P-adic L-functions for modular forms

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We construct "many variabled" S-adic L-function for modular forms over arbitrary number field k. We take for our form a weight 2 Hecke eigenform (on $GL(2)$, of level $\Gamma_0(\mathfrak{a})$) and for simplicity assume it is cuspidal at infinity. S is a finite set of primes away from the level of our form, and (if we want boundedness) is such that for $p \in S$ we can choose a root ρ_p of the p'th Euler polynomial that is a p- unit. The S-adic L-function is given by a measure on the Galois group of the maximal unramified-outside-S abelian extension of k; the measure obtained by playing the modular symbol game in an adelic setting. We prove that the S-adic L-function interpolates the critical values of the classical zeta function of the twists of our form by finite characters of conductor supported at S, and that it satisfies a similar functional equation. The gist of the p-adic continuation is the proof that a certain module in which our distribution takes its values is finitely generated, and the idea is to give this module a geometric interpretation as periods of a harmonic form against certain cycles. From our modular form we get an $r_1 + r_2$ harmonic form on the $2r_1+3r_2$ dimensional symmetric space

$$X = GL(2;k)\backslash GL(2;k_A)/\mathcal{K}_A\cdot\mathcal{Z}_\infty$$

where r_1 (resp. r_2) is the number of real (resp. complex) primes of k; $\mathcal{K}_A$ the level groups, $\mathcal{Z}_\infty$ the center at infinity. It turns out that one needs to work with an associated $[k:\mathbb{Q}] = r_1 + 2r_2$ form on the $2\cdot[k:\mathbb{Q}]$ dimensional symmetric space

$$X^{sgn} = GL(2;k)\backslash GL(2;k_A)/\mathcal{K}_A\cdot\mathcal{Z}_\infty^{\pm}$$

where $\mathcal{Z}_\infty^{\pm}$ consist of the real and totally positive elements in $\mathcal{Z}_\infty$; only in X^{sgn} can one define the appropriate cycles for a field k which is not totally real or CM. See [M,S-D] for the origin of all this, where the case $k = \mathbb{Q}$ is treated; [M] for totally real k; $[K]$ and [T] for CM fields. In order to keep everything in half their size we assume all the places at infinity of k are complex (the necessary adjustments needed for a field k having both real and complex places are indicated at the end of the paper).

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§1. In this section we recall the adelic definition of a modular form and fix our notations following mainly those of [W].

Let k denote our totally complex number field, $[k:\mathbb{Q}] = 2n$, $\mathcal{O}$ the integers of k, k_v the completion of k at a place v, $\mathcal{O}_v$ the integers of k_v, $k_A = k_{fin} \times k_\infty$ the adeles, and fix a character $\psi: k_A/k \to \mathbb{C}^*$, $\psi = \otimes_v \psi_v$, by $\psi_v(x) = \exp[-2\pi i(x+\bar{x})]$ for $v \mid \infty$. We write $k_\infty^* = k_\infty^{sgn} \times k_\infty^+$ where $k_\infty^{sgn} = \prod_{v\mid\infty} k_v^{sgn}$ is the maximal compact subgroup of the infinite ideles k_∞^*, $k_\infty^+ = \prod_{v\mid\infty} k_v^+$, k_v^+ the positive reals inside k_v^*, and we let $x = sgn(x)\cdot|x|$ denote the respective decomposition of $x \in k_\infty^*$. We fix ideles $\tau_1 \cdots \tau_h$ representing $Cl(k)$, the class group of k; ∂ representing the absolute different of k; $\mathfrak{a}$ representing the level of our modular form (i.e. the classical $\Gamma_0(\mathfrak{a})$); $\mathfrak{f}$ representing the conductor of a grossencharacter ω; usually $\mathfrak{a}, \mathfrak{f}$ (and the τ_i's) will be taken relative prime. Let $G = GL(2)/k$ and G_k, G_v, $G_A = G_{fin} \times G_\infty$ its points with values in k, k_v, k_A respectively; $\mathcal{Z}_k, \mathcal{Z}_v$, $\mathcal{Z}_A = \mathcal{Z}_{fin} \times \mathcal{Z}_\infty$ the centers of the above groups, $\mathcal{Z}_\infty^+$ the real and totally positive elements of $\mathcal{Z}_\infty$; $B = \{(x,y) \overset{def}{=} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\} = G_m \times G_a$ and B_k, B_v, $B_A = B_{fin} \times B_\infty$ its rational points, $B_\infty^+ = \{(x,y) \in B_\infty \text{ with } x \in k_\infty^+\}$. We define our level groups by $\mathcal{K}_v = SU(2;k_v)$ for $v \mid \infty$, for $v \nmid \infty$ we set

$$\mathcal{K}_v = \left\{ \begin{pmatrix} x & \partial^{-1} y \\ a\,\partial z & w \end{pmatrix},\ x,y,z,w \in \mathcal{O}_v,\ \det \in \mathcal{O}_v^{\bullet} \right\}$$

and we write $\mathcal{K}_A = \mathcal{K}_{fin} \times \mathcal{K}_\infty$ for the associated adelic group. Let $\mathcal{V} = \bigotimes_{v|\infty} \mathcal{V}_v$ the value space of our form, where $\mathcal{V}_v$ is a 3-dimensional complex vector space with basis V_v^1, V_v^0, V_v^{-1}, so $\mathcal{V}$ has basis V^e, $e = \{e_v\}$, $e_v \in \{1,0,-1\}$. We let $\mathcal{K}_\infty$ act on the right on $\mathcal{V}$ via the symmetric square representation M:

$$M\begin{pmatrix} c & b \\ -\bar{b} & \bar{c} \end{pmatrix} = \bigotimes_{v|\infty} \begin{pmatrix} c_v^2 & 2c_v b_v & b_v^2 \\ -c_v \bar{b}_v & |c_v|^2 - |b_v|^2 & \bar{c}_v b_v \\ \bar{b}_v^2 & -2\bar{c}_v \bar{b}_v & \bar{c}_v^2 \end{pmatrix}$$

and we extend this action to all of $\mathcal{K}_A \mathcal{Z}_A$ by trivial $\mathcal{K}_{fin}\mathcal{Z}_A$ action. We define $W : k_\infty^\bullet \to \mathcal{V}$, $\quad W(x) = \bigotimes_{v|\infty} W_v(x_v)$, $\quad W_v(x_v) = \sum_{j=1,0,-1} W_{v,j}(x_v)\cdot V_v^j$, with

$W_{v,0}(x) = |x|^2 K_0(4\pi|x|)$, $W_{v,\pm1}(x) = \frac{1}{2}[\frac{1}{i}\, sgn(x)]^{\pm 1}\cdot|x|^2 K_1(4\pi|x|)$, where K_0, K_1 are Hankel's functions [F].

