The Weil-Petersson Geometry of the moduli space of SU($n \geq 3$) (Calabi-Yau) manifolds I.

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§ 0.1. Introduction

In this paper we are going to study some differential-geometric properties of the moduli space of compact complex manifolds of $\dim_{\mathbb{C}} \geq 3$ which admit non-flat metrics g with holonomy groups $H(g) \neq \{0\}$ and $H(g) \subseteq SU(n)$. Such manifolds we will call $SU(n)$ or Calabi-Yau manifolds. Before stating the main results, we will make several remarks.

<u>Remark 0.1.1.</u> It is not difficult to see that a metric on a compact complex manifold whose holonomy $H^0 \neq \{0\}$ and $H \subseteq SU(n)$, will be Kähler and Ricci flat. We will call it Calabi-Yau metric. (See [2]).

<u>Remark 0.1.2.</u> If M is a Calabi-Yau manifold, then from the theory of invariants of the group $SU(n)$ and the fact that holonomy group $H^0 \neq \{0\}$ and $H \subseteq SU(n)$ it follows $H^0(M,\Omega^i) = 0$ for $1 < i < n$ and $H^0(M,\Omega^n)$ is spanned by a holomorphic n-form w_0, which has no zeroes and no poles. This implies that $c_1(M) = 0$ Constructions of Calabi-Yau manifolds are based on the solution of Calabi conjecture by Yau. See [15].

Recently SU(3) manifolds have attracted the interests of physicists working on string theory and algebraic geometers working on the classification of threefolds and on algebraic cycles.

Let me state the results that are contained in this paper. In §1 the following theorem is proved:

Theorem 1. Let M be a Calabi-Yau $(SU(n\geq 3))$ manifold, where $n = \dim_{\mathbb{C}} M$. Let $\pi : X \longrightarrow S \ni 0$, $\pi^{-1}(0) = M$ be the Kuranishi family of M, then S is a non-singular complex analytic space such that

$$\dim_{\mathbb{C}} S = \dim_{\mathbb{C}} H^1(M,\Theta) = \dim_{\mathbb{C}} H^1(\Omega^{n-1}).$$ See also [13].

More precisely we have proved Theorem 1'. From Theorem 1' follows Theorem 1 and our curvature computations are based on Theorem 1'.

Theorem 1' Let M be a Calabi-Yau $(SU(n \geq 3))$ manifold. Let $(g_{\alpha\bar{\beta}})$ be a Calabi-Yau metric on M. Let $\mathbb{H}^1(M,\Theta)$ denote the harmonic elements of $H^1(M,\Theta)$ with respect to $(g_{\alpha\bar{\beta}})$, let φ_1 be any element of $\mathbb{H}^1(M,\Theta)$, then there exists a unique power series

$$\varphi(t) = \varphi_1 t + \varphi_2 t^2 + \dots + \varphi_N t^N + \dots$$

such that for $|t| < \varepsilon$

a) $\varphi(t) \in C^\infty(M,\Omega^{0,1} \otimes \Theta_M)$ b) $\bar{\partial}^*\varphi(t) = 0$, where $\bar{\partial}^*$ is the adjoint operator of $\bar{\partial}$ with respect to $(g_{\alpha\bar{\beta}})$.
c) $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t),\varphi(t)$ $= 0$ d) for each $K \geq 2$ $\varphi_K \lrcorner\, \omega_0 = \partial\Psi_K$, where $\omega_0 \in H^0(M,\Omega^n)$ and ω_0 has no zeroes.

Theorem 1 was first anounced by F.A. Bogomolov in [18]. Later P. Candelas, G. Horowith, A. Strominger and E. Witten proved theorem 1 under the assumption that

$$H^2(M,\mathbb{Z}) \cong \mathbb{Z}. \quad ([17])$$

Theorem 1 was also proved by Tian independently. Next we are going to describe the results in §2. So we need some definitions in order to formulate the results.

Definition. A pair (M,L) where M is a Calabi-Yau manifold and $L \in H^2(M,\mathbb{R})$ will be called a polarized $SU(n)$ manifold if $L = [\operatorname{Im} g^1_{\alpha\overline{\beta}}]$, where $(g^1_{\alpha\overline{\beta}})$ is a Kähler metric on M.

With $[\omega]$ we will denote the class of cohomology of a form ω. From now on we will suppose that L is fixed.

Suppose that $M \to S$ is the Kuranishi family of polarized Calabi-Yau manifold (M,L), so may be after shrinking S we may suppose that for each $s \in S$ on $M_s = \pi^{-1}(s)$ there exists a unique Ricci-flat Kähler metric $g_{\alpha\overline{\beta}}(s)$ such that $[\operatorname{Im} g_{\alpha\overline{\beta}}(s)] = L$. The last fact follows from Yau's solution of Calabi's conjecture, Kodaira's stability theorem, which states that small deformations of Kähler manifold is Kähler and the fact that for $SU(n \geq 3)$ manifolds $H^2(X,O_X) = 0$. From $h^{2,0} = 0$ it follows that M is an algebraic manifold. Here we use the fact $n \geq 3$, since if $n = 2$, $h^{2,0} = 1$. Now we can identify the tangent space at $s \in S$, $T_{s,S}$ with $\mathbb{H}^1(M_s,\Theta_s)$, where $\mathbb{H}^1(M_s,\Theta_s)$ is the harmonic part of $H^1(M_s,\Theta_s)$ with respect to $g_{\alpha\overline{\beta}}(s)$ and Θ_s is the sheaf of holomorphic vector fields.

Now we are ready to define Weil-Petersson metric on S-the local moduli space of (M,L).

Definition. Let $\varphi_1, \varphi_2 \in T_{s,S} = \mathbb{H}^1(M_s, \Theta_s)$, then

$$\langle \varphi_1, \varphi_2 \rangle_{w.p.} := \int_{M_s} \varphi_{1\bar{\alpha}}^{\mu} \overline{\varphi_{\bar{\beta}}^{\nu}}\, g_{\mu\bar{\nu}} g^{\beta\bar{\alpha}} \operatorname{vol}(g_{\alpha\bar{\beta}}(s)) .$$

Here we are using the usual Einstein's conventions for summation.

In §2 we calculated the Weil-Petersson metric on the moduli space of polarized $SU(n \geq 3)$ manifolds in terms of the standard cup product on $H^{n-2,2}$, i.e.

$$\langle u,v \rangle = (-1)^{\frac{n(n-2)}{2}} (i)^{n-4} \int_M u \wedge \bar{v}, \quad u,v \in H^{n-2,2} .$$

In order to simplify the computation of the curvature tensor $R_{\alpha\bar{\beta},\mu\bar{\nu}}$ of the Weil-Petersson metric $(h_{\mu\bar{\nu}})$ we need to find "good" local coordinates $(t^1,\ldots,t^K)$ in S so that

$$h_{\mu\bar{\nu}} = \delta_{\mu\bar{\nu}} + \Sigma h_{\mu\bar{\nu},\alpha\bar{\beta}} t^{\alpha} \bar{t}^{\beta} + \text{(higher order terms)} .$$

In §2 it is proved that such a coordinate system exists and so $\{h_{\mu\bar{\nu}}\}$, i.e. the Weil-Petersson metric is a Kähler metric. Let me describe how one fixes such "good" coordinate system which we call "Kodaira-Spencer-Kuranishi" local in S. Let $\{\eta_\nu\}$ $\nu = 1,\ldots,K$ be a basis in $\mathbb{H}^1(M,\Theta)$ and let

$$\varphi_\alpha(t^\alpha) = \eta_\alpha t^\alpha + \varphi_{\alpha,t}(t^\alpha)^2 + \varphi_{\alpha,3}(t^\alpha)^3 + \ldots$$

be the power series with the properties stated in Theorem 1', then it is proved in §2, that $(t^1,\dots,t^K)$ will be a good local coordinate system, namely the following lemma is proved.

Lemma. Let $(h_{\alpha\bar\beta})$ be the Weil-Petersson metric on S, then with respect to Kodaira-Spencer-Kuranishi local coordinates the following formula is true:

$$(*)\quad h_{\alpha\bar\beta}(t,\bar t)=(-1)^{\frac{n(n-1)}{2}}(i)^{n-2}\Big[\int_M(\eta_\alpha\lrcorner\omega_0)\wedge(\overline{\eta_\beta\lrcorner\omega_0})+4t^\alpha\bar t^\beta\int_M[\Lambda^2h_\alpha\lrcorner\omega_0]\wedge\overline{[\Lambda^2h_\beta\lrcorner\omega_0]}$$

$$+\ (\text{terms of order } \geq 3)\Big],\text{ where }\ \Lambda^2\eta_\alpha : \Lambda^2\Omega^{1,0}\to\Lambda^2\Omega^0\ \text{ and }\ \Lambda^2\eta_\alpha(u\wedge v)=$$

$$=\eta_\alpha(u)\wedge\eta_\alpha(V)\quad\text{and}\quad[\Lambda^2h_\alpha\lrcorner\omega_0]$$

denote the cohomology class of $\mathbb{H}(\Lambda^2h_\alpha\lrcorner\omega_0)$ in $H^{n-2,2}\subset H^n(M,\mathbb{C})$, where $\mathbb{H}$ is the harmonic projection. From this lemma we derive the following theorem

Theorem 2. a) The following formulas are true for the curvature tensor $R_{\alpha\bar\beta,\mu\bar\nu}$ of the Weil-Petersson metric on the moduli space of SU(n≥3) manifolds

$$R_{\alpha\bar\beta,\mu\bar\nu}=0\quad\text{if}\quad\alpha\neq\mu\quad\text{or}\quad\nu\neq\beta$$

$$R_{\alpha\bar\beta,\alpha\bar\beta}=+(-1)^{\frac{n(n+1)}{2}}(i)^{n-2}8\int[\Lambda^2\eta_\alpha\lrcorner\omega_0]\wedge\overline{[\Lambda^2\eta_\beta\lrcorner\omega_0]}$$

b) The biholomorphic sectional curvature of the Weil-Petersson metric on the moduli space of SU(n≥3) manifolds is negative.

