STRINGY K-THEORY

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1. Introduction

In recent years there has been an interest in extending classical functors such as K–theory, the Chow ring and cohomology to orbifolds and stacks. The new ingredient in this theory is a new ring structure, called stringy, which traces its way back to orbifold conformal field theory [DW, DVVV, IV, V] and string theory [DHVW]. Of course there are mathematical versions of the algebraic properties of the expected ring structures given via co-bordisms (see e.g. [Ka1]).

There are basically two flavors of this geometry. One for general stacks and one specifically for global quotients, e.g. a pair (X, G) of a smooth projective variety X and a finite group G that acts on it. In the former situation one uses the inertia stack while in the latter one uses the geometry of the inertia variety, that is the disjoint union over all fixed point sets.

Initially ring string structures of the above type were introduced using Gromov–Witten theory [CR], that is in terms of a moduli space of suitable maps from orbi–curves. Even on the classical level, that is genus 0 and mapping to a point, which is all we will deal with in the following, this defines new product structures. The GW approach yielded a new product on H^* of the inertia orbifold and which is known as Chen–Ruan cohomology. The main character here is the obstruction bundle defined by the moduli space. In [FG] the obstruction bundle was given using Galois covers establishing a product for H^* on the inertia variety level, i.e. a G–Frobenius algebra as defined in [Ka1], which is commonly referred to as the Fantechi–Göttsche ring. In [JKK1], we put this global structure back into a moduli space setting and proved the so–called trace axiom.

In the algebraic category, the multiplication on the Chow ring A^* for the inertia stack was defined in [AGV].

In [JKK2] we were able to give a representation of the obstruction bundle in terms of representation theoretically defined elements of rational K theory called $\mathscr{S}_m$. This allowed us to define a stringy K–theory in both the global quotient and the stack setting. Furthermore using the classes $\mathscr{S}_m$ we could

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even give a Chern character. Since this definition is essentially *curve free* it is particularly useful in situations where it would otherwise not be possible to say what the maps of curves should be. It for instance allows one to pass to topological K–theory of stably almost complex manifolds and to the deRham setting. It also has found its use on Hochschild cohomology, singularity theory and string topology.

2. Stringy K–Theory

This paragraph is mainly joint work with Tyler Jarvis and Takashi Kimura [JKK2].

2.1. **Global Quotient Case.** In the global quotient case the setup is as follows.

We fix G a finite group and regard the following data and objects

1. We fix (X, G) variety with a G action.
2. We let X^g be fixed points of $g \in G$, or more generally X^H, $H \subset G$.
3. We define the *inertia variety*: $I((X,G)) = \amalg_{m \in G} X^m$.

Notice we have a map $\vee : I((X,G)) \to I((X,G))$ via $\vee : X^g \to X^{g^{-1}}$ identically.

The main observation is now that for $m_1 m_2 m_3 = e$ we have the inclusion maps

$$\begin{array}{ccc} X^{m_1} & X^{m_2} & X^{m_3^{-1}} \\ e_1 \nwarrow & \uparrow e_2 & \nearrow \check{e}_3 \\ & X^{\mathbf{m}} & \end{array} \tag{2.1}$$

Let's fix coefficients to lie in $\mathbb{Q}$, then using the diagram above, one defines the stringy versions of the functors $\mathcal{F}$ using the following procedure.

Given (X, G) and a functor, which has a multiplication (E.g. K, A^*, K^*_{top}, H^*), we will use $\mathcal{F} = K$ for concreteness.

