AN INFINITE DIMENSIONAL POINT OF VIEW ON TEICHMÜLLER THEORY (THE MOMENT MAP, BERS IMBEDDING AND WEIL–PETERSSON METRIC)

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AN INFINITE DIMENSIONAL POINT OF VIEW ON TEICHMÜLLER THEORY
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#0. "BALKAN POINT OF VIEW" ON TEICHMÜLLER THEORY (INTRODUCTION.)

The main object of this article is

$$\mathrm{Diff}^{+}(S^1)/\mathbb{P}SU_{1,1}:=\mathfrak{X}$$

This can be interpreted as all possible "C^{∞}" complex structures on the **unit disc.** It is very natural to expect that the *TEICHMÜLLER SPACES* $\mathcal{T}_g$ for compact Riemann surfaces of genus $g\geq 2$ can be imbedded in $\hat{\mathfrak{X}}$, where $\hat{\mathfrak{X}}$ is the space of quasi-symmetric homeomorphisms of the circle. The definition of a quasi-symmetric homeomorphism of $\mathbb{R}^1$ is the following one:

DEFINITION.

An increasing homoemorphism $h:\bar{\mathbb{R}}\to\bar{\mathbb{R}}$ with $h(\infty)=\infty$ is said to be *k-quasisymmetric* if

$$\frac{1}{k}\leq\frac{h(x+t)-h(x)}{h(x)-h(x-t)}\leq k$$

for all $x\in\bar{\mathbb{R}}$ and all $t>0$. A function is quasisymmetric if it is k-quasisymmetric for some k.

We recall the definition of the *TEICHMÜLLER SPACES* $\mathcal{T}_g$ for compact Riemann surfaces.

DEFINITION.

Let Γ be a compact *Riemann* surface of genus $g\geq 2$ and let

$$I(\Gamma):=\left\{\text{all possible complex structures on } \Gamma\right\}$$

Then the *TEICHMÜLLER SPACE of* $\mathcal{T}_g(\Gamma)$ is defined in the following way:

$$\mathcal{T}_g(\Gamma):=I(\Gamma)/\mathrm{Diff}_0(\Gamma)$$

where

$$\mathrm{Diff}_0(\Gamma):=\{\text{all diffeomorphisms of } \Gamma \text{ isotopic to the identity}\}$$

and $\mathrm{Diff}_0(\Gamma)$ acts on $\mathrm{I}(\Gamma)$ by pulling back the complex structures on Γ.

This imbedding is given in the following way: each element of $g\in\mathfrak{T}_g(\Gamma)$ can be represented by a quasiconformal map $\tilde{g}$ of the unit disk in $\mathbb{C}$. By a THEOREM of TEICHMÜLLER there exixsts a unique extremal quasiconformal map $\phi(g)$ in the class of equivelences introduced by TEICHMÜLLER. Let me recall this class of equivelences; we call two tripples (Γ',ϕ,Γ) and (Γ'',ψ,Γ) equivalent if $(\phi^{-1})\cdot\psi:\Gamma'\to\Gamma''$ is homotopic to a conformal map, where $\phi:\Gamma'\to\Gamma$ and $\psi:\Gamma''\to\Gamma$ are quasiconformal maps. This unique "extremal" quasiconformal map $h(\phi)$ can be prolonged to a map $\overline{\phi(g)}:\bar{D}\to\bar{D}$, where $\bar{D}$ is the closure of D. The restriction of $\overline{\phi(g)}$ on S^1 is a qusisymmetric homeomorphism of S^1. This is the imbedding of $\mathfrak{T}_g(\Gamma)$ in $\hat{\mathfrak{T}}$.

The main THEOREMS in ***TEICHMÜLLER THEORY***, basically due to AHLFORS and BERS, are that ***TEICHMÜLLER SPACE of*** $\mathcal{T}_g(\Gamma)$ for $g\geq 2$ is a domain in $\mathbb{R}^{6g-6}$ and moreover is a STEIN manifold. Later different proofs of these two THEOREMS, using the WEIL-PETERSSON metric were given by A. Fischer and A. Tromba, A. Tromba, J. Jost, S. Wolpert and M. Wolf.

One should also mention the deep work of ROYDEN. He proved that the ***TEICHMÜLLER METRIC*** is exactly the ***KOBAYASHI METRIC.*** From here he derived that the group of the automorphisms of the ***TEICHMÜLLER SPACE of*** $\mathcal{T}_g(\Gamma)$ for $g\geq 2$ is exactly the mapping class group, i.e. $\mathrm{Diff}^+(\Gamma)/\mathrm{Diff}_0(\Gamma)$.

Excellent books on ***TEICHMÜLLER THEORY*** are [01], [10], [12], [17] and [19].

DESCRIPTION OF THE RESULTS OF THIS ARTICLE.

In #1 using KIRILLOV'S classification of coadjoint orbits of BOTT-VIRASORO group, we see that $Diff(S^1)/\mathbb{P}SU_{1,1}$ is isomorphic to one of these orbits. Namely we remind the following THEOREM:

THEOREM 1.1. (KIRILLOV [16]).
The corresponding central extension of $\mathbb{P}SU_{1,1}$ in Vir is the coadjoint isotropy of

$$p_0(2(dt)^{\otimes 2},-1)\in Vir^* \text{ for any } p_0\in\mathbb{R}$$

Thus

$$Diff(S^1)/\mathbb{P}SU_{1,1}$$

is diffeomorphic to the coadjoint orbit of Vir in vir^* through the point

$$(2(dt)^{\otimes 2},-1).$$

Let us remind that Vir is a central extension of $Diff^+(S^1)$ via the so called BOTT-VIRASSORO cocycle with $\mathbb{R}/\mathbb{Z}$. Vir^* is the dual of the Lie algebra Vir of the VIRASSORO LIE GROUP. This Lie algebra is just the central extension of the LIE ALGEBRA $Vect(S^1)$ of $Diff^+(S^1)$ via the GELFAND-FUCHS cocycle: Namely

Gelfand, I. and Fuks, D. in [11] have shown that

$$c\left(X\frac{\partial}{\partial t},Y\frac{\partial}{\partial t}\right)=\frac{1}{2\pi}\int_0^{2\pi} X'(t)Y''(t)dt$$

is a two cycle which is uniquely defined up to a constant multiple up to the addition of a coboundary, i.e. $H^2(Vect(S^1),\mathbb{R})$ is one dimensional. Therefore there is a unique central extension of $Vect(S^1)$ by $\mathbb{R}$ which we shall denote by

$$Vir=Vect(S^1)\oplus\mathbb{R}$$

and whose bracket operation is given by:

$$\left[(X\frac{\partial}{\partial t},a),(Y\frac{\partial}{\partial t},b)\right]=\left(-\left[(X\frac{\partial}{\partial t},a),(Y\frac{\partial}{\partial t},b)\right],c\left(X\frac{\partial}{\partial t},Y\frac{\partial}{\partial t}\right)\right)$$

$$=\left\{-(XY'-X'Y)\frac{\partial}{\partial t},\frac{1}{2\pi}\int_0^{2\pi} X'(t)Y''(t)dt\right\}$$

for $a,b\in\mathbb{R}$, $X\frac{\partial}{\partial t}, Y\frac{\partial}{\partial t}\in \mathrm{Vect}(S^1)$.

In #2 we remind how one can construct a map from all elements $\phi\in \mathrm{QSDiff}^+(S^1)$ to a pair of unique univalent functions (f_ϕ, g_ϕ) respectively inside the unit disk and outside the unit disk in $\mathbb{C}$ such that:

a) $g_\phi(\infty)=\infty$ and $g'_\infty(\infty)=1$

b) $g_\phi \cdot f_\phi^{-1}|_{S^1}=\phi$

This map is due to SULLIVAN and KIRILLOV. Using this map we define the BERS IMBEDDING; $\phi\to S(f_\phi)$, where

$$S(f_\phi)=\frac{f'_\phi f'''_\phi-\frac{3}{2}(f''_\phi)^2}{(f'_\phi)^2}$$

From this we define the topology on $\hat{\mathfrak{T}}$, namely this is the induced topology on $\hat{\mathfrak{T}}$ from the L^∞ norm on the quadratic differentials, namely

$$\|S(f_\phi)\|=\sup_{z\in D}|S(f_\phi)(z)|(1-|z|^2)^2$$

In #3 we remind how KIRILLOV constructed an integrable complex structure on $\mathrm{Diff}^+(S^1)/\mathbb{P}SU_{1,1}$. We call this complex strucrture **the KIRILLOV complex structure.** Combining this complex structure with **KIRILLOV-KONSTANT-SAURIAU form,** we get a unique left invariant KÄHLER METRIC on $\mathrm{Diff}(S^1)/\mathbb{P}SU_{1,1}$. This metric we call THE WEIL-PETERSSON METRIC. We have the following formulas for the WEIL-PETERSSON metric:

THEOREM 3.

A) Let v and w be two left invariant vector fields on $\mathrm{Diff}^+(S^1)/\mathbb{P}SU_{1,1}$ of type (1,0), $v=\sum_{n=2}^{\infty}v_nL_n$ and $w=\sum_{n=2}^{\infty}w_nL_n$ where $L_n=\exp(\pi n i)\frac{\partial}{\partial t}$,

THEN the WEIL-PETERSSON metric is given by

$$<v,w>=\frac{c}{12}\sum_{n=2}^{\infty}v_n\overline{w_n}\,(n^3-n),\ \text{where } c>0$$

B) Let $H_1(S^1):=\{$all vector fields v on S^1 of type (1,0)| v is at least $C^{2+\epsilon}\}$, then $H_1(S^1)$ is a complete HILBERT SPACE with respect to the metric defined in *A)*.

This metric was studied by NAG and VERJOVSKY in [20]. They proved that it is defined just on vector fields on S^1 which are $C^{2+\epsilon}$. Since the imbedding of the TEICHMÜLLER SPACE for genus g>1 gives only quasisymmetric homeomorphisms, the WEIL-PETTERSSON metric defined as above cannot be defined on the TEICHMÜLLER SPACE for genus g>1. NAG and VERJOVSKY found a beautifull way to regularized the above defined metric on $\mathfrak{T}_g$. So we will explain how they do this.

Review of the results of NAG and VERJOVSKI (See [20].)

First we will recall that the tangent space of $\hat{\mathfrak{T}}$ can be identified with the space of all BELTRAMI differentials with finite L^∞ norm. Let $L^\infty(D)$ be this space, i.e.the space of all BELTRAMI differentials with finite L^∞ norm on D. Let $\upsilon\in L^\infty(D)$ Then the corresponding "quasisymmetric" vector field on S^1 is given by

$$v(t)\frac{\partial}{\partial t}=\frac{\dot{w}[\upsilon](e^{it})}{ie^{it}}\frac{\partial}{\partial t}$$

where $w[\upsilon]$ is the solution of the BELTRAMI equation

$$w_{\bar{z}}=\upsilon w_z$$

and $\dot{w}[\upsilon]$ is the first variational term in the solution of the BELTRAMI equation, i.e.

$$w_{t\upsilon}(z)=z+t\dot{w}[\upsilon](z)+o(t),\ t\to 0$$

THEOREM 1. (NAG and VERJOVSKI) (See [20].)

Let μ and $\upsilon\in L^\infty(D)$ represent two tangent vectors at the origin of $\hat{\mathfrak{T}}$, then the WEIL-PETERSSON metric is given formally as:

$$g(\mu,\upsilon)=-\frac{ia}{3\pi^2}\int\!\!\int_D x\int\!\!\int_D\frac{\mu(z)\overline{\upsilon(\varsigma)}}{(1-z\bar{\varsigma})^4}(d\varsigma\wedge d\bar{\varsigma})\cdot(dz\wedge d\bar{z})-$$

$$-\frac{ia}{3\pi^2}\int_D(1-|z|^2)^2\mu(z)\overline{\upsilon(\varsigma)}dz\wedge d\bar{z}$$

THEOREM 2. (NAG and VERJOVSKI) (See [20].)

Let Γ=D/G be a compact RIEMANN surface of genus $g>1$. Let $\nu_0 \in L^{\infty}(G)/N(G)$ be any tangent vector with unit length with respect to the WEIL-PETERSSON metric on Γ, then we have for any two vectors μ and υ as in THEOREM 1 the WEIL-PETERSSON metric on Γ is given by

$$g(\mu,\upsilon)=\lim_{r\to 1^-}\frac{g_r(\mu,\upsilon)}{g_r(\nu_0,\upsilon_0)}$$

where

$$g_r(\mu,\upsilon)=-\frac{ia}{3\pi^2}\iint_{D_r} x\int_{D}\frac{\mu(z)\overline{\upsilon(\varsigma)}}{(1-z\bar{\varsigma})^4}(d\varsigma\wedge\overline{d\varsigma})\cdot(dz\wedge\overline{dz})$$

and $D_r=\{t\in\mathbb{C}\,|\,|t|<r\}$.

