^0_\bullet(X_0)$.

Let us try to compute an extension of δ_y to the neighbourhood of the distinguished point of the second order $\mathbf{C}[t]/(t^3)$. Set

$$\widetilde{\delta}_y = \delta_y + tV^1,$$

where $V^1 = A\partial_x + B\partial_y$, is a differential operator of the first order, and

$$f_t = f + te, \; e = x^2y.$$

Then f_t defines the family of germs X_t over the minimal versal deformation along the direction $e \in T^1(X_0) \cong T_*(S)$. Thus

$$\widetilde{\delta}_y(f_t) = 21yf + t(\delta_y(e) + V^1(f)) + t^2V^1(e),$$

and one has to obtain $\widetilde{\delta}_y(f_t) \equiv \gamma_t \cdot f_t \,(\mathrm{mod}\, t^3)$, where $\gamma_t \in \mathcal{O}_{X_t}$. One can compute

$$\delta_y(e) = \Big((\frac{85}{3} - \frac{25}{3}cy + 532c)x + (\frac{34}{3} - \frac{5}{3}cy)y^2\Big)f'_x +$$
$$\Big((\frac{34}{3} + 231c)y - \frac{10}{3}cy^2\Big)f'_y + (-85 - 1617c + 25cy)f.$$

Let us substitute the expression for $\delta_y(e)$ in the previous equation. This gives

$$\widetilde{\delta}_y(f_t) = \Big(21y + (-85 - 1617c + 25cy)t\Big)f + t^2\Big(V^1(e) + (85 + 1617c - 25cy)e\Big),$$

where

$$V^1 = \Big(\big(\frac{85}{3} - \frac{25}{3}cy + 532c\big)x + \big(\frac{34}{3} - \frac{5}{3}cy\big)y^2\Big)\partial_x + \Big(\big(\frac{34}{3} + 231c\big)y - \frac{10}{3}cy^2\Big)\partial_y.$$

Further calculations show that

$$V^1(e) + (85 + 1617c - 25cy)e =$$

$$= (11291x^2y + 1666xy^3 + 280xy^4 + 15792x^2y^2 + 2100x^2y^3)c \neq 0,$$

and that the image of this element in $T^1(X_0)$ is equal to

$$11291x^2y + 1666xy^3 \neq 0.$$

It is a non-trivial obstruction to the extension of δ_y to the neighbourhood of the second order. Thus the vector field δ_y has no extension to the second neighbourhood. Consequently, every element of the restriction of the defining ideal of the maximal modular stratum in the direction $x^2y \in T^1(X_0)$ is divisible t^2, that is, this ideal defines a non-reduced germ.

Remark 2.8. A priory, there is also another reason for a modular germ to have singularities: it is the existence of a nontrivial obstruction in $T^2(X_0)$ for the extension of infinitesimal deformations. However, in the hypersurface case or, more generally, for complete intersections $T^2(X_0) = 0$.

Suppose that X_0 is the germ of an isolated singularity. Then $\dim_{\mathbf{C}} T^1(X_0)$ is called the Tjurina number of X_0. It is denoted by $\tau(X_0)$. Let $\varphi : (X, X_0) \to (S, *)$ be the minimal versal deformation of the germ X_0. For any $s \in S$ the fiber X_s is a *multigerm* consisting of finite number of germs $X_i,\ i = 1, 2, \dots$, and the space $T^1(X_s)$ is the direct sum of $T^1(X_i)$:

$$\dim_{\mathbf{C}} T^1(X_s) = \sum \dim_{\mathbf{C}} T^1(X_i), \text{ or } \tau(X_s) = \sum \tau(X_i).$$

Corollary 2.9. *Let X_0 be the germ of an isolated complete intersection singularity. Then the support of the maximal modular germ M_{red} coincides with the support of the stratum $\{\tau_{\max} = \mathrm{const}\} \subset S$, i.e. with the set of points $s \in S$ for which*

$$\dim_{\mathbf{C}} T^1(X_s) = \dim_{\mathbf{C}} T^1(X_0) = \tau(X_0).$$

Proof. In the case of complete intersection $T^2(X_0) = 0$. The statement follows directly from the condition 4) of Theorem (2.3) and the properties of the minimal versal deformation.

Definition 2.10. The dimension of the maximal modular germ $M \subset S$ is called the *modularity* of the singularity X_0 and denoted by $\wp = \wp(X_0)$.

Remark 2.11. Let $\mu(X_0)$ be the Milnor number of the singularity X_0. Then the dimension of the stratum $\{\mu_{\max} = \text{const}\}$ of the minimal versal deformation of the singularity X_0 is usually called the *modality* of the singularity X_0 (see e.g. [5]). It consists of points $s \in S$ for which $\mu(X_s) = \mu(X_0)$.

3. Maximal modular germ for graded singularities

We now assume that the germ X_0 is $\mathbf{Z}_+$-*graded* or, equivalently, there is a good $\mathbf{C}^*$-action on X_0. Any $\mathbf{C}^*$-action on the germ X_0 induces the $\mathbf{C}^*$-action on the tangent cohomology $T^*(X_0)$. In particular, in the case of an *isolated* singularity we have a decomposition $T^1(X_0) = \oplus_{\nu \in \mathbf{Z}} T^1(X_0)_\nu$ into a direct sum of the homogeneous components of the *finite-dimensional* space $T^1(X_0)$.