Let $F : G_A = B_A \mathcal{Z}_A \mathcal{K}_A \to \mathcal{V}$ denote our modular form, so $F(gkz) = F(g)M(k)$ for $k \in \mathcal{K}_A$, $z \in \mathcal{Z}_A$, and $F\left(g \cdot \begin{pmatrix} 0 & -\partial^{-1} \\ \partial a & 0 \end{pmatrix}_{fin}\right) = \varepsilon_F \cdot F(g)$, $\varepsilon_F = \pm 1$. We assume for simplicity that F is cuspidal at infinity and so has Fourier expansion $F(x,y) = \sum_{\xi \in k^\bullet} C((\xi x))\, W(\xi x_\infty)\psi(\xi y)$, and we write $L_F(\omega) = \sum_b C(b)\cdot\omega(b)$ for the associated L-function. We assume that F is an eigenform of all the Hecke operators T_p, thus $L_F(\omega)$ has an Euler expansion $L_F(\omega) = \prod_{v\nmid\infty} P_v(\mathrm{N}v^{-1}\cdot\omega(v))^{-1}$ with Euler polynomial $P_v(t) = 1 - \lambda_v t + \mathrm{N}v \cdot t^2 = (1-\rho_v t)\cdot(1-\tilde{\rho}_v t)$ for $v \nmid (a)\infty$. Note that everything is normalized so that the functional equation for finite ω has the form $L_F(\omega) = (-1)^n \cdot \varepsilon_F \cdot \omega((a))\cdot\tau(\omega)^2 \cdot L_F(\omega^{-1})$, i.e. the critical value is at "$s = 0$"; here the Gaussian sums are defined by $\tau(\omega) = \prod_{v\nmid\infty} \tau_v(\omega)$, $\quad \tau_v(\omega) = \omega_v(\partial)$ for $v \nmid (f)$, and for $v \mid (f)$: $\tau_v(\omega) = |f|_v^{1/2} \cdot \sum_{\eta \in (\mathcal{O}_v/(f_v))^\bullet} \omega_v^{-1}(\partial^{-1} f^{-1}\eta)\psi_v(\partial^{-1} f^{-1}\eta)$.

§2. In this section we define the harmonic form, on the symmetric space X, associated with our modular form, following [W], and introduce the new symmetric space X^{sgn}.

Let $X = G_k \backslash G_A / \mathcal{K}_A \cdot \mathcal{Z}_\infty$. Decomposing it into connected components we get $X = \bigcup_{i=1}^{h} X_{r_i}$ with $X_{r_i} = \Gamma_{r_i} \backslash G_\infty / \mathcal{K}_\infty \mathcal{Z}_\infty$, $\Gamma_{r_i} = G_k \cap [(r_i,0)\mathcal{K}_{fin}(r_i^{-1},0) \times G_\infty]$. We have coordinates (x,y) on $G_\infty / \mathcal{K}_\infty \mathcal{Z}_\infty$ via the map $B_\infty^+ \tilde{\to} G_\infty / \mathcal{K}_\infty \mathcal{Z}_\infty$ and the Riemannian structure is the usual $ds^2 = \frac{1}{x^2}(dx^2 + dy d\bar{y})$, so each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty$ acts as an isometry on B_∞^+; we denote this action by $\gamma \circ (x,y)$ and define $J(\gamma;(x,y)) = \begin{pmatrix} sgn(\gamma)\overline{(cy+d)} & -sgn(\gamma)\overline{cx} \\ cx & (cy+d) \end{pmatrix} \in \mathcal{K}_\infty \mathcal{Z}_\infty$ where $sgn(\gamma) = sgn(\det(\gamma)) \in k_\infty^{sgn}$. We have $\gamma \circ (x,y) = \gamma \cdot \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \cdot J(\gamma ; (x,y))^{-1}$ from which we derive the *automorphy relation*

$$J(\gamma_1\gamma_2;(x,y)) = J(\gamma_1;\gamma_2 \circ (x,y)) \cdot J(\gamma_2;(x,y)).$$

On B_∞^+ we define an n-form with values in $\mathcal{V}'$, the vector space dual to $\mathcal{V}$, by $\beta = \sum_a \beta^a \cdot V_a$, where $\{V_a\}$ is the dual basis of $\{V^a\}$, and $\beta^a = \bigwedge_{v|\infty} \beta_v^{a_v}$, $\beta_v^{a_v} = -\frac{dy_v}{x_v}$, $\frac{dx_v}{x_v}$, $\frac{d\bar{y}_v}{x_v}$ for $a_v = 1, 0, -1$ respectively.

Claim: $\beta|_\gamma(x,y) = \beta(x,y)^t M(J(\gamma;(x,y)))$, $\gamma \in G_\infty$.

Using the automorphy relation and the decomposition $G_\infty = B_\infty^+ \mathcal{K}_\infty \mathcal{Z}_\infty$ it is sufficient to consider the cases:

(i) $\gamma \in Z_\infty$ where $J(\gamma ; (x,y)) = \gamma$, $M(J(\gamma ; (x,y))) = 1$;

(ii) $\gamma \in B_\infty^+$ where $J(\gamma ; (x,y)) = 1$;

(iii) $\gamma \in \mathcal{K}_\infty$ and $(x,y) = (0,1)$ where $J(\gamma ; (1,0)) = \gamma$.

The cases (i) and (ii) are trivial, and (iii) is a straightforward calculation.

Claim: $F(\gamma\circ(x,y)\cdot(r_i,0)) = F((x,y)\cdot(r_i,0))\cdot M(J(\gamma;(x,y))^{-1}, \gamma\in\Gamma_{r_i}$.