The proof of the lemma is based on the following two observations.

<u>Observation 1.</u> $\langle\varphi_1,\varphi_2\rangle_{w.p.} = (-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int(\varphi_1\lrcorner\omega_0)\wedge(\overline{\varphi_2\lrcorner\omega_0})$,

where $\int\omega_0\wedge\bar{\omega}_0 = \int \mathrm{vol}\ (g_{\alpha\bar{\beta}})$.

This formula says that in case of $SU(n)$ manifold we do not need Calabi-Yau metric in order to define Weil-Petersson metric. We only need the polarization class since if $\int\omega_t\wedge\bar{\omega}_t = \int L^n$, then we have a canonical isomorphism $\alpha : H^1(M,\Theta) \xrightarrow{\sim} H^1(\Omega^{n-1})\alpha(\varphi) = \varphi\lrcorner\omega_0$. On $H^{n-1,1}$ we have a canonical metric; $\langle a,b\rangle = (-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int_M a\wedge\bar{b}$. This is so since if $n \geq 3$ all elements of $H^{n-1,1}$ are primitive and $H^{n-2,0} \cong H^0(M,\Omega^{n-2}) = 0$.

<u>Observation 2.</u> Let $D_\alpha \subset S$ be the disc define by $t^1 = 0,\ldots,t^{\alpha-1} = 0,t^{\alpha+1} = 0,\ldots,t^K = 0$. Let $M_\alpha \longrightarrow D_\alpha$ be the restriction of the Kuranishi family on D_α . Let ω_{t^α} be the holomorphic form on $\pi^{-1}(t^\alpha) = M_{t^\alpha}$ such that $\int\omega_{t^\alpha}\wedge\bar{\omega}_{t^\alpha} = \int\omega_0\wedge\bar{\omega}_0$, then ω_{t^α} as C^∞ n-form on M_0 can be expressed in the following way

$$(**)\quad \omega_{t^\alpha} = \omega_0 + \sum_{l=1}^{n}(-1)^{\frac{l(l-1)}{2}}\Lambda^l\varphi_\alpha(t^\alpha)\lrcorner\omega_0\ ,\ \text{where}\ \Lambda^l\varphi_\alpha(t^\alpha) \in$$
$$\Gamma(M,\mathrm{Hom}(\Lambda^l\Omega^{\Lambda,0},\Lambda^l\Omega^{0,1}))\ \text{and}\ \Lambda^l\varphi_\alpha(t^\alpha)(u_1\wedge\ldots\wedge u_l) =$$
$$= \varphi_\alpha(t^\alpha)(u_1)\wedge\ldots\wedge\varphi_\alpha(t^\alpha)(u_l)\ .$$

From observation 1 and 2 it follows that

$$(***)\quad h_{\alpha\bar{\beta}}(t,\bar{t}) = (-1)^{\frac{n(n-1)}{2}}(i)^{n-2}\left(\int_M\frac{d\omega_{t^\alpha}}{dt^\alpha}\wedge\overline{\frac{d\omega_{t^\beta}}{dt^\beta}}\right)\ .$$

From (***) Theorem 2 follows almost directly.

Remark 1. It is a well known fact that the moduli space of marked polarized K3 surfaces is $SO(2,19)/SO(2) \times SO(19)$. From observation 1 it follows that the Weil-Petersson metric is the Bergmann metric on $SO(2,19)/SO(2) \times SO(19)$. (See also [11]).

Some historical notes. The purpose of introducing of invariant metric on the moduli space (in case of Riemann surfaces on the Teichmüller space), is to provide information on the intrinsic properties of the space. The Weil-Petersson metric has successfully filled this role in case of Riemann surfaces of genus $g \geq 2$.

Ahlfors was the first to consider the curvature of the Weil-Petersson metric in case of Riemann surfaces, i.e. on the Teichmüller space. See [1]. He obtained singular integral formulas for the Riemann curvature tensor. As an application he found that the Ricci, holomorphic sectional and scalar curvatures are all negative. Royden later showed that the holomorphic sectional curvature is bounded away from zero. Tromba gave a complete formula for the curvature of Weil-Petersson metric on Teichmüller space and found that the general sectional curvature is negative. See [14].

Later Scott Wolpert gave another formulas for the curvature tensor of the Weil-Petersson metric on the Teichmüller space of Riemann surfaces of genus $g \geq 2$. From his formulas S. Wolpert showed that the holomorphic sectional and Ricci curvatures are bounded above by $\frac{-1}{2\pi(g-1)}$ and the scalar curvature is bounded above by $\frac{-3(3g-1)}{4\pi}$. S. Wolpert showed that the curvatures are governed by the spectrum of the Laplacian. See [16]. J. Royden also obtained similar results. Later S. Wolpert used his calculations of the curvature tensor of the Weil-Petersson metric to get some information of the global structure of the moduli space of Riemann surfaces.

Siu generalized the formulas of S. Wolpert in case of algebraic manifolds with Ricci < 0 and complex dimension ≥ 2. See [12]. Nannacini used Siu's method and obtain similar to Siu's formulas in case of $SU(n \geq 2)$ polarized manifolds. See [9]. In his introduction of [12] Siu announced that Royden also obtained some formulas for the curvature tensor of the Weil-Petersson metric on the moduli space of compact complex manifolds with $\dim \geq 2$ and Ricci < 0. Koiso was the first to introduce Weil-Petersson metric in case of $\dim \geq 2$, [7], [4].

Recently the author found some applications of the results of the present paper. Namely we proved the global Torelli theorem for complete intersections of $SU(n \geq 3)$ manifolds. The proof is based on the observations that the Discs D_α defined in Observation 2 are totally geodesic submanifolds. After allowing certain singularity we may construct a complete moduli space of $SU(n \geq 3)$ manifolds with respect to Weil-Petersson metric. We also use some results of A. Beauville on the image of $\mathrm{Diff}_+(M)$ = {group of diffeomorphisms preserving the orientation of complete intersections in $\mathbb{P}^N$, if $\dim_{\mathbb{C}} M \geq 3$} in $\mathrm{Aut}H^n(M,\mathbb{Z})$. See [3].

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§ 0.2 Conventions on some relations

0.2.1.a. $(z^1,\ldots,z^n)$ will denote a system of local coordinates on a compact complex manifold.

0.2.1.b. $dz^1\wedge\ldots\wedge \widehat{dz^{i_1}}\wedge\ldots\wedge\widehat{dz^{i_K}}\wedge\ldots\wedge dz^n$ means that if $i_1<\ldots<i_K$ then $dz^{i_1},\ldots,dz^{i_K}$ are omitted.

0.2.2 Given a Hermitian metric $ds^2 = h_{\alpha\bar\beta}dt^\alpha \wedge d\bar t^\beta$ on a complex manifold, we say that it is Kähler if

$$(0.2.2.1)\qquad \frac{\partial h_{\alpha\bar\beta}}{\partial t^\gamma} = \frac{\partial h_{\gamma\bar\beta}}{\partial t^\alpha}\ .$$

A metric is Kähler if and only if we can find normal coordinates at each point, i.e. holomorphic coordinates such that at the point the metric tensor has the development $h_{\alpha\bar\beta} = \delta_{\alpha\bar\beta} + 0(|t|^2)$. If the metric is Kähler and real analytic, one can introduce a set of canonical coordinates at a point which are characterized by the property that the power series for $h_{\alpha\bar\beta}$ contains no terms which are products only of unbarred (or only of barred variables). In terms of canonical coordinates

$$(0.2.2.2)\qquad h_{\alpha\bar\beta} = \delta_{\alpha\bar\beta} + \frac{1}{2}R_{\alpha\bar\beta,\gamma\bar\delta}t^\gamma t^{\bar\delta} + 0\,(|t|^3)$$

where $h_{\alpha\bar\beta,\gamma\bar\delta}$ is the Riemann curvature tensor. If $(\xi^1,\ldots,\xi^K)$ and $(\eta^1,\ldots,\eta^K)$ are unit tangent vectors, the holomorphic bisectional curvature in direction ξ,η is given by:

$$(0.2.2.3)\qquad K_{\xi\eta} = R_{\alpha\bar\beta,\gamma\bar\delta}\xi^\alpha\bar\xi^\beta\bar\xi^\gamma\bar\xi^\delta\ .$$ See [10]

and the holomorphic scetional curvature in direction ξ is given by

(0.2.2.4) $K_{\xi\xi} = R_{\alpha\bar{\beta},\gamma\bar{\delta}}\xi^{\alpha}\bar{\xi}^{\beta}\bar{\xi}^{\gamma}\bar{\xi}^{\delta}$. See [10].

So we have proved in § 2 that Kodaira-Spencer-Kuranishi coordinates are normal coordinates. So we apply (0.2.2.3) and (0.2.2.4) in order to get Theorem 2.

§1. The Kuranishi space of a SU(n) manifold M is unobstructed

1.1. Remark a) From now on we will suppose that M is an $SU(n)$ manifold with a fixed Calabi-Yau metric $(g_{\alpha\bar{\beta}})$, i.e. $g_{\alpha\bar{\beta}}$ is a Kähler, Ricci-flat metric on M.

b) If φ is any element of $H^j(M,\Lambda^K\Theta)$, then by $\mathbb{H}\varphi$ we will denote the harmonic part of φ and by $\mathbb{H}^j(M,\Lambda^K\Theta)$ all harmonic tensors on M whch are elements of $H^j(M,\Lambda^K\Theta)$ with respect to the Calabi-Yau metric.

c) For any point $x \in M$ from now on we will chose the local coordinates $(z^1,\ldots,z^n)$ in $U \ni x$ in such a way that

$$\omega_0|_U = dz^1 \wedge \ldots \wedge dz^n$$

where ω_0 is the holomorphic form without zeroes on M.