1. We define the additive structure $\mathscr{K}(X,G) := K(I(X,G)) = \bigoplus_{g \in G} K(X^g)$
2. We define a *stringy G–graded* multiplication which is expected from the stringy (cobordism/"twist field") via a push–pull formalism using an obstruction bundle.

$$K(X^g) \otimes K(X^h) \to K(X^{gh})$$

For $\mathscr{F}_g \in K(X^g), \mathscr{F}_h \in K(X^h)$

$$\mathscr{F}_g \cdot \mathscr{F}_h := e_{3*}(e_1^*(\mathscr{F}_g) \otimes e_2^*(\mathscr{F}_h) \otimes Obs_K(g,h))$$

Remark 1. The above formula without $Obs_K(g,h)$ will not give something associative in general. There are also natural gradings which would not be respected without $Obs_K(g,h)$ either.

2.1.1. **Two sources for the obstruction bundle $\mathscr{R}(g,h)$.** As mentioned in the introduction there are basically two sources for the obstruction bundle:

1. $\mathscr{R}(g,h)$ from GW theory/mapping of curves [CR, FG, JKK1] (Initially only for H^*).
2. $\mathscr{R}(g,h)$ from K-theory and representation theory [JKK2].

Of course in situations where both definitions apply, they agree.

2.1.2. **The obstruction Bundle through an Eigenspace decomposition.** Let $g \in G$ have the order r. $\langle g\rangle \subset G$ acts on X and leaves X^g invariant. So $\langle g\rangle$ acts on the restriction of the tangent bundle $TX|_{X^g}$ and the latter decomposes into Eigenbundles $W_{g,k}$ whose Eigenvalue is $\exp(2\pi ki/r)$ for the action of g.

$$TX|_{X^g} = \bigoplus_{k=1}^{r-1} W_{g,k}$$

The bundles $\mathscr{S}$

$$\mathscr{S}_g := \bigoplus_{k=1}^{r-1} \frac{k}{r} W_{g,k} \in K(X^g)$$

Theorem 2 (JKK)**.** *Let X be a smooth projective variety with an action of a finite group G, then*

$$\mathscr{R}(g,h) = TX^{\langle g,h\rangle} \ominus TX|_{X^{\langle g,h\rangle}} \oplus \mathscr{S}_g|_{X^{\langle g,h\rangle}} \oplus \mathscr{S}_h|_{X^{\langle g,h\rangle}} \oplus \mathscr{S}_{(gh)^{-1}}\big|_{X^{\langle g,h\rangle}}$$

defines a stringy multiplication on $\mathscr{K}(X,G)$ – actually a G–Frobenius algebra object.

2.2. **The functors A and H^*.** In the case of the functors A and H^*, we use essentially the same formula, only that $\lambda_{-1}(\mathscr{R}^*)$ gets appropriately replaced by the top Chern class $c_{\mathrm{top}}(\mathscr{R})$.

$$v_g \cdot v_h := e_{3*}(e_1^*(v_g) \cdot e_2^*(v_h) \cdot c_{\mathrm{top}}(\mathscr{R}(g,h)))$$

2.3. **Notation.** In all the situations the resulting ring for the pair (X,G) with the stringy multiplication will be denoted by math-script letters: $\mathscr{K}, \mathscr{A}, \mathscr{H}$.

2.4. **The Chern Character.** Let X be a smooth, projective variety with an action of G.

Define $\mathscr{C}\mathbf{h} : \mathscr{K}(X,G) \to \mathscr{A}(X,G)$ via

$$\mathscr{C}\mathbf{h}(\mathscr{F}_g) := \mathfrak{C}h(\mathscr{F}_g) \cup \mathbf{td}^{-1}(\mathscr{S}_g)$$

Here $\mathscr{F}_g \in K(X^g)$, $\mathbf{td}$ is the Todd class and $\mathfrak{C}h$ is the usual Chern–Character.

Theorem 3 (JKK)**.** *$\mathscr{C}\mathbf{h} : \mathscr{K}(X,G) \to \mathscr{A}(X,G)$ is an isomorphism — to be precise allometric isomorphism G–Frobenius algebras objects. Here allometric means that the co–unit does not have to be preserved.*

2.5. **Interlude: G–Frobenius algebra objects.** Let $D(k[G])$ be the Drinfel'd double of the group ring $k[G]$, see Definition 15 with $\beta \equiv 1$. It is a triangular Hopf algebra with an R matrix.