Indeed, we have,

$$F(\gamma\circ(x,y)\cdot(r_i,0)) = F(\gamma^{-1}\cdot\gamma\circ(x,y)\cdot(r_i,0)) \qquad \text{by left } G_k\text{-invariance}$$

$$\begin{aligned}
&= F(\gamma_\infty^{-1}\cdot\gamma\circ(x,y)\cdot(r_i,0)\cdot(r_i^{-1},0)\cdot\gamma_{fin}^{-1}\cdot(r_i,0))\\
&= F(\gamma_\infty^{-1}\cdot\gamma\circ(x,y)\cdot(r_i,0)) \qquad \textit{since}\ (r_i^{-1},0)\gamma_{fin}^{-1}(r_i,0)\in\mathcal{K}_{fin}\\
&= F((x,y)\cdot J(\gamma;(x,y))^{-1}\cdot(r_i,0))\\
&= F((x,y)\cdot(r_i,0))\cdot M(J(\gamma;(x,y))^{-1}
\end{aligned}$$

Now let $\Omega_{r_i}(x,y) = F(r_i x,y)\cdot\beta(x,y)$. Using the above two claims we observe that Ω_{r_i} is Γ_{r_i}- invariant, and can be viewed as a $\mathbb{C}$-valued n-form on $X_{r_i} = \Gamma_{r_i}\backslash B_\infty^+$. (note that elliptic elements in Γ_{r_i} give whole geodesics that are singular, and X_{r_i} is not a manifold; strictly speaking we should view Ω_{r_i} as a Γ_{r_i}/Γ_0-invariant form on $\Gamma_0\backslash B_\infty^+$, where $\Gamma_0\subset\Gamma_{r_i}$ is a subgroup of finite index having no torsion). The properties of Hankel's functions, $xK_0'' + K'_0 = xK_0$, $K_1 = -K'_0$, imply that the 1-form

$$\sum_{e=1,0,-1} W_{v,e}(x)\psi_v(y)\beta_v^e = xK_0(4\pi x)e^{-2\pi i(y+\bar{y})}dx + \frac{i}{2}xK_1(4\pi x)e^{-2\pi i(y+\bar{y})}(dy+d\bar{y})$$

is closed and $*$-closed. Hence we see that Ω_{r_i} is harmonic, and so we have a cohomology class $[\Omega]\in H^n(X,\mathbb{C})$ represented by the n-form Ω with $\Omega|X_{r_i} = \Omega_{r_i}$.

Letting $H = B_\infty^+ \cup \mathbb{P}^1(k)$, and taking for neighborhoods of $\eta \in k$ the sets $\{\eta\} \cup \{(x,y) \mid \prod_{v|\infty} \frac{1}{x_v}(|\eta - y|_v^2 + |x|_v^2) < r\}$ and for ∞ the sets $\{\infty\} \cup \{(x,y) \mid \prod_{v|\infty} \frac{1}{x_v} < r\}$, for all $r > 0$, we see that G_k acts continuously on the Hausdorff space H, and we get the compactification $\bar{X} = \bigcup_{i=1}^{h} \bar{X}_{r_i}$ of X, where $\bar{X}_{r_i} = \Gamma_{r_i} \setminus H$.

Let $X^{sgn} = G_k \setminus G_A / \mathcal{K}_A \cdot \mathcal{Z}_\infty^+ = \bigcup_{i=1}^{h} \Gamma_{r_i} \setminus B_\infty$, where similarly to above we put coordinates via $B_\infty \xrightarrow{\sim} G_\infty / \mathcal{K}_\infty \mathcal{Z}_\infty^+$, and note that the canonical projection $G_\infty / \mathcal{K}_\infty \mathcal{Z}_\infty^+ \to G_\infty / \mathcal{K}_\infty \mathcal{Z}_\infty$ is given by $B_\infty \to B_\infty^+$, $(x,y) \to (|x|,y)$. On B_∞ we define an $\mathbb{R}$-valued n-form $\Theta = \bigwedge_{v|\infty} \Theta_v$, by $\Theta_v(x,y) = \frac{1}{2\pi i} d \log(sgn(x_v))$; this is G_k-invariant since $sgn(\gamma \circ (x,y)) = sgn(\gamma) \cdot sgn(x)$, and so we have a closed n-form Θ on X^{sgn}. We denote by $\bar{X}^{sgn} = \bigcup_{i=1}^{h} \Gamma_{r_i} \setminus B_\infty \cup \mathbb{P}^1(k)$ the obvious compactification of X^{sgn} induced by the Seifert-fibration $X^{sgn} \to X$. (this becomes an actual fibration after passage to subgroups $\Gamma_0 \subseteq \Gamma_{r_i}$ having no torsion).

Fixing an infinite place $v \mid \infty$, one can look at the action of G_v on B_v in the following way. Denote by j the element of the quaternions H, and identify B_v with $H \setminus \mathbb{C}$ via $(x,y) \to z = x + yj$. The action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_v$ on $H \setminus \mathbb{C}$ becomes the Möbius action $\gamma \circ z = (az + b) \cdot (cz + d)^{-1}$.

§3. In this section we study the periods $L(r,\eta)$; these are first introduced as an adelic integral, then after Lemma 1, we transform it to an archimeadian integral, and finally after Lemma 2, we show it is given by an integral of our harmonic form pulled back to X^{sgn} against a relative cycle going from the cusp at infinity to the cusp "(r,η)". Besides giving us a geometrical intuition, we can deduce from this interpretation the crucial result that the module generated by these periods is finitely generated.

For $r \in k_A^*$, $\eta \in k_{fin}$, such that $|\eta|_v < |r|_v$ for $v|(a)$, we define the "*periods*":

$$L(r,\eta) = \frac{1}{[\mathcal{O}^*:\mathcal{E}]} \int_{k^* \backslash \prod_{v\nmid\infty} \mathcal{O}_v^*/\mathcal{E}} F_0(\partial r x, -\eta) d^* x$$

where $F_0 : G_A \to \mathbb{C}$ is the $\otimes_{v|\infty} V_v^0$-component of F, $\mathcal{E}$ the subgroup of $e \in \mathcal{O}^*$ satisfying $e \equiv 1 \; mod(r_v/\eta_v)$, (which holds trivially when $|\eta|_v \le |r|_v$, i.e. for almost all v' s), and the Haar measure $d^*x = \otimes_v d^*x_v$ being normalized by $\int_{\mathcal{O}_v^*} d^*x_v = 1$ for $v \nmid \infty$, and $d^*x_v = \dfrac{d \; sgn(x_v) \wedge d|x|_v}{2\pi i \cdot x_v}$ for $v|\infty$.

Lemma 1:

(0) $L(r,\eta)$ is well defined.

(1) $L(r,\eta)$ depends only on the ideal $((r)) \overset{def}{=} k \cap (r)$, $(r) \overset{def}{=} \prod_{v \nmid \infty} r_v \mathcal{O}_v$.

(2) $L(r,\eta)$ depends only on the image $\eta \in k_{fin}/(r)$.

(3) $L(r,\eta) = L(r\xi, \eta\xi)$ for $\xi \in k^*$.

(4) $L(r,\eta) = (-1)^n \varepsilon_F L(a r_S^2 r^{-1}, -\eta^{-1})$ for $\eta_v = 0$, $v \notin S$; and $\eta_v \in \mathcal{O}_v^*$, $|r|_v < 1$, $v \in S$.