1.2. Theorem. Let M be a $SU(n)$ manifold and let $\begin{array}{ccc} M & \subset & X \\ \downarrow & & \downarrow\pi \\ 0 & \in & S \end{array}$ be the Kuranishi family of M, then

a) S is a non-singular complex manifold

b) $\dim_{\mathbb{C}} S = \dim_{\mathbb{C}} H^1(M,\Theta)$.

Proof: Let us first remember how the Kuranishi family is defined. We define $\bar{\partial}^*$ to be the adjoint of $\bar{\partial}$ with respect to the Calabi-Yau metric, $\square$ to be the Laplace operator, and G to be the Green operator. Let $\{\eta_\nu | \nu = 1,\ldots,m\}$ be a base for $\mathbb{H}^1(M,\Theta)$. Kuranishi proved that the power series solution of the equation

$$\varphi(t) = \eta(t) + \frac{1}{2}\bar{\partial}^* G[\varphi(t),\varphi(t)]$$

where $\dot{\eta}(t) = \sum_{\nu=1}^{m} t_\nu \dot{\eta}_\nu$ has a unique convergent power series solution. And this $\varphi(t)$ satisfies

$$\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t),\varphi(t)] = 0$$

if and only if

$$\mathbb{H}[\varphi(t),\varphi(t)] = 0.$$

Let $\{\beta_\lambda | \lambda = 1,\ldots, Z\}$ be an orthonormal base of $\mathbb{H}^2(M,\Theta)$ and let $<,>$ be the inner product in

$$A^2 = \Gamma(\Omega^{0,2} \otimes \Theta)..$$

Then

$$\mathbb{H}[\varphi(t),\varphi(t)] = \sum_{\lambda=1}^{\tau} <[\varphi(t),\varphi(t)],\beta_\nu>\beta_\nu .$$

Hence $\mathbb{H}[\varphi(t),\varphi(t)] = 0$ iff $<[\varphi(t),\varphi(t)],\beta_\nu> = 0$ for $\lambda = 1,\ldots,\tau$. Since $\lambda = 1,\ldots,\tau$. Since $\varphi(t)$ is a power series in t so is $<[\varphi(t),\varphi(t)],\beta_\nu> = b_\nu(t)$. Thus $b_\nu(t)$ is holomorphic in t for $\lambda = 1,\ldots,\tau$ and $|t| < \varepsilon$. Then Kuranishi proved that S is defined as follows

$$S = \{t \,|\, |t| < \varepsilon,\ b_\lambda(t) = 0,\quad \lambda = 1,\ldots,\tau\} .$$

We have a family $X \xrightarrow{\pi} S$ such that it is locally complete and $\pi^{-1}(0) = M$. From all this it follows that if we prove that for each η_ν, $\nu = 1,\ldots,r$, there exists a power series (convergent)

$$\varphi_\nu(t) = \dot{\eta}_\nu t + \varphi_2^\nu t^2 + \ldots + \varphi_K^\nu t^K \ldots$$

such that:

a) $$\overline{\partial}\varphi_\nu(t) = \frac{1}{2}[\varphi_\nu(t),\varphi_\nu(t)]$$

b) $\varphi_\nu(t)$ fulfills the following equation

$$\varphi_\nu(t) = \eta_\nu t + \frac{1}{2}\overline{\partial}^* G[\varphi_\nu(t),\varphi_\nu(t)]$$

then $b_\nu(t) \equiv 0$ and so S is an open subset in $\mathbb{H}^1(M,\Theta)$, i.e. S is an non-singular manifold of dimension equal to the $\dim H^1(M,\Theta)$.

So we need to prove that for each $\eta_\nu \in \mathbb{H}^1(M,\Theta)$ $\nu = 1,\dots,z$ we can find a power series

$$\varphi_\nu(t) = \eta_\nu t + \varphi_2^\nu t^2 + \dots + \varphi_K^\nu t^K + \dots$$

such that

a) $$\overline{\partial}\varphi_\nu(t) = \frac{1}{2}[\varphi_\nu(t),\varphi_\nu(t)]$$

b) $$\varphi_\nu(t) = \eta_\nu t + \frac{1}{2}\overline{\partial}^* G[\varphi_\nu(t),\varphi_\nu(t)].$$

<u>Lemma 1.2.1.</u> Let $\varphi_\nu(t) = \eta_\nu t + \varphi_2^\nu t^2 + \dots + \varphi_K^\nu t^K + \dots$ be convergent power series such that

a) $\overline{\partial}^*\varphi_\nu(t) = 0$ b) $\overline{\partial}\varphi_\nu(t) = \frac{1}{2}[\varphi_\nu(t),\varphi_\nu(t)]$, then

$$\varphi_\nu(t) = \eta_\nu t + \frac{1}{2}\overline{\partial}^* G[\varphi_\nu(t),\varphi_\nu(t)]$$

<u>Proof:</u> $\varphi_\nu(t) - \mathbb{H}\varphi_\nu(t) = G\Box\varphi_\nu(t) = G(\overline{\partial}^*\overline{\partial} + \overline{\partial}\,\overline{\partial}^*)\varphi_\nu(t) = G\overline{\partial}^*\overline{\partial}\varphi_\nu(t)$. This is so since $\overline{\partial}^*\varphi_\nu(t) = 0$. From the equality

$$\varphi_\nu(t) - \mathbb{H}\varphi_\nu(t) = G\overline{\partial}^*(\overline{\partial}\varphi_\nu(t))$$

and from $\bar{\partial}\varphi_\nu(t) = \frac{1}{2}[\varphi_\nu(t),\varphi_\nu(t)]$ we get

$$\varphi_\nu(t) = \mathbb{H}\varphi_\nu(t) + \frac{1}{2}\,\bar{\partial}^*G[\varphi_\upsilon(t),\varphi_\nu(t)] = \eta_\nu t + \frac{1}{2}\,\bar{\partial}^*G[\varphi_\nu(t),\varphi_\nu(t)]$$

Q.E.D.

From all these fact it follows that we need to solve by induction the following equations:

$$(*)\qquad \begin{aligned} &\bar{\partial}\varphi_2 = \frac{1}{2}\,[\varphi_1,\varphi_1], \quad \text{where} \quad \bar{\partial}^*\varphi_2 = 0 \\ &\vdots \\ &\bar{\partial}\varphi_{N+1} = \frac{1}{2}([\varphi_N,\varphi_1] + [\varphi_{N-1},\varphi_2] + \ldots + [\varphi_1,\varphi_N]) \end{aligned}$$

where $\bar{\partial}^*\varphi_{N+1} = 0$.

The solutions of (*) is based on the following lemmas

<u>Lemma 1.2.2.</u> For each $\eta \in \mathbb{H}^1(M,\Theta)$ $\eta \perp \omega_0$ is a harmonic form of type $(n-1,1)$.

<u>Proof:</u> Let $\eta|_U = \Sigma\, \eta^\mu_{\bar{\alpha}}\, d\bar{z}^\alpha \otimes \frac{\partial}{\partial z^0}$ and $\omega_0|_U = dz^1\wedge\ldots\wedge dz^n$, then

$$\eta \perp \omega_0|_U = \Sigma(-1)^{\mu-1}\eta^\mu_{\bar{\alpha}}\; d\bar{z}^\alpha\wedge\ldots\wedge \widehat{dz^\mu}\wedge\ldots\wedge dz^n$$

now clearly

$$\bar{\partial}\eta = 0 \Rightarrow \bar{\partial}(\eta \perp \omega_0) = 0 .$$

Next we need to prove that

$$\bar{\partial}^*(\eta \perp \omega_0) = 0 \quad .$$

The proof of this fact is based on the following fact

$$(**) \qquad (\bar{\partial}^*\varphi)_{A_p,\bar{B}_q} = (-1)^{p+1}\sum_{\beta} g^{\bar{\beta}\alpha}\nabla_\alpha \varphi_{A_p,\bar{\beta}\,\bar{B}_q} \quad , \quad \text{see} \quad [8] \; .$$

From the formula

$$\nabla(\eta \perp \omega_0) = \nabla\eta \perp \omega_0 \pm \eta \perp \nabla\omega_0$$

and from the Bochner principle, that on any Ricci flat compact complex manifolds any holomorphic tensor is parallel, we get that $\nabla\omega_0 = 0$. See [8]. So

$$\nabla_\alpha(\eta \perp \omega_0) = \nabla_\alpha \eta \perp \omega_0 \quad .$$

From this formula we get that

$$\bar{\partial}^*(\eta \perp \omega_0) = (\bar{\partial}^*\eta) \perp \omega_0 = 0.$$

Q.E.D.

Lemma 1.2.3. For each $\eta \in \mathbb{H}^1(M,\Theta)$ we have that if $\eta|_U = \Sigma \eta^\mu_{\bar{\alpha}}\, d\bar{z}^\alpha \otimes \frac{\partial}{\partial z^\mu}$, then

$$\sum_{\mu=1}^{\mu} \partial_\mu \varphi^\mu_{\bar{\alpha}} = 0 \qquad \forall\alpha = 1,\ldots,n.$$

Proof: We know that $\eta \perp \omega_0$ is a harmonic form on a Kähler manifold so

$$\partial(\eta \perp \omega_0) = 0 \; .$$

On the other hand

$$\eta \perp \omega_0|_U = \Sigma(-1)^{\mu-1}\eta^{\mu}_{\bar\alpha}\, d\bar z^{\alpha}\wedge dz^1\wedge\ldots\wedge d\hat z^{\mu}\wedge\ldots\wedge dz^{\mu} .$$

So

$$\partial(\eta\perp\omega_0|_U) = \sum_{\alpha}(\sum_{\mu}\partial_{\mu}\varphi^{\mu}_{\bar\alpha})d\bar z^{\alpha}\wedge dz^1\wedge\ldots\wedge dz^{\mu}\wedge\ldots\wedge dz^{\mu} \equiv 0 .$$

From here $\Rightarrow \Sigma\partial_{\mu}\varphi^{\mu}_{\bar\alpha} = 0$

Q.E.D.