Via R and the coproduct Δ, $D(k[G])$–Mod is a braided tensor category.

Definition 4. *A Frobenius algebra object in a braided tensor category $\mathscr{C}$ is an object A together with*

(1) *a multiplication $\mu_A : A \otimes A \to A$*

(2) *a co-multiplication* $\Delta_A : A \to A \otimes A$,
(3) *a unit* $\eta : 1_{\mathscr{C}} \to A$
(4) *a co-unit* $\epsilon : A \to 1_{\mathscr{C}}$

such that

(1) *It is braided commutative and associative.*
(2) $(id \otimes \Delta_A) \circ (\mu_A \otimes id) = \mu_A \circ \Delta_A = (\Delta_A \otimes id)(id \circ \mu_A)$

Definition 5. *For an endomorphsims of a Frobenius object* $\phi : A \to A$ *we define the F–trace as* $Tr(\phi) := \epsilon(\mu(\phi \otimes id)(\Delta \circ \eta(1))$. *Here we additionally assume that* $1_{\mathscr{C}} = k$.

Definition 6. *A (strict) G–Frobenius algebra object is a Frobenius algebra object in the braided category* $D(k[G])$*–mod, which additionally satisfies:*

(T) $\rho(v^{-1}) = id$, *where* v *is the element which expresses* S^2 *as an inner automorphisms in* $D(k[G])$*, i.e.* $S^2(u) = vuv^{-1}$. *It is given by* $\sum_g g\llcorner_{g}$

(S) $\forall v \in A$:

$$\mathrm{Tr}(l_v \circ \rho(hgh^{-1}\llcorner_{h})) = \chi_{g^{-1}} \mathrm{Tr}(\rho(h \llcorner_{g^{-1}}) \circ l_v)$$

where $\lambda_v(a) = va$.

Remark 7. This definition is actually necessary in the case of $\mathscr{A}$ and $\mathscr{K}$ since we will not get G–Frobenius algebras in general. The reason is that the spaces might be infinite dimensional and hence the form might turn out to be degenerate. To amend the situation, we introduced trace elements in [JKK2]. Here these traces appear naturally via the structural morphisms. Checking this claim of course amounts to a calculation, see [KP].

2.6. Stack case: Stringy K-theory for stacks K_{full}.

2.6.1. **Inertia stack.** For a DM stack, the inertia stack is defined via

$$\mathfrak{I}_{\mathscr{X}} := \mathscr{X} \times_{\mathscr{X} \times \mathscr{X}} \mathscr{X}$$

One can show that $\mathfrak{I}_{\mathscr{X}} = \coprod_{(g)} \mathscr{X}_{(g)}$, where the indices run over conjugacy classes of local automorphisms, and $\mathscr{X}_{(g)} = \{(x,(g)) | g \in G_x\}/Z_{G_x}(g)$.

For a nice (see below) stack $\mathscr{X}$ we [JKK] defined a full orbifold K–theory which has as underlying space

$$K_{\mathrm{full}}(\mathscr{X}) := K(\mathfrak{I}_{\mathscr{X}})$$

This construction is the technical generalization to stacks of the push–pull formulas and the obstruction bundle of the global quotient case discussed above.

Remark 8. The full technical assumption is as follows:

We say that a stack $\mathscr{X}$ *satisfies the KG-condition* if the Grothendieck group $K^{\mathrm{naive}}(\mathscr{X}, \mathbb{Z})$ of (orbi-)vector bundles is isomorphic to the Grothendieck group $K^0(\mathscr{X}, \mathbb{Z})$ of perfect complexes and to the Grothendieck group $G_0(\mathscr{X}, \mathbb{Z})$ of coherent sheaves on $\mathscr{X}$.