Proof: (1) is clear, (2) follows since by right $\mathcal{K}_{fin}$- invariance, for $\mu \in (r)$, $F(\partial r x, -\eta) = F((\partial r x, -\eta)\;(1, -\partial^{-1} r^{-1} x \mu)) = F(\partial r x, -\eta - \mu)$; (3) follows since by left G_k-invariance, for $\xi \in k^*$, $F(\partial r x, -\eta) = F((\xi,0)(\partial r x, -\eta)) = F(\partial r \xi x, -\eta\xi)$. As for (0), using (1), (2), (3) it's easily seen that the integrand in the definition of $L(r,\eta)$ is $\mathcal{E}$-invariant so integration $mod\; \mathcal{E}$ is o.k. if it converges. For convergence, we first use (2) and assume $\eta_v = 0$ for v outside a finite set of places S, then using (3) we can assume $\eta_v \in \mathcal{O}_v^*$ and $|r|_v < 1$ for $v \in S$, now

$$(\partial rx,-\eta)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}\partial a\eta^{-1}&-ar_S^2r^{-1}x^{-1}\\\partial a&0\end{pmatrix}\cdot\cdot\begin{pmatrix}r&-\partial^{-1}x^{-1}\eta\\\partial x\eta^{-1}&0\end{pmatrix}_S\cdot a^{-1}r_S^{-1}rx$$

(where $r_S = r_v$ (resp. 1), $\eta^{-1} = \eta_v^{-1}$ (resp. 0) for $v \in S$ (resp $v \not\in S$)) and so

$$F_0(\partial rx,-\eta)=F_0\begin{pmatrix}\partial a\eta^{-1}&-ar_S^2r^{-1}x^{-1}\\\partial a&0\end{pmatrix}=F_0\left(\begin{pmatrix}\partial ar_S^2r^{-1}x^{-1}&\eta^{-1}\\0&1\end{pmatrix}\begin{pmatrix}0&-\partial^{-1}\\\partial a&0\end{pmatrix}\right)=$$

$$=(-1)^n\varepsilon_F\cdot F_0(\partial ar_S^2r^{-1}x^{-1},\eta^{-1}).$$

By using the fact that F is cuspidal at infinity and trivial estimates on Hankel's K_0, we get $|F_0(\partial rx,-\eta)| = O(|x|^\sigma)$ for all $\sigma \in \mathbb{R}$ as $|x| \to \infty$, and from the above formula also when $|x| \to 0$; this proves convergence (that is, our condition, $|\eta|_v < |r|_v$ for $v|(a)$, imply that the cusp (r,η) is congruent to the cusp at infinity). Integrating the above formula over $k_\infty^\bullet \prod_{v|\infty}\mathcal{O}_v^\bullet/\mathcal{E}$ we obtain (4).

Note that by part (3) of the lemma we can translate any $L(r,\eta)$ into some $L(\partial^{-1}r_i,\eta')$, and then using part (2) we can assume $\eta' = \alpha_{fin}$ for some $\alpha \in k^\bullet$, finally using left G_k-invariance we obtain the *archimedian integral expression:*

$$L(r,\eta) = L(\partial^{-1}r_i,\alpha_{fin}) = \frac{1}{[\mathcal{O}^\bullet,\mathcal{E}]}\int_{k_\infty^\bullet/\mathcal{E}} F_0(r_ix,\alpha_\infty)d^\bullet x.$$

We shall now define our relative cycles. Let $z(r_i,\alpha) : k_\infty^\bullet \to X_{r_i}^{sgn}$, $z(r_i,\alpha)(x) =$ image of (x,α_∞) in $X_{r_i}^{sgn}$. Note that for $e \in \mathcal{E}$, $(1-e)\alpha \in (r_i\partial^{-1})$, hence $(e,(1-e)\alpha) \in \Gamma_{r_i}$, and we get $z(r_i,\alpha)(ex) =$ image of $(e,(1-e)\alpha)\circ(x,\alpha) = z(r_i,\alpha)(x)$, so we can view $z(r_i,\alpha)$ as a smooth map $k_\infty^\bullet/\mathcal{E} \to X_{r_i}^{sgn}$. Moreover, let $I(\mathcal{E})$ denote the obvious compactification of $k_\infty^\bullet/\mathcal{E} \approx (0,\infty)\times(\mathbb{R}/\mathbb{Z})^{2n-1}$ obtained by adding $0\times(\mathbb{R}/\mathbb{Z})^{2n-1}$ and $\infty\times(\mathbb{R}/\mathbb{Z})^{2n-1}$, so that $I(\mathcal{E}) \approx [0,\infty]\times(\mathbb{R}/\mathbb{Z})^{2n-1}$. Setting $z(r_i,\alpha)[0\times(\mathbb{R}/\mathbb{Z})^{2n-1}] = \alpha$, $z(r_i,\alpha)[\infty\times(\mathbb{R}/\mathbb{Z})^{2n-1}] = \infty$, we get a continuous $2n$ relative cycle, $z(r_i,\alpha) : I(\mathcal{E}) \to \overline{X}_{r_i}^{sgn}$, with $\partial z(r_i,\alpha)$ supported on $\{\alpha,\infty\} \subseteq \partial\overline{X}_{r_i}^{sgn}$.

Lemma 2: $[\mathcal{O}^\bullet:\mathcal{E}]L(r,\eta) = \int_{z(r_i,\alpha)} \Omega_{r_i}^{sgn}\wedge\Theta$

where $\Omega_{r_i}^{sgn}$ is the pull-back of Ω_{r_i} along $\pi : X_{r_i}^{sgn} \to X_{r_i}$.

Proof: We have: $\int_{z(r_i,\alpha)} \Omega_{r_i}^{sgn} \wedge \Theta$

$$= \int_{k_-^\cdot/\mathcal{E}} [F(r_i|x|,\alpha_\infty)\cdot(z(r_i,\alpha)^*\pi^*\beta)] \wedge \bigwedge_{v|\infty} [\frac{1}{2\pi i} d \log(sgn(x_v))]$$

but since all the "y-components" of $z(r_i,\alpha)$ are constant, $y_v = \alpha_v$, the above simplify to

$$\int_{k_-^\cdot/\mathcal{E}} F_0(r_i x,\alpha_\infty) \bigwedge_{v|\infty} \frac{d|x|_v}{|x|_v} \wedge \bigwedge_{v|\infty} \frac{d\ sgn(x_v)}{2\pi i\cdot sgn(x_v)} = [\mathcal{O}^\cdot:\mathcal{E}]L(r,\eta)$$

by the above archimedian integral expression.