Further, the tangent cohomology $T^*(X_0)$ is endowed with a natural Lie algebra structure [14], and $T^0(X_0)$ acts on $T^1(X_0)$ via the Lie bracket. Let $\xi \in T^0(X_0)$ be the *Euler* vector field and let $\ell_\xi : T^1(X_0) \to T^1(X_0)$ denote the linear operator defined by $\ell_\xi(t) = [\xi, t]$. It is easy to prove the following three assertions (see [1]).

Lemma 3.1. *Let X_0 be the $\mathbf{Z}_+$-graded germ of isolated singularity. Then the $\mathbf{Z}$-grading on $T^1(X_0)$ defined by the $\mathbf{C}^*$-action coincides with the $\mathbf{Z}$-grading defined by the Euler vector field.*

Proof. It is easy to check that $T^1(X_0)_\nu$ are the eigenspaces of ℓ_ξ corresponding to the eigenvalues $\nu \in \mathbf{Z}$, so that the dimension of $T^1(X_0)_\nu$ is equal to the multiplicity of ν in the spectrum of this linear operator.

Proposition 3.2. *Let $\varphi : (X, X_0) \to (S, *)$ be the minimal versal deformation of the graded isolated singularity X_0, and let $M \subset S$ be the maximal modular germ with respect to φ. Then there exist canonical embeddings*

$$\mathcal{T}_*(M) \hookrightarrow T^1(X_0)_0 \hookrightarrow \mathcal{T}_*(S),$$

where the tangent spaces of the germs M and S are denoted by $\mathcal{T}_*(M)$ and $\mathcal{T}_*(S)$, respectively. In particular, $\wp(X_0) \leq \dim_{\mathbf{C}} T^1(X_0)_0$.

Proof. Our statement follows immediately from the bijectivity of the differential D_f and the functoriality of the notion of maximal modular germ M.

Proposition 3.3. *Let X_0 be a graded singularity. Then the obstruction to extending an infinitesimal deformation of the $(n-1)$-order, $n \geq 2$, corresponding to an element from $T^1(X_0)_\nu$, $\nu \in \mathbf{Z}$, to an infinitesimal deformation of the n-order belongs to $T^2(X_0)_{n\nu}$.*

Proof. Recall that the cohomology $H_*(\mathcal{L})$ of any graded Lie algebra $\mathcal{L}$ whose differential is a differentiation of degree one inherits the Lie algebra structure. There exists a partial cohomological operation $H^1(\mathcal{L}) \to H^2(\mathcal{L})$ defined by the formula $h \longmapsto [h,h]$ and called *Massey square.* In the deformation theory the tangent complex of X_0 plays the role of $\mathcal{L}$, so that the Massey square $T^1(X_0) \to T^2(X_0)$ is defined on the tangent cohomology. Further, it is well known that the obstruction to extending an infinitesimal deformation of the first order corresponding to an element $t \in T^1(X_0)$, to an infinitesimal deformation of the second order is equal to $\frac{1}{2}[t,t] \in T^2(X_0)$ (see e.g. [9],(5.1), or [19]). In the graded case it is easy to verify the equality

$$\deg[t,t] = 2\deg(t) = 2\nu, \quad t \in T^1(X_0)_\nu.$$

If the Massey square is equal to zero, then a desirable extension exists. In such a case the obstruction to extending an infinitesimal deformation of the second order corresponding to the element $t \in T^1(X_0)$ to an infinitesimal deformation of the third order is equal to $\dfrac{1}{2^2}[t,[t,t]] \in T^2(X_0)$, $\deg[t,[t,t]] = 3\nu$, and so on.

Remark 3.4. Thus the obstruction to extending infinitesimal deformations of the first order, corresponding to an element from $T^1(X_0)_0$, to *complete* ones lies in $T^2(X_0)_0$. If there are no such obstructions (for instance, if $T^2(X_0)_0 = 0$), then it is easy to see that the dimension of the tangent space $\mathcal{T}_*(M)$ is *equal* to the multiplicity of zero in the spectrum of the linear operator on $T^1(X_0)$ defined by the Euler vector field $\xi \in T^0(X_0)$ (cf. Proposition (3.2)). For complete intersections, this remark can be made more precise (see [2], (1.3)).

Theorem 3.5. *For a graded isolated complete intersection singularity the maximal modular germ M is reduced and smooth. The modularity $\wp$ of such a germ is equal to the multiplicity of zero in the spectrum of the linear operator on $T^1(X_0)$ defined by the Euler vector field $\xi \in T^0(X_0)$.*

Proof. From the theorem in [3], (6.1) it follows that for such a singularity the image of $T^0(X/S) \to T^0(X_0)$ is generated over $\mathcal{O}_{X_0}$ by Hamiltonian derivations, while the finite-dimensional vector space $T^0_\bullet(X_0)$ is generated by fields which are multipliers of ξ. The Euler field ξ acts on $T^1(X_0)$ with no kernel, so that under our assumptions there is an isomorphism $\mathcal{T}_*(M) \simeq T^1(X_0)_0$. Now let $Z \subset Y$ be any Artinian subgerm such that $\mathcal{T}_*(Z) \subset \mathcal{T}_*(M)$. Since $\mathcal{T}_*(M) \simeq T^1(X_0)_0$, it is evident that the field ξ (and, consequently, any element of $T^0_\bullet(X_0)$) extends to a vertical vector field on the restriction $f\mid_Z$. For complete intersections $T^2(X_0) = T^2(X_0)_0 = 0$, so it is clear that the above considerations imply

$$\wp(X_0) = \dim M = \dim_{\mathbf{C}} \mathcal{T}_*(M) = \dim_{\mathbf{C}} T^1(X_0)_0,$$

that is, M is *smooth.*

Corollary 3.6. *For a graded isolated complete intersection singularity the stratum* $\{\tau_{\max} = \text{const}\}$ *is reduced and smooth.*