In #4 we prove the following THEOREM, using the idea of NAG and VERJOVSKI how to regularized the WEIL-PETERSSON metric on $\hat{\mathfrak{T}}$.

THEOREM 4.

A) Suppose that $\mu(z)$ is a holomorphic function in D and

$$\frac{1}{\pi i}\int_{D}((1-|z|^2)^4|\mu|^2)(dz\wedge\overline{dz})<\infty$$

Let

$$g_r(\mu,\mu)=\frac{1}{\pi i}\iint_{D_r}((1-|z|^2)^4|\mu|^2)\frac{(dz\wedge\overline{dz})}{(1-|z|^2)^2}=-\frac{ia}{3\pi^2}\iint_{D_r} x\int_{D}\frac{\mu(z)\overline{\mu(\varsigma)}}{(1-z\bar{\varsigma})^4}(d\varsigma\wedge\overline{d\varsigma})\cdot(dz\wedge\overline{dz})$$

Suppose that $\lim_{r\to 1^-} g_r(\mu,\mu)=\infty$

If we define

(*) $$g(\mu,\nu)=\lim_{r\to 1^-}\frac{g_r(\mu,\nu)}{\mathfrak{P}_r}$$

where

$$\mathcal{P}_r = \frac{1}{\pi i}\iint_{D_r} \frac{dz\wedge d\bar{z}}{(1-|z|^2)^2} \text{ and } D_r := \{t\in\mathbb{C} \mid |t|<r\}$$

THEN

$g(\mu,\mu)$ exists and

$$g(\mu,\mu)>0$$

B) Let $H_2(S^1)$ be the space of all all vector fields on S^1 such that they fulfill the conditions 1), 2) and 3) stated bellow

1) $$v\frac{\partial}{\partial t}=\sum_{n=2}^{\infty} c_n e^{int}\frac{\partial}{\partial t}$$

in other words $v\frac{\partial}{\partial t}$ is of type (1,0) with respect to the KIRILLOV'S COMPLEX STRUCTURE on $\hat{\mathfrak{T}}$

2) v is at most C^2

3) If we define $f_v(z):=\sum_{n=2}^{\infty} c_n z^n$ in D then we require that $g(f_v,f_v)>0$

THEN $H_2(S^1)$ is a complete HILBERT SPACE with respect to the metric $g(v,v)$ as defined in *A)*.

C) The restriction of $g(\mu,\mu)$ defined on $H_2(S^1)$ as in *A)* on the imbedded TEICHMÜLLER SPACE $\mathfrak{T}_g$ is just the WEIL-PETERSSON METRIC.

DEFINITION.

The metric $g(\mu,\nu)$ defined by THEOREM 4 we will call the WEIL-PETERSSON METRIC on quasisymmetric vector field that are at most C^2.

This is another way to prove that the above defined CANONICAL METRIC is not only defined on diffeomorphisms of S^1 of $C^{2+\epsilon}$ but on all $QSDiff^+(S^1)$, i.e. on all quasisymmetric homeomorphisms of S^1.

REMARK.

The tangent space of all "qusisymmetric" vector fields on S^1 we can split into an orthogonal sum of two HILBERT SPACES, namely $H_1(S^1)\dot{+}H_2(S^1)$, where $H_1(S^1)$ consists of all vector fields on S^1 which are at least $C^{2+\epsilon}$ and the metric on $H_1(S^1)$ is defined as in THEOREM 3, while the metric on $H_2(S^1)$ is defined as in THEOREM 4, i.e. $g(\mu,\mu)$ for $\mu \in H_2(S^1)$.

In #5 we show how to find a potential of this metric on $\hat{\mathfrak{T}}$. This potential is defined by the following procedure: We can interpret $Diff(S^1)/\mathbb{P}SU_{1,1}$ as all possible complex structures on $D=\{t \mid t\in\mathbb{C}$ and $|t|^2<1\}$. So each point t of $Diff(S^1)/\mathbb{P}SU_{1,1}$ defines an elliptic operator $\bar{\partial}_t$ which acts on the space of all quadratic differentials on S^1. Then using QUILLEN'S construction we get the so called determinant line bundle $det(\bar{\partial})$. This line bundle has a natural section $det(\bar{\partial}_t)$. Next we define the so called QUILLEN'S metric on $det(\bar{\partial})$ in the following way:

$$\|det(\bar{\partial}_\phi)\|^2 := exp(-\zeta'_{\Delta_\phi}(0))$$

where $\Delta_\phi := \bar{\partial}^*_\phi \bar{\partial}_\phi$, $\bar{\partial}^*_\phi$ is the conjugate of $\bar{\partial}_\phi$ with respect to the metric g(t) and

$$\zeta'_{\Delta_\phi}(s) = \sum_i \lambda_i^{-s}$$

where λ_i are the eigen values of Δ_ϕ.

"Almost" copying the proof of a THEOREM of QUILLEN in [18] we prove that:

THEOREM 5.

A) $\partial\bar{\partial}\log(exp(-\zeta'_{\Delta^+_\mu}(0))) = \partial\bar{\partial}\log\|det\bar{\partial}_t\|_Q$ is just the WEIL-PETRSSON METRIC on $\hat{\mathfrak{T}}$.

This turns out to be part of ***the more general principle,*** which was observed also by FUJIKI and SCHUMACHER. Namely, let $\mathfrak{X} \to \mathfrak{U}$ be the KURANISHI family of KÄHLER manifolds with a fixed class of polarization $L \in H^{1,1}(X_0,\mathbb{R})$. Suppose that for each $t\in\mathfrak{U}$ we can find a KÄHLER metric g_t depending C^∞ on t and such that the cohomology class $[Im g_t]=L$. Then we can define the WEIL-PETERSSON metric in the following way: We know from KURANISHI theory that the ZARISKI tangent space $T_t\mathfrak{U} \approx \mathbb{H}^1(X_t,\Theta_t)$, where $\mathbb{H}^1(X_t,\Theta_t)$ is the space of harmonic (0,1) forms with coefficients in the tangent space. Let ϕ and $\sigma \in \mathbb{H}^1(X_t,\Theta_t)$; then

$$\phi|_U = \sum \phi^\alpha_{\bar{\mu}} d\bar{z}^\mu \otimes \frac{\partial}{\partial z^\alpha} \quad \sigma|_U = \sum \sigma^\nu_{\bar{\beta}} d\bar{z}^\beta \otimes \frac{\partial}{\partial z^\nu}$$

Then we define the WEIL-PETERSSON metric in the following way:

$$<\phi,\sigma> = \int_{X_t} \phi^\nu_{\bar{\beta}} \, \overline{\sigma^{\bar{\mu}}_{\bar{\alpha}}} \, g_{\nu\bar{\mu}} g^{\alpha\bar{\beta}} vol(g(t))$$

We can interpret $\mathfrak{U}$ as the space that parametrizes the operators $\bar{\partial}_t$. Let $\bar{\partial}_t^*\bar{\partial}_t=\nabla_t$, where $\bar{\partial}_t^*$ is the conjugate of $\bar{\partial}_t$ with respect to the metric g(t), then the following THEOREM is true:

THEOREM 5.1.

$\partial\bar{\partial}\log(\exp(-\zeta'_{\nabla_t}(0)))= \partial\bar{\partial}\log\|\det\bar{\partial}_t\|_Q$ is just the WEIL-PETRSSON METRIC on U.

The WEIL-PETERSSON metric on $\mathrm{Diff}^+(S^1)/\mathbb{P}SU_{1,1}$ was studied also by some phisists in [5], [6], [7] and [8].

We should mention also that we define a determinant holomorphic function on $\hat{\mathfrak{T}}$. It is closely connected with τ function defined by G. WILSON and G. SIEGAL in their beautiful paper [21].

From the above THEOREM it is not difficult to show that the WEIL-PETERSSON metric has a negative curvature operator. Even more there are strong indications that the global potensial of the WEIL-PETERSSON metric is the RUELLE'S zeta functions at $s=\frac{1}{2}$ for each $\phi\in\hat{\mathfrak{T}}$. The curvature computations will be considered in a future paper.

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CONTENT.

#1. COADJOINT ORBITS OF THE BOTT-VIRASORO GROUP (SEE [16] and [22]).

Let $\mathcal{G}$ be a Lie and $\mathcal{G}^*$be its dual. $\mathcal{G}^*$ is a POISSON manifold relative to the LIE-POISSON structure given by:

$$\{f,h\}(\mu)=-<\mu,\left[\frac{\delta f}{\delta\mu},\frac{\delta h}{\delta\mu}\right]>$$

where $f,h:\mathcal{G}^*\to\mathbb{R}$, $\mu\in\mathcal{G}^*$, $<\ ,\ >:\mathcal{G}\times\mathcal{G}^*\to\mathbb{R}$ is a weakly non-degenerate pairing and

$$\frac{\delta f}{\delta\mu} \text{ and } \frac{\delta h}{\delta\mu} \in\mathcal{G}$$

are the functional derivatives of f and g respectively, computed at μ via the formula:

$$\mathbb{D}f(\mu)\nu:=<\nu,\frac{\delta f}{\delta\mu}>$$

for any $\nu\in\mathcal{G}^*$; here $\mathbb{D}$ denotes the usual *Frechet* derivative of functions on $\mathcal{G}^*$. The symplectic leaves of this linear *POISSON* manifold are the coadjoint orbits of the underlying *LIE* group G of $\mathcal{G}$. They are endowed therefore with a **symplectic** structure ω_O called the ***orbit symplectic structure*** or the ***KIRILLOV-KONSTANT-SOURIAU*** symplectic structure:

$$\omega_O(\mu)((ad\xi)\mu^*,(ad\eta)\mu^*)=-<\mu,[\xi,\eta]>$$

where $\mu\in O\subset\mathcal{G}^*$, O is a coadjoint orbit, $\xi,\eta\in\mathcal{G}$, and $(ad\xi)\mu^*$, $(ad\eta)\mu^*\in T_\mu O$ are arbitrary tangent vectors to the orbit O at μ.

In this paper we shall be concerned with a very particular example of a coadjoint orbit, defined by a central extension. Let us denote by $G:=\mathrm{Diff}^+(S^1)$ the oriented preserving diffeomorphisms of the circle relative to the length form $dt\in\Omega^1(S^1)$. The Lie algebra $\mathrm{Diff}^+(S^1)$ consists of $\mathrm{Vect}(S^1)$, the space of vector fields on S^1, endowed with minus the usual Lie bracket on vector fields. Relative to the L^2- pairing, the "dual" of $\mathrm{Vect}(S^1)$ consists of quadratic differentials

$$p(t)(dt)^{\otimes 2}$$

$$<p(t)(dt)^{\otimes 2}, X(t)\frac{\partial}{\partial t}> = \frac{1}{2\pi}\int_0^{2\pi} p(t)X(t)dt$$

The geometric interpretation of quadratic differentials as dual to $Vect(S^1)$ is the following: They are one-form densities on S^1. We denote, as usual, by $Vect(S^1)^*$ the dual of $Vect(S^1)$ relative to < , >.

Gelfand, I. and Fuks, D. in [11] have shown that

$$c\left(X\frac{\partial}{\partial t}, Y\frac{\partial}{\partial t}\right) = \frac{1}{2\pi}\int_0^{2\pi} X'(t)Y''(t)dt$$

is a two cycle which is uniquely defined up to a constant multiple up to the addition of a coboundary, i.e. $H^2(Vect(S^1),\mathbb{R})$ is one dimensional. Therefore there is a unique central extension of $Vect(S^1)$ by $\mathbb{R}$ which we shall denote by

$$vir = Vect(S^1)\oplus\mathbb{R}$$

and whose bracket operation is given by:

$$\left[(X\frac{\partial}{\partial t},a),(Y\frac{\partial}{\partial t},b)\right] = \left(-\left[(X\frac{\partial}{\partial t},a),(Y\frac{\partial}{\partial t},b)\right], c\left(X\frac{\partial}{\partial t}, Y\frac{\partial}{\partial t}\right)\right)$$

$$= \left(-(XY'-X'Y)\frac{\partial}{\partial t}, \frac{1}{2\pi}\int_0^{2\pi} X'(t)Y''(t)dt\right)$$

for $a,b\in\mathbb{R}$, $X\frac{\partial}{\partial t}, Y\frac{\partial}{\partial t}\in Vect(S^1)$.