<u>Lemma 1.2.4.</u> Let $\varphi,\psi \in \Gamma(M,\Omega^{0,1}\otimes\Theta) = \Gamma(M,\mathrm{Hom}(\Omega^{1,0},\Omega^{0,1}))$ and $\partial(\varphi\perp\omega_0) = \partial(\Psi\perp\omega_0) = 0$, then

$$2\partial(\varphi\wedge\Psi\perp\omega_0) = ([\varphi,\Psi]\perp\omega_0)$$

where $\varphi\wedge\Psi\in\Gamma(M,\mathrm{Hom}(\Lambda^2\Omega^{1,0},\Lambda^2\Omega^{0,1}))$ and

$$(\varphi\wedge\Psi)(u\wedge v) := \varphi(u)\wedge\Psi(v) .$$

<u>Proof:</u> We have:

$$2\varphi\wedge\Psi|_U = (\sum_{i<j}\varphi^i\wedge\Psi^j - \Psi^i\wedge\varphi^j)\otimes\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^j} .\ \text{Here}$$

$$\varphi|_U = \sum\varphi^i_{\bar\mu}\, d\bar z^{\mu}\otimes\frac{\partial}{\partial z^i},\quad \Psi|_U = \sum\varphi^j_{\bar\nu}\, d\bar z^{\nu}\otimes\frac{\partial}{\partial z^j}\quad\text{and}$$

$$\varphi^i = \sum_{\mu}\varphi^i_{\bar\mu}\, d\bar z^{\mu},\quad \Psi^j = \sum_{\nu}\Psi^j_{\bar\nu} d\bar z^{\nu} .$$

From these formulas we get:

$$2(\varphi\wedge\Psi\perp\omega_0)|_U = \sum_{i<j}(-1)^{i+j-2}(\varphi^i\wedge\Psi^j - \Psi^i\wedge\varphi^j)dz^1\wedge\ldots\wedge d\hat z^i\wedge\ldots\wedge d\hat z^j\wedge\ldots\wedge dz^n .$$

Let us compute the coefficient of $2\partial(\varphi\wedge\Psi\perp\omega_0)$ in front of $dz^1\wedge\ldots\wedge dz^i\wedge\ldots\wedge \hat{dz}^j\wedge\ldots\wedge dz^\mu$. So we have

$$(1.2.4.1)\quad 2\partial(\varphi\wedge\Psi\perp\omega_0)|_U = \sum_j[\sum_i(-1)^{j-1}(\partial_i\varphi^i\wedge\Psi^j-\partial_i\Psi^i\wedge\varphi^j+\varphi^i\partial_i\Psi^j-\Psi^i\wedge\partial_i\varphi^j)\wedge$$

$$\wedge\, dz^1\wedge\ldots\wedge dz^i\wedge\ldots\wedge d\hat{z}^j\wedge\ldots\wedge dz^\mu],\qquad \text{where}$$

$$\partial_i\varphi^i = \sum_{\mu=1}^{n}(\partial_i\varphi^i_{\bar\mu})d\bar z\quad ,\qquad \partial_i\Psi^i = \sum_{\nu=1}^{\mu}\partial_i\Psi^i_{\bar\nu}d\bar z^\nu\quad .$$

From $$\partial(\varphi\perp\omega_0) = \partial(\Psi\perp\omega_0) = 0 \Rightarrow \sum_{i=1}^{\mu}\partial_i\varphi^i = \sum_{i=1}^{\mu}\partial_i\Psi^i = 0.$$

So from these formulas and (1.2.4.1) it follows that:

$$(1.2.4.2)\quad 2\partial(\varphi\wedge\Psi\perp\omega_0)|_U = \sum_{j=1}^{n}(\sum_{i=1}^{n}(-1)^{1+j}[\varphi^i\partial_i\Psi^j-\Psi^i\partial_i\varphi^j]\wedge dz^1\wedge\ldots\wedge d\hat{z}^j\wedge\ldots\wedge dz^n$$

From the definition of $[\varphi,\Psi]$, i.e.

$$(1.2.4.3)\quad [\varphi,\Psi]|_U = \sum_j(\sum_i(\varphi^i\partial_i\Psi^j-\Psi^i\partial_i\varphi^j))\otimes\frac{\partial}{\partial z^j}$$

and (1.2.4.2) we get that

$$2\partial[(\varphi\wedge\Psi)\perp\omega_0] = [\varphi,\Psi]\perp\omega_0$$

Q.E.D.

Lemma 1.2.5. Let $\varphi_i\in\Gamma(M,\mathrm{Hom}(\Omega^{1,0},\Omega^{0,1}))$ for $2\leq i\leq N$ and

a) $\bar\partial\varphi_i = \frac{1}{2}([\varphi_{i-1},\varphi_1]+[\varphi_{i-2},\varphi_2]+\ldots+[\varphi_2,\varphi_{i-1}]+[\varphi_1,\varphi_{i-1}]$,

b) $\bar\partial\varphi_1 = 0$

then

$$\bar\partial\left(\frac{1}{2}([\varphi_N,\varphi_1] + [\varphi_{N-1},\varphi_2] + \ldots + [\varphi_2,\varphi_{N-1}] + [\varphi_1,\varphi_N]\right) = 0.$$

<u>Proof:</u>

Clearly we have:

$$(1.2.5.1) \qquad \frac{1}{2}\bar\partial\left(\sum_{K=0}^{N-1}[\varphi_{N-K},\varphi_{K+1}]\right) =$$

$$= \frac{1}{2}\left(\sum_{K=0}^{N-1}\left([\bar\partial\varphi_{N-K},\varphi_{K+1}] - [\varphi_{N-K},\bar\partial\varphi_{K+1}]\right)\right) .$$

From $-[\varphi_j,\bar\partial\varphi_i] = [\bar\partial\varphi_i,\varphi_j]$ $[\varphi_i,\varphi_j] = [\varphi_j,\varphi_i]$ (see [8]) and

$$\bar\partial\varphi_{K+1} = \frac{1}{2}([\varphi_K,\varphi_1] + \ldots + [\varphi_1,\varphi_K]) = \frac{1}{2}\left(\sum_{i=1}^{K}[\varphi_i,\varphi_{K-i+1}]\right)$$

we get

$$(1.2.5.2.) \qquad \frac{1}{2}\,\bar\partial\left(\sum_{K=0}^{N-1}[\varphi_{N-K},\varphi_{K+1}]\right) =$$

$$= (2[[\varphi_{N-1},\varphi_1],\varphi_1] + [[\varphi_1,\varphi_2],\varphi_{N-1}]) +$$

$$+ \sum_{\substack{i>1,j>1,K>1\\ i\neq j\neq K\\ i+j+K=N+1}} [[\varphi_i,\varphi_j],\varphi_K] + [[\varphi_K,\varphi_i],\varphi_j] + [[\varphi_j,\varphi_K],\varphi_i])$$

$$+ \sum_{\mu\neq N-2\mu+1}([[\varphi_{N-2\mu+1},\varphi_\mu,\varphi_\mu] + \frac{1}{2}[[\varphi_\mu,\varphi_\mu],\varphi_{N-2\mu+1}]) + (\frac{1}{2}[[\varphi_K,\varphi_K],\varphi_K]])$$

if $N+1 = 3K$) . From (1.2.5.2.) and Jacobi identity we get that

$$\frac{1}{2}\,\overline{\partial}\Big(\sum_{K=0}^{W-1}[\varphi_{N-K},\varphi_{K+1}]\Big] = 0$$

Q.E.D.

1.2.6. Now we are ready to solve the equations (*) on page 4. We will solve them inductively.

Induction hypothesis. Suppose that for any $2 \leqq \kappa \leqq N$, we have

a) $\overline{\partial}\varphi_K = \frac{1}{2}\Big(\sum_{i=1}^{K-1}[\varphi_{K-1},\varphi_i]\Big)$

b) $\overline{\partial}^*\varphi_K = 0$

c) $\varphi_K \perp \omega_0 = \partial\Psi_K$ and so $\partial(\varphi_K \perp \omega_0) = 0$.

We must find φ_{N+1} such that

a) $\overline{\partial}\varphi_{N+1} = \frac{1}{2}\Big(\sum_{i=1}^{N}[\varphi_{N-i+1},\varphi_i]\Big)$

b) $\overline{\partial}^*\varphi_{N+1} = 0$

c) $\partial(\varphi_{N+1} \perp \omega_0) = 0$ and moreover

$$\varphi_{N+1} \perp \omega_0 = \partial\Psi_{N+1} \quad .$$

From lemma 1.2.4. it follows that

1.2.6.1. $\frac{1}{2}\Big(\sum_{i=1}^{N}[\varphi_{N-1+1},\varphi_i]\Big) \perp \omega_0 = \partial\Big(\sum_{i+k=N+1}(\varphi_i \wedge \varphi_k) \perp \omega_0\Big)$

From lemma 1.2.5. it follows that

1.2.6.2. $\frac{1}{2}\overline{\partial}\Big(\sum_{i=1}^{N}[\varphi_{N-i+1},\varphi_i]\Big) \perp \omega_0 = \overline{\partial}\partial\Big(\sum_{i+K=N+1}(\varphi_i \wedge \varphi_K) \perp \omega_0\Big) = 0.$

From Hodge theorem, the fact that M is a Kähler manifold we get

1.2.6.3. $\partial[(\sum_{i+K=N+1} \varphi_i \wedge \varphi_K) \perp \omega_0] = \partial\bar{\partial}\text{-}\Psi^1_{N+1} = \bar{\partial}\partial\Psi^1_{N+1}$.