Remark 3.7. The assertion of the proposition was conjectured by V.P. Palamodov, and it was proved for hypersurfaces in [13], Theorem 9. In the case when X_0 is semi-quasihomogeneous isolated hypersurfaces singularity the maximal modular germ may be non-reduced (see §2, Example 2.7). Among them there are $T_{p,q,r}$-singularities given by the equation $x^p + y^q + z^r + xyz = 0$, where $1/p + 1/q + 1/r < 1$ (see [16], Ex. 3.2).

4. Modular space

Let us recall some notions from the deformation theory of complex spaces (see [14], [16]). For the construction of modular spaces we have to consider families of germs over arbitrary complex base spaces.

Definition 4.1. A *modular family* is a proper flat mapping of complex analytic spaces $f : X \to M$ such that for every point $s \in M$ the restriction of f to the germ (M, s) is a *modular deformation* of the fiber X_s.

Let X_0 be an admissible representative of the germ of a complex analytic space and $f : X \to M$ be its *modular deformation.* By definition, for every point $s \in M$ there is an open neighbourhood $s \in V \subset M$ satisfying the following condition: *For any $v \in V$ the fibers X_v and X_s are not isomorphic.*

However, some fibers over V may be isomorphic to each other. Assume that such fibers belong to one-dimensional families. Then there are distinct morphisms of germs $\phi : (\mathbf{C}, 0) \to (M, s)$ and $\psi : (\mathbf{C}, 0) \to (M, s)$ so that the fibers of the induced deformations $\phi^*(f)$ and $\psi^*(f)$ pairwise isomorphic. Besides, these isomorphisms form a holomorphic family over $(\mathbf{C}, 0)$. Obviously, the isomorphism of the distinguished fibers $g : X_0 \simeq X_0$ is not equal to the identity; otherwise this would contradict the basic property of the modular deformation.

Let $\mathrm{Aut}(X_0)$ be the complex Lie group of holomorphic automorphisms of X_0. It is evident that the above isomorphism g does not belong to the connected component of the identity $\mathrm{Aut}_e(X_0) \subset \mathrm{Aut}(X_0)$. On the contrary, given a fixed automorphism $g \in \mathrm{Aut}(X_0)$ there is a unique morphism $g_M : M \to M$ such that the induced deformation $g_M^*(f)$ and the deformation f with the fixed isomorphism a of the distinguished fiber are isomorphic. It is clear that g_M is a biholomorphic mapping from the germ M into itself. Thus, the action of the group $\mathrm{Aut}(X_0)$ lifts to an action in the bundle $f : X \to M$. It turns out that the subgroup $\mathrm{Aut}_e(X_0) \subset \mathrm{Aut}(X_0)$ as well as the *discrete* group $\Gamma(X_0) = \mathrm{Aut}(X_0)/\mathrm{Aut}_e(X_0)$ act on the base M.

Let us also assume that the modular deformation of X_0 satisfies the following condition (see [16], Problem 4.3). For any points $s, t \in M$ which are sufficiently near the distinguished point of M the following is true: *The fibers X_s and X_t are isomorphic if and only if both points $s, t \in M$ lie in the same orbit of the action Γ on M.*

Definition 4.2. Let X_0 be the germ of an isolated singularity satisfied the above assumptions. Let M be the *maximal modular germ*, and let $\Gamma = \Gamma(X_0)$ be the discrete group as defined before. Then the quotient germ $\mathcal{M} = M/\Gamma$ is called the germ of a *modular* space for X_0.

It seems that in many cases quite a clear description of modular spaces is given by the collections of charts or finite coverings, that is, by proper multivalent mappings $\varphi_U : U \to \mathcal{M}$, where $U \subset \mathbf{C}^m$ are the open subsets. The next sections aim at computing modular spaces for quasi-homogeneous space curve-singularities.

5. Unimodular curve-singularities

According to the classification theorems, [2], Theorem 5.4 and Theorem 5.10, there are, in all the 14 types of unimodular families, three 2-modular ones and only one 3-modular family of graded curve singularities, which are complete intersection with Milnor numbers less than seventeen (cf. also [6], [21]). We denote the normal form of such a germ and the corresponding one-parameter family by the same symbol. Any member of such a family X_λ is embedded in $(\mathbf{C}^3, 0)$. In the notations [2], we obtain the following assertion.

Theorem 5.1. *Let X_λ be an unimodular family of a graded complete intersection curve singularity. Then there exists a modular space $\mathcal{M}$ containing the base of such a family. Moreover, there is a covering $\mathcal{M}' \to \mathcal{M}$ such that $\mathcal{M}'$ is biholomorphically equivalent to the complex plane $\mathbf{C}^1$, except for three cases of families Γ^1_{15}, $\mathcal{Q}^1_{16}$, $\dot{\mathcal{Q}}^1_{16}$. In the latter cases $\mathcal{M}'$ is biholomorphically equivalent to the punctured complex plane.*

Proof. The method of the proof is the case-by-case checking.