Correspondingly, there exixts a central extension Vir of $Diff(S^1)$ by S^1 whose defining two-cocycle B was determined by Bott [4]:

$$B(\eta,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \left[\ln(\eta\cdot\phi)'\right]d(\ln\phi')\Big)$$

Again, B is uniquely determined up to a constant and the addition of a coboundary. The group operation in Vir is given by

$$(\eta,t)(\phi,s) = (\eta\cdot\phi, t+s+B(\eta,\phi)(\text{mod}\,2\pi))$$

Vir is called the ***BOTT-VIRASORO group***. The coadjoint actions of Lie group and Lie algebra are given by

$$Ad^*_{(\eta,\upsilon)^{-1}}(p(dt)^{\otimes 2},\tau)=((p\cdot\eta^{-1})(dt)^{\otimes 2}-\tau S(\eta^{-1}),\tau)$$

(*)
$$ad^*_{(X\frac{\partial}{\partial t},a)^{-1}}(p(dt)^{\otimes 2},\tau)=(\tau\frac{\partial^3 X}{\partial t^3}-(p\cdot\frac{\partial}{\partial t}+\frac{\partial}{\partial t}\cdot p)X,0)$$

where

$$S(\phi)=\frac{\phi'\phi''-\frac{3}{2}(\phi'')^2}{(\phi')^2}$$

is the ***Schwartzian derivative*** of ϕ. Using these formulas it is easy to see that $\mathbb{P}SU_{1,1}$ is not the isotropy group in $Diff(S^1)$ of any element in $Vect(S^1)^*$. However, we have the following:

THEOREM 1.1. (KIRILLOV [16]).

The corresponding central extension of $\mathbb{P}SU_{1,1}$ in Vir is the coadjoint isotropy of

$$p_o(2(dt)^{\otimes 2},-1)\in Vir^* \text{ for any } p_o\in\mathbb{R}$$

Thus

$$Diff(S^1)/\mathbb{P}SU_{1,1}$$

is diffeomorphic to the coadjoint orbit of Vir in vir^* through the point

$$(2(dt)^{\otimes 2},-1).$$

Let us close this section with a few words about the manifold structure of $Diff^+(S^1)/\mathbb{P}SU_{1,1}$. Endow $Diff^+(S^1)$ with H^s-topology for $s>\frac{3}{2}$ so that all elements of $Diff^+(S^1)$ are at least C^1 according to SOBOLEV'S IMBEDDING THEOREM. Then, it can be shown that $Diff^+(S^1)/\mathbb{P}SU_{1,1}$ carries the corresponding quotient structure. To understand this better, it is convinient to proceed in a slightly different fashion. Denote by $Rot(S^1)$ the orintation preserving of S^1. Then, as shown by KIRILLOV, $Diff^+(S^1)/Rot(S^1)$ is a contractible space. Since $Diff^+(S^1)/\mathbb{P}SU_{1,1}$ fibres over $Diff^+(S^1)/Rot(S^1)$ with fibre the POINCARE disk D, it follows that the manifold structure of $Diff^+(S^1)/\mathbb{P}SU_{1,1}$ is that of a product of $Diff^+(S^1)/Rot(S^1)$ with D.

THEOREM 1.2.

$\mathrm{Diff}^+(S^1)/\mathrm{Rot}(S^1) \to \mathrm{Diff}^+(S^1)/\mathbb{P}SU_{1,1}$ is a fibre bundle of H^s-manifolds for $s > \frac{3}{2}$, whose fibre is the POINCARE disk.

REMARK.

The expression occuring in (∗) is the second POISSON structure for the KdV equation, or equivalently, the operator appearing in the squared eigenfunction relation of the HILL operator.

#2. THE CHOICE OF TOPOLOGY OF $\hat{\mathfrak{T}}$ AND BERS IMBEDDING.

DEFINITION 5.1.

Let us denote by $QSDiff^{+}(S^1)$ the group of quasisymmetric homoemorphisms of the circle S^1.

LEMMA 2.2.(KIRILLOV (See [15]).

Let $\phi \in QSDiff^{+}(S^1)$, i.e. ϕ is a quasisymmetric homoemorphism of S^1, then ϕ defines a pair of univalent functions (f,g), where f is defined in in $D^{+}:=\{t\in\mathbb{C} \mid |t|\leq 1\}$, g is defined in $D^{-}:=\{t\in\mathbb{C} \mid |t|\geq 1\}$ and (f,g) have the following properties:

A) $g(\infty)=\infty$ and $g'(\infty)=1$ and so g is uniquely defined

B) $g^{-1}\cdot f_{|S^1}=\phi$

C) f is uniquely defined

Proof: Let

$$D^{+}\cup_{\phi}D^{-}=\mathbb{CP}^{1}_{\phi}$$

be the glueing of the two disc via ϕ along the boundary S^1. So we get that

$$D^{+}\cup_{\phi}D^{-}=\mathbb{CP}^{1}_{\phi}$$

is the S^2, i.e. the two-dimensional sphere. Since ϕ is a quasisymmetric homeomorphism of S^1 it follows from a THEOREM of AHLFORS and BEURING that ϕ can be prolonged to a quasiconformal map of D. (See [1].) So from here and according to a THEOREM of BERS and AHLFORS, there exists a unique complex structure on $D^{+}\cup_{\phi}D^{-}=\mathbb{CP}^{1}_{\phi}$ which coincides with standard ones on both D^{+} and D^{-}. (See [1].) Since all compex structures on S^2 are equivalent, there exists a holomorphic map

$$F:\mathbb{CP}^{1}_{\phi}\to\mathbb{CP}^{1}$$

such that $F(\infty)=\infty$, $F'(\infty)=1$. Recalling the **definition** of $\mathbb{CP}^{1}_{\phi}$ we see that the *definition* of F is equivelent to the *definition* of a pair of functions (f,g) defined on D^+ and D^- respectively. From *THE RIEMANN MAPPING THEOREM* we get that g is uniquely defined. So *A). B)* and *C)* follow automatically.

Q.E.D.

COR. 2.2.1.

$f(\phi)$ can be prolonged continuously to a quasiconformal function $\tilde{f}(\phi)$ outside D, i.e $\tilde{f}(\phi)$ is defined in D^-.

Proof: From the THEOREM of AHLFORS and BEURLING ϕ can be prolonged to a quasiconformal map $\tilde{\phi}:D^- \to D^-$. (See [1].) Clearly, $f(\tilde{\phi})=g(\tilde{\phi}(z))$ is a quasiconformal map with the requiered properties.

Q.E.D.

REMARK 2.3.

Up to now we have constructed an injective map, namely to each

$$\phi \in QSDiff(S^1) \to (f,g)$$

where (f,g) are defined by LEMMA 2.1., and f is an univalent function in the closed disk $\bar{D}$ uniquely defined by $\phi \in QSDiff^+(S^1)$. Next to f we assign its *Schwarcian derivative,* i.e we get a map

(2.3.1.) $\mathfrak{B}: \phi \to f(\phi) \to S(f(\phi))$

where

$$S(f)=\frac{f'f'''-\frac{3}{2}(f'')^2}{(f')^2}$$

The main properties of the *Schwarcian derivative* S(f) are:

(2.3.2.) $$S(f\cdot g))=\{S(f)\cdot g\}\left(\frac{d^2g}{dz^2}\right)^2+S(g)$$

(2.3.3.) $S(h)=0$ iff h is a Möbius transformation

So from (2.3.2.) and (2.3.3.) it follows that we can interpret $\mathfrak{B}(QSDiff^+(S^1))=\hat{\mathfrak{T}}:=QSDiff^+(S^1)/PSU_{1,1}$ and for $\phi \in \hat{\mathfrak{T}}$ and $S(f(\phi))$, defined as in LEMMA 2.2., as a quadratic differential on D. This is BERS IMBEDDING.

LEMMA 2.4. (BERS IMBEDDING.) (See [3].)

The map (2.3.1.) gives an imbedding of $QSDiff^{+}(S^1)/\mathbb{P}SU_{1,1}$ into the space of univalent holomorphic functions in D such that the NEHARI norm

$$\|S(f(\phi))\|_D := \sup_{z\in D} |S(f(\phi)(z))|(1-|z|^2)^2 < 2.$$

PROOF: In [1] it is proved that if f is a univalent function in D and

$$\|S(f)\|_D := \sup_{z\in D} |S(f(z))|(1-|z|^2)^2 < 2.$$

then f can be continued to a quasiconformal map outside D. So from this result and (2.2.3.) and (2.2.4.), it follows that $\hat{\mathfrak{T}} \subset \mathcal{A}(2)$, where $\mathcal{A}(2)$ is the ball in the Banach space $\mathfrak{B}^2$ of radius 2, where $\mathfrak{B}^2$ is the space of all holomorphic functions in D with finite NEHARI norm

$$\|f\|_D := \sup_{z\in D} |f(z)|(1-|z|^2)^2 < \infty$$

Even more it was proved that $\hat{\mathfrak{T}}$ is exactly the interior of $\mathfrak{B}^2$. LEMMA 2.4. follows from here and LEMMA 2.2.

Q.E.D.

REMARK .

It is easy to see that $\hat{\mathfrak{T}}$ contains $F^{-1}(D_r)$ for every $r<2$, where F maps the space of holomorphic functions f such that

$$\|f\|^2 = \frac{1}{2\pi i}\int_D (1-|z|^2)^4 f\ \bar{f}(dz\wedge d\bar{z}) < \infty$$

to $\mathbb{C}$ and it is defined in the following manner: $F(f)=f(0)$, $D_r := \{t\in\mathbb{C}|\ |t|<r\}$.

2.5. DEFINITION.

We will give a definition of left invariant vector fields on $\hat{\mathfrak{T}}$. First we will recall that the tangent space of $\hat{\mathfrak{T}}$ can be identified with the space of all BELTRAMI differentials with finite L^{∞} norm. Let $L^{\infty}(D)$ be this space, i.e.the space of all BELTRAMI differentials with finite L^{∞} norm on D. Let $\upsilon\in L^{\infty}(D)$ Then the corresponding "quasisymmetric" vector field on S^1 is given by

$$v(t)\frac{\partial}{\partial t}=\frac{\dot{w}[\upsilon](e^{it})}{ie^{it}}\frac{\partial}{\partial t}$$

where $w[\upsilon]$ is the solution of the BELTRAMI equation

$$w_{\bar{z}}=\upsilon w_z$$

and $\dot{w}[\upsilon]$ is the first variational term in the solution of the BELTRAMI equation, i.e.

$$w_{t\upsilon}(z)=z+t\dot{w}[\upsilon](z)+o(t),\ t\to 0$$

2.6. DEFINITION.

From now on $\hat{\mathfrak{T}}$ we will intoduce the topology induced by the L^{∞} NEHARI **norm** on the space of holomorphic functions on D, namely

$$\|f\|_D:=\sup_{z\in D}|f(z)|(1-|z|^2)^2<\infty$$

We need to use the BERS IMBEDDING , i.e. $\hat{\mathfrak{T}}\subset\mathfrak{B}(D)$ and then the L^{∞} NEHARI NORM induces a BANACH MANIFOLD SSTRUCTURE on $\hat{\mathfrak{T}}:=QSDiff^{+}(S^1)/PSU_{1,1}$.

#3. KIRILLOV COMPLEX STRUCTURE AND THE KÄHLER METRIC ON $\hat{\mathfrak{T}}$.

We know that if M is a real even dimensional manifold, then there are two equivalent definitions of complex structures on M, namely:

DEFINITON 3.1.A.

There exists $I \in \Gamma(M, \mathrm{Hom}(T^*M, T^*M))$ such that $I^2 = -\mathrm{id}$ plus the integrability condition.

DEFINITON 3.1.B.

There exists a global splitting of the complexified cotangent bundle

$$T^*M \otimes \mathbb{C} = \Omega^{1,0} \oplus \Omega^{0,1}$$

such that $\overline{\Omega^{1,0}} = \Omega^{0,1}$ plus the integrability condition.