From Hodge theorem and the fact that M is a Kähler manifold we get

1.2.6.4. $\partial\Psi^1_{N+1} = \bar{\partial}\Psi''_{N+1} + \bar{\partial}^*\Psi'''_{N+1}$

where $\bar{\partial}^*\Psi'''_{N+1} = \partial\Psi_{N+1}$ and so $\bar{\partial}^*\partial\Psi_{N+1} = 0$ since $\bar{\partial}^* \circ \bar{\partial}^* = 0$.
Define

1.2.6.2. $\varphi_{N+1} = \partial\Psi_{N+1} \perp \omega_0^*$, where

$\omega_0^* \in \Gamma(M,\Lambda^n\Theta)$ and $<\omega_0^*,\omega_0> = 1$ pointwise, i.e.

$$\omega_0^*|_U = \frac{\partial}{\partial z^1} \ \cdots \ \frac{\partial}{\partial z^n} .$$

Clearly from the fact that

$\nabla_\alpha \omega_0^* = 0$ (Bochner principle)

we get immediately that

$\bar{\partial}^*(\partial\Psi_{N+1} \perp \omega_0^*) = (\bar{\partial}^*\partial\Psi_{N+1} \perp \omega_0) = 0$. (For more details see lemma 1.2.2.). So

$\bar{\partial}^*\varphi_{N+1} = 0$ and $\bar{\partial}\varphi_{N+1} = \frac{1}{2}(\sum_{i=1}^{N} [\varphi_{N+1-i}, \varphi_i])$

The theorem is proved.

Q.E.D.

We have proved the following theorem:

Theorem 1.2'. Let M be a $SU(N)$ manifold and let $\eta \in \mathbb{H}^1(M,\Theta)$, then there exists a convergent power series in norms defined in [8]

$$\varphi(t) = \eta t + \varphi_2 t^2 + \ldots + \varphi_K t^K + \ldots$$

such that

1) $\varphi_i \in \Gamma(M,\Omega^{0,1} \otimes \Theta)$
2) $\bar{\partial}^* \varphi_i = 0$
3) $\varphi_i \perp \omega_0 = \partial \Psi_i$
4) $\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t),\varphi(t)]$.

Remark. It is proved in [8] that if $\varphi(t)$ fulfills 1),2) & 4), then $\varphi(t) \in C^\infty(M,\Omega^{0,1} \otimes \Theta)$.

§2. Computations of the curvature tensor of the Weil-Petersson metric

2.1. Definition. We know that the tangent space $T_{s,S}$ at the point $s \in S$, where S is the Kuranishi space, can be identified with $\mathbb{H}^1(M_s,\Theta_s)$. $\mathbb{H}^1(M_s,\Theta_s)$ is the harmonic part of $H^1(M,\Theta_s)$ with respect to Calabi-Yau metric $g_{\alpha\bar{\beta}}(s)$ on $\pi^{-1}(s) = M$, where the cohomology class $[[mg_{\alpha\bar{\beta}}(s)] = [[mg_{\alpha\bar{\beta}}(0)]$. We know that $g_{\alpha\bar{\beta}}(s)$ is the unique Kähler Ricci-flat metric such that

$$[\mathrm{Im}(g_{\alpha\bar{\beta}}(s))] = [\mathrm{Im}(g_{\alpha\bar{\beta}}(0))] = L$$

see [15].

Let $\varphi_1,\varphi_2 \in T_{s,S} = \mathbb{H}^1(M_s,\Theta_s)$, then

2.1.1. $$\langle\varphi_1,\varphi_2\rangle_{w.p.} = \int \varphi^{\mu}_{1,\bar{\alpha}}\ \bar{\varphi}^{\bar{\nu}}_{1,\bar{\beta}}\ g_{\mu\bar{\nu}}\cdot g^{\beta\bar{\alpha}}\ \mathrm{vol}(g_{\alpha\bar{\beta}}(s))$$

2.2. Lemma. $\langle\varphi_1,\varphi_2\rangle = (-1)^{\frac{n(n+1)}{n}}(i)^{n-2}\int(\varphi_1 \perp \varphi_s) \wedge (\overline{\varphi_2 \perp \varphi_\alpha})$, where $\int\omega_s \wedge \bar{\omega}_s = \int \mathrm{vol}(g_{\alpha\bar{\beta}}(s))$ and ω_s is a holomorphic n-form on M_s.

Proof: From $\nabla\omega_s = 0$ (Bochner is principle) $\Rightarrow \nabla(\omega_s \wedge \bar{\omega}_s) = 0$. On the other hand $\nabla(\mathrm{vol}(g_{\alpha\bar{\beta}}(s)) = 0$, so we may assume

(2.2.1) $$\omega_s \wedge \bar{\omega}_s = \mathrm{vol}(g_{\alpha\bar{\beta}}(s)).$$

From (2.2.1) and $\varphi_{\bar{\alpha}\bar{\beta}} = \varphi_{\bar{\beta}\bar{\alpha}}$ (See [9]) it follows that

$$\langle\varphi_1,\varphi_2\rangle_{w.p.} = (-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int_{M_s}(\varphi_1 \perp \omega_s) \wedge (\overline{\varphi_2 \perp \omega_s})$$

Q.E.D.

2.3. Let us define locally

$$\Theta_t^\alpha = dz^\alpha + \sum \varphi_{\overline{\mu}}^\alpha(t)\, d\overline{z}^\mu \qquad \alpha = 1,\dots,n$$

where

$$\varphi(t) = \varphi_1 t + \varphi_2 t^2 + \dots + \varphi_n t^n + \dots$$

and

1) $\varphi_1 \in \mathbb{H}^1(M,\Theta)$

2) $\overline{\partial}^* \varphi(t) = 0$

3) $\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t),\varphi(t)]$

4) $\varphi_i \perp \omega_0 = \partial \Psi_i$ for $i \geq 2$.

Clearly for each t $\{\Theta_t^\alpha\}$ is a basis for $\Omega_t^{1,0}|_U$. Of course we suppose that

$$\omega_0|_U = dz^1 \wedge \dots \wedge dz^u \ .$$

<u>2.4. Lemma.</u> $d(\Theta_t^1 \wedge \dots \wedge \Theta_t^n) = 0$ (See also [19] and A. Weil, Collected work vol. 2.)

<u>Proof:</u> Since $\varphi(t) \in \Gamma(M,\mathrm{Hom}(\Omega^{1,0},\Omega^{0,1}))$, then for each $K \leq n$ $K > 0$ we can define

$$\Lambda^K \varphi(t) \in \Gamma(M,\mathrm{Hom}(\Lambda^K\Omega^{1,0},\Lambda^K\Omega^{0,1}))$$

where

$$(\Lambda^K\varphi(t))(u_1\wedge\ldots\wedge u_K) = \varphi(t)(u_1)\wedge\ldots\wedge\varphi(t)(u_K) \ .$$

Next we have the following formula

(2.4.1.) $$\theta_t^1\wedge\ldots\wedge\theta_t^n = dz^1\wedge\ldots\wedge dz^n + \sum_{K=1}^{n}(-1)^{\frac{K(K-1)}{2}}\Lambda^K\varphi\lrcorner dz^1\wedge\ldots\wedge dz^n =$$
$$= \omega_0\big|_U + \sum(-1)^{\frac{K(K-1)}{2}}(\Lambda^K\varphi\lrcorner\omega_0)\big|_U \ .$$

Formula (2.4.1) follows from the definitions of $\Lambda^K\varphi(t)$ and $\theta_t^1\wedge\ldots\wedge\theta_t^n$.

<u>Proposition 2.4.2.</u> $(-1)^{\frac{K(K-1)}{2}}\bar\partial(\Lambda^K\varphi\lrcorner\omega_0)+(-1)^{\frac{K(K+1)}{2}}\partial(\Lambda^{K+1}\varphi\lrcorner\omega_0) = 0$.