Corollary: The $\mathbf{Z}$-module $\mathcal{L}^0 \subseteq \mathbf{C}$ generated by all the numbers $\{[\mathcal{O}^\cdot:\mathcal{E}]L(r,\eta)\}$, $r \in k_A^\cdot$, $\eta \in k_{fin}$, $|\eta|_v < |r|_v$ for $v|(a)$, is finitely generated.

Proof: The forms $\Omega_{r_i}^{sgn} \wedge \Theta$ are closed and so the integral in Lemma 2 depends only on the homology class of $z(r_i,\alpha)$ in $H_{2n}(\overline{X}^{sgn},\partial\overline{X}^{sgn};\mathbf{Z})$.

§4. In this section, following [M]'s and [K]'s generalization of the basic idea of [B], we prove "Birch's Lemma" expressing the critical values of the L-functions as linear combinations of our periods.

Let ω denote now a *finite* grossencharacter primitive of conductor (f), and set $F_0^\omega(x) = \sum_{\xi\in k^\cdot} C((\xi x))\omega((\xi x))\cdot W_0(\xi x_\infty)$, where $W_0(x)$ is the $\bigotimes_{v|\infty} V_v^0$-component of $W(x)$. An easy calculation gives

$$\Gamma(s)\cdot L_F(\omega_s) = \int_{k_A^\cdot/k^\cdot} F_0^\omega(x)\ |x|_A^s d^\cdot x$$

where $\omega_s(x) = \omega(x)\cdot|x|_A^s$, $\Gamma(s) = (4\pi)^{-2n}(2\pi)^{-2ns}\Gamma(s+1)^{2n}$, and Re(s) is large. Decomposing the above integral into ideal classes, we get

$$\Gamma(s)\cdot L_F(\omega_s) = \sum_{i=1}^{h} |\tau_i|_A^s \cdot \frac{1}{[\mathcal{O}^*:\mathcal{E}]} \int_{k_\infty^*/\mathcal{E}} F_0^\omega(\tau_i x_\infty)\cdot |x|_\infty^s \, d^*x_\infty.$$

An application of finite Fourier inversion gives for $\xi \in k^*$:

$$\omega((\xi)) = \tau(\omega)\cdot |f|_A^{1/2} \cdot \sum_{\eta\in(\mathcal{O}/(f))^*} \omega(\partial^{-1}f^{-1}\eta)\psi(-\partial^{-1}f^{-1}\eta\xi)$$

where in any multiplicative context (e.g. in $\omega(\ldots)$) we view η as an idele equal to 1 outside (f), and in any additive content (e.g. in $\psi(\ldots)$) we view η as an adele equal to 0 outside (f). Using this we get:

$$F_0^\omega(\tau_i x) = \tau(\omega)\cdot |f|_A^{1/2} \sum_{\eta\in(\mathcal{O}/(f))^*} \omega(\tau_i\partial^{-1}f^{-1}\eta)F_0(\tau_i x, -\partial^{-1}f^{-1}\eta)$$

substituting this in the above and evaluating at $s = 0$, we obtain:

$$L_F(\omega)=\tau(\omega)|f|_A^{1/2}(4\pi)^{2n}\sum_{i=1}^{h}\sum_{\eta\in(\mathcal{O}/(f))^*}\omega(\tau_i\partial^{-1}f^{-1}\eta)\frac{1}{[\mathcal{O}^*:\mathcal{E}]}\int_{k_\infty^*/\mathcal{E}} F_0(\tau_i x, -\partial^{-1}f^{-1}\eta)d^*x.$$

Letting $\xi \in k^*$ be such that $|\xi|_v = |\partial f|_v^{-1}$ for $v\,|\,(f)$, putting $\eta\xi\partial f$ for η, and $x_\infty\cdot\xi_\infty$ for x_∞, then using left G_k-invariance to multiply the argument of F_0 by $(\xi^{-1},0)$, and finally substituting $\tau_i(\xi\partial)_{(f)}^{-1}\partial(\xi)$ for τ_i, we have the following

Birch Lemma [B]: For finite character ω, primitive of conductor (f),

$$L_F(\omega) = \tau(\omega)\cdot |f|_A^{1/2}(4\pi)^{2n}\sum_{i=1}^{h}\sum_{\eta\in(\mathcal{O}/(f))^*}\omega(\tau_i\eta)L(\tau_i f,\eta)$$

§5. In this section, following [M]'s adelization of [M,S-D], we construct distributions μ_τ by specifying its values on open sets to be a certain linear combination of our periods. The additivity of μ_τ follows from the Hecke Relations among the periods.

Fix S, a finite set of primes of k away from $(a)\infty$. Denote by $\mathcal{L}_S^0$ the $\mathbf{Z}[\rho_v^{-1};v \in S]$-module generated by $[\mathcal{O}^*:\mathcal{E}_{d,\eta}]\cdot L(\tau d,\eta)$ with τ prime-to-S, d supported-on-S, $\eta \in k_S$, and recall that $\mathcal{E}_{d,\eta} = \{e \in \mathcal{O}^* \mid |(e-1)\eta|_v \le |d|_v$ for $v \in S\}$, and that ρ_v is *one of the two* roots of the v'th Euler polynomial; we also set $\rho_d = \prod_{v\in S}\rho_v^{\mathrm{ord}_v d}$. Whenever $\eta \in k_S/(d)$ is given by the context as $\eta \in k_S/(bd)$, (e.g. when b^{-1} is integral), we can define a formal operator $\mathcal{R}_b L(rd,\eta) = L(rbd,\eta)$; these are only formal conveniences and whenever we have an expression involving $\mathcal{R}_b$'s and $L(rd,\eta)$'s we first apply the R_d's and only thereafter can evaluate the periods. We define the operators $\mathcal{U}_p$ for $p \in S$ by $\mathcal{U}_p L(r,\eta) = \sum_{u \in \mathcal{O}/p} L(rp,\eta+u)$, and extend this to all the $L(rd,\eta)$' s by using Lemma 1,(3).

Hecke Lemma: When acting on $L(r,\eta)$, r prime-to-p, we have the following relations:

(1) $(\rho_p + \tilde{\rho}_p) = \mathcal{R}_{p^{-1}} + \mathcal{U}_p$

(2) $\rho_p \cdot \tilde{\rho}_p = \mathcal{R}_{p^{-1}} \cdot \mathcal{U}_p$.

Proof: (2) is clear since $\rho_p \cdot \tilde{\rho}_p = \mathbb{N}p$ and $\mathcal{R}_{p^{-1}}\mathcal{U}_p$ also equal $\mathbb{N}p$ since for all u $L(r,\eta+u) = L(r,\eta)$ by Lemma 1, (2). For (1) we use the fact that F is a Hecke eigenform with eigenvalue $\rho_p + \tilde{\rho}_p$, and the fact that $T_p = \mathcal{R}_{p^{-1}} + \mathcal{U}_p$ when acting on any $L(r,\eta)$ with r prime-to-p.