In order to define the complex structure on $\mathrm{Diff}(S^1)/\mathbb{P}SU_{1,1}$ we will use DEFINITION 3.1.B.. Let $\mathrm{Vect}(S^1)\otimes\mathbb{C}$ be all complex valued vector fields on S^1. It is easy to see that

$$\mathrm{Vect}(S^1)\otimes\mathbb{C} = \left[\sum_{n=-\infty}^{\infty} a_n L_n\right]$$

where

$$L_n = e^{2\pi i n}\frac{\partial}{\partial t}.$$

The Lie algebra of $\mathbb{P}SU_{1,1}$ is naturally a subalgebra in $\mathrm{Vect}(S^1)\otimes\mathbb{C}$ spanned by

$$\left\{L_{-1}, L_0, L_1\right\}.$$

Thus the complexified cotangent space at $\mathrm{id}(\mathrm{mod}\,\mathbb{P}SU_{1,1})$ in $\mathrm{Diff}^+(S^1)/\mathbb{P}SU_{1,1}$ can be identified with

$$\left\{\sum_{n \neq -1,0,1} a_n L_n\right\}$$

DEFINITON 3.2.

$$\text{Let } \Omega^{1,0}_{id(mod\ \mathbb{P}SU_{1,1})} := \sum_{n>1} a_n L_n$$

Clearly we defined a left invariant complex structure on $\mathfrak{T}$. Then it is very easy to check that we have the following; Let u and $v \in \Omega^{1,0}$, then

$$[u,v] \in \Omega^{1,0}$$

This is just the **integrability condition.**

DEFINITON 3.3.

Let v and w be two left invariant vector fields on $Diff^+(S^1)$, then the metric defined in the following way:

$$<v,w> = \omega(v, Iw)$$

where ω is the KIRILLOV-KONSTANT-SAURIAU form on the **coadjoint orbit** of BOTT-VIRASORO group that passes through $\eta = (2(dt)^{\otimes 2}, -1)$ and so is isomorphic to $\mathfrak{T} = Diff^+(S^1)/\mathbb{P}SU_{1,1}$ by the results of #1 and I is the KIRILLOV **complex structure** operator on $\mathfrak{T}$. We will call this metric the WEIL-PETERSSON **metric.**

3.4.

THEOREM 3.

*A)*Let v and w be two left invariant vector fields on $Diff^+(S^1)/\mathbb{P}SU_{1,1}$ of type (1,0), i.e.

$$v = \sum_{n=2}^{\infty} v_n L_n \text{ and } w = \sum_{n=2}^{\infty} w_n L_n$$

where $L_n = \exp(\pi n i)\frac{\partial}{\partial t}$, *THEN* the WEIL-PETERSSON metric defined in #3.3. is given by

$$<v,w> = \frac{c}{12}\sum_{n=2}^{\infty} v_n \overline{w_n}\ (n^3 - n), \text{ where } c > 0$$

B) Let $H_1(S^1):=\{$all vector fields v on S^1 of type (1,0)| v is at least $C^{2+\epsilon}\}$, then $H_1(S^1)$ is a complete HILBERT SPACE with respect to the metric defined in *A)*.

REMARK.

Let us remind that the WEIL-PETRSSON METRIC defined in 3.3. is obtained in the following manner:

$$<v,w>=\omega(u,Iw)$$

where ω is the **KIRILLOV'S form.** Moreover clearly the WEIL-PETRSSON **metric** is a KÄHLER **metric.**

PROOF OF *A:*

Part *A* of this THEOREM follows from the definition of the VIRASORO **algebra,** i.e.:

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m,-n}$$

and more precisely from the way we defined the KIRILLOV-KONSTANT-SAURIAU form ω on the **coadjoint orbit** of THE BOTT-VIRASORO group through $(2(dt)^{\otimes 2},-1)$ which is isomorphic to $Diff^+(S^1)/\mathbb{P}SU_{1,1}$

$$\omega(L_n,L_m)=2b(dt)^{\otimes 2}\perp\{[L_m,L_n]\}=\frac{c}{6}(m^3-m)\delta_{m,-n}+\int_0^{2\pi}2b\exp(2\pi i(n+m))dt$$

From the fact that m and n are different from 1,0,-1 we get that the above formula is just

(3.4.1.) $$\omega(L_n,L_m)=\frac{c}{6}(m^3-m)\delta_{m,-n}$$

From (3.4.1.) the definition of KIRILLOV'S **complex structure** on $\hat{\mathfrak{X}}$ we get that

(3.4.2.) $$\omega(u,Iv)=\frac{c}{6}\sum_{m=2}^{\infty}(m^3-m)u_m\bar{v}_m$$

So (3.4.2.) shows that $\omega(u,Iv)$ is a metric on $Diff^+(S^1)/\mathbb{P}SU_{1,1}$. The fact that the this metric is a KÄHLER **metric** follows from the fact that ω is a closed form on $\hat{\mathfrak{X}}$. So part *A* of THEOREM 3 is proved.

PROOF OF PART *B* of THEOREM 3:

Before we prove PART *B* of THEOREM 3, let us recall the following facts:

Fact 1.

First we will recall that the tangent space of $\hat{\mathfrak{T}}$ can be identified with the space of all BELTRAMI differentials with finite L^{∞} norm. Let $L^{\infty}(D)$ be this space, i.e.the space of all BELTRAMI differentials with finite L^{∞} norm on D. Let $\upsilon \in L^{\infty}(D)$ Then the corresponding "quasisymmetric" vector field on S^1 is given by

(*) $$v(t)\frac{\partial}{\partial t}=\frac{\dot{w}[\upsilon](e^{it})}{ie^{it}}\frac{\partial}{\partial t}$$

where $w[\upsilon]$ is the solution of the BELTRAMI equation

$$w_{\bar{z}}=\upsilon w_z$$

and $\dot{w}[\upsilon]$ is the first variational term in the solution of the BELTRAMI equation, i.e.

$$w_{t\upsilon}(z)=z+t\dot{w}[\upsilon](z)+o(t),\ t\to 0$$

FACT 2. (See [20].).

Let $\mu \in L^{\infty}(D)$ be a BELTRAMI differential and it corresponds to a vector field $v(t)\frac{\partial}{\partial t}$ on S^1 and it is $C^{2+\epsilon}$ as (*). The FOURIER coefficients of v(t) are given by

$$v_k=\frac{i}{\pi}\int_D \overline{\mu(z)}\ \bar{z}^{k-2}dxdy \text{ for } k\geq 2$$

and $v_k=\overline{v_{-k}}$ for $k\leq -2$.

PROOF: It follows directly from the definition of FOURIER coefficients. (See [20].)

Q.E.D.

FACT 3.(NAG and VERJOVSKY in [20].)

Let $\nu(z)$ and $\mu(z)\in L^{\infty}(D)$ be a BELTRAMI **differentials** and the corresponding vector fields $v(t)\frac{\partial}{\partial t}$ and $w(t)\frac{\partial}{\partial t}$ on S^1 by (*) are of type $C^{2+\epsilon}$.

THEN the following formula is true:

$$\sum_2^{\infty}\overline{v_m}w_m(m^3-m)=-\frac{1}{\pi^2}\int_D x\int_D \mu(z)\overline{\nu(\varsigma)}\Big(\sum_2^{\infty}z^{m-2}\bar{\varsigma}^{m-2}(m^3-m)d\xi d\eta\cdot dxdy=$$

PROOF OF FACT 3:

The formula

$$\sum_2^{\infty}\overline{v_m}w_m(m^3-m)=$$

$$-\frac{1}{\pi^2}\int_D x\int_D \mu(z)\overline{\nu(\varsigma)}\Big(\sum_2^{\infty}z^{m-2}\bar{\varsigma}^{m-2}(m^3-m)d\xi d\eta\cdot dxdy=$$

$$\int_D x\int_D \frac{\mu(z)\overline{\nu(\varsigma)}}{(1-z\bar{\varsigma})^4}d\xi d\eta\cdot dxdy$$

follows from the fact that

$$\sum_2^{\infty}(m^3-m)y^{m-2}=\frac{-1}{6(1-y)^4},\ \text{for } |y|<1$$

as can be easily proved by differentiating

$$\sum_0^{\infty}y^m=(1-y)^{-1}$$

three times. So from here FACT 3 will follow if we prove the following formula:

(**) $$\int_D x \int_D \frac{\mu(z)\overline{\nu(\zeta)}}{(1-z\bar{\zeta})^4} d\xi d\eta \cdot dx dy = \frac{1}{\pi i} \int_D (1-|z|^2)^4 \mu(z)\overline{\nu(z)} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}$$

PROOF OF (**):

This formula follows from the fact that

$$\frac{c_2}{(1-z\bar{\zeta})^4}$$

is the BERGAMAN kernal function for the quadratic differentials in D. For this fact see [19]. So from here (**) follows immediately. See [20]. So FACT 3 is proved.

Q.E.D.

Let $L^2(D)$ denote the HILBERT space of complex functions on D for which

$$\int_D |f|^2 d\mu < \infty$$

where

$$d\mu = \frac{1}{2\pi i}(1-|z|^2)^2 dz \wedge d\bar{z}$$

In [14] it is proved that the subspace of all holomorphic functions with finite L^2 norm of the type, i.e.

$$\int_D |f|^2 d\mu < \infty$$

form a closed HILBERT subspace. So THEOREM 3 is proved.

Q.E.D.

So we will end this paragraph with the following PROPOSITION:

PROPOSITION 3.6.

For each genus $g \geq 2$ the *TEICHMÜLLER SPACE* $\mathfrak{T}_g(\Gamma)$ of the compact RIEMANN surface Γ of genus g can be imbedded in $QSDiff^+(S^1)$.

PROOF:

Let $f_1,\ldots,f_{3g-3}$ be a basis in $H^0(\Gamma,(\Omega^1_\Gamma)^{\otimes 2})$, i.e. for each j, f_j is a globally defined holomorphic quadratic differentials on Γ. Let $\Gamma=D/G$, where $G \subset \mathbb{P}SU_{1,1}$. We can lift each f_j to a automorphic holomorphic form of weght two in D with respect to G. We will denote by

$$\mu_i := \frac{|f_i|}{f_i}$$

Clearly $\mu_i \in H^1(\Gamma,\Theta)$, i.e. μ_i is a BELTRAMI differential on Γ and $\|\mu_i\|_\infty = 1$. Now we can identified $\mathfrak{T}_g(\Gamma)$ with the subset $\mathcal{T}_g$ in $H^1(\Gamma,\Theta)$, where

$$\mathcal{T}_g := \left\{ \sum_{i=1}^{3g-3} \tau_i \mu_i \mid \sum_{i=1}^{3g-3} |\tau_i| < 1 \text{ and } \tau_i \in \mathbb{C} \right\}$$

Let

$$\tau = \sum_{i=1}^{3g-3} \tau_i \mu_i \in \mathcal{T}_g$$

be a BELTRAMI differential on Γ and $\|\tau\|_\infty < 1$. We can lift τ to $\tilde{\tau}$ in D and solve the equation

$$\bar{\partial}_Z \phi = \tau \partial_Z \phi$$

From the THEOREM of AHLFORS and BERS we can find a solution of the above equation ϕ_τ, which gives a quasiconformal map of D. By a standard facts from the quasiconformal maps, it follows that ϕ_τ can be prolonged to a map of $\bar{D}$. The restriction of ϕ_τ to $\partial D = S^1$ will be a qusisymmetric homeomorphism of S^1, which we will denote by ϕ. (See [1].) This is the desired imbedding.

Q.E.D.

#4. THE REGULARIZATION OF THE WEIL-PETRSSON METRIC.

THEOREM 4.

A) Suppose that $\mu(z)$ is a holomorphic function in D and

$$\frac{1}{\pi i}\int_D ((1-|z|^2)^4|\mu|^2)(dz\wedge\overline{dz})<\infty$$

Let

$$g_r(\mu,\mu)=\frac{1}{\pi i}\iint_{D_r}((1-|z|^2)^4|\mu|^2)\frac{(dz\wedge\overline{dz})}{(1-|z|^2)^2}-\frac{ia}{3\pi^2}\iint_{D_r} x\int_D\frac{\mu(z)\overline{\mu(\varsigma)}}{(1-z\overline{\varsigma})^4}(d\varsigma\wedge\overline{d\varsigma})\cdot(dz\wedge\overline{dz})$$

Suppose that $\lim_{r\to 1^-} g_r(\mu,\mu)=\infty$

If we define

(*)
$$g(\mu,\nu)=\lim_{r\to 1^-}\frac{g_r(\mu,\nu)}{\mathcal{P}_r}$$

where

$$\mathcal{P}_r=\frac{1}{\pi i}\iint_{D_r}\frac{dz\wedge\overline{dz}}{(1-|z|^2)^2}\text{ and } D_r:=\{t\in\mathbb{C}\,|\,|t|<r\}$$

THEN

$g(\mu,\mu)$ exists and

$$g(\mu,\mu)>0$$

B) Let $H_2(S^1)$ be the space of all all vector fields on S^1 such that they fulfill the conditions 1), 2) and 3) stated bellow

1)
$$v\frac{\partial}{\partial t}=\sum_{n=2}^{\infty}c_n e^{int}\frac{\partial}{\partial t}$$

in other words $v\frac{\partial}{\partial t}$ is of type (1,0) with respect to the KIRILLOV'S COMPLEX STRUCTURE on $\hat{\mathfrak{T}}$

2) v is at most C^2

3) If we define $f_v(z)=\sum_{n=2}^{\infty} c_n z^n$ in D and suppose that $g(f_v,f_v)>0$

THEN $H_2(S^1)$ is a complete HILBERT SPACE with respect to **the metric** $g(v,v)$ as defined in *A)*.