If $\mathscr{X}$ satisfies the KG-condition, we will simply write $K(\mathscr{X})$ to denote this group with rational coefficients:

$$K(\mathscr{X}) := K^0(\mathscr{X}, \mathbb{Z}) \otimes \mathbb{Q} \cong K^{\mathrm{naive}}(\mathscr{X}, \mathbb{Z}) \otimes \mathbb{Q} \cong G_0(\mathscr{X}, \mathbb{Z}) \otimes \mathbb{Q}.$$

We assume that the stack $\mathscr{X}$, its inertia stack $\mathfrak{I}_{\mathscr{X}}$, and its double inertia stack $\mathfrak{II}_{\mathscr{X}}$ all satisfy the KG-condition.

The condition is for instance satisfied if $\mathscr{X}$ its inertia and double inertia are smooth with resolution property. This happens e.g. if $\mathscr{X}$ is a smooth DM stack with finite stabilizers.

Hence special cases of interest are $[X/\mathscr{G}]$, where $\mathscr{G}$ is a Lie group which operates with finite stabilizers.

2.7. **Variations and Compatibility.**

(1) If X is a stable complex manifold, then analogous results hold for the topological K–theory K^*_{top} and cohomology H^* yielding an isomorphism of G–Frobenius algebras.

(2) For a smooth Deligne-Mumford stack $\mathscr{X}$ with a projective coarse moduli space there are respective versions for the K–theory, the Chow Rings and the Chern–Characters using the inertia *stacks*.

In this situation the Chern character is only an injective ring *homomorphism* as the K–theory is bigger than the Chow ring. In the case of a global quotient $[X/G]$ we [JKK2] have identified the inverse image of the Chow ring as a subring of the full orbifold K–theory given by $K_{\mathrm{small}}([X/G]) = \mathscr{K}(X,G)^G$, that is the G–invariants of the global orbifold theory for a pair that defines a presentation of the stack.

2.8. **Comparing the Theories.** In particular for a global quotient three theories where introduced which are *additively* over $\mathbb{C}$ given as follows.

$$\mathscr{K}((X,G)) := \quad K(I(X,G))) \quad \simeq \bigoplus_{g\in G} K(X^g) \tag{2.2}$$

$$K_{full}([X/G]) := \quad K(\mathfrak{I}[X/G]) \quad \simeq \bigoplus_{[g]} K([X^g/Z(g)]) \tag{2.3}$$

$$K_{small}([X/G]) := \quad K_{global}((X,G))^G \quad \simeq \bigoplus_{[g]} K(X^g)^{Z(g)} \tag{2.4}$$

Notice that we proved in [JKK] that K_{small} is isomorphic to the Chen–Ruan Cohomology while $\mathscr{K}$ is isomorphic to the Fantechi–Göttsche ring via a Chern character.

Proposition 9. *Additively we have:*

$$\begin{aligned} \mathscr{K}(I(X,G),G) &= K(\amalg_{x\in G}(\amalg_{g\in G}X^g)^x \\ &= \bigoplus_{g\in G, x\in Z(g)} K(X^{\langle g,x\rangle}) \end{aligned} \tag{2.5}$$

and for $\prod x_i = 1$, *and* $g : x \in Z(g), h : y \in Z(h)$ *the multiplication is given by*

$$\mathcal{F}_{g,x_1} * \mathcal{F}_{h,x_2} = \check{e}_{x_3 *}(e^*_{x_1}(\mathcal{F}_{g,x})e^*_{x_2}(\mathcal{F}_{h,y})\mathcal{R}(x_1,x_2)) = \delta_{g,h}\, \mathcal{F}_{g,x_1} *_g \mathcal{F}_{g,x_2} \tag{2.6}$$

where $*_g$ *is the multiplication on* $\mathscr{K}(X^g, Z(g))$, *that is as rings*