<u>Proof:</u> So it is enough to prove

$$\bar\partial(\Lambda^K\varphi\lrcorner\omega_0) + (-1)^K\partial(\Lambda^{K+1}\varphi\lrcorner\omega_0) = 0 \ .$$

From $\bar\partial\varphi(t) = \frac{1}{2}[\varphi(t),\varphi(t)]$ it follows that

(2.4.2.1.) $$\bar\partial\varphi^i(t) = \sum_{j=1}^{n}\varphi^j\partial_j\varphi^i$$

since

(2.4.2.2.) $$\Lambda^K\varphi = \sum_{i_1<\ldots<i_K}\varphi^{i_1}\wedge\ldots\wedge\varphi^{i_K}\otimes\frac{\partial}{\partial z^{i_1}}\wedge\ldots\wedge\frac{\partial}{\partial z^{i_K}}$$

we get

(2.4.2.3.) $$(\Lambda^K\varphi\lrcorner\omega_0)\big|_U = \sum(-1)^{(i_1-1)+\ldots+(i_K-1)}\varphi^{i_1}\wedge\ldots\wedge\varphi^{i_K}\wedge dz^1\wedge\ldots\wedge\widehat{dz^{i_1}}\wedge\ldots\wedge\widehat{dz^{i_K}}\wedge\ldots\wedge dz^n$$

$$(2.4.2.4)\quad \bar{\partial}(\Lambda^K\varphi\lrcorner\omega_0)\Big|_v=\sum(-1)^{(i_1-1)+..+(i_K-1)+(\mu-1)}\varphi^{i_1}\wedge..\wedge\bar{\partial}\varphi^{i_\mu}\wedge..\wedge\varphi^{i_K}\wedge dz^1\wedge..\wedge \wedge..\wedge\widehat{dz^{i_1}}\wedge..\wedge\widehat{dz^{i_K}}\wedge..\wedge dz^n .$$

From (2.4.2.1) and (2.4.2.3) we get

$$(2.4.2.5)\quad \bar{\partial}(\Lambda^K\varphi\lrcorner\omega_0)\Big|_v=\sum(-1)^{(i_1-1)+..+(i_K-1)+(\mu-1)}\varphi^{i_1}\wedge..\wedge\sum_j\varphi^j\wedge\partial_j\varphi^{i_\mu}\wedge..\wedge d\varphi^{i_K}\wedge \wedge dz^1\wedge..\wedge\widehat{dz^{i_1}}\wedge..\wedge\widehat{dz^{i_K}}\wedge..\wedge dz^n =$$

$$=\sum(-1)^{(i_1-1)+..+(i_K-1)}(\sum_j\varphi^j\wedge\varphi^{i_1}\wedge..\wedge\partial_j\varphi^i\wedge..\wedge\varphi^{i_K})\wedge dz^1\wedge..\wedge\widehat{dz^{i_1}}\wedge..\wedge \wedge..\wedge\widehat{dz^{i_K}}\wedge..\wedge dz^n .$$

Next we must compute $(-1)^K\partial(\Lambda^{K+1}\varphi\lrcorner\omega_0) = ?$

Suppose that $i_1 < i_2 < \ldots < i_{l-1} \leq j < i_l < \ldots < i_K$

$$(2.4.2.6)\quad \partial((-1)^K\Lambda^{K+1}\varphi\lrcorner\omega_0)=(-1)^K\partial(\sum_{i_1<..<j<..<i_x}(-1)^{(i_1-1)+..+(j-1)+..+(i_K-1)}\varphi^{i_1}\wedge..\wedge\varphi^j\wedge..\wedge \wedge..\wedge\varphi^{i_K}\wedge dz^1\wedge..\wedge\widehat{dz^{i_1}}\wedge..\wedge\widehat{dz^j}\wedge..\wedge\widehat{dz^{i_K}}\wedge..\wedge dz^n=$$

$$=(-1)^K\sum_j\sum(-1)^{(i_1-1)+..+(j-1)+..+(i_K-1)+(l-1)+(j-1)+(K+1)}\varphi^{i_1}\wedge..\wedge\partial_j\varphi^{i_\mu}\wedge..\wedge \wedge..\wedge\varphi^{i_K}\wedge dz^1\wedge..\wedge\widehat{dz^{i_1}}\wedge..\wedge dz^j\wedge..\wedge\widehat{dz^{i_K}}\wedge..\wedge dz^n .$$

From $\sum\partial_j\varphi^j = 0$ and (2.4.2.6) we get that

$$(2.4.2.7)\quad \partial((-1)^K\Lambda^{K+1}\varphi\lrcorner\omega_0) =$$

$$=-(\sum_{i_1<..<i_K}(-1)^{(i_1-1)+..+(i_K-1)}(\sum_{j=1}^n\varphi^j)\wedge\varphi^{i_1}\wedge..\wedge\partial_j\varphi^{i_\mu}\wedge..\wedge\varphi^{i_K}\wedge dz^1\wedge..\wedge \wedge..\wedge\widehat{dz^{i_1}}\wedge..\wedge\widehat{dz^{i_K}}\wedge..\wedge dz^n .$$

From (2.4.2.5) and (2.4.2.7) we get that

$$\bar{\partial}(\Lambda^K\varphi \perp \omega_0) + (-1)^K \partial(\Lambda^{K+1}\varphi \perp \omega_0) = 0 .$$

Q.E.D.

From 2.4.2 it follows that

$$d(\Theta_t^1 \wedge .. \wedge \Theta_t^n) = 0 .$$

Q.E.D.

Remark 2.4.9. Since ω_0 and for all K $\Lambda^K\varphi$ are globally defined tensors it follows that

$$(2.4.9.1) \qquad \omega_t = \omega_0 + \sum_{K=1}^{n} (-1)^{\frac{K(K-1)}{2}} (\Lambda^K \varphi \perp \omega_0)$$

is also globally defined. From (2.4.1) it follows that ω_t is a holomorphic n-form on M_t, since $d\omega_t = 0$ and ω_t is of type $(n,0)$ on M_t.

Remark 2.5. Let $\{\eta_\alpha\}$ be a basis of $\mathbb{H}^1(M,\Theta)$. Let

$$\varphi_\alpha(t^\alpha) = \eta_\alpha t^\alpha + \varphi_{\alpha,2}(t^\alpha)^2 + .. + \varphi_{\alpha,N}(t^\alpha)^N + ..$$

be an element of $\Gamma(M,\Omega^{0,1} \otimes \Theta)$ such that

a) $\bar{\partial}^* \varphi_\alpha(t^\alpha) = 0$

b) $\bar{\partial}\varphi_\alpha(t^\alpha) = \frac{1}{2}[\varphi_\alpha(t^\alpha), \varphi_\alpha(t^\alpha)]$

c) $\varphi_{\alpha,K} \perp \omega_0 = \partial\varphi_{\alpha,K} \quad \forall K \geq 2$

then $(t^1,\dots,t^K)$ will be a local coordinate system in S. From Theorem 1.2' we know that $\varphi_\alpha(t^\alpha)$ exists for each α. We will call such coordinates Kodaira-Spencer-Kuranishi coordinates. From now on we will fix these local coordinates.

2.6. $\varphi_\alpha(t^\alpha)$ defines a disc D_α in S and we have a family of SU(n) manifolds over D_α $\pi_\alpha : X_\alpha \longrightarrow D_\alpha$. Let ω_{t_α} be the holomorphic n-form on $M_{t^\alpha} = \pi^{-1}(t^\alpha)$ defined by (2.4.9.1), i.e.

$$(2.6.1) \qquad \omega_{t^\alpha} = \omega_0 + \sum (-1)^{\frac{K(K-1)}{2}} \Lambda^K \varphi_\alpha(t^\alpha) \lrcorner \omega_0$$

then we have:

$$(2.6.2) \qquad \int_{M_0} \omega_{t^\alpha} \wedge \bar{\omega}_{t^\alpha} = \int_{M_0} \omega_0 \wedge \overline{\omega_0} .$$

<u>Proof of (2.6.2):</u> Let $f(t^\alpha,\overline{t^\alpha}) := \int_M \omega_{t^\alpha} \wedge \bar{\omega}_{t^\alpha}$. If we prove that $\frac{df}{dt^\alpha} = \frac{df}{d\bar{t}^\alpha} = 0$ at each point, then (2.6.2) will be proved. From (2.6.1) it follows that:

$$(2.6.2.1) \qquad \omega_{t^\alpha} = \omega_0 + t\eta_\alpha + 0(t^2) .$$

So

$$(2.6.2.2)\ f(t^\alpha,\bar{t}^\alpha) = \int_{M_0} \omega_{t^\alpha} \wedge \bar{\omega}_{t^\alpha} = \int_{M_0} \omega_0 \wedge \bar{\omega}_0 + |t^\alpha|^2 \int (\eta_\alpha \lrcorner \omega_0) \wedge (\overline{\eta_\alpha \lrcorner \omega_0}) + 0(|t^\alpha|^2) .$$

Let $A_{t^\alpha} \in \Gamma(M, \mathrm{Hom}(T^*(M)\otimes\mathbb{C}, T^*(M\otimes\mathbb{C}))$, where locally

$$(2.6.2.3)\qquad A_{t^\alpha}(dz^K) = dz^K + \sum\varphi_\alpha(t^\alpha)^K_{\bar\mu} d\bar z^\mu \quad K = 1,\dots,n$$

$$A_{t^\alpha}(\overline{dz^K}) = \overline{dz^K} + \overline{\sum\varphi_\upsilon(t^\alpha)^K_{\bar\mu} d\bar z^\mu} \ .$$

From the definition of A_{t^α} it follows that

$$(2.6.2.4)\qquad \omega_{t^\alpha}\big|_U = (A_{t^\alpha} dz^1) \wedge\dots\wedge (A_{t^\alpha} dz^n) \ .$$

Let us fix t_0^α, where t_0^α is any point of D_α. We want to find Taylor series expression of $f(t^\alpha,\bar t^\alpha) = \int_{M_0} \omega_{t^\alpha} \wedge \bar\omega_{t^\alpha}$ in the form:

$f(t^\alpha,\bar t^\alpha) = f(t_0^\alpha,\bar t_0^\alpha) + (t-t_0)f_1(t_0^2,\bar t_0^\alpha) + \dots$. In order to get this expression we notice that

$$(2.6.2.5)\qquad dz^K = A^{-1}_{t_0^\alpha}\, \Theta^K_{t_0^\alpha} \ .$$

So we have

$$(2.6.2.6)\qquad \omega_{t^\alpha}\big|_U = \left(A_{t^\alpha} A^{-1}_{t_0^\alpha}\, \Theta^1_{t_0^\alpha}\right) \wedge\dots\wedge \left(A_{t^\alpha} A^{-1}_{t_0^\alpha}\, \Theta^n_{t_0^\alpha}\right) \ .$$

From the definition of A_t we get that

$$(2.6.2.7)\qquad A^{-1}_{t_0^\alpha}\, \Theta^K_{t_0^\alpha} = \Theta^K_{t_0^\alpha} - t_0 \sum\eta^K_{\alpha,\bar\mu}\, \Theta^{\bar\mu}_{t_0^\alpha} \otimes (\Theta^K_{t_0})^* + 0(t_0^2) \ .$$