C) The restriction of $g(\mu,\mu)$ defined on $H_2(S^1)$ as in *A)* on the imbedded TEICHMÜLLER SPACE $\mathfrak{T}_g$ is just the WEIL-PETERSSON METRIC.

PROOF OF PART *A* of THEOREM 4.:

The proof consists of several steps

PROPOSITION A.

Suppose that μ belongs to $\mathcal{L}^2(D)$, i.e. μ is holomorphic quadratic differential in D, i.e. we can view $\mu(z)$ as a holomorphic function in D and

$$\frac{1}{\pi i}\int_D (1-|z|^2)^4|\mu|^2(dz\wedge d\bar{z})<\infty$$

THEN

(*) $$g(\mu,\mu)=\lim_{r\to 1^-}\frac{g_r(\mu,\mu)}{\mathcal{P}_r} \textit{ exists}$$

$$(\mathcal{P}_r=\frac{1}{\pi i}\iint_{D_r}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}\text{ , } D_r:=\{t\in\mathbb{C}\,|\,|t|<r\}\text{ and } g_r(\mu,\nu)=\frac{1}{\pi i}\iint_{D_r}((1-|z|^2)^4\mu\bar{\nu})\frac{(dz\wedge d\bar{z})}{(1-|z|^2)^2})$$

Proof of PROPOSITION A:

First we will introduce some notations. Let us denote by $f_\mu:=(1-|z|^2)^4|\mu|^2$. From

$$\frac{1}{\pi i}\int_D (1-|z|^2)^4|\mu|^2(dz\wedge d\bar{z})<\infty$$

we get that

(4.1.) $$\frac{1}{\pi i}\int_D f_\mu(dz\wedge d\bar{z})<\infty$$

Q.E.D.

PROPOSITION 4.2.

Let

$$\tau_r(\mu,\mu)=\iint_{D_r} f_\mu(dz\wedge d\bar{z})$$

then

$$\tau_r(\mu,\mu)=\int_0^r \frac{\phi_\mu}{(1-r)^\alpha}dr,\text{ where } r\in[0,1]\subset\mathbb{R}$$

where ϕ_μ is a bounded function in $[0,1]\subset\mathbb{R}$ and α iis a real number such that $0\leq\alpha<1$, i.e. α is the same as in the assumptions of the THEOREM 4.

PROOF:

From the definition of $f_\mu=(1-|z|^2)^2|\mu|^2$, where $\mu(z)$ is a complex-analytic function in D we get that we have for f_μ the following formula:

(*) $$f_\mu=(1-|z|^2)^2\Big(\sum_{n\geq 0}a_nz^n\Big)\Big(\sum_{n\geq 0}\overline{a_nz^n}\Big)$$

From the definition of $\tau_r(\mu,\mu)$ and (*) we get

(**) $$\tau_r(\mu,\mu)=\frac{1}{\pi i}\iint_{D_r}(1-|z|^2)^2\Big(\sum_{n\geq 0}a_nz^n\Big)\Big(\sum_{n\geq 0}\overline{a_nz^n}\Big)(dz\wedge d\bar{z})$$

If we make change of the coordinates in (**) $z=Re^{i\phi}$ we will get after elementary calculations that

(***) $$\tau_r(\mu,\mu)=\frac{1}{\pi i}\int_0^r F_\mu(R)dR$$

where $F_\mu(R)$ is a real analytic function of R. Now our PROPOSITION follows immediately from (***) and the fact that $\lim_{r\to 1}\tau_r(\mu,\mu)$ exists .

Notice that α is determined in the following way:

a) $0\leq\alpha\in\mathbb{R}$ ***b)*** $\lim_{R\to 1}(1-R)^\alpha F_\mu(R)=c$ and $c\neq 0$

Q.E.D.

The end of the proof of PROPOSITION A.

From the definition of $\mathcal{P}_r$, i.e.

$$\mathcal{P}_r=\frac{1}{\pi i}\iint_{D_r}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}\text{ and } D_r:=\{t\in\mathbb{C}\,|\,|t|<r\}$$

after making the change of the coordinates $z=Re^{i\phi}$ we get that

(A) $$\mathcal{P}_r=\int_0^r \frac{dR}{(1-R)^2}$$

Again if we make the change of the coordinates $z=Re^{i\phi}$ and using PROPSITION 4.1. we will get that

(B) $$g_r(\mu,\mu)=\frac{1}{\pi i}\iint_{D_r} \frac{(1-|z|^2)^4|\mu|^2}{(1-|z|^2)^2}dz\wedge d\bar{z}=\int_0^r \frac{\phi_\mu(R)}{(1-R)^{2+\alpha}}dR$$

where $\phi_\mu(R)$ is a bounded function on $[0,1]\subset\mathbb{R}$ and $0\leq\alpha<1$. From (B) and the fact that $\phi_\mu(R)$ is a bounded function we get that

(C) $$\lim_{r\to 1^-}\frac{g_r(\mu,\mu)}{\mathcal{P}_r}=\frac{\int_0^r \frac{\phi_\mu(R)}{(1-R)^{2+\alpha}}dR}{\int_0^r \frac{dR}{(1-R)^2}}=c>0$$

exists and $c>0$.

So PROPOSITION A is proved. This proves PART A of THEOREM 4.

Q.E.D.

PROOF OF PART B OF THEOREM 4.

Let $\mathcal{H}(D)$ denotes all holomorphic functions f in D such that $g(f,f)<\infty$. Let $\{f_n\}$ be a sequence in $\mathcal{H}(D)$ which converges to an element $f\in L^2(D)$, where $L^2(D)$ is the space of all complex valued functions in D such that $g(f,f)<\infty$. Then from the definition of $g(f,f)$, i.e.

$$\lim_{r\to 1^-}\frac{g_r(f,f)}{\mathcal{P}_r}=\lim_{r\to 1^-}\frac{\frac{1}{2\pi i}\int_{D_r}(1-|\varsigma|^4)|f|^2\frac{d\varsigma\wedge d\bar{\varsigma}}{(1-|\varsigma|^2)^2}}{\frac{1}{2\pi i}\int_{D_r}\frac{d\varsigma\wedge d\bar{\varsigma}}{(1-|\varsigma|^2)^2}}$$

we get that for each $0<r<1$ and "sufficiently" close to 1

$$\lim_{n\to\infty}\|f_n\|_r=\|f\|_r$$

where

$$\|f\|_r^2 = \frac{g_r(f,f)}{\varphi_r} = \frac{\frac{1}{2\pi i}\int_{D_r}(1-|\varsigma|^4)|f|^2\frac{d\varsigma\wedge d\bar{\varsigma}}{(1-|\varsigma|^2)^2}}{\frac{1}{2\pi i}\int_{D_r}\frac{d\varsigma\wedge d\bar{\varsigma}}{(1-|\varsigma|^2)^2}}$$

The proof is based on the following PROPOSITION B_1:

PROPOSITION B_1.

Let A be a compact subset of D, *THEN* there exists a number $N_A(r)$ such that

$$|f(z)| \leq N_A(r)\|f\|_r$$

for all $z \in A$ and all holomorphic functions in D such that $g(f,f)<\infty$, where g is defined in THEOREM 4 part ***A*** and $A \subset D_r$, where $D_r := \{z \in \mathbb{C} \mid |z| < r < 1\}$.

REMARK.

Since A is a compact subset in D then from the definition of compactness, namely that A is a closed and bounded subset in D it follows that such D_r exists.

PROOF OF PROPOSITION B_1:

Repeat the arguments in [14] of PROPOSITION 3.1. on page 364.

Q.E.D.

PROPOSITION B_2.

From Proposition B_1 follows PART B of THEOREM 4.

PROOF of B_2:

Let A be a compact subset of D and let $A \subset D_r$. Then by PROPOSITION B_1we get that

$(B_2 1)$ $$|f_n(\varsigma) - f_m(\varsigma)| \leq N_A(r)\|f_m - f_n\|_r$$

for all $\varsigma \in A$. It follows that there exists a function h on D such that $f_n \to h$ uniformly on each compact subset A of D_r. Hence h is holomorphic on D_r. From here we get that h is defined in D. By $(B_2 1)$ we have for $\varsigma \in A \subset D_r$.

$(B_2 2)$ $$|f_n(\varsigma) - h(\varsigma)| \leq N_A(r)\|f_n - h\|_r$$

Given A, there exists an integer K such that the right-hand side of $(B_2 2)$ is ≤ 1 for $n \geq K$ and such that $\|f_K\|_r \leq \|f\|_r + 1$. Then we must prove that $g(h,h) < \infty$. For the proof of this fact we will remind the DEFINITION of $g(h,h)$, i.e.

$$g(h,h) := \lim_{r \to 1^-} g_r(h,h) := \lim_{r \to 1^-} \|h\|_r^2$$

From this definition and the easy inequality

$$\|h\|_r \leq \|f_K - h\|_r + \|f_K\|_r \leq c + \|f\|_r \text{ (c is a positive constant for)}$$

we get that h is a holomorphic function on each D_r and so in D. Finally since on each compact subset $A \subset D$ we have

$$\lim \frac{1}{2\pi i} \int_A |f_n(z) - f(z)|^2 \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2} = 0$$

and

$$\lim \frac{1}{2\pi i} \int_A |f_n(z) - h(z)|^2 \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2} = 0$$

So it follows that f=h almost everywhere.

Q.E.D.

PROOF OF PART C OF THEOREM 4.:

The proof is based on the following PROPOSITION:

PROPOSITION C.

Let $\Gamma = D/G$ and let ϕ and ψ be two holomorphic quadratic differential on Γ, then

$$W.P.\langle\phi,\psi\rangle = \lim_{r \to 1^-} \frac{g_r(\tilde{\phi},\tilde{\psi})}{\mathcal{P}_r}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ are holomorphic forms of weight two with respect to G on D and $W.P.\langle\phi,\psi\rangle$ is the WEIL-PETERSSON inner product on $\Gamma = D/G$.

PROOF of PROPOSITION C OF THEOREM 4 (NAG and VERJOVSKY) (See [20]):

It is easy to see that if $\tilde{\phi}$ and $\tilde{\psi}$ are holomorphic forms of weight two with respect to G on D, then

$$g_r(\tilde{\phi},\tilde{\psi}) = N_r(W.P.(\phi,\psi))$$

where N_r=number of copies (tiles) of D/G in D_r. No compact subset of D can meet infinitely many copies (tiles). This is proved in [9]. So N_r is a finite number. On the other hand we have that

$$\mathcal{P}_r = \frac{1}{\pi i}\iint\limits_{D_r} \frac{dz \wedge \overline{dz}}{(1-|z|^2)^2} \text{ and } D_r := \{t \in \mathbb{C} \mid |t| < r\}$$

is equal to N_r(area of $D_r \cap U_G$) with respect to the POINCARE metric, where U_G is the fundamental domain of G. From here everything follows directly.

Q.E.D.

So THEOREM 4 is proved.

Q.E.D.

CORR.

$QSDiff^{+}(S^1)/\mathbb{P}SU_{1,1}$ is a KÄHLER MANIFOLD.

#5. THE DETERMINANT LINE BUNDLE AND THE QUILLEN METRIC.

5.0. Some Definitions and Notations.

We can interpret $QSDiff^{+}(S^1)/\mathbb{P}SU_{1,1}$ as all *"Possible quasiconformal complex structures"* on the unit disk, so $QSDiff^{+}(S^1)/\mathbb{P}SU_{1,1}$ parametrizes different $\bar{\partial}_\phi$ for $\phi \in QSDiff^{+}(S^1)/\mathbb{P}SU_{1,1}$. Now we will define precisely $\bar{\partial}_\phi$

5.1. THE DEFINITION of $\bar{\partial}_\phi$.

A) We know from LEMMA 2.2. that to each

$$\phi \in QSDiff^{+}(S^1)/\mathbb{P}SU_{1,1}$$

there corresponds a unique pair of univalent holomorphic functions $(f(\phi),g(\phi))$ defined respectively in D and outside D, where

$$g(\phi)(\infty)=\infty \text{ and } g(\phi)'(\infty)=1$$

and

$$f(\phi)\cdot(g(\phi)^{-1})|_{S^1}=\phi$$

Let $\tau(\phi)$ be the BELTRAMI **differential** defined in the unit disk in the following manner:

$$\tau(\phi):=-(1-|z|^2)^2\overline{S(f(\phi))}$$

where $S(f(\phi))$ is the SCHARTZIAN **derivative** of $f(\phi)$. Now we define $\bar{\partial}_\phi$ in the following manner:

$$\bar{\partial}_\phi=\bar{\partial}_Z-\tau(\phi)\partial_Z$$

REMARK.