We define a $\mathbb{Q} \otimes_{\mathcal{O}} \mathcal{L}_S^g$-valued distribution μ_r on $\mathcal{O}_S^\bullet$ by giving its values on "elementary sets" as follows. We write $S = S_0 \cup S_1$, and denote by p's (resp. q's) the primes in S_1 (resp. S_0); we let $f = \prod p^{e_p}$ with $e_p > 0$, and let $\eta \in \mathcal{O}_{S_1}^\bullet$ be extended to $\eta \in \mathcal{O}_S$ by decreeing that $\eta_q = 0$; we set $\eta + f^\bullet \overset{def}{=} \mathcal{O}_{S_0}^\bullet \times \prod_p (\eta + p^{e_p}) \subset \mathcal{O}_S^\bullet$. Every open set is a finite union of such elementary open sets.

Definition:

$$\mu_r(\eta+f^\bullet) = \prod_q (1-\rho_q^{-1}\mathcal{R}_q)(1-\rho_q^{-1}\mathcal{R}_{q^{-1}})\prod_p(1-\rho_p^{-1}\mathcal{R}_{p^{-1}})\cdot\rho_f^{-1}\mathcal{R}_f L(r,\eta).$$

Note that this depends only on the image of η in $\mathcal{O}_{S_1}^\bullet/(1+(f))$ by Lemma 1,(2).

Lemma 3. μ_r is indeed a distribution,

$$\mu_r(\bigcup_{j=1}^{N} u_j) = \sum_{j=1}^{N} \mu_r(u_j) \text{ for disjoint open sets } u_j \subseteq \mathcal{O}_S^{\bullet}.$$

Proof: It's enough to check that

$$\text{(I)} \qquad \sum_{\substack{u_q \bmod q \\ u_q \neq 0}} \mu_r(\eta+\sum_q u_q + (\Pi q)f^{\bullet}) = \mu_r(\eta+f^{\bullet})$$

and to check that for f divisible by all $p \in S$, $\eta \in \mathcal{O}_S^{\bullet}$, and any $p_0 \in S$

$$\text{(II)} \qquad \sum_{\substack{\eta' \bmod fp_0 \\ \eta' \equiv \eta \bmod f}} \mu_r(\eta'+p_0 \cdot f^{\bullet}) = \mu_r(\eta+f^{\bullet}).$$

Letting $(-1)^d$ denote the Mobious function we have the additive expression

$$\mu_r(\eta+f^{\bullet}) = \rho_f^{-1} \sum_{d|f} (-1)^d \rho_d^{-1} \mathcal{R}_{fd^{-1}} L(r,\eta)$$

whenever f is divisible by all places in S.

Using this expression for the left hand side of (I), then grouping terms back into a multiplicative form, we obtain

$$\rho_{f\pi q}^{-1} \prod_q (\mathcal{R}_q - \rho_q^{-1})(\mathcal{R}_{q^{-1}} \mathcal{U}_q - 1) \prod_p (1-\rho_p^{-1} \mathcal{R}_{p^{-1}}) \cdot \mathcal{R}_f \; L(r,\eta)$$

and (I) follows upon invoking the Hecke Lemma to put

$$\rho_q^{-1}(\mathcal{R}_q - \rho_q^{-1})(\mathcal{R}_{q^{-1}} \mathcal{U}_q - 1) = (1-\rho_q^{-1} \mathcal{R}_{q^{-1}})(1-\rho_q^{-1} \mathcal{R}_q).$$

For (II) we choose $\xi \in k^{\bullet}$, such that $(\xi)_S = f$, and writing $\eta' = \eta + \xi u$, with $u \in \mathcal{O}_{p_0}$ running through a complete set of representatives of $\mathcal{O}/p_0$, we use again the additive expression for the left hand side of (II), and we obtain

$$\rho_{fp_0}^{-1} \sum_{u \bmod p_0} \sum_{d|fp_0} (-1)^d \rho_d^{-1} \mathcal{R}_{fp_0 d^{-1}} L(r,\eta+u\xi).$$

Writing $\sum_{d|fp_0}$ as $\sum_{\substack{d|f \\ p_0 \nmid d}} + \sum_{\substack{d|fp_0 \\ p_0|d}}$, and substituting dp_0 for d in the second sum, then using Lemma 1,(3) to divide by ξ, we get

$$\rho_f^{-1}\sum_{\substack{d\mid f\\ p_0\nmid d}}(-1)^d\rho_d^{-1}\sum_{u \bmod p_0}[\rho_{p_0}^{-1}L(r\xi^{-1}fp_0d^{-1},\eta\xi^{-1}+u)-\rho_{p_0}^{-2}L(r\xi^{-1}fd^{-1},\eta\xi^{-1}+u)].$$

Using Hecke Lemma (1) and (2) for the first and second terms inside the brackets respectively, then using Lemma 1,(3) to multiply back by ξ, we get the additive expression for $\mu_r(\eta+f^\bullet)$ upon canceling terms inside the brackets.

Note that by Lemma 1,(1) and (3), we have for $e\in\mathcal{O}^\bullet$, $L(r,e\eta)=L(r,\eta)$, hence $\mu_r(e\cdot u)=\mu_r(u)$ for all $u\subset\mathcal{O}_S^\bullet$, and we view μ_r as a distribution on $\mathcal{O}_S^\bullet/\bar{\mathcal{O}}^\bullet$, where $\bar{\mathcal{O}}^\bullet$ denotes the closure of $\mathcal{O}^\bullet$ in $\mathcal{O}_S^\bullet$. As such, μ_r takes its values in $\mathcal{L}_S^0$; indeed, if $u\subset\mathcal{O}_S^\bullet$ is stable under multiplication by $\mathcal{O}^\bullet$, it can be written as a disjoint union $u=\bigcup_{e\in\mathcal{O}^\bullet/\mathcal{E}_{f,\eta}}(e\eta+f^\bullet)$, and hence $\mu_r(u)=\sum_{e\in\mathcal{O}^\bullet/\mathcal{E}_{f,\eta}}\mu_r(e\eta+f^\bullet)=[\mathcal{O}^\bullet:\mathcal{E}_{f,\eta}]\cdot\mu_r(\eta+f^\bullet)\in\mathcal{L}_S^0$. Using the corollary to Lemma 2, we have:

Theorem 1. μ_r is a distribution on $\mathcal{O}_S^\bullet/\bar{\mathcal{O}}^\bullet$ with values in the finitely generated $\mathbb{Z}[\rho_p^{-1};p\in S]$-module $\mathcal{L}_S^0$.