1) *The family* $\mathcal{R}_9$. Let $\epsilon \in \mathbf{C}^*$, so that $\epsilon^4 = 1/4$. Set $U = \mathbf{C}^1 \setminus \{\epsilon, \epsilon^2, \epsilon^3, \epsilon^4\}$. Then the family $\mathcal{R}_9$ is given by the mapping $f : X \to U$, so that the fiber $X_a \subset \mathbf{C}^3$, $a \in U$, is defined by the following system of equations:

$$\begin{cases} x^2 + y^4 + z^4 &= 0 \\ yz + ax &= 0. \end{cases}$$

Let us define a chart $\varphi_U : U \to \mathbf{CP}^1$ by $\varphi_U(a) = a^4$. It is clear that the chart φ_U covers the whole Riemann sphere $\mathbf{CP}^1$, except for two points $a^4 = 1/4$ and ∞. The latter may be covered by another chart. To see this, let us consider the family $\mathcal{X}_9$ of *hypersurfaces* given by the equation $y^4 + z^4 + by^2z^2 = 0$, $b \in V = \mathbf{C}^1 \setminus \{\pm 2\}$. Then $\varphi_V : V \to \mathbf{CP}^1$, defined by $\varphi_V(b) = b^2$, will be the desirable chart. Besides φ_V maps the member of the family $\mathcal{X}_9$ with the parameter $b = 0$ to ∞. It is not difficult to verify that both charts are glued along $\varphi_U^{-1}(N)$ and $\varphi_V^{-1}(N)$, where $N = \varphi_U(U) \cap \varphi_V(V)$, to the complex plane $\mathbf{C}^1$. This implies $\mathcal{M}' \cong \mathbf{C}^1$. In fact, the gluing isomorphism is given by the formula $a^2 = 1/b$.

It also implies that the remaining point of $\mathbf{CP}^1$, which is not covered by two charts φ_U, φ_V, corresponds to *non-reduced* singularities defined by $(y \pm z^2)^2 = 0$. The following question arises: does there exist the third chart corresponding to a family of *isolated* curve singularities (perhaps, non-complete intersection) that covers the remaining point of $\mathbf{CP}^1$? In other words, is it possible to compactify the family $\mathcal{R}_9$ of germs with constant Milnor and Tjurina numbers ? (see also [16], Problem 4.4).

2) *The family* $\mathcal{T}_{10}$. Let $\epsilon \in \mathbf{C}^*$, so that $\epsilon^6 = -4/27$. Set $U = \mathbf{C}^1 \setminus \bigcup_{q=1}^{6} \{\epsilon^q\}$. Then the family $\mathcal{T}_{10}$ is defined by the following system of equations:

$$\begin{cases} x^2 + y^3 + z^6 &= 0 \\ yz + ax &= 0, \end{cases}$$

where $a \in U$. Define a mapping $\varphi_U : U \to \mathbf{CP}^1$ by $\varphi_U(a) = a^6$. Then the chart φ_U covers the whole projective line $\mathbf{CP}^1$, except for two points $a^6 = -4/27$ and ∞. The latter may be covered by another chart corresponding to the modular family $\mathcal{J}_{10}$ of hypersurfaces $\{y^3 + z^6 + by^2z^2 = 0\}$, where $b \in V = \mathbf{C}^1 \setminus \bigcup_{q=1}^{3} \{\nu^q\}$, $\nu^3 = -27/4$. While $\varphi_V : V \to \mathbf{CP}^1$ is defined by $\varphi_V(b) = b^3$. As before both charts are glued to the complex plane $\mathbf{C}^1$ with the aid of the isomorphism given by the formula $a^2 = 1/b$.

3) *The family* $\mathcal{U}_{11}$. In this case the family $\mathcal{U}_{11}^*$ from the list [5], p.132 is very suitable to describe coverings. Namely, let $\epsilon^3 = 27/4$ and set $U = \mathbf{C}^1 \setminus \{\epsilon, \epsilon^2, \epsilon^3\}$. Then the family $\mathcal{U}_{11}^*$ is given by the following system:

$$\begin{cases} x^2 + yz &= 0 \\ xy + z^5 + axz^3 &= 0, \end{cases}$$

where $a \in U$. A mapping $\varphi_U : U \to \mathbf{CP}^1$, defined by $\varphi_U(a) = a^3$, covers the whole projective line $\mathbf{CP}^1$, except for two points $a^3 = 27/4$ and ∞. The latter may be covered by another chart. Consider an open set $V = \mathbf{C}^1 \setminus \{\pm 2/3\sqrt{3}\}$, the modular family over this open set looks as follows:

$$\left\{ \begin{array}{ccc} x^2 + yz & = & 0 \\ xy + bz^5 + xz^3 & = & 0. \end{array} \right.$$

Then $\varphi_V : V \to \mathbf{CP}^1$, defined by $\varphi_V(b) = b^2$, covers the point ∞ by the element corresponding to $b = 0$. As before, we see that $\mathcal{M}' \cong \mathbf{C}^1$ and the gluing isomorphism is given by the formula $a^3 = 1/b^2$.

Remark 5.2. In fact, there exists the third chart corresponding to the family $\mathcal{U}_{11}$ from [2], Table 2) of Theorem 5.4. In order to show this, let us consider an open set $W = \mathbf{C}^1 \setminus \{\pm 2\}$, so that the modular family $\mathcal{U}_{11}$ over W is given by the system of equations:

$$\left\{ \begin{array}{ccc} x^2 + yz + z^4 + cxz^2 & = & 0 \\ xy & = & 0. \end{array} \right.$$

Then $\varphi_W : W \to \mathbf{CP}^1$, defined by the formula $\varphi_W(c) = c^2$, covers the whole $\mathbf{CP}^1$, except for two points $c^2 = 4$ and ∞. One can verify that the gluing isomorphism between the families over V and W is given by the formula

$$b^2 = c^2(9 - 2c^2)^2/27(c^2 - 3)^2.$$

It is not difficult to see that our third chart paired with the first or the second one cover the whole modular space $\mathcal{M}$.