We will give another **definition** of $\bar{\partial}_\phi$, which we will use later. This **Deinition** is based on KADAIRA-SPENCER-KURANISHI THEOREY.

DEFINITION *B* OF $\bar{\partial}_{\phi}$.

Let $\phi \in QSDiff^{+}(S^1)$. From the **definition** of $QSDiff^{+}(S^1)$ it follows that ϕ can be prolonged to a quasiconformal map of D. Let us denote this map by $w(\phi)$. We may suppose that $w(\phi)$ satisfies the equation

$$\bar{\partial}_{\phi}(w(\phi))=\bar{\partial}_Z(w(\phi))-\tau(\phi)\partial_Z(w(\phi))=0$$

(For thwe proof of this fact see [1].)

Let I_{ϕ} be the pullback of the standard complex structure I_Z of D by $w(\phi)$, i. e.

$$I_{\phi}=\phi^{*}(I_Z)$$

We know from KODAIRA-SPENCER-KURANISHI DEFORMATION THEORY that I_{ϕ} defines a map

$$\tau(\phi):\Omega^{1,0}\to\overline{\Omega^{1,0}}$$

Notice that

$$\tau(\phi)\in\Gamma(D,Hom(\Omega^{1,0},\overline{\Omega^{1,0}}\ )=\Gamma(D,\Theta\otimes\Omega^{0,1})$$

so $\tau(\phi)$ is a BELTRAMI DIFFERENTIAL.

REMARK 1.

From now on we will use the second DEFINITION. Notice that $\tau(\phi)$ when restricted to S^1 is given by

$$\tau(\phi)|_{S^1}=\frac{\tau(\phi)(e^{it})}{e^{it}}\frac{\partial}{\partial t}$$

REMARK 2.

In order to justify the DEFINITION of $\bar{\partial}_{\phi}$ we will recall BERS THEORY of simulatanious uniformization for RIEMANN surfaces. A nice exposition of this THEORY, one can find in the beautiful paper by PH. A. GRIFFITHS and the review paper by L. I. BERS. (See[13] and [03].)

5.2. BERS THEORY. (See [3] & [13].)

Let Γ be a **RIEMANN surface** of genus $g>1$ and let R_O be a fixed quadratic differential on Γ. Let us consider the 2^{nd} order differential equation on Γ

(5.2.1.) $$\frac{d^2\mu}{dx^2}+R_O\mu=0$$

Next we will show how to associate an imbedding $\mu_O:\tilde{\Gamma}\to\mathbb{CP}^1$, where $\tilde{\Gamma}$ is the universal covering of Γ.

5.2.2. CONSTRUCTION OF μ_O.(See [13]).

We choose a point $z_O\in\Gamma$ and consider a basis μ_1, μ_2 for the solution of (5.2.1.) which may be assumed to exists in a neighborhood of z_O. By the principle of analytic continuation, we may extend the domain of definition of μ_1 and μ_2 to obtain single-valued functions $\tilde{\mu}_1$, $\tilde{\mu}_2$ on the universal covering $\tilde{\Gamma}$ of Γ. Furthermore, if we let $\pi_1(\Gamma)$ operate as a group of covering transformations on $\tilde{\Gamma}$, then we will find a transformation rule:

$$\tilde{\mu}_1(\gamma\cdot z)=a_\gamma\tilde{\mu}_1(z)+b_\gamma\tilde{\mu}_2(z)$$

$$\tilde{\mu}_2(\gamma\cdot z)=c_\gamma\tilde{\mu}_1(z)+d_\gamma\tilde{\mu}_2(z)$$

for $\gamma\in\pi_1(\Gamma)$ and all $z\in\tilde{\Gamma}$, and where the transformation matrix

$$M_\gamma=\begin{bmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{bmatrix}\in SL(2,\mathbb{C})$$

because the FUCKSIAN differential equation (5.2.1.) has no term involving $\frac{d\mu}{dz}$ in it.

Let

$$\mu_O=\frac{\tilde{\mu}_1}{\tilde{\mu}_2}$$

then $\mu_O:\tilde{\Gamma}\to\mathbb{CP}^1$ gives an imbedding. The equation (5.2.1.) gives a monodromy representation

(5.2.2.1.) $$\delta_O:\pi_1(\Gamma)\to SL(2,\mathbb{C})$$

which defines:

(5.2.2.2.) $$\mu_O(\gamma z)=\delta(\gamma)\mu_O(z) \text{ for } z\in\tilde{\Gamma}.$$

Conversely, suppose we are given an ėtale map:

$$\mu_Q:\tilde{\Gamma}\to\mathbb{CP}^1$$

which have the properties (5.2.2.1.) and (5.2.2.2.). We may think μ_Q as being meromorphic function on $\tilde{\Gamma}$, and we consider the ***Schwarzian derivative***:

$$S(\mu_Q)=\frac{\mu_Q'''}{\mu_Q'}-\frac{3}{2}\left[\frac{\mu_Q''}{\mu_Q'}\right]^2.$$

From the standrd properties of the Schwarzian derivative we get that $S(\mu_Q)=Q$ is a quadratic differential on Γ. From (5.2.2.2.) we deduce that μ_Q satisfies the differential equation:

$$\frac{d^2\mu_Q}{dx^2}+Q\mu_Q=0$$

So the following PROPOPOSITION holds:

<u>PROPOSITIN 5.2.2.3. (See [3].)</u>
There is one to one map between the maps $\mu:\tilde{\Gamma}\to\mathbb{CP}^1$ such that μ is an imbedding and have properties (5.2.2.1.) and (5.2.2.2.) and all equations of the type

$$\frac{d^2\mu_Q}{dx^2}+Q\mu_Q=0$$

where Q is a holomorphic quadratic differential on Γ with NEHARI **norm** $\|Q\|<6$.

REMARK I.

Suppose that the NEHARI NORM of Q $\|Q\|<2$, then the map $\mu_Q:\tilde{\Gamma}\to\mathbb{CP}^1$ can be prolonged to qusiconformal function $h(\mu_Q)$ such that

$$\bar{\partial}_Z(h(\mu_Q)) = -(1-|z|^2)^2\overline{S(\mu_Q)}\ \partial_Z(h(\mu_Q))$$

we suppose that $\tilde{\Gamma}\cong D$. See [1].

REMARK II.

It is easy to see that $-(1-|z|^2)^2\overline{S(\mu_Q)}$ is defined in D. Using the usual change of coorfinates $z\to\frac{1}{z}$ we will get that it is defined outside D.

REMARK 5.2.3. (See [3].)

Notice that all holomorphic quadratic diferentials on Γ with NEHARI norm less then 2 can be identified with the TEICHMÜLLER SPACE $\mathfrak{T}_g(\Gamma)$. This can be done in the following manner: Let $\Gamma=H/G$, where H is the upper-half plane and let $\Gamma^*=H^*/G^*$be the conjugate curve of Γ. Here H^* is the lower-half plane. Consider all pairs (C,h) where

$$h:\Gamma^*\to C$$

is a quasiconformal homeomorphism from the RIEMANN surface Γ^* to the RIEMANN surface C. Introduce the equivalence relation

$$(C_1,h_1)\cong(C_2,h_2)$$

whenever $h_2\cdot h_2^{-1}$ is homotopic to a conformal mapping of C_1 onto C_2. We denote by $\mathfrak{T}_g(\Gamma)$ the set of all such equibalence classes of pairs (C,h) and this is the usual definition of the TEICHMÜLLER space. Let us denote by $\mathcal{T}_g(\Gamma)$ the set of all quadratic holomorphic with NEHARI norm less then 2. Let $Q\in\mathcal{T}_g(\Gamma)$ and choose a quasiconformal extension U_Q of the univalent holomorphic mapping

$$\mu_Q:\tilde{\Gamma}\to\mathbb{CP}^1$$

Notice that we have the following commutative diagramm

$$\begin{array}{llcl} U_Q: & H^* & \to & D\text{-}(Q) \\ & \downarrow & & \downarrow \\ h_Q: & \Gamma^* = H^*/G^* & \to & D\text{-}(Q)/G_Q = C_Q \end{array}$$

yields a point (C_Q,h_Q) in $\mathfrak{T}_g(\Gamma)$. So we got a map

$$\mu:\mathcal{T}_g(\Gamma)\to\mathfrak{T}_g(\Gamma)$$

BERS proved in [2] that μ is an isomorphisms of sets.

REMARK 5.2.4.

Notice that if $\phi\in QSDiff(S^1)/\mathbb{P}SU_{1,1}$ is obtained from h_Q, then μ_Q is just $f(\phi)$ of LEMMA 2.2.

5.3. SOME NOTATIONS AND THE DOMAIN OF THE ACTION OF $\bar\partial_\phi$.

5.3.1. If $\phi\in QSDiff^+(S^1)/\mathbb{P}SU_{1,1}$ and ϕ is of class C^∞ then $\bar\partial_\phi$ acts on $C^\infty(S^1)$, i.e. on all C^∞ vector fields on S^1 in the obvious ways, namely let $\tilde v$ be a vector field in D such that when restricted to S^1 is v and both $\tilde v$ and v are of type C^∞.Then $\bar\partial_\phi v$ is just the restriction of $\bar\partial_\phi\tilde v$ on S^1.

We will recall the DEFINITION of the tangent space of

$$\hat{\mathfrak{T}}:=QSDiff^+(S^1)/\mathbb{P}SU_{1,1}$$

The tangent space can be identified with the space of all BELTRAMI differentials with finite L^∞ norm. Let $L^\infty(D)$ be this space, i.e. this is the space of all BELTRAMI differentials with finite L^∞ norm on D. Let $\upsilon\in L^\infty(D)$ Then the corresponding "quasisymmetric" vector field on S^1 is given by

(5.3.1.1.) $$v(t)\frac{\partial}{\partial t}=\frac{\dot w[\upsilon](e^{it})}{ie^{it}}\frac{\partial}{\partial t}$$

where $w[\upsilon]$ is the solution of the BELTRAMI equation

$$w_{\bar{z}}=\upsilon w_z$$

and $\dot{w}[\upsilon]$ is the first variational term in the solution of the BELTRAMI equation, i.e.

$$w_{t\upsilon}(z)=z+t\dot{w}[\upsilon](z)+o(t),\ t\to 0$$

5.3.1.A. Definition.

In THEOREM 4 we proved that if $v(t)\frac{\partial}{\partial t}$ is a vector field on S^1 at least of class $C^{2+\epsilon}$, then we can define a structure of HILBERT SPACE on all those vector fields, namely

$$<v(t),w(t)>=\frac{c}{12}\sum_{m=2}^{\infty}m^2(m-1)v_m\overline{w_m}$$

where

$$v(t)\frac{\partial}{\partial t}=\sum_{m=2}^{\infty}v_m e^{\pi mi}\frac{\partial}{\partial t} \quad \text{and} \quad w(t)\frac{\partial}{\partial t}=\sum_{m=2}^{\infty}w_m e^{\pi mi}\frac{\partial}{\partial t}$$

We denoted this space by $H_1(S^1)$.

5.3.1.B. Definition.

Next suppose that

$$\psi(t)\frac{\partial}{\partial t}=\sum_{m=2}^{\infty}\psi_m e^{\pi mi}\frac{\partial}{\partial t}$$

is a "quasisymmetric" vector field on S^1 and $\psi(t)\frac{\partial}{\partial t}$ is at most of class C^2, ***THEN*** we defined on this space aHILBERT SPACE STRUCTURE in the following manner:

Suppose that

$$f_\psi(z):=\sum_{m=2}^{\infty}\psi_m z^m$$

Clearly $f_\psi(z)$ is a holomorphic function in D

$$f_\psi(z)|_{S^1}=\sum_{m=2}^{\infty}\psi_m e^{\pi mi}$$

From the definition of a "quasisymmetric" vector field on S^1 we get that $f_\psi(z)$ must have a finite NEHARI NORM, namely

$$\sup_{z\in D}(1-|z|^2)^2|f_{\psi}(z)|<\infty$$

(For the proof of this fact see [19].)