§6. In this section we average the distributions μ_r over all ideal classes, and use class field theory to get a measure on the Galois group. The "Mellin-transform" of this measure is the S-adic L-function. We prove the interpolation property relating the S-adic L-function to its classical counterpart, and the functional equation.

Let $k(1)$ denote the Hilbert class field of k, and let $k(S)$ denote the maximal abelian extension of k unramified outside S. By means of the Artin symbol we have identifications:

$$\mathcal{O}_S^\bullet/\bar{\mathcal{O}}^\bullet \cong \frac{k^\bullet \prod_{v+\infty} \mathcal{O}_v^\bullet \cdot k_\infty^\bullet}{\overline{k^\bullet \prod_{v+S^\infty} \mathcal{O}_v^\bullet \cdot k_\infty^\bullet}} \tilde{\to} Gal(k(S)/k(1))$$

$$k_A^\bullet / \overline{k^\bullet \prod_{v+S^\infty} \mathcal{O}_v^\bullet k_\infty^\bullet} \tilde{\to} Gal(k(S)/k)$$

$$Cl(k) \cong k_A^\bullet / k^\bullet \prod_{v+\infty} \mathcal{O}_v^\bullet \cdot k_\infty^\bullet \tilde{\to} Gal(k(1)/k)$$

We define a distribution on $G_S = Gal(k(S)/k)$ by $\mu_F = \sum_{i=1}^{h} \delta_{r_i} * \mu_{r_i}$; that is for a locally constant function g on G_S, we have

$$\int_{G_S} g \; d\mu_F = \sum_{i=1}^{h} \int_{\mathcal{O}_S^\bullet/\mathcal{O}^\bullet} g(r_i\eta) d\mu_{r_i}(\eta).$$

The distribution μ_F is determined by its values on finite characters ω, we let $\mathcal{L}_S^0[\omega]$ denote the $\mathbf{Z}[\omega]$-module generated by $\mathcal{L}_S^0$, where $\mathbf{Z}[\omega] \subset \mathbf{C}$ denotes the subring generating by the values of ω.

Theorem 2. For a finite character $\omega : G_S \to \mathbf{Z}[\omega]$, primitive of conductor f, we have inside $\mathcal{L}_S^0[\omega]$:

$$\int_{G_S} \omega \; d\mu_F = (\tau(\omega)(4\pi)^{2n} \cdot \rho_f)^{-1} \cdot \mathbb{N}f^{\frac{1}{2}} \prod_{q \in S} (1-\rho_q^{-1}\omega(q))(1-\rho_q^{-1}\omega^{-1}(q)) \cdot L_F(\omega).$$

Proof: Using an additive expression for our measure we have

$$\int_{G_S} \omega d\mu_F = \sum_{i=1}^{h} \sum_{\eta \subset (\mathcal{O}/f)^\bullet} \omega(r_i\eta)\rho_f^{-1} \sum_{d \mid \pi q \pi p} (-1)^d \rho_d^{-1} \mathcal{R}_{d^{-1}} \sum_{d' \mid \pi q} (-1)^{d'} \rho_{d'}^{-1} \mathcal{R}_{d'} \cdot \mathcal{R}_f L(r_i\eta).$$

By invoking Lemma 1,(2) we see that we may assume $(d, f) = 1$ and take the summation only over $d \mid \Pi q$, then substituting $r_i d' d^{-1}$ for r_i, we get

$$\rho_f^{-1} \sum_{d \mid \pi q} (-1)^d \rho_d^{-1} \omega(d) \sum_{d' \mid \pi q} (-1)^{d'} \omega^{-1}(d') \cdot \sum_{i=1}^{h} \sum_{\eta \in (\mathcal{O}/f)^\bullet} \omega(r_i\eta) L(r_i f, \eta)$$

and the expression in the theorem follows from Birch Lemma upon transforming the additive $\sum_{d \mid \pi q} \cdots \sum_{d' \mid \pi q} \cdots$ into the Euler product $\prod_q (\ldots)(\ldots)$.

Assume that the ρ_p's, $p \in S$, can be chosen to be p-units (hence S-units). Let $\mathcal{L}_S = \mathbf{Z}_S \otimes \mathcal{L}_S^0$ denote the S-adic completion of $\mathcal{L}_S^0$; where $\mathbf{Z}_S = \prod_p \mathbf{Z}_p$ the product taken over all rational primes p such that there exists a prime $p \in S$ above p. $\mathcal{L}_S$ is a finitely generated $\mathbf{Z}_S$-module; and so if $\tilde{O}$ is any S-adically complete and separated flat $\mathbf{Z}_S$-algebra, we can associate to every *continuous* function $g : G_S \to \tilde{O}$ the well define integral of g with respect to μ_F,

$$\int_{G_S} g \; d\mu_F \in \tilde{O} \underset{\mathbf{Z}_S}{\otimes} \mathcal{L}_S.$$

In particular, for any continuous S-adic character $\omega : G_S \to \tilde{O}^{\bullet}$, we can define the S-adic L-function:

$$L_{F,S}(\omega) = \int_{G_S} \omega \; d\mu_F \in \tilde{O} \underset{\mathbf{Z}_S}{\otimes} \mathcal{L}_S.$$

Theorem 2 gives the precise sense in which the $L_{F,S}$ interpolates the classical L_F.

Theorem 3: We have the functional equation

$$L_{F,S}(\omega) = (-1)^n \varepsilon_F \omega(a) \cdot L_{F,S}(\omega^{-1}).$$

Proof: By Lemma 1(4)

$$L(rf, \eta) = (-1)^n \varepsilon_F \; L(ar^{-1}f, -\eta^{-1}).$$

This implies a functional equation for our measures

$$\mu_r(\eta) = (-1)^n \varepsilon_F \mu_{r^{-1}a}(-\eta^{-1})$$

from which the functional equation for $L_{F,S}(\omega)$ follows immediately.

§7. We end this paper with a few remarks.

