EXPLICIT FORMULAS AS TRACE FORMULAS, QUANTIZED CALCULUS, AND SPECTRAL INTERPRETATION OF THE ZEROS OF THE RIEMANN ZETA FUNCTION

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Global fields k provide a natural context for the Riemann Hypothesis on the zeros of the zeta function and its generalization to Hecke L-functions. When the characteristic of k is non zero this conjecture was proved by A. Weil. His proof relies on the following dictionary (put in modern language) which gives a geometric meaning in terms of algebraic geometry over finite fields, to the function theoretic properties of the zeta functions. Recall that k is a function field over a curve Σ defined over $\mathbb{F}_q$,

Algebraic Geometry	*Function Theory*
Eigenvalues of action of Frobenius on $H^*_{\text{et}}(\overline{\Sigma}, \mathbb{Q}_\ell)$	Zeros and poles of ζ
Poincaré duality in ℓ-adic cohomology	Functional equation
Lefschetz formula for the Frobenius	Explicit formulas
Castelnuovo positivity	Riemann Hypothesis

We shall describe a third column in this dictionary, which will make sense for any global field. It is based on the geometry of the Adele class space,

$$X = A/k^* \,, \;\; A = \text{Adeles of } k\,. \tag{1}$$

This space is of the same nature as the space of leaves of the horocycle foliation of a Riemann surface and the same geometry will be used to analyse it.

Our spectral interpretation of the zeros of zeta involves Hilbert space. The reasons why Hilbert space (apparently invented by Hilbert for this purpose) should be involved are manifold, let us mention three,

(A) The discovery of Hugh Montgomery ([M]) about the statistical fluctuations of the spacings of zeros of zeta. Numerical tests by Odlyzko ([O]) and further theoretical work by Katz-Sarnak ([KS]) give overwhelming evidence that zeros of zeta should be the eigenvalues of a hermitian matrix.

(B) The equivalence between RH and the positivity of the Weil distribution on the Idele class group C_k which shows that Hilbert space is implicitly present.

(C) The deep arithmetic significance of the work of A. Selberg on the spectral analysis of the Laplacian on $L^2(G/\Gamma)$ where Γ is an arithmetic subgroup of a semi simple Lie group G.

Direct attempts (cf. [B]) to construct the Polya-Hilbert space giving a spectral realization of the zeros of ζ using quantum mechanics, meet a serious minus sign problem explained in [B].

The very same $-$ sign appears in the Riemann-Weil explicit formula,

$$\sum_{L(\chi,\rho)=0} \widehat{h}(\chi,\rho) - \widehat{h}(0) - \widehat{h}(1) = -\sum_v \int_{k_v^*}' \frac{h(u^{-1})}{|1-u|}\, d^*u\,, \tag{2}$$

where h is a test function on the Idele class group C_k, $\widehat{h}$ is its Fourier transform,

$$\widehat{h}(\chi,z) = \int_{C_k} h(u)\,\chi(u)\,|u|^z\, d^*u\,, \tag{3}$$

and the finite values $\int'$ are suitably normalized. If we use the above dictionary when $\mathrm{char}(k) \neq 0$, the geometric origin of this $-$ sign becomes clear, the formula (2) is the Lefschetz formula,

$$\#\text{ of fixed points of } \varphi = \mathrm{Trace}\,\varphi/H^0 - \mathrm{Trace}\,\varphi/H^1 + \mathrm{Trace}\,\varphi/H^2 \tag{4}$$

in which the space $H^1_{\mathrm{et}}(\overline{\Sigma},\mathbb{Q}_\ell)$ which provides the spectral realization of the zeros appears with a $-$ sign. This indicates that the spectral realization of zeros of zeta should be of cohomological nature or to be more specific, that the Polya-Hilbert space providing this realization should appear as the last term of an exact sequence of Hilbert spaces,

$$0 \to \mathcal{H}_0 \xrightarrow{T} \mathcal{H}_1 \to \mathcal{H} \to 0\,. \tag{5}$$

Let $X = A/k^*$ be the Adele class space. Our basic idea is to take for $\mathcal{H}_0$ a suitable completion of the codimension 2 subspace of functions on X such that,

$$f(0) = 0\,, \quad \int f\,dx = 0\,, \tag{6}$$

while $\mathcal{H}_1 = L^2(C_k)$ and T is the restriction map coming from the inclusion $C_k \to X$, multiplied by $|a|^{1/2}$,

$$(Tf)(a) = |a|^{1/2}\, f(a)\,. \tag{7}$$

The action of the Idele class group C_k which gives the spectral realization is then the obvious one, for $\mathcal{H}_0$

$$(U(g)f)(x) = f(g^{-1}\,x) \qquad \forall\, g \in C_k \tag{8}$$

using the action by multiplication of C_k on X, and similarly the regular representation V for $\mathcal{H}_1$.