4) *The family* $\mathcal{V}_{10}$. Let $\epsilon^3 = -27/4$. Set $U = \mathbf{C}^1 \setminus \{\epsilon, \epsilon^2, \epsilon^3\}$, so that the modular family $\mathcal{V}_{10}$ is given over U by the system of equations:

$$\left\{ \begin{array}{ccc} x^3 + yz + z^3 + ax^2z & = & 0 \\ xy & = & 0. \end{array} \right.$$

Then $\varphi_U : U \to \mathbf{CP}^1$, acting by formula $\varphi_U(a) = a^3$, covers the whole projective line $\mathbf{CP}^1$ except for two points $a^3 = -27/4$ and ∞. The latter may be covered by another chart $\varphi_V : V \to \mathbf{CP}^1$, where $V = \mathbf{C}^1 \setminus \{\pm 2\sqrt{-1}/27\}$. The corresponding modular family over V is defined by the following system of equations:

$$\left\{ \begin{array}{ccc} bx^3 + yz + x^2z + z^3 & = & 0 \\ xy & = & 0, \end{array} \right.$$

and $\varphi_V(b) = b^2$. As before, we get $\mathcal{M}' \cong \mathbf{C}^1$ and the gluing isomorphism $a^3 = 1/b^2$.

Remark 5.3. The first chart coincides with the modular family $\mathcal{U}_{10}^{**}$ described in the Table [2].

5) *The family* $\mathcal{Y}_{11}$. This case is considered in the same manner as the latter case. More precisely, in the same notations, the first chart is constructed as the modular family $\mathcal{Y}_{11}$ given over U by the system of equations:

$$\begin{cases} x^2 + y^3 + z^3 + ay^2z &= 0 \\ xy &= 0, \end{cases}$$

while the second one is constructed as the modular family over V defined by the equations:

$$\begin{cases} x^2 + by^3 + z^3 + y^2z &= 0 \\ xy &= 0. \end{cases}$$

The charts φ_U and φ_V and the gluing isomorphism are described in the same manner as in the preceding case.

Remark 5.4. The modular family $\mathcal{Y}_{11}$ completes the list [5], p.132, where the surrounding genotypes $(\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ have been recounted.

6) *The family* $\mathcal{M}_{11}$. Let $U = \mathbf{C}^1 \setminus \{\pm\sqrt{-1}\}$ and $f : X \to U$ be the modular family $\mathcal{M}_{11}$ defined over U by the system of equations:

$$\begin{cases} x^2 + z^4 &= 0 \\ y^2 + z^3 + axz &= 0. \end{cases}$$

Then $\varphi_U : U \to \mathbf{CP}^1$, acting by the formula $\varphi_U(a) = a^2$, covers the whole Riemann sphere $\mathbf{CP}^1$, except for two points $a^2 = -1$ and ∞. The latter point may be covered by another chart. Set $V = \mathbf{C}^1 \setminus \{\pm\sqrt{-1}\}$ and consider the modular family over V, given by the equations:

$$\begin{cases} x^2 + z^4 &= 0 \\ y^2 + bz^3 + xz &= 0, \end{cases}$$

where $b \in V$. Then $\varphi_V : V \to \mathbf{CP}^1$ is defined by the formula $\varphi_V(b) = b^2$. Besides, φ_V maps the member of the family $\mathcal{M}_{11}$ with the parameter $b = 0$ to ∞. The gluing isomorphism is given by the formula $a = 1/b$.

It should be noted that one can construct the third chart, corresponding to the family $\mathcal{W}^*_{11}$ from the list [5], p.132. In fact, this family is given by the system:

$$\begin{cases} x^2 + z^4 + cxz^2 &= 0 \\ y^2 + xz &= 0, \end{cases}$$

where $c \in W = \mathbf{C}^1 \setminus \{\pm 2\}$. Then $\varphi_W : W \to \mathbf{CP}^1$ is defined by the formula $\varphi_W(c) = c^2$. It is not difficult to see that this chart paired with the first or the second one covers the whole modular space $\mathcal{M}' \cong \mathbf{C}^1$. The gluing isomorphism of the families over U and W (over V and W) is given by the formula $c^2 = 4/(a^2+1)$

($c^2 = 4b^2/(b^2+1)$, respectively). For completeness, we describe the fourth chart for $\mathcal{M}$. Set $Z = \mathbf{C}^1 \setminus \{1\}$, and consider the modular family

$$\left\{ \begin{array}{rcl} x^2 + xz^2 & = & 0 \\ y^2 + dxz + z^3 & = & 0, \end{array} \right.$$

where $d \in Z$. The gluing isomorphism for the families over U and Z (over V and Z) is given by the formula $a = d\sqrt{-1}/(d-2)$ ($c^2 = (d-2)^2/(d-1)$, respectively). In conclusion, we note that the member of the modular family $\mathcal{M}_{11}$ with the parameter $a = 0$ is denoted in the list [5], p.132 by $\mathcal{Z}^*_{11}$.