$$\mathscr{K}(I(X,G),G) = \bigoplus_{g\in G} \mathscr{K}(X^g, Z(g)) \tag{2.7}$$

Corollary 10. *Additively:*

$$\begin{aligned} K_{\text{small}}(I(X,G),G) &= \mathscr{K}(I(X,G),G)^G \\ &= [\bigoplus_{(g,x)\in G\times G, x\in Z(g)} K((X^g)^x)]^G \\ &= \bigoplus_{[g,x]\in C^2(G), x\in Z(g)} K(X^{\langle g,x\rangle})^{Z(g,x)} \qquad (2.8) \end{aligned}$$

and as rings

$$K_{\text{small}}(I(X,G),G) = \bigoplus_{[g]\in C(G)} K_{\text{small}}(X^g, Z(g)) \qquad (2.9)$$

On the other hand we have *additively*

$$\begin{aligned} K_{\text{full}}([X/G]) &= \bigoplus_{[g]\in C(G)} K([X^g/Z(g)]) \\ &= \bigoplus_{[g]\in C(G)} K_{Z(g)}(X^g) \\ &= \bigoplus_{[g]\in C(G), [x]\in C(Z(g))} K((X^g)^x)^{Z(g,x)} \\ &= \bigoplus_{[g,x]\in C^2(G)} K(X^{\langle g,x\rangle})^{Z(g,x)} \qquad (2.10) \end{aligned}$$

The multiplication is more complicated however, see e.g. Theorem 11.

2.9. **Examples.** Let's look at some simple examples like (pt, G)

$$\begin{aligned} \mathscr{K}((pt,G)) &= k[G] \\ K_{\text{small}}((pt,G)) &= k[G]^G = Z(k[G]) \\ K_{AS}([pt/G]) &= K_G(pt) = Rep(G) \qquad (2.11) \end{aligned}$$

Here the latter is the Atiyah–Segal K–theory. We wish to point out that although the two latter theories are additively isomorphic they have decidedly different ring structures.

Theorem 11. *[KP] Let $D(k[G])$ be the Drinfel'd double of $k[G]$, then $K_{\text{full}}([pt/G]) \simeq Rep(D(k[G]))$.*

If G is Abelian then $\mathscr{K}((I(pt,G),G)) = D(k[G])$, else it is a subalgebra that is Morita equivalent to $D(k[G])$ in the sense that they have the same category of representations.

We were informed by C. Teleman, that a similar formula for the case of $[pt/G]$ can be deduced from the work of Freed-Hopkins-Teleman [FHT].

3. Applications

As mentioned previously the new description of the obstruction bundle opens up a plethora of possibilities to introduce stringy multiplications. Here are a few examples.

3.1. **DeRham theory.** In [Ka5], we used the definition of the obstruction bundle in terms of the $\mathscr{S}_g$ to give a pull–push formula suitable for de Rham–chains using a variant of the Thom–map. This gives a new stringy product on the chain level.

3.2. **Hochschild.** Similar results were obtained afterwards by [PPTT] in the S^1–equivariant setting using a Hochschild description for the groupoids.

3.3. **String Topology.** In [GLSU] the authors used our description of the obstruction bundle to define a string topology for orbifolds.

3.4. **Singularities.** In the case of singularities [Ka6] with an Abelian symmetry group we can also now solve the stringy multiplication problem [Ka1, Ka2] in terms of Hessians of the obstruction bundle using the results of [Ka5].

4. Twisting K–Theory with Gerbes

This paragraph is basically joint work with David Pham [KP].

We will consider twisting the various versions of K–theory for global quotients (X, G) with gerbes pulled back from $[pt/G]$. In particular, we will consider $0, 1, 2$ gerbes on (X, G) pulled back from $[pt/G]$ which are given by group co–cycles.

Following Hitchin and Thaddeus, we can think of these gerbes as *trivial gerbes* that are *not equivariantly trivial*.