This idea works but there are two subtle points; first since X is a delicate quotient space the function spaces for X are naturally obtained by starting with function spaces on A and modding out by the "gauge transformations"

$$f \to f_q\,,\ \ f_q(x) = f(xq)\,, \quad \forall\, q \in k^*\,. \tag{9}$$

Here the natural function space is the Bruhat-Schwarz space $\mathcal{S}(A)$ and by (6) the codimension 2 subspace,

$$\mathcal{S}(A)_0 = \left\{ f \in \mathcal{S}(A)\,;\ f(0) = 0\,,\ \int f\,dx = 0 \right\}\,. \tag{10}$$

The restriction map T is then given by,

$$T(f)(a) = |a|^{1/2} \sum_{q\,\in\,k^*} f(aq) \qquad \forall\, a \in C_k\,. \tag{11}$$

The corresponding function $T(f)$ belongs to $\mathcal{S}(C_k)$ and all functions $f - f_q$ are in the kernel of T.

The second subtle point is that since C_k is abelian and non compact, its regular representation does not contain any finite dimensional subrepresentation so that the Polya-Hilbert space cannot be a subrepresentation (or unitary quotient) of V. There is an easy way out (which will be improved later) which is to replace $L^2(C_k)$ by $L^2_\delta(C_k)$ using the polynomial weight $(\log^2|a|)^{\delta/2}$, i.e. the norm,

$$\|\xi\|^2_\delta = \int_{C_k} |\xi(a)|^2 \, (1+\log^2|a|)^{\delta/2} \, d^*a \,. \tag{12}$$

Let $\mathrm{char}(k) = 0$ so that $\mathrm{Mod}\, k = \mathbb{R}^*_+$ and $C_k = K \times \mathbb{R}^*_+$ where K is the compact group $C_{k,1} = \{a \in C_k \, ; \; |a| = 1\}$.

Theorem. ([Co]) *Let $\delta > 1$, $\mathcal{H}$ be the cokernel of T in $L^2_\delta(C_k)$ and W the quotient representation of C_k. Let χ be a character of K, $\widetilde{\chi} = \chi \times 1$ the corresponding character of C_k. Let $\mathcal{H}_\chi = \{\xi \in \mathcal{H} \, ; \; W(g)\,\xi = \chi(g)\,\xi \quad \forall\, g \in K\}$ and $D_\chi = \lim\limits_{\epsilon \to 0} \frac{1}{\epsilon}\,(W(e^\epsilon) - 1)$. Then D_χ is an unbounded closed operator with discrete spectrum, $\mathrm{Sp}\, D_\chi \subset i\,\mathbb{R}$ is the set of imaginary parts of zeros of the L function with Grössencharakter $\widetilde{\chi}$ which have real part $1/2$. Moreover the spectral multiplicity of ρ is the largest integer $n < \frac{1+\delta}{2}$ in $\{1, \ldots,$ multiplicity as a zero of $L\}$.*

A similar result holds for $\mathrm{char}(k) > 0$. This allows to compute the character of the representation W as,

$$\mathrm{Trace}(W(h)) = \sum_{\substack{L\left(\chi, \frac{1}{2}+\rho\right) = 0 \\ \rho \in i\,\mathbb{R}/N^\perp}} \widehat{h}(\chi, \rho) \tag{13}$$

where $N = \mathrm{Mod}(k)$, $W(h) = \int W(g)\, h(g)\, d^*g$, $h \in \mathcal{S}(C_k)$, $\widehat{h}$ is defined in (3) and the multiplicity is counted as in the theorem.

This result is only preliminary because of the unwanted parameter δ which artificially restricts the multiplicities. The analogue of the Hodge $*$ operation is given on $\mathcal{H}_0$ by the Fourier transform,

$$(Ff)(x) = \int_A f(y)\, \alpha(xy)\, dy \qquad \forall\, f \in \mathcal{S}(A)_0 \tag{14}$$

which, because we take the quotient by (9), is independent of the choice of additive character α of A such that $\alpha \neq 1$ and $\alpha(q) = 1 \quad \forall\, q \in k$. Note also that $F^2 = 1$ on the quotient. On $\mathcal{H}_1$ the Hodge $*$ is given by,