Thus the six modular families described in 1) - 6) are contained in the list 2) of Theorem 5.4 from [2]. They all have the *inner modality* one. Below we shall describe the remaining 8 modular families, from the above mentioned list, with the inner modality greater one.

7) *The family* $\mathcal{Y}_{15}$. Analogously to the cases 4) and 5). More precisely, in the same notations, the first chart is constructed as the modular family $\mathcal{Y}_{15}$ given over U by the system of equations:

$$\left\{ \begin{array}{rcl} x^3 + y^4 + z^3 + ay^2z & = & 0 \\ xy & = & 0, \end{array} \right.$$

while the second one is constructed as the modular family over V defined by the system of equations:

$$\left\{ \begin{array}{rcl} bx^3 + y^4 + z^3 + x^2z & = & 0 \\ xy & = & 0. \end{array} \right.$$

The charts φ_U and φ_V and the gluing isomorphism are given by exactly the same formulae as in 4) and 5). Thus, $\mathcal{M}' \cong \mathbf{C}^1$.

8) *The family* $\mathcal{V}_{16}$. Again, in the same notations, the first chart is constructed as the modular family $\mathcal{V}_{16}$ given over U by the equations:

$$\left\{ \begin{array}{rcl} x^6 + yz + z^3 + x^4z & = & 0 \\ xy & = & 0, \end{array} \right.$$

while the second one is given as the modular family over V defined by the system of equations:

$$\left\{ \begin{array}{rcl} bx^6 + yz + z^3 + x^4z & = & 0 \\ xy & = & 0. \end{array} \right.$$

The charts φ_U and φ_V and the gluing isomorphism are given by exactly the same formulae as in the preceding case.

9) *The family* $\mathcal{G}_{16}$. Once more, in the same notations, the first chart is constructed as the modular family $\mathcal{G}_{16}$ given over U by the system of equations:

$$\left\{ \begin{array}{rcl} x^6 + y^3z + z^3 + ax^2z & = & 0 \\ xy & = & 0, \end{array} \right.$$

while the second one is given as the modular family over V defined by the system of equations:

$$\begin{cases} bx^3 + y^3 z + z^3 + x^2 z & = & 0 \\ xy & = & 0. \end{cases}$$

The charts φ_U and φ_V and the gluing isomorphism are given by exactly the same formulae as in the preceding case.

10) *The family* $\mathcal{M}_{15}$. Set $U = \mathbf{C}^1 \setminus \{\pm\sqrt{-1}\}$ and consider the modular family $\mathcal{M}_{15}$ defined over U by the system of equations:

$$\begin{cases} x^2 + z^6 & = & 0 \\ y^2 + z^3 + ax & = & 0. \end{cases}$$

Then $\varphi_U : U \to \mathbf{CP}^1$, acting by formula $\varphi_U(a) = a^2$, covers the whole projective line $\mathbf{CP}^1$, except for two points $a^2 = -1$ and ∞. The latter may be covered by another chart, corresponding to the modular family $\mathcal{W}_{15}$ of *hypersurfaces* defined by the equation $y^4 + z^6 + by^2z^2 = 0$, where $b \in V = \mathbf{C}^1 \setminus \{\pm 2\}$. The chart $\varphi_V : V \to \mathbf{CP}^1$ is described by the formula $\varphi_V(b) = b^2$, while the gluing isomorphism is described by the formula $b^2 = 4/(1 + a^2)$. As a result $\mathcal{M}' \cong \mathbf{C}^1$.

11) *The family* $\mathcal{N}_{15}$. Set $U = \mathbf{C}^1 \setminus \{\pm\sqrt{-1}\}$ and consider the modular family $\mathcal{N}_{15}$ defined over U by the system of equations:

$$\begin{cases} x^2 + z^5 + ayz^3 & = & 0 \\ y^2 + z^4 & = & 0. \end{cases}$$

Then $\varphi_U : U \to \mathbf{CP}^1$, acting by formula $\varphi_U(a) = a^2$, covers the whole projective line $\mathbf{CP}^1$, except for two points $a^2 = -1$ and ∞. The latter may be covered by another chart, corresponding to the modular family defined over $V = \mathbf{C}^1 \setminus \{\pm\sqrt{-1}\}$ by the system of equations:

$$\begin{cases} x^2 + bz^5 + yz^3 & = & 0 \\ y^2 + z^4 & = & 0. \end{cases}$$

The chart $\varphi_V : V \to \mathbf{CP}^1$ is described by the formula $\varphi_V(b) = b^2$, while the gluing isomorphism of the charts φ_U and φ_V is described by the formula $a = 1/b$. Hence $\mathcal{M}' \cong \mathbf{C}^1$.

In conclusion, let us consider the exceptional cases.

12) *The family* Γ_{15}. In this case, there is only one chart constructed with the aid of the modular family given over $U = \mathbf{C}^1 \setminus \{\pm 2\}$ by the system of equations:

$$\begin{cases} x^2 + y^2 z + z^4 + axz^2 & = & 0 \\ xy & = & 0. \end{cases}$$

The chart $\varphi_U : U \to \mathbf{CP}^1$, defined by the formula $\varphi_U(a) = a^2$, covers the whole projective line $\mathbf{CP}^1$, except for two points $a^2 = 4$ and ∞. Thus, $\mathcal{M}' \cong \mathbf{C}^1 \setminus \{4\}$.