4.1. **Gerbes over** $[pt/G]$**.** For $i \in \{0, 1, 2\}$ an i–gerbe is given by an element of $Z^i(G, k^*)$. That is we will actually consider trivialized gerbes.

4.2. **Twisting with** 0**–Gerbes aka. line bundles.** A 0–gerbe over $[pt/G]$ is nothing but an equivariant line bundle over a point, that is a 1–dim representation χ of G.

If we pull–back the line–bundle to X we get a trivial line bundle that is not equivariantly trivial. That is there are isomorphisms $g^*(\mathscr{L}) \simeq \mathscr{L}$ for each $g \in G$. Trivializing the line bundle we get back the representation χ of G.

We can use this to either

a) Twist Cohomology: $H^*(X, \mathscr{L})$.
b) Twist K-theory: $K_\chi(X) = K(X) \otimes \mathscr{L}$. That is we actually have an gauge endomorphism of $K(X)$ given by “twisting” with $\mathscr{L}$.

Remark 12. Notice that in this case there is no obvious product. The considerations are useful in the following, however, and in the context of non–strict G–Frobenius algebras and the Ramond twisting of [Ka1]. This theory plays a special role when considering singularities with symmetries [Ka2] where it corresponds to passing from the Milnor ring to the differentials by choosing a primitive form.

4.3. **Twisting with 1–Gerbes.** A 1–gerbe over $[pt/G]$ is a trivial gerbe over pt but the pull back under g need not be trivial. In this case the isomorphism of two trivial gerbes under pull–back by g is given by a line bundle.

If we pull–back the gerbe X we get a trivial gerbe $\mathscr{G}$ that is not equivariantly trivial. That is there are isomorphisms $g^*(\mathscr{G}) \simeq \mathscr{G}$ for each $g \in G$ given by (trivial) line bundles $\mathscr{L}_g$.

Moreover we have isomorphisms $\mathscr{L}_g \otimes \mathscr{L}_h \to \mathscr{L}_{gh}$ that satisfy associativity.

4.4. **Co-cycles.** Notice that if we trivialize the $\mathscr{L}_g$ we obtain a co-cycle $\alpha \in Z^2(G, (U(1))$.

4.5. **Transgression.** Another way to view the line bundles is via transgression to $\mathfrak{I}_{\mathscr{X}}$. Here we get a 0 gerbe that is a collection of line bundles $\mathscr{L}_g|_{\mathscr{X}_{(g)}}$.

We could also transgress to $I((X, G))$ or $\Lambda\mathrm{grp}(\mathscr{X})$, that is the loop groupoid of the groupoid associated to the stack.

We can use this to either

a) Twist Orbifold Cohomology: $H^*(X^g, \mathscr{L}_g)$ as was considered in [CR].
b) Twist K-theory: $K_\chi(X) = K(X^g) \otimes \mathscr{L}_g$. That is we actually have an gauge endomorphism of $K(X)$ given by "twisting" with $\mathscr{L}$.

Notice in both cases we can use the isomorphisms $\mathscr{L}_g \otimes \mathscr{L}_h \to \mathscr{L}_{gh}$ to get a multiplication on $\mathcal{F}_{global}((X, G))$ or $\mathcal{F}_{stringy}([X/G])$.

Note: Twists of the type b) were also considered in [AR], but since they do not consider a string product their twisted theory carries no internal product.

4.6. **Discrete torsion.**

4.6.1. **Twisted group ring.** Given a co-cycle $\alpha \in Z^2(g, k^*)$ we can define the twisted group ring $k^\alpha[G]$ by giving $k[G]$ a new multiplication defined by: $\hat{g} \cdot \hat{h} = \alpha(g, h)\widehat{gh}$

4.6.2. **A product for G–graded modules.** Given two G graded modules $A = \bigoplus_{g \in G} A_g$, $B = \bigoplus_{g \in G} B_g$, we define

$$A \hat{\otimes} B := \bigoplus_{g \in G} (A_g \otimes B_g)$$

Proposition 13. *Given $\alpha \in Z^2(g, k^*)$ the α twisted theories satisfy $\mathcal{F}^\alpha_{global}((X, G)) = \mathcal{F}_{global}((X, G)) \hat{\otimes} k^\alpha[G]$.*

This essentially identifies the geometric twists above with the algebraic twists of [JKK2, Ka4].