This is so since $A_t = \mathrm{id} + (t\eta_\alpha + \bar t\bar\eta_\alpha) + \dots$. From (2.6.2.7) and (2.6.2.6) we obtain that

$$(2.6.2.8)\quad \omega_{t^\alpha}\Big|_U = \theta^1_{t_0^\alpha}\wedge\ldots\wedge\theta^e_{t_0^\alpha}\wedge\ldots\wedge\theta^n_{t_0^\alpha}\Big|_U + (t^\alpha - t_0^\alpha)\sum(-1)^{\mu-1}\eta^\mu_{\alpha,\bar\gamma}\overline{d\theta^\gamma_{t_0^\alpha}}\wedge$$

$$\wedge\theta^1_{t_0^\alpha}\wedge\ldots\wedge\hat{\theta}^\mu_{t_0^\alpha}\wedge\ldots\wedge\theta^n_{t_0^\alpha} + 0((t^\alpha - t_0^\alpha)^2)\ .$$

So from (2.6.2.8) it follows that

$$(2.6.2.9)\quad \omega_{t^\alpha}\wedge\bar\omega_{t^\alpha} = \omega_{t_0^\alpha}\wedge\bar\omega_{t_0^\alpha} + |t^\alpha - t_0^\alpha|^2(\omega_{t^\alpha}\lrcorner\eta_\alpha)\wedge(\overline{\omega_{t^\alpha}\lrcorner\eta_\alpha}) + 0((t^\alpha - t_0^\alpha)^3)\ .$$

Here η_α stands for an element of $H^1(M_{t_0^\alpha},\Theta_{t_0^\alpha})$. So we get that:

$$(2.6.2.10)\quad f(t^\alpha,\overline{t^\alpha}) = f(t_0^\alpha,\overline{t_0^\alpha}) + |t^\alpha - t_0^\alpha|^2 f_1(t_0^\alpha,\overline{t_0^\alpha}) + 0((t^\alpha - t_0^\alpha)^2)\ .$$

From (2.6.2.10) $\Rightarrow \dfrac{\partial f(t^\alpha,\overline{t^\alpha})}{\partial t^\alpha}\Big|_{t^\alpha = t_0^\alpha} = \dfrac{\bar\partial f(t^\alpha,t_0^\alpha)}{\bar\partial t^\alpha}\Big|_{t^\alpha = t_0^\alpha} = 0$.

So $f(t^\alpha,\bar t^\alpha) = \text{const.}$

Q.E.D.

<u>Lemma 2.7.</u> Let $(h_{\mu\bar\nu}(t,\bar t))$ be the Weil-Petersson metric, then

$$h_{\mu\bar\nu}(t,\bar t) = (-1)^{\frac{n(n-1)}{2}}(i)^{n-2}\int_M \frac{d\omega_{t^\mu}}{dt^\mu}\wedge\overline{\frac{d\omega_{t^\nu}}{dt^\nu}}\ .$$

<u>Proof:</u> Lemma 2.7 follows from lemma 2.2, Remark 2.6 and $\dfrac{d\omega_{t^\mu}}{dt^\mu}$ is a form of type $(n-1,1)$ on M_{t^μ} . Moreover

$$\frac{d\omega_{t^\mu}}{dt^\mu}\Big|_{M_{t^\mu}} = \frac{d\varphi_\mu(t^\mu)}{dt^\mu}\lrcorner\,\omega_{t^\mu}\ .$$

This is proved by Griffiths in [19]. So from the last equality we get 2.7.

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Lemma 2.8. $h_{\mu\bar{\nu}}(t,\bar{t}) = (-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\Big(\int_M(\eta_\mu \lrcorner \omega_0)+(\overline{\eta_\nu \lrcorner \omega_0})+$

$+4t^\mu t^{\bar{\nu}}(\int_M[\Lambda^2\eta_\mu \lrcorner \omega_0]\Lambda[\overline{\Lambda^2\eta_\nu \lrcorner \omega_0}])$ + (terms of higher order ≥ 3 in $t^\mu, t^\nu, \bar{t}^\mu, \bar{t}^\nu$)], here $[\Lambda^2\eta_\mu \lrcorner \omega_0]$ and $[\Lambda^2\eta_\nu \lrcorner \omega_0]$ denote the classes of cohomology of $[\Lambda^2\eta_\mu \lrcorner \omega_0]$ and $[\Lambda^2\eta_\nu \lrcorner \omega_0]$ in $H^{n-2,2} \subset H^n(M,\mathbb{C})$, namely $[\Lambda^2\eta_\mu \lrcorner \omega_0] = [\mathbb{H}(\Lambda^2\eta_\mu \lrcorner \omega_0)]$.

Proof: From (2.4.1) and the fact that $\varphi_\nu(t^\nu) = \eta_\nu t^\nu + \varphi_{\nu,2}(t^\nu)^2 + \varphi_{\nu,3}(t^\nu)^3 + \ldots$ we get that

$$(2.8.1)\quad \frac{d\omega_{t^\mu}}{dt^\mu} = \eta_\mu \lrcorner \omega_0 + 2t^\mu(\varphi_{\mu,2}\lrcorner\bar{\omega}_0 - \Lambda^2\eta_\mu \lrcorner \omega_0) + 3(t^\mu)^2(\varphi_{\nu,3}\lrcorner\omega_0 -$$

$$- \varphi_{\nu,2}\Lambda\eta_\nu \lrcorner \omega_0 - \Lambda^3\eta_\nu \lrcorner \omega_0) + \ldots .$$

Clearly we have:

$$\varphi_\nu(t^\nu) = \eta_\nu t^\nu + \varphi_{\nu,2}(t^\nu)^2 + \ldots + \varphi_{\nu,N}(t^\nu)^N + \ldots .$$

So

$$(2.8.1)\quad \frac{d\omega_{t^\nu}}{dt^\nu} = \eta_\nu \lrcorner \omega_0 + 2t^\nu(\varphi_{\nu,2}\lrcorner\omega_0 - \Lambda^2\eta_\nu \lrcorner \omega_0) + 3(t^\nu)^2(\varphi_{\nu,3}\lrcorner\omega_0 -$$

$$- \varphi_{\nu,2}\Lambda\eta_\nu \lrcorner \omega - \Lambda^3\eta_\nu \lrcorner \omega_0) + \ldots .$$

So we have

$$(2.8.2)\quad h_{\mu\bar{\nu}}(t,\bar{t}) = (-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\int_M \frac{d\omega_t{}^{\mu}}{dt^{\mu}} \wedge \overline{\frac{\omega_t{}^{\nu}}{dt^{\nu}}} =$$

$$= (-1)^{\frac{n(n+1)}{2}}(i)^{n-2}\Big[\int_M (\eta_\mu \lrcorner \omega_0)\wedge(\overline{\eta_\nu \lrcorner \omega_0}) + 2t^{\mu}\int_M (\varphi_{\mu,2}\lrcorner\omega_0)\wedge(\overline{\eta_\nu\lrcorner\omega_0}) +$$

$$+ 2\bar{t}^{\nu}\int_M (\eta_\mu\lrcorner\omega_0)\wedge(\overline{\varphi_{\nu,2}\lrcorner\omega_0}) + 4t^{\mu}\bar{t}^{\nu}\int_M (\varphi_{\mu,2}\lrcorner\omega_0 - \Lambda^2\eta_\mu\lrcorner\omega_0)\wedge(\overline{\varphi_{\nu,2}\lrcorner\omega_0 - \Lambda^2\eta_{\nu,2}\lrcorner\omega_0})$$

$$+ 3(t^{\mu})^2\int_M (\varphi_{\mu,3}\lrcorner\omega_0)\wedge(\overline{\eta_\nu\lrcorner\omega_0}) + 3(\overline{t^{\nu}})^2\int_M (\eta_\mu\lrcorner\omega_0)\wedge(\overline{\varphi_{\nu,3}\lrcorner\omega_0}) +$$

$$+ (\text{terms of order } \geq 3)\ .$$

Since $(\varphi_{\nu,K}\lrcorner\omega_0) = \partial\psi_{\nu,K}$ for $K \geq 1$ and the fact that for any μ $\eta_\mu\lrcorner\omega_0$ is a harmonic form by 1.2.2 and so $d(\eta_\nu\lrcorner\omega_0) = 0 =$ $= \partial(\eta_\mu\lrcorner\omega_0) = \bar{\partial}(\eta_\mu\lrcorner\omega_0)$ we get that for any τ,α and $K \geq 2$

$$(2.8.3)\quad \begin{aligned} &(\eta_\tau\lrcorner\omega_0)\wedge(\overline{\varphi_{\alpha,n}\lrcorner\omega_0}) = d((\eta_\tau\lrcorner\omega_0)\wedge\bar{\psi}_{\alpha,n}) \quad \text{and} \\ &(\varphi_{\alpha,K}\lrcorner\omega_0)\wedge(\overline{\eta_\tau\lrcorner\omega_0}) = d(\psi_{\alpha,K}\wedge(\overline{\eta_\tau\lrcorner\omega_0}))\ . \end{aligned}$$

From Stroke's theorem and 2.8.3 we get that

$$(2.8.4)\quad \int_M (\eta_\tau\lrcorner\omega_0)\wedge(\overline{\varphi_{\alpha,K}\lrcorner\omega_0}) = \int_M d((\eta_\tau\lrcorner\omega_0)\wedge\bar{\psi}_{\alpha,K}) = \int(\varphi_{\alpha,K}\lrcorner\omega_0)\wedge(\overline{\eta_\tau\lrcorner\omega_0}) =$$

$$= \int_M d(\psi_{\alpha,K}\wedge(\overline{\eta_\tau\lrcorner\omega_0}) = 0\ .$$

Next we must prove that

$$(2.8.5)\quad \int_M ((\varphi_{\mu,2}\lrcorner\omega_0) - \Lambda^2\eta_\mu\lrcorner\omega_0)\wedge(\overline{\varphi_{\nu,2}\lrcorner\omega_0 - \Lambda^2\eta_\nu\lrcorner\omega_0}) =$$

$$= \int_M \mathbb{H}(\Lambda^2\eta_\mu\lrcorner\omega_0)\wedge\mathbb{H}(\Lambda^2\eta_\nu\lrcorner\omega_0)\ ,\ \text{where } \mathbb{H} \text{ is the harmonic}$$

projection with respect to Calabi-Yau metric.