13) *The family* $\mathcal{Q}_{16}$. This case is the same as before. The only chart may be constructed with the aid of the modular family:

$$\left\{ \begin{array}{lcl} x^3 + y^2 + z^4 + ayz^2 & = & 0 \\ xy & = & 0. \end{array} \right.$$

14) *The family* $\dot{\mathcal{Q}}_{16}$. Set $U = \mathbf{C}^1 \setminus \{\pm 2\sqrt{-1}\}$. The only chart of the modular space is constructed with the aid of the modular family given over U by the system of equations:

$$\left\{ \begin{array}{lcl} x^3 + yz + axz^3 & = & 0 \\ xy + z^5 & = & 0. \end{array} \right.$$

The chart $\varphi_U : U \to \mathbf{CP}^1$, defined by the formula $\varphi_U(a) = a^2$, covers the whole Riemann sphere $\mathbf{CP}^1$, except for two points $a^2 = -4$ and ∞. Thus, as before $\mathcal{M}' \cong \mathbf{C}^1 \setminus \{-4\}$.

6. Higher modularity and adjacency

According to the classification of the quasi-homogeneous singular germs of curves which are complete intersection (see [2], Theorem 5.4 and Theorem 5.10), there are three 2-modular families and only one 3-modular family with Milnor numbers less than seventeen.

1) 2-*modular family of hypersurfaces* $\dot{N}_{16}$. Set $U = \mathbf{C}^2 \setminus \{\Delta = 0\}$, where

$$\Delta = (ab - 9)(4a^3 + 4b^3 - a^2b^2 - 18ab + 27).$$

Consider the modular family defined by the equation

$$x^4y + xy^4 + ax^3y^2 + bx^2y^3 = 0,$$

where $a, b \in U$. Then the chart $\varphi_U : U \to \mathbf{CP}^2$, given by the formula

$$\varphi_U(a, b) = (ab, 4(a^3 + b^3) - a^2b^2 - 18ab),$$

covers the whole projective plane $\mathbf{CP}^2$, except for three curves

$$(ab, -27), \quad (9, 4(a^3 + b^3) - a^2b^2 - 18ab), \quad a, b \in U,$$

and the line at infinity.

2) 2-*modular family* Γ_{13}. Let $U = \mathbf{C}^2 \setminus \{(a^2 - 4)(b^2 - 4) = 0\}$ and $f : X \to U$ be the modular family Γ_{13} defined over U by the system of equations:

$$\left\{ \begin{array}{lcl} x^2 + y^2 + z^4 + axz^2 + byz^2 & = & 0 \\ xy & = & 0. \end{array} \right.$$

Then $\varphi_U : U \to \mathbf{CP}^2$, acting by formula $\varphi_U(a,b) = (a^2, b^2)$, covers the whole projective plane $\mathbf{CP}^2$, except for two lines $(a^2, 4)$, $(4, b^2)$ and the infinite line.

In order to find another chart let us consider the modular family Z_{13}^* from the list of [5], p.132 $f : X \to V$ defined over $V = \mathbf{C}^2 \setminus \{\delta = 0\}$, where

$$\delta = 256(a^3 + b^3 + a^2b^2) + 288ab - 27,$$

by the system of equations:

$$\begin{cases} x^2 + yz^2 + az^4 &= 0 \\ y^2 + xz^2 + bz^4 &= 0, \end{cases}$$

where $a, b \in V$. Then $\varphi_V : V \to \mathbf{CP}^2$ is defined by the formula $\varphi_V(b) = b^2$.

3) 2-*modular family* H_{16}. Let $U = \mathbf{C}^2 \setminus \{(a+b)^3 + (1+a)^3 = 0\}$ and $f : X \to U$ be the modular family H_{16} defined over U by the system of equations:

$$\begin{cases} x^2 + y^3 + ay^2z + byz^2 &= 0 \\ x^3 + y^3 &= 0. \end{cases}$$

Then $\varphi_U : U \to \mathbf{CP}^2$, acting by formula $\varphi_U(a,b) = (2a+b, a^2+ab+b^2+a-b)$, covers the whole projective plane $\mathbf{CP}^2$, except for two lines

$$(2a+b, -1), \quad (-1, a^2+ab+b^2+a-b), \quad a, b \in U,$$

and the infinite line.

4) 3-*modular family* Y_{13}. Let $U = \mathbf{C}^2 \setminus \{\Delta = 0\}$ and $f : X \to U$ be the modular family Y_{13} defined over U by the system of equations:

$$\begin{cases} x^3 + y^3 + z^3 + ax^2z + by^2z &= 0 \\ xy + cz^2 &= 0. \end{cases}$$

One can verify that there is a chart $\varphi_U : U \to \mathbf{CP}^3$, acting by rather complicated formula, covers the whole projective space $\mathbf{CP}^3$, except for 4 hyperplanes. One of them is the infinite hyperplane.

In order to find another chart, let us consider the modular family W_{13}^{**} from the list of [5], p.132 $f : X \to V$ defined over $V = \mathbf{C}^2 \setminus \{\delta = 0\}$, where

$$\delta = (a^2 - 4)(b^2 - 4)(c^2 - 4)(a^2 + b^2 + c^2 - 2abc),$$

by the system of equations:

$$\begin{cases} x^2 + y^2 + z^2 + axy + byz + cxz &= 0 \\ xyz &= 0, \end{cases}$$

where $a, b, c \in V$. Then $\varphi_V : V \to \mathbf{CP}^3$, defined by the formula $\varphi_V(a,b,c) = (a^2, b^2, c^2)$, covers the whole projective space $\mathbf{CP}^3$, except for three hyperplanes and the infinite hyperplane.