4.7. **Twisting with 2 Gerbes.**

Remark 14. The situation for 2–Gerbes is as follows:

(1) The 2–gerbes over $[\widetilde{p}/G]$ are of the type $\beta \in Z^3(G, U(1))$ and this type of gerbe can also the be used to twist the Drinfel'd double. And indeed there is a connection.

(2) We can transgress the equivariant 2–gerbe to an equivariant 1–gerbe $\mathscr{G}$ on $\mathfrak{I}_{\mathscr{X}}$ and actually even to a 1–gerbe over $(I(X,G),G)$. Here the gerbe is characterized by a set of line bundles, which provide the isomorphisms $\mathscr{L}_{g,x} : x^*(\mathscr{G}|_{X^g}) \xrightarrow{\sim} \mathscr{G}|_{X^{x^{-1}gx}}$ together with associativity isomorphisms $\theta_g(x,y) : \mathscr{L}_{g,x} \otimes \mathscr{L}_{h,y} \to \mathscr{L}_{g,xy}$ if $g = x^{-1}gx$.

(3) The condition of coming from a 2-gerbe expresses itself in a constraint on the θ_g. See below.

Definition 15. *for a finite group G and an element $\beta \in Z^3(G,k^*)$, the twisted Drinfel'd double $D^\beta(k[G])$ is the quasi-triangular quasi-Hopf algebra whose*

(1) *underlying vector space has the basis $g\underset{x}{\llcorner}$ with $x, g \in G$*

$$D^\beta(k[G]) = \bigoplus k\, g\underset{x}{\llcorner}$$

(2) *algebra structure is given by*

$$g\underset{x}{\llcorner}\, h\underset{y}{\llcorner} = \delta_{g,xhx^{-1}} \theta_g(x,y)\, g\underset{xy}{\llcorner}$$

where $\theta_g(x,y) = \frac{\beta(g,x,y)\beta(x,y,(xy)^{-1}g(xy))}{\beta(x,x^{-1}gx,y)}$

(3) *The co-algebra structure is given by*

$$\Delta(\, g\underset{x}{\llcorner}) = \sum_{g_1g_2=g} \gamma_x(g_1,g_2)\, g_1\underset{x}{\llcorner} \otimes\, g_2\underset{x}{\llcorner}$$

where $\gamma_x(g_1,g_2) = \frac{\beta(g_1,g_2,x)\beta(x,x^{-1}g_1x,x^{-1}g_2x)}{\beta(g_1,x,x^{-1}g_2x)}$

(4) *The R matrix is given by*

$$R = \sum_{g\in G} g\underset{e}{\llcorner} \otimes \mathbf{1}\underset{g}{\llcorner}, \text{ where } \mathbf{1}\underset{g}{\llcorner} = \sum_{h\in G} h\underset{g}{\llcorner}$$

(5) *The antipode S is given by*

$$S(\, g\underset{x}{\llcorner}) = \frac{1}{\theta_{g^{-1}}(x,x^{-1})\gamma_x(g,g^{-1})}\ x^{-1}g^{-1}x\underset{x^{-1}}{\llcorner}$$

(6) *The Drinfel'd associator Φ is given by*

$$\Phi = \sum_{g,h,k\in G} \beta(g,h,k)^{-1}\, g\underset{e}{\llcorner} \otimes h\underset{e}{\llcorner} \otimes k\underset{e}{\llcorner}$$

4.7.1. **Category of modules.** Since $D^\beta(k[G])$ is a quasi–triangular quasi Hopf algebra we have a *braided* category of $D^\beta(k[G])$ modules.