Remark 1: Let $E = \mathbb{Q}(\rho_v) \subseteq \mathbb{C}$ denote the subfield generated by all the ρ_v's, $v \neq \infty$. Assume F is a new form so that $E \cdot \mathcal{L}_S^0 \approx E \cdot t$ is a one dimensional E-vector space. Take for S a set of finite primes away from (a) and containing all the p-

places of k, p a "good" rational prime. (i.e., such that we can find ρ_v's which are p-units for $v \mid p$). Let $\widetilde{E}$ denote the field generated over E by all roots of unity of order dividing $\mathbb{N}v-1$, or some power of $\mathbb{N}v$, for all $v \in S$. Choose a place $\mathfrak{p}$ of $\widetilde{E}$ above p and let $\widetilde{E}_{\mathfrak{p}}$ denote the completion of $\widetilde{E}$ at $\mathfrak{p}$. G_S is the Galois group of the maximal S-ramified abelian extension of k, and for each continuouis character $\omega : G_S \to \widetilde{E}_{\mathfrak{p}}^{\bullet}$, we obtain for the value of the p-*adic* L-*function* at ω: $L_{F,S,p}(\omega) \in \widetilde{E}_{\mathfrak{p}} \cdot t$. (this is the "$p$-component" of the above $L_{F,S}(\omega) \in \mathbb{Z}_S \otimes \widetilde{E} \cdot t$).

Remark 2: If the ρ_p's were not S-adic units the μ_F defined above would still be a distribution but would not be bounded. Nevertheless, it would have a "moderate growth" [i.e. $\rho_f \mu_{r_i}$(image of $(\eta + f^{\bullet})$ *mod* $\mathcal{O}^{\bullet}$) takes values in a finitely generated $\mathbb{Z}_S$-module, and ρ_p is a p-unit such that at worst $|\rho_p|_p = |\mathbb{N}p|_p^{k/2}$] and hence analytic functions (e.g. S-adic characters) could be integrated against it. But continuous function could not be integrated and our S-adic L- functions might have infinitely many zeros, cf. [V].

Remark 3: The presence of real spaces v slightly complicates the situation, since for finite grossencharacter ω, ω_v need not be trivial, $\omega_v(-1) = \pm 1$, and so now we have to keep track of the "directions" at the real places. We shall indicate the needed modifications in the order of their appearance above. We let $\psi_v(x) = \exp[-2\pi x]$; r_i representing the *wide* class group $k^{\bullet} \backslash k_A^{\bullet} / \prod_{v \neq \infty} \mathcal{O}_v^{\bullet} k_{\infty}^0$, $k_{\infty}^0 =$ the connected component of $k_{\infty}^{\bullet}$; $\mathcal{K}_v = O(2;k_v)$; $\mathcal{V}_v$ a 2-dimensional complex vector space with basis V_v^1, V_v^{-1}, on which $\mathcal{K}_v$ acts on the right via the representation $M_v \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} = \begin{bmatrix} e^{-2i\vartheta} & 0 \\ 0 & e^{2i\vartheta} \end{bmatrix}$, $M_v \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $W_k : k_v^{\bullet} \to \mathcal{V}_v$ is given by $W_v(x) = |x| \exp(2\pi|x|) V_v^{\mathrm{sgn}\, x}$; $\beta_v^1 = \frac{1}{x}(dy + idx)$, $\beta_v^{-1} = \frac{1}{x}(-dy + idx)$, so that the v-component of our form is: $W_v(x)\psi_v(y)\beta_v = \exp[2\pi(|x| - iy)]$

$(dy + i\, sgn(x)dx)$. Note that since $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{K}_v$ there is no difference between X and X^{sgn} from the point of view of a real place v, that is: $B_v^+ \tilde{\to} G_v/\mathcal{K}_v\mathcal{Z}_v^+ \tilde{\to} G_v/\mathcal{K}_v\mathcal{Z}_v$, and $X_{r_i}^{sgn}$ depends only on the ideal class of (r_i), but Ω_{r_i} will depend also on $sgn(r_i)_\infty$; $\Theta = \bigwedge_v \Theta_v$, the product is taken only over the complex v's. Now fix a direction $d = \{d_v \mid v \text{ real}\}$, $d_v = \pm 1$. The definition of the periods is altered by replacing k_∞^* by k_∞^0, V_v^0 by $V_v^{-1} + d_v V_v^{-1}$, and requiring the units in $\mathcal{E}$ to be *positive* in all the *real* places v: Lemma 1(1); $L(r,\eta)$ depends also on the sign of r_v, $L(r\cdot(-1)_v,\eta) = d_v \cdot L(r,\eta)$ (and of course also on our choice of d); Lemma 1(4): $(-1)^n$ is replaced by $(-1)^{r_1+r_2}$. In the definition of the cycles replace again k_∞^* by k_∞^0, so that $I(\mathcal{E}) \approx [0,\infty] \times (\mathbb{R}/\mathbf{Z})^{[k:\mathbb{Q}]-1}$, and note again that $z(r_i,\alpha)$ depends only on the ideal class of (r_i); Lemma 2 and its corollary remain unchanged. The proof of Birch Lemma needs the obvious modification of keeping track of the directions, but its statement remains true for all finite characters ω satisfying $\omega_v(-1) = d_v$ (where we sum over the wide ideal class representatives r_i's, and replace $(4\pi)^{2n}$ by $(4\pi)^{[k:\mathbb{Q}]}$). From this point onwards everything remains the same if only we replace "class-group" by "*wide class-group*", k_∞^* by k_∞^0, and we obtain a distribution μ_F on G_S, such that for finite characters $\omega : G_S \to \mathbf{Z}[\omega]$, $L_{F,S}(\omega) = \int_{G_S} \omega\, d\mu_F$ interpolates the classical $L_F(\omega)$ in the sense of Theorem 2 (replacing $(4\pi)^{2n}$ by $(4\pi)^{[k:\mathbb{Q}]}$) and satisfies the functional equation $L_{F,S}(\omega) = \omega_\infty(-1)(-1)^{r_1+r_2}\varepsilon_F\, \omega(a)\, L_{F,S}(\omega^{-1})$.

Remark 4: Having started with a modular form corresponding to a harmonic form on X we pulled it back to X^{sgn} in order to construct the p-adic L-functions. Thus, from the "p-adic point of view", it seems more natural to *start* with a modular form corresponding to a harmonic form on $X^{sgn} = \bigcup_{i=1}^{h} \Gamma_{r_i} \setminus \prod_{v \text{ complex}} (H \setminus \mathbb{C}) \prod_{v \text{ real}} (\mathbb{C} \setminus \mathbb{R})$. Such forms when written adelically take values in $\bigotimes_{v|\infty} \mathcal{V}_v$. For v complex $\mathcal{V}_v$ is the complexification of $T_j(H)$, the tangent space to the quaternions at j. Note that under the action of the maximal compact subgroup $SU(2;\mathbb{C})$, $T_j(H)$ splits as a direct sum of two irreducible representations, one 3 dimensional and the other 1 dimensional. In particular, $H \setminus \mathbb{C}$ has no complex structure, invariant under the G_v action, and hence X^{sgn} has no natural complex structure. It seems interesting to inquire what further structure X^{sgn} possess (besides the Riemannian structure), and what kind of moduli interpretation X^{sgn} admits.

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