$$(*\,\xi)(a) = \xi(a^{-1}) \qquad \forall\, a \in C_k \,. \tag{15}$$

The Poisson formula means exactly that T commutes with the $*$ operation. This is just a reformulation of the work of Tate and Iwasawa on the proof of the functional equation and should be compared with [Z], but we shall now see that if we follow the proof by Atiyah-Bott ([AB]) of the Lefschetz formula we do obtain a clear geometric meaning for the Weil distribution. One can of course as in ([G]) define inner products on function spaces on C_k using the Weil distribution, but as long as the latter is put by hands and does not appear naturally one has very little chance to understand why it should be positive. Now, let φ be a diffeomorphism of a smooth manifold Σ and assume that the graph of φ is transverse to the diagonal, one can then easily define and compute (cf. [AB]) the distribution theoretic trace of the permutation U of functions on Σ associated to φ,

$$(U\xi)(x) = \xi(\varphi(x)) \qquad \forall\, x \in \Sigma \,. \tag{16}$$

One has "Trace" $(U) = \int k(x,x)\,dx$, where $k(x,y)\,dy$ is the Schwarz kernel associated to U, i.e. the distribution on $\Sigma \times \Sigma$ such that,

$$(U\xi)(x) = \int k(x,y)\,\xi(y)\,dy \,. \tag{17}$$

Now near the diagonal and in local coordinates one has,

$$k(x,y) = \delta(y - \varphi(x)) \,, \tag{18}$$

where δ is the Dirac distribution. One then obtains,

$$\text{“Trace”}\ (U) = \sum_{\varphi(x)\,=\,x} \frac{1}{|1 - \varphi'(x)|} \,, \tag{19}$$

where φ' is the Jacobian of φ and $|\ |$ stands for the absolute value of the determinant.

With more work ([GS]) one obtains a similar formula for the distributional trace of the action of a flow,

$$(U_t\,\xi)(x) = \xi(F_t(x)) \qquad \forall\, x \in \Sigma \,,\ t \in \mathbb{R} \,. \tag{20}$$

It is given, under suitable transversality hypothesis, by

$$\text{“Trace”}\ (U(h)) = \sum_{\gamma} \int_{I_\gamma} \frac{h(u)}{|1 - (F_u)_*|}\ d^*u \,, \tag{21}$$

where $U(h) = \int h(t)\,U(t)\,dt$, h is a test function on $\mathbb{R}$, the γ labels the periodic orbits of the flow, including the fixed points, I_γ is the corresponding isotropy subgroup, and $(F_u)_*$ is the tangent map to F_u on the transverse space to the orbits, and finally d^*u is the unique Haar measure on I_γ which is of covolume 1 in $(\mathbb{R}, dt)$.

Now it is truly remarkable that when one analyzes the periodic orbits of the action of C_k on X one finds that (21) becomes,

$$\text{(22)} \qquad \text{"Trace"} \ (U(h)) = \sum_v \int_{k_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u \, .$$

Thus, the isotropy subgroups I_γ are parametrized by the places v of k and coincide with the natural cocompact inclusion $k_v^* \subset C_k$ which relates local to global in class field theory. The denominator $|1-u|$ is for the module of the local field k_v and the u^{-1} in $h(u^{-1})$ comes from the discrepancy between notations (8) and (16). It turns out that if one normalizes the Haar measure d^*u of modulated groups as in Weil ([W3]) , by,

$$\text{(23)} \qquad \int_{1 \leq |u| \leq \Lambda} d^*u \sim \log \Lambda \qquad \text{for } \Lambda \to \infty \, ,$$

one gets the same covolume 1 condition as in (21).

The transversality condition imposes the condition $h(1) = 0$. The distributional trace for the action of C_k on C_k by translations vanishes under the condition $h(1) = 0$.

Thus equating the alternate sum of traces on $\mathcal{H}_0$, $\mathcal{H}_1$ with the trace on the cohomology should thus provide the geometric understanding of the Riemann-Weil explicit formula (2) and in fact of RH using (13) if it could be justified for some value of δ.
The trace of permutation matrices is positive and this explains the Hadamard positivity,

$$\text{(24)} \qquad \text{"Trace"} \ (U(h)) \geq 0 \qquad \forall \, h \, , \ h(1) = 0 \, , \ h(u) \geq 0 \quad \forall \, u \in C$$

(not to be confused with Weil positivity).