So from here we get that

$$(5.3.1.2) \qquad \int_D (1-|z|^2)^4|f|^2(dz\wedge\overline{dz})<\infty$$

then we proved in #4 that

$$g(f_{\psi},f_{\psi})=\lim_{r\to 1^-}\frac{\iint_{D_r}(1-|z|^2)^4|f|^2\frac{(dz\wedge\overline{dz})}{(1-|z|^2)^2}1}{\iint_{D_r}\frac{dz\wedge\overline{dz}}{(1-|z|^2)^2}}<\infty$$

and $g(f_{\psi},f_{\psi})\neq 0$.

So $g(f_{\psi},f_{\psi})$ defines a HILBERT STRUCTURE on all vector fields of type (1,0) on S^1 which are at most C^2. This was proved in THEOREM 4. We will denote this HILBERT SPACE of vector fields on S^1 by $H_2(S^1)$.

5.3.1.C. Definition.

So we can identified the space of real left invariant vector fields on S^1 with the direct sum of the HILBERT SPACES $H_1(S^1)\oplus H_2(S^2)$. (Here we denote by $H_1(S^1)$ and by $H_2(S^1)$ the direct sum of the holomorphic and antiholomorphic vector fields on S^1) This HILBERT SPACE we will denote by $\mathcal{H}^2(S^1)$.

PROPOSITION 5.3.2.

For each $\phi \in QSDiff^{+}(S^1)/\mathbb{P}SU_{1,1}$ $\bar{\partial}_\phi$ defines an elliptic operetor on $\mathcal{H}^2(S^1)$.

PROOF:

First we will suppose that $\bar{\partial}_\phi$ is a well defined linear operator on $\mathcal{H}^2(S^1)$ and under this assumption we will prove that it is an elleiptic operator.

Proof of the fact that $\bar{\partial}_\phi$ is an elliptic operator:

Notice that if w_ϕ is quasiconformal solution solution of

$$\bar{\partial}_\phi w_\phi = 0$$

then w_ϕ defines a ***new complex structure on D.*** This new complex structure is just the pull back of the standard complex structure on D. With respect to this new complex structure $\bar{\partial}_\phi$ is just the usual $\bar{\partial}$ operator. Its symbol is just $i\xi$. So $\bar{\partial}_\phi$ is elliptic operator.

Q.E.D.

Definition of the action of $\bar{\partial}_\phi$:

First we will define how $\bar{\partial}$ acts on all the "quasisymmetric" vector fields on S^1. This action is given in the following way:

Let $\overline{f(z)(dz)^{\otimes 2}}$ be an antiholomorphic quadratic differential on D and let

$$\dot{w}[\nu] = \frac{(1-|z|^2)^2}{dz \otimes \overline{dz}} \overline{f(z)(dz)^{\otimes 2}}$$

be the BELTRAMI DIFFERENTIAL obtained from $\overline{f(z)(dz)^{\otimes 2}}$ then

$$(*) \qquad \bar{\partial}(\dot{w}[\nu]) = \frac{(1-|z|^2)^2}{dz \otimes \overline{dz}} \overline{\partial(f(z))(dz)^{\otimes 2}}$$

Next we must show that from (*) it follows that the restriction of $\bar{\partial}$ on S^1 is defined correctly on $\mathcal{H}^2(S^1)$. This follows from the fact that $\bar{\partial}$ is a closed operator and now we can invoke closed graph THEOREM.(See [23].) Notice that $\bar{\partial}$ is defined on both $H_1(S^1)$ and $H_2(S^2)$. This is so since in $H_1(S^1)$ all C^∞ vector fields are an everywhwere dense subset and C^2 vector fields are an everywhere dense subset in $H_2(S^1)$. Here we define $H_1(S^1)$ and $H_2(S^1)$ here as real vector fields, i.e. both of them are sum of holomorphic and anti-holomorphic vector fields on S^1.

So up to now we have defined only the action of $\bar{\partial}$ on $\mathcal{H}^2(S^1)$. Remember that we can view $\mathcal{H}^2(S^1)$ as left invariant vector fields on $QSDiff^+(S^1)/\mathbb{P}SU_{1,1}$. In the same manner we can define the action of ∂ on $\mathcal{H}^2(S^1)$.

From the DEFINITION of $\bar{\partial}_\phi$

$$\bar{\partial}_\phi=\partial-\tau(\phi)\bar{\partial}$$

it follows that we must consider two different cases namely, when $\tau(\phi)$ is of at least of class $C^{2+\epsilon}$ and $\tau(\phi)$ is at most of class C^2. In the first case $\bar{\partial}_\phi$ is defined on $H_1(S^1)$ and is just zero on $H_2(S^1)$. In the second case $\bar{\partial}_\phi$ is defined on $H_2(S^1)$ and is zero on $H_1(S^1)$. In the first case we must use the fact that all vector fields of class $C^{3+\epsilon}$ are everywhere dense subset in $H_1(S^1)$ and in the second case that all vector fields of type C^2 are dense in $H_2(S^1)$. Then in both cases $\bar{\partial}_\phi$ is a close operator so applying CLOSED GRAPH THEOREM, we get that $\bar{\partial}_\phi$ is defined on $\mathcal{H}^2(S^1)$.

Q.E.D.

REMARK.

From now on we will consider the action of $\bar{\partial}_\phi$ on $\mathcal{H}^2(S^1)$ vectorfields on S^1.

5.4. THE DETERMINANT LINE BUNDLE.

5.4.1. Definition.

Let

$$\Delta_\phi^+ := \bar\partial_\phi^* \bar\partial_\phi \text{ and } \Delta_\phi^- := \bar\partial_\phi \bar\partial_\phi^*$$

where $\bar\partial_\phi^*$ is the conjugate of $\bar\partial_\phi$ with respect to the metric $g(\mu,\nu)$ defined in THEOREM 4. Both operators Δ_ϕ^+ and Δ_ϕ^- are positive and so all their eigenvalues are ≥ 0.

5.4.2. Definition.

Let $\{f_n^+\}$ be an orthonormal basis of eigen sections of Δ_ϕ^+, i.e.

$$\Delta_\phi^+(f_n^+) = \lambda_n$$

and $\{f_n^-\}$ be an orthonormal basis of eigen sections of Δ_ϕ^-, i.e.

$$\Delta_\phi^-(f_n^-) = \lambda_n$$

5.4.3. REMARK.

Let $\bar\partial_\phi f_n^+$, then

$$\Delta_\phi^-(\bar\partial_\phi f_n^+) = (\bar\partial_\phi((\bar\partial_\phi^* \bar\partial_\phi)(f_n^+)) = \bar\partial_\phi(\lambda_n f_n^+) = \lambda_n(\bar\partial_\phi f_n^+)$$

so the operators Δ_ϕ^+ and Δ_ϕ^- have the same eigen values and

$$\{f_n^-\} = \{\bar\partial_\phi f_n^+\} \in (\ker \bar\partial_\phi)^\perp$$

Therefore, to every eigensection f_n^+ of Δ_ϕ^+ with eigenvalue $\lambda_n > 0$, there corresponds an eigensection $f_\phi^- = \bar\partial_\phi f_n^+$ with eigen value λ_n.

5.4.3. REMARK.

For $a>0$, let U_a be the set of points in $QSDiff^+(S^1)/\mathbb{P}SU_{1,1}$ where a is not an eigenvalue of Δ^+_ϕ. This is an open set. Clearly since the operators $\bar{\partial}_\phi$ depends holomorphically on $\phi \in QSDiff^+(S^1)/\mathbb{P}SU_{1,1}$, the number of eigen balues of Δ^+_ϕ less then a is constant in U_a. Note that it is important that $a>0$, because the spectrum is bounded below by zero.

5.4.4. The construction of the determinant line bundle.

Let $(E^{\pm}_\phi)_a$ be the subspace of $\mathcal{H}(S^1)$ spanned by all eigensections of $\Delta^{\pm}_\phi$ with eigen values less then a. The determinants of these spaces will be pieced together over $QSDiff^+(S^1)/\mathbb{P}SU_{1,1}$ in order to construct the ***determinant line bundle.*** The trivialization of the ***determinant line bundle*** over U_a will be

$$U_a x((\det(E^+_\phi)_a)\otimes(\det(E^+_\phi)_a)^*)$$

and the fibre over $\phi \in QSDiff^+(S^1)/\mathbb{P}SU_{1,1}$ is $((\det(E^+_\phi)_a)\otimes(\det(E^+_\phi)_a)^*)$.

Similarly, one considers U_b for $b>a>0$. We know that every line bundle is defined by a cocycle $\{\sigma_{b,a}\}$, where

$$\{\sigma_{b,a}\} \in \prod_{a<b}(U_a \cap U_b, \mathcal{O}^*_{\mathfrak{T}})$$

and

$$\sigma_{c,a}=\sigma_{c,b}\sigma_{b,a} \text{ for } c>b>a.$$

Definition of $\sigma_{b,a}$.

Let

$$\{\psi^{+(a,b)}_{i,\phi};i=1,\dots,k\}$$

be the maximal number of orthonormal eigensections of the operator Δ^+_ϕ with eigen values in the interval (a,b), where a and b are positive real numbers. Let

$$\{\psi^{-(a,b)}_{i,\phi};i=1,\dots,k\}$$

be the maximal number of orthonormal eigensections of the operator Δ_ϕ^- with eigen values in the interval (a,b), where a and be are positive real numbers. We know that from REMARK 5.4.3. that the number of $\psi_{i,\phi}^{-(a,b)}$ is equal to the number of $\psi_{i,\phi}^{+(a,b)}$ and we have the following:

$$\bar{\partial}_\phi(\psi_{i,\phi}^{+(a,b)})=\sum_{j=1}^{k}\alpha_{ij,\phi}^{(a,b)}\psi_{j,\phi}^{-(a,b)} \text{ and } i=1,\dots,k$$

Since the operator $\bar{\partial}_\phi$ depends on ϕ holomorphically we obtain that $\det(\alpha_{ij,\phi}^{(a,b)})$ depends on ϕ holomorphically. So let us define

$$\sigma_{a,b}:=\det(\alpha_{ij,\phi}^{(a,b)})$$

Clearly $\{\sigma_{a,b}\}$ fulfills the cocycle condition. This will define the ***determinant line bundle.*** We will denote this line bundle by $\det(\bar{\partial})$.

5.5. Definition of the *QUILLEN metric*.

Let $\{\bar{\partial}_\phi\}$ be the set of $\bar{\partial}$ operators parametrized holomorphically by $QSDiff^+(S^1)/PSU_{1,1}$. Then the determinant line bundle $\det(\bar{\partial})$ has a canonical section over U_a, namely

$$(\det\bar{\partial})_a(\phi)=\frac{(\bar{\partial}_\phi\psi_1\wedge\dots\wedge\bar{\partial}_\phi\psi_k)}{(\psi_1(\phi)\wedge\dots\wedge\psi_k(\phi))}$$

where $\phi\in U_a$ and $\psi_1(\phi),..,\psi_k(\phi)$ is the maximal number of eigensections for Δ_ϕ^+ that corresponds to the non-zero eigenvalues $<a$. Now we are ready to define

$$\|\det\bar{\partial}_\phi\|_Q^2:=\exp(--\zeta'_{\Delta_\phi^+}(0))$$

where

$$\zeta'_{\Delta_\phi^+}(s)=\sum_{i=1}^{\infty}\lambda_i^{-s} \text{ and } \lambda_i \text{ are eigen values of } \Delta_\phi^+ \text{ and } Re(s)>1.$$

Sometimes we will denote $\zeta'_{\Delta_\phi^+}(s)$ by $\zeta(s)$.

REMARK.

Let $\mathfrak{B}$ be the space of all $\bar{\partial}_\phi=\bar{\partial}-\tau(\phi)\partial$ for $\phi\in\mathfrak{T}$. Then $\|\bar{\partial}_\phi\|^2=<\tau(\phi),\tau(\phi)>$, where $<\ ,\ >$ is the scalar product on $\mathcal{H}^2(S^1):=H_1(S^1)\dot{+}H_2(S^1)$ defined as above.

5.6. FORMULATION OF QUILLEN'S THEOREM.

THEOREM 5. (QUILLEN) (See [18].)