Notice that there is the operation of forming the tensor product which is neither commutative nor associative. But there are *isomorphisms* for the commutation and the association.

4.7.2. **2–Gerbe twisting.** First of all there is a naïve twisting on $\mathscr{K}((I(X,G),G)$ by the various θ_g transgressed from β. In the case of (pt,G) with G Abelian this yields a geometric incarnation of $D^\beta(k[G])$. In the general group case, we get a Morita equivalent subalgebra.

More importantly, however, there is a twisting for the full K–theory.

Definition 16. *Given $\beta \in Z^3(G,(U(1))$ we define the twisted full K–theory $K^{\beta}_{\text{full}}([X/G])$ using the co–product and the obstruction: That is the multiplcation that is induced by:*

$$\mathscr{F}_g \cdot \mathscr{F}_h := e_{3*}(e_1^*(\mathscr{F}_g) \otimes^{\gamma} e_2^*(\mathscr{F}_h) \otimes Obs_K(g,h))$$

Here we use the co–product in $D^{\beta}(k[G])$ which is given by γ defined above to define the action of $Z(g,h)$ on the tensored bundle.

Remark 17. There is a different twist considered in [ARZ], which is on the full K-theory of the inertia stack: $K_{\text{full}}(\mathfrak{I}_{\mathscr{X}})$ and does not seem to use a co–product structure.

Theorem 18. $K_{\text{full}}([pt/G])^{\beta} \simeq Rep(D^{\beta}(k[G]))$.

4.7.3. **Philosophical remarks.** This theorem is astonishing in the sense that the resulting structure is neither commutative nor associative in general. But it is of course associative and commutative in the sense of braided monoidal categories. We hope that we have motivated the appearance of braided monoidal categories already through the definition of Frobenius traces and objects. Moreover, if one reads for instance Moore and Seiberg's work on classical and quantum field theory one sees that the fusion ring is actually not expected to be associative and commutative. Only the dimensions of the intertwiners lead to such an algebra on the nose. In case of the objects themselves one should actually expect that one has to go to the braided picture.

4.7.4. **Alternative description in the trivial G–action case.** If we have no obstruction, like in the case $[pt/G]$, we can give a different simpler description:

For this we first recall the setup of G–equivariant K–theory in terms of modules: $K_G(X) \simeq B_{\text{proj.,fin. gen.}} - mod$ where B is $C^{\infty}(X) \rtimes G$ with the multiplication $(a,g)\cdot(a',g') = (ag(a'), gg')$.

In order to twist with a 1–gerbe $\alpha \in Z^2(G,(U(1))$ following Atiyah-Segal, we give a new multiplication on B via

$$(a,g)\cdot(a',g') = (ag(a'), \alpha(g,g')gg')$$

Now the twisted K–theory are just the projective finitely generated B–modules.

For the 2-gerbe β twisted K–theory, we twist the G action on $V \otimes W$ by using the co–product. In the free G–action case this amounts to the following. If $A = C^{\infty}(X)$ *and* the action is free then $C^{\infty}(I(X,G)) = \bigoplus_{g\in G} A$. Now although the G action on X is trivial, it is not trivial on IX, since it permutes the components.

It is easy to check that in the trivial G action case the algebra is

$$B = C^{\infty}(I(X,G)) \rtimes k[G] \simeq A \otimes D(k[G])$$

and hence we see that we can twist with β to obtain

$$B^{\beta} = A \otimes D^{\beta}(k[G])$$

whence we get the generalization of the Theorem of $[\widetilde{p}/G]$ to the case of a trivial G–action by considering the braided category projective finitely generated B^{β} modules.

Remark 19. In the algebraic category, we can use $\mathcal{O}_X$ instead of A.

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[!] Due to the time constraints the bibliography is abbreviated. Full references can be found in the proper papers.

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