Proof of 2.8.5.: We know that

$$(2.8.5.1)\quad \omega_{t^\mu} = \omega_0 + \cdot\sum(-1)^{\frac{K(K-1)}{2}}\Lambda^K\varphi_\mu(t^\mu) = \omega_0 + t^\mu\eta_\mu\perp\omega_0 +$$

$$+ (t^\mu)^2(\varphi_{\mu,2}\perp\omega_0 - \Lambda^2\eta_\mu\perp\omega_0) + o((t^\mu)^3)$$

and $d\omega_{t^\mu} = 0$, so we get that each coefficient, which is a complex valued form, in front of $(t^\mu)^N$ must be closed, namely $d(\varphi_{\mu,2}\perp\omega_0 - \Lambda^2\eta_\mu\perp\omega_0) = \bar\partial(\varphi_{\mu,2}\perp\omega_0) - \partial(\Lambda^2\eta_\mu\perp\omega_0) = 0$. This is so since $\partial(\varphi_{\mu,2}\perp\omega_0) = \partial\psi_{\mu,2} = 0$, $\bar\partial(\Lambda^2\eta_\mu\perp\omega_0) = 0$. From Hodge theorem and the fact that M is a Kähler variety, i.e. $\Box_d = \frac{1}{2}\Box_{\bar\partial} = \frac{1}{2}\Box_\partial$ we get that

$$(2.8.5.2)\quad \mathbb{H}(\varphi_{\mu,2}\perp\omega_0)-\Lambda^2\eta_\mu\perp\omega_0) = \mathbb{H}\varphi_{\mu,2}\perp\omega_0 - \mathbb{H}(\Lambda^2\eta_\mu\perp\omega_0) = \mathbb{H}(\Lambda^2\eta_\mu\perp\omega_0)$$

since $\varphi_{\mu,\alpha}\perp\omega_0 = \partial\psi_{\mu,2}$ and so $\mathbb{H}\partial\psi_{\mu,2} = 0$.
So we have

$$(2.8.5.3)\quad (\varphi_{\mu,2}\perp\omega_0 - \Lambda^2\eta_\mu\perp\omega_0) = \mathbb{H}(\Lambda^2\eta_\mu\perp\omega_0) + df_{\mu,2} ,$$

where $f_{\mu,2}$ is $(n-1)$ complex valued form on M . From (2.8.5.3) and Stoke's theorem (2.8.5) follows

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(2.8.5) proves lemma 2.8.

Q.E.D.

Cor. 2.8.6. The Weil-Petersson metric on the moduli space of SU($n \geq 3$) manifolds is a Kähler metric, moreover $h_{\mu\bar{\nu}}$ are real analytic functions with respect to $(t^1,\ldots,t^K,\bar{t}^1,\ldots,\bar{t}^K)$.

Proof: It is a well known fact, that if $(g_{\mu\bar{\nu}})$ is a Hermitian metric on a complex manifold M and if around any point $m \in M$ we can find local coordinates $(t^1,\ldots,t^K)$ such that

$$g_{\mu\bar{\nu}}(t,\bar{t}) = \delta_{\mu\bar{\nu}} + \sum h_{\mu\bar{\nu},\alpha\bar{\beta}} t^\alpha \bar{t}^\beta + (\text{terms of order } \geq 3)$$

where $h_{\mu\bar{\nu},\alpha\bar{\beta}}$ are constants, then $g_{\mu\bar{\nu}}(t,\bar{t})$ will be a Kähler metric. Cor. 2.8.6 follows directly from 2.8 using this criterium.

Q.E.D.

Theorem 2.9. a) We have the following formulas for the curvature tensor of the Weil-Petersson metric on the moduli space of SU($n \geq 3$) manifolds

$$(2.9.1) \quad R_{\alpha\bar{\beta},\mu\bar{\nu}} = 0 \quad \text{if} \quad \alpha \neq \mu \quad \text{or} \quad \beta \neq \nu$$

$$R_{\alpha\bar{\beta},\alpha\bar{\beta}} = +(-1)^{\frac{n(n-1)}{2}} (i)^{n-2} 8 \int \left[\Lambda^2 \eta_\alpha \lrcorner \omega_0\right] \wedge \left[\Lambda^2 \eta_\beta \lrcorner \omega_0\right].$$

b) The biholomorphic sectional curvature of the Weil-Petersson metric on the moduli space of SU($n \geq 3$) manifolds is negative.

Proof of a): From lemma 2.8 and the fact that Weil-Petersson metric is a Kähler metric we get that

$$(2.9.1.1) \quad R_{\alpha\bar{\beta},\mu\bar{\nu}} = + \frac{\partial^2 h_{\alpha\bar{\beta}}}{2\partial t^\mu \partial \bar{t}^\nu} = 0 \quad \text{if} \quad \alpha \neq \mu \quad \text{or} \quad \beta \neq \nu$$

$$R_{\alpha\bar{\beta},\alpha\bar{\beta}} = + \frac{\partial^2 h_{\alpha\bar{\beta}}}{2\partial t^\alpha \partial \bar{t}^\beta} = +(-1)^{\frac{n(n-1)}{2}} i^{n-2} 8 \int_M \left[\Lambda^2 \eta_\alpha \lrcorner \omega_0\right] \wedge \overline{\left[\Lambda^2 \eta_\beta \lrcorner \omega_0\right]}.$$

So 2.9.a is proved.

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Proof of 2.9.b. From the definition of biholomorphic sectional curvature it follows that we need to prove that for any two vectors $(\xi^1,\ldots,\xi^K)$ and $(\eta^1,\ldots,\eta^K)$ we have

$$(2.9.1.2) \quad +8\sum R_{\alpha\bar\beta,\alpha\bar\beta}\xi^\alpha\bar\xi^\beta\eta^\alpha\bar\eta^\alpha < 0 \ .$$

Proof of (2.9.1.2): First we will prove that $[\Lambda^2\eta_\mu \lrcorner \omega_0]$ for any μ are primitive classes of cohomology with respect to the polarization class $L = [\mathrm{Im}(g_{\alpha\bar\beta})]$. So we need to show that $[\Lambda^2\eta_\mu \lrcorner \omega_0]$ can not be represented as, i.e.

$$(2.9.1.2.1) \quad [\Lambda^2\eta_\mu \lrcorner \omega_0] = L\Lambda[\varphi] \ , \text{ where } [\varphi] \in H^{n-3,1} \ .$$

Suppose that (2.9.1.2.1) holds then from the formula

$$(2.9.1.2.2) \quad [\omega_{t^\mu}] = [\omega_0] + t^\mu[\eta_\mu \lrcorner \omega_0] + (t^\mu)^2[\Lambda^2\eta_\mu \lrcorner \omega_0] + \ldots$$

$$= [\omega_0] + t^\mu[\eta_\mu \lrcorner \omega_0] + (t^\mu)^2[\varphi]\Lambda L + \ldots \ .$$

Follows that ω_{t^μ} is not a primitive class of cohomology, this contradicts the fact that ω_{t^μ} is a form of type $(n,0)$ on M_{t^μ} and so ω_{t^μ} is a primitive form. Notice that we use the fact that L is $(1,1)$ class of cohomology for all M_{t^μ} .

So we have proved that the subspace $E \subset H^{n-2,2}$ which is spanned by $\{[\Lambda^2\eta_\mu \lrcorner \omega_0]\}$ is contained in the space of all primitive classes of cohomology of type $(n-2,2)$, which we will denote by $H_0^{n-2,2}$. From this fact and the following formula:

(2.9.1.2.3) if $a \in H^{n-2,2}$, then

$$a = (-1)^{\frac{n(n+1)}{2}} (i)^{n-4} a$$

it follows that the Hermitian form on $H_0^{n-2,2}$

$$\langle a,b\rangle = (-1)^{\frac{n(n+1)}{2}} (i)^{n-4} \int_M a \wedge b = \int a \wedge * b \qquad (2.9.1.2.4)$$

is definite positive and so comparing (2.9.1) and (2.9.1.2.4) we get that

$$\pm 8 \sum R_{\alpha\bar{\beta},\alpha\bar{\beta}} \xi^{\alpha} \bar{\xi}^{\beta} \eta^{\alpha} \bar{\eta}^{\beta} = +8 \sum \xi^{\alpha} \eta^{\alpha} H_{\alpha\bar{\beta}} \bar{\xi}^{\beta} \bar{\eta}^{\beta}$$

$$= 8 (-1)^{\frac{n(n+1)}{2}} (i)^{n-2} \left(\sum \xi^{\alpha} \eta \cdot \left(\int_M [\Lambda^2 \eta_{\alpha} \lrcorner \omega_0] \wedge \overline{[\Lambda^2 \eta_{\beta} \lrcorner \omega_0]} \right) \bar{\xi}^{\beta} \bar{\eta}^{\beta} \right) < 0$$

where $H_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta},\alpha\bar{\beta}}$.

Q.E.D.

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