To eliminate the artificial parameter δ and give rigorous meaning, as a Hilbert space trace, to the distributional "trace", one proceeds as in the Selberg trace formula and introduces a cutoff. In physics terminology the divergence of the trace is both infrared and ultraviolet as is seen in the simplest case of the action of K^* on $L^2(K)$ for a local field K. In this local case one lets,

$$\text{(25)} \qquad R_\Lambda = \widehat{P}_\Lambda \, P_\Lambda \, , \ \Lambda \in \mathbb{R}_+ \, ,$$

where P_Λ is the orthogonal projection on the subspace,

$$\{\xi \in L^2(K)\,;\ \xi(x) = 0 \qquad \forall\, x, |x| > \Lambda\}\,, \tag{26}$$

while $\widehat{P}_\Lambda = F\,P_\Lambda\,F^{-1}$, F the Fourier transform.

One proves ([Co]) in this local case the following analogue of the Selberg trace formula,

$$\mathrm{Trace}\,(R_\Lambda\,U(h)) = 2\,h(1)\log'(\Lambda) + \int' \frac{h(u^{-1})}{|1-u|}\,d^*u + o(1) \tag{27}$$

where $h \in \mathcal{S}(K^*)$ has compact support, $2\log'(\Lambda) = \int_{\lambda\,\in\,K^*,\,|\lambda|\,\in\,[\Lambda^{-1},\Lambda]} d^*\lambda$, and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on K which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform vanishes at 1.

As it turns out this principal value agrees with that of Weil for the choice of Fourier transform F associated to the standard character of K.

Let k be a global field and let first S be a finite set of places of k containing all the infinite places. To S corresponds the following localized version of the action of C_k on X. One replaces C_k by

$$C_S = \prod_{v \in S} k_v^*/O_S^*\,, \tag{28}$$

where $O_S^* \subset k^*$ is the group of S-units. One replaces X by

$$X_S = \prod_{v \in S} k_v/O_S^*\,. \tag{29}$$

The Hilbert space $L^2(X_S)$, its Fourier transform F and the orthogonal projections P_Λ, $\widehat{P}_\Lambda = F\,P_\Lambda\,F^{-1}$ continue to make sense, with

$$\mathrm{Im}\,P_\Lambda = \{\xi \in L^2(X_S)\,;\ \xi(x) = 0 \quad \forall\, x\,,\ |x| > \Lambda\}\,. \tag{30}$$

As soon as S contains more than 3 elements, (e.g. $\{2,3,\infty\}$ for $k = \mathbb{Q}$) the space X_S is an extremely delicate quotient space. It is thus quite remarkable that the *trace formula* holds,

Theorem. ([Co]) *For any $h \in \mathcal{S}_c(C_S)$ one has, with $R_\Lambda = \widehat{P}_\Lambda\,P_\Lambda$,*

$$\mathrm{Trace}\,(R_\Lambda\,U(h)) = 2\log'(\Lambda)\,h(1) + \sum_{v \in S} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\,d^*u + o(1)$$

where the notations are as above and the finite values $\int'$ depend on the additive character of Πk_v defining the Fourier transform F. When $\mathrm{Char}\,(k) \neq 0$ the projectors P_Λ, $\widehat{P}_\Lambda$ commute on L^2_χ for Λ large enough so that one can replace R_Λ by the orthogonal projection Q_Λ on $\mathrm{Im}\,P_\Lambda \cap \mathrm{Im}\,\widehat{P}_\Lambda$. This follows from an explicit formula for the product $R_\Lambda = \widehat{P}_\Lambda\, P_\Lambda$ in terms of the quantized calculus on the group C_S and the Kronecker theorem characterizing rational fractions from the finite rank of their quantized differential ([C]).

The situation for $\mathrm{Char}\,(k) = 0$ is more delicate since P_Λ and $\widehat{P}_\Lambda$ do not commute (for Λ large) even in the local Archimedian case. But fortunately these operators commute ([LPS]) with a specific second order differential operator, whose eigenfunctions, the Prolate Spheroidal Wave functions provide the right filtration Q_Λ. This allows to replace R_Λ by Q_Λ and to state the global trace formula

$$\mathrm{Trace}\,(Q_\Lambda\, U(h)) = 2\log'(\Lambda)\, h(1) + \sum_v \int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)\,.$$

Our final result is that the validity of this trace formula implies (in fact is equivalent to) the positivity of the Weil distribution, i.e. RH for all L-functions with Grössencharakter. The global trace formula remains unproved in general but its validity for certain test functions such as those supported by the group of ideles of unit modulus in the function field case, is easy to check. Moreover the filtration by Q_Λ allows to define the Adelic cohomology and to complete the dictionary between the function theory and the geometry of the Adele class space X.

Function Theory	*Geometry*
Zeros and poles of Zeta	Action of C_k on Adelic cohomology
Functional Equation	$*$ operation
Explicit formulas	Lefschetz formula for action of C_k on X
Riemann Hypothesis?	Trace formula?

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