A) $\partial\bar{\partial}\log(\exp(-\varsigma'_{\Delta^{+}_{\mu}}(0)))=\partial\bar{\partial}\log\|\det\bar{\partial}_t\|_Q$ is just the **WEIL-PETRSSON METRIC on $\hat{\mathfrak{T}}$.**

PROOF:

It suffices to check that the curvature form of the **QUILLEN metric** ***coinsides*** with the **WEIL-PETERSSON metric** over one-parameter family $\bar{\partial}_{\phi(\mu)}$ of invertible operators depending on μ holomorphically on the complex variable $\mu\in\mathbb{C}$ and $\phi(\mu)\in\hat{\mathfrak{T}}$. We may suppose that $\phi(\mu)=\mu\phi\in\hat{\mathfrak{T}}$. *THE curvature form* is given by:

$$\bar{\partial}\partial\log\|\det\bar{\partial}_\mu\|^2_Q=\frac{\partial^2}{\partial\mu\bar{\partial}\bar{\mu}}(\varsigma'_{\Delta^{+}_{\mu}}(0))d\mu\wedge d\bar{\mu}$$

We recall that

$$\varsigma(s):=\varsigma'_{\Delta^{+}_{\mu}}(s):=\mathrm{Tr}((\Delta^{+}_{\mu})^{-s})\text{ and }\Delta^{+}_{\mu}:=\bar{\partial}^{*}_{\mu}\bar{\partial}_{\mu}$$

So easy computations show that we have:

$$-\frac{\partial}{\partial\mu}\varsigma(s)=s\mathrm{Tr}((\Delta^{+}_{\mu})^{-s-1}\frac{\partial}{\partial\mu}(\Delta^{+}_{\mu}))=s\mathrm{Tr}((\Delta^{+}_{\mu})^{-s}\partial^{-1}_{\mu}(\frac{\partial}{\partial\mu}(\Delta^{+}_{\mu})))=$$

$$\frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(e^{-t\Delta^{+}_{\mu}}\partial^{-1}_{\mu}(\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu}))t^{s-1}dt$$

Let us denote by

(5.6.0.) $$\psi(s):=\int_0^\infty \mathrm{Tr}(e^{-t(\Delta^{+}_{\mu})}\partial^{-1}_{\mu}((\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu})))t^{s-1}dt$$

Let

(5.6.1.) $$\psi(s)=\psi(0)+0(s)$$

then we need to compute $\psi(0)$ in (5.6.1.) in terms of the ***parametrix*** of the invertible operator $\bar{\partial}_\phi$ $G_0(z,z')$ and its ***SCHWARTZ kernal*** $G(z,z')$. Namely we will prove the following LEMMA:

LEMMA 5.6.2.

For $\psi(0)$ defined as in (5.6.1.) we have:

$$\psi(0):=\langle J,(\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu\phi})\rangle$$

where $J:=\lim_{z\to z'}(G(z,z')-G_0(z,z')$, and $\langle\ ,\ \rangle$ is the scalar product on $\mathcal{H}^2(S^1)$.

PROOF:

From the definition of the function $\psi(z)$ we get that:

$$\psi(s):-\int_0^\infty \mathrm{Tr}(e^{-t(\Delta_\mu^+)}\partial_\mu^{-1}((\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu\phi})))t^{s-1}dt \tag{5.6.0.}$$

so from this expression we get that

$$\psi(0):=\lim_{t\to 0}\mathrm{Tr}(e^{-t(\Delta_\mu^+)}\partial_\mu^{-1}((\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu\phi})))=? \tag{5.6.2.1.}$$

QUILLEN in [18] proved the following THEOREM:

THEOREM (QUILLEN) (See [18]).

One has

$$\lim_{t\to 0}\langle z|e^{-t(\Delta_\mu^+)}G|z\rangle=J(z)$$

uniformly in z, and consequently for any $B\in\mathcal{B}$

$$\lim_{t\to 0}\mathrm{Tr}(e^{-t(\Delta_\mu^+)}(\bar{\partial}_\mu^{-1})(B))=\langle J,B\rangle$$

where $\mathcal{B}$ is the space of operators $\{\bar{\partial}_\phi|\phi\in\hat{\mathfrak{T}}\}$ acting on The HILBERT SPACE $\mathcal{H}^2(S^1)$ with inner product $\langle\hat{\partial}_\phi,\hat{\partial}_\psi\rangle=\langle\tau(\phi),\tau(\psi)\rangle$, where $\langle\ ,\ \rangle$ is the scalar product in $\mathcal{H}^2(S^1)$. Remember that $\tau(\phi)$ was defined via BERS IMBEDDING.

This THEOREM follows from the continuity of $G-G_0$ along the diagonal and the formula

$$\lim_{t\to 0}\langle z|e^{-t(\Delta_\mu^+)}G_0|z\rangle=0$$

which is derived by calculating the asymptotic expansion of the ***heat kernal***. For more details see [18].

From this THEOREM of QUILLEN it follows that

(5.6.2.2.) $$\lim_{t\to 0} \mathrm{Tr}(e^{-t(\Delta_{\mu}^{+})}\partial_{\mu}^{-1}((\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu\phi})))=\langle J,(\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu\phi})\rangle$$

From (5.6.2.1.) and (5.6.2.2.) LEMMA 5.6.2.follows.

Q.E.D.

5.6.3. The end of the proof of the THEOREM 5.

Up to now we have proved that

(5.6.3.1.) $$-\frac{\partial}{\partial\mu}\varsigma(s)=s\left\{\langle J,\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu\phi})\rangle+0(s)\right\}$$

So we get that

(5.6.3.2.) $$\frac{\partial}{\partial\mu}\varsigma(0)=0$$

and

(5.6.3.2.) $$-\frac{\partial}{\partial\mu}\varsigma'(0)=\langle J,\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu\phi})\rangle$$

So in order to finish the proof of the THEOREM we need to compute J and also $\frac{\bar{\partial}}{\partial\mu}J$. Notice that

(5.6.3.3.) $$\frac{\partial}{\partial\mu}(\bar{\partial}_{\mu\phi})=\tau(\phi)$$

This follows directly from the definition of $\bar{\partial}_{\mu\phi}$, i.e.$\bar{\partial}_{\mu}:=\bar{\partial}_{z}-\mu\tau(\phi)\partial_{z}$, where $\mu\in\mathbb{C}$ and $\phi\in\hat{\mathfrak{X}}$.

On the other hand from (5.6.3.2.) we get that

(5.6.3.4.) $$-\frac{\partial^{2}\varsigma'(0)}{\partial\mu\overline{\partial\mu}}=\langle\frac{\bar{\partial}}{\partial\mu}J,\tau(\phi)\rangle$$

So we need to prove

LEMMA 5.6.3.5. $\frac{\bar{\partial}}{\partial\mu}(J)=\overline{\tau(\phi)}$

PROOF:

We need to compute explicetely the operator J. For this reason let us remind you the definition of J:

$$J:=\lim_{z\to z'}(G(z,z')-G_0(z,z'))$$

where $G(z,z')$ is the ***SCHWARZ kernal*** of $\bar{\partial}_\phi$ and $G_0(z,z')$ is ***the parametrix*** of $\bar{\partial}_\phi$. So we need to find explicit formulas for $G(z,z')$ and $G_0(z,z')$.

5.6.3.5.1. Explicit computation of $G_0(z,z')$ and $G(z,z')$.

REMARK.

Let $w(z,\bar{z})$ be a quasiconformal solution of the BELTARMI equation:

$$\bar{\partial}_{\bar{z}}w=\tau(\phi)\partial_z$$

i.e. w is real analytic homeomorphism of D which is quasiconformal. Clearly that if we make change of the coordinates $z\to w(z,\bar{z})$ we will get a new complex structure on D, where all holomorphic functions to this new complex structure are power series in w. Clearly that

$$\bar{\partial}_{\bar{z}}-\tau(\phi)\partial_z=\bar{\partial}_w$$

After this remark it is easy to compute the *parametrix* and *the SCHWARZ kernal* of $\bar{\partial}_w$. Remember that w depends on μ in a complex analytic manner, i.e. w_μ is a solution of the equation

$$\bar{\partial}_{\bar{z}}-\tau(\mu)\phi\partial_z=\bar{\partial}_{w_\mu}$$

(For the proof of this see [1].)

Computation of the *SCHWARZ kernal* G.

The operator $\bar{\partial}_{\bar{z}}-\tau(\mu)\phi\partial_z=\bar{\partial}_{w_\mu}$ has a fundamental solution

$$P_{\mu\phi}:\Gamma(D,\Omega^{0,1})\to\Gamma(D,\Omega^{0,0})$$

given in coordinate w_μ by the formula:

$$P_{\mu\phi}\omega:=\frac{1}{2\pi i}\int_D\frac{d\xi}{\xi-w_\mu}\wedge\omega(\xi)$$

So from here we obtain that ***SCHWARZ kernal*** $G(w'_\mu,w_\mu)$ of $\bar{\partial}_{w_\mu}$ is given by:

(1) $$G(\xi'_\mu,\xi_\mu)=\frac{d\xi'_\mu}{\xi'_\mu-\xi_\mu}$$

Computation of the *parametrix* G_o of $\bar{\partial}_{w_\mu}$.

We will construct the ***parametix*** of $\bar{\partial}_{w_\mu}$ in the following way. Let ∇ be the unique connection on the tangent bundle of D, i.e. on the trivial bundle, compatible with the pull back of the ***Eucleadian metric*** via w_μ compatible with this metric and the operator $\bar{\partial}_{w_\mu}$. Let $\mathcal{F}(\xi_\mu,\xi'_\mu)$ be the parallel transport with respect to ∇ along the geodesic from ξ_μ to ξ'_μ. Let $r^2(\xi_\mu,\xi'_\mu)$ be the distance between w_μ and w'_μ. Put

(A) $$G_o(\xi_\mu,\xi'_\mu):=\partial_{\xi'_\mu}\log(r^2(\xi_\mu,\xi'_\mu))\mathcal{F}(\xi_\mu,\xi'_\mu)$$

where

(B) $$\mathcal{F}(\xi_\mu,\xi'_\mu)=1+(\xi_\mu-\xi'_\mu)d\xi'_\mu+\overline{(\xi_\mu-\xi'_\mu)d\xi'_\mu}+\ldots$$

So we get from (A) and (B)

(II) $$G_o(\xi_\mu,\xi'_\mu)=\frac{1}{2\pi i}\frac{1}{(\xi_\mu-\xi'_\mu)}(\xi_\mu-\xi'_\mu)d\xi'_\mu+\overline{(\xi_\mu-\xi'_\mu)d\xi'_\mu}+\ldots$$

Computation of J and of $\frac{\partial}{\partial\mu}J(z)$.

From the definition of J:

$$J:=\lim_{z\to z'}(G(z,z')-G_o(z,z')$$

and from (A) and (B) we get that

(III) $$J(z)=d\xi'_\mu=(\partial_z\xi'_\mu+\overline{\mu\tau(\phi)\bar{\partial}_z\xi'_\mu})dz$$

From (III) we get that

(IV) $$\frac{\partial}{\partial\mu}J(z)=0 \text{ and } \frac{\bar{\partial}}{\partial\mu}J(z)=\overline{\tau(\phi)}$$

From

(5.6.3.4.) $$-\frac{\partial^2\zeta'(0)}{\partial\mu\bar{\partial}\mu}=<\tau(\phi),\tau(\phi)>$$

This proves QUILLEN'S THEOREM.

Q.E.D.

FINAL REMARK.

Just in the same manner one can prove ***more general principle,*** which was observed also by FUJIKI and SCHUMACHER. Namely, let $\mathfrak{X} \to \mathfrak{U}$ be the KURANISHI family of KÄHLER manifolds with a fixed class of polarization $L \in H^{1,1}(X_0, \mathbb{R})$. Suppose that for each $t \in \mathfrak{U}$ we can find a KÄHLER metric g_t depending C^∞ on t and such that the cohomology class $[\operatorname{Im} g_t] = L$. Then we can define the WEIL-PETERSSON metric in the following way: We know from KURANISHI theory that the ZARISKI tangent space $T_t\mathfrak{U} \approx \mathbb{H}^1(X_t, \Theta_t)$, where $\mathbb{H}^1(X_t, \Theta_t)$ is the space of harmonic (0,1) forms with coefficients in the tangent space. Let ϕ and $\sigma \in \mathbb{H}^1(X_t, \Theta_t)$; then

$$\phi_{|U} = \sum \phi^{\alpha}_{\bar{\mu}} d\bar{z}^{\mu} \otimes \frac{\partial}{\partial z^{\alpha}}$$

$$\sigma_{|U} = \sum \sigma^{\nu}_{\bar{\beta}} d\bar{z}^{\beta} \otimes \frac{\partial}{\partial z^{\nu}}$$

Then we define the WEIL-PETERSSON metric in the following way:

$$<\phi, \sigma> = \int_{X_t} \phi^{\nu}_{\bar{\beta}} \, \overline{\sigma^{\mu}_{\bar{\alpha}}} \, g_{\nu\bar{\mu}} g^{\alpha\bar{\beta}} \operatorname{vol}(g(t))$$

We can interpret $\mathfrak{U}$ as the space that parametrizes the operators $\bar{\partial}_t$ where $\bar{\partial}_t$ acts on the following