Let us show that the modularity reflects the properties of adjacency of singularities (cf. [2]).

Definition 6.1. An isolated singularity X_0 is called a *surrounding* singularity for isolated singularities with modality m if $m(X_0) = m + 1$ and $\wp(X_0) \geq 1$.

Remark 6.2. Let X_0 be a surrounding singularity. Suppose that M is the maximal modular germ. Then $T^1(X_0)_- \neq 0$. That is, M does not coincide with the base space of minimal versal deformation, and X_0 is adjacent to a singularity with different type along the "negative" direction in the base. In all the known examples, a surrounding complete intersection singularity with $m(X_0) = m + 1$ actually "demarcates", in the sense of the adjacency diagram, between complete intersections with modalities m and $m + 1$.

Proposition 6.3. *Let $X_0 \to X_1$ be an adjacency of two complete intersection germs. Suppose that the adjacency is "general". Then we have the following inequalities for Milnor and Tjurina numbers:*

$$\mu(X_0) \geq \mu(X_1) + \wp(X_0) + 1, \qquad \tau(X_0) \geq \tau(X_1) + \wp(X_0) + 1.$$

Proof. Consider the germ H transversal to the modular germ at the "general" point. Then $\dim H = \mu(X_0) - \wp(X_0)$, and the germ H is a versal deformation at any of its point. By definition, the distinguished point $\{*\} \in H$. If $\mu(X_1) = \dim H$, then H is the minimal versal deformation (because a versal deformation contains the minimal versal deformation). Since X_0 and X_1 are not isomorphic and the Milnor number is upper semi-continuous, $\mu(X_1) > \mu(X_0)$. Hence $\mu(X_1) \leq \dim H - 1$. The same inequality is also true for Tjurina numbers.

Remark 6.4. It easy to see that, for adjacencies of hypersurface singularities, it commonly happens that the general fiber of the maximal modular germ for X_0 decomposes into singularities of the set X_1 along a "negative" direction in the base of the minimal versal deformation. In the case of complete intersections, which are not hypersurfaces, this is already so. For example, the 1-modular singularity $M_{11} = (x^2 + z^4, y^2 + z^3 + axz)$ is adjacent to $M_{10} = (x^2 + yz^2, y^2 + z^2)$ at the single point $\{a = 0\} \in \mathcal{M}(M_{11})$. Here $\Delta(\mu) = 1 < 2$. Compare also the adjacency $M_{11} \to M_9$.

Remark 6.5. Consider also another type of adjacency of surrounding singularities (along the "null-direction" $T^1(X_0)_0$ in the base of the minimal versal deformation). It is easily seen that the 1-modular family R_9 defined by two equations $x^2 + y^4 + z^4 = yz + ax = 0$ is adjacent at the general point $\{a \neq 0\} \in \mathcal{M}(R_9)$ to the plane hypersurface singularity X_9 given by the equation $y^4 + z^4 + \lambda y^2 z^2 = 0$. Hence, in this case $\Delta(\mu) = 0$. In other words, the stratum $\{\mu = \mathrm{const}\}$ of the singularity with embedding dimension two is compactified by the germ with embedding dimension three. A similar situation occurs for the adjacencies $T_{10} \to J_{10}$ and $M_{15} \to W_{15}$.

Our approach enables us to give the following interpretation of the list [5], p.132, where surrounding genotypes $(\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ are recounted. In the notations of [5], the singularities $T_9^*, T_{10}^*, U_{11}^*, U_{10}^{**}$ define only one chart of the corresponding modular family. The singularities W_{11}^*, Z_{11}^* define two different charts of one and the same unimodular family ($\mathcal{M}_{11}^1$ in our notations). Besides, in that list the unimodular singularity Y_{11}^1 was left off. (The corresponding family is given by two charts, one of them is defined by two equations $x^2 + y^3 + z^3 + ay^2 z = xy = 0$, where $4a^3 + 27 \neq 0$). Note also that in the list [21] a member of the family $\mathcal{Y}_{11}^1$ is denoted by HA_{11}. Finally, the singularity Z_{13}^* defines a chart of the 2-modular family Γ_{13}^2, and W_{13}^{**} defines a chart of the 3-modular family Y_{13}^3.

In conclusion, let us consider the case of one-dimensional non-complete intersections. It is known that graded monomial curves and bouquets of such singularities have modularity zero (see [17], (12.5), and [1], respectively). Further, using considerations [17] it is not difficult to prove that if X_0 is the cone over the set of d points in $\mathbf{P}^n$, $d \geq n+2$, then $\wp(X_0) = n(d-n-2)$. As a result, the cone over the set of $d \leq n+2$ points in $\mathbf{P}^n$ is a 0-modular singularity. Further, the cone over the set of $n+3$ points in $\mathbf{P}^n$ is an unimodular singularity. In particular, in the case $n=1$ we obtain the famous unimodular family of hypersurfaces $\mathcal{X}_9$. This family contains modular deformations of the germ of four lines in the complex plane $\mathbf{C}^2$ parameterized by the classical *harmonic ratio* (see [16], Example 3.1). In fact, there are non-complete intersection singularities of dimension greater than one, whose maximal modular germ coincides with the whole base of the minimal versal deformation. For example, the vertex of the affine cone over the Veronese embedding of high enough degree of a smooth hypersurface in the 3-dimensional projective space gives us such a singularity (see [18]). However, it seems to be a very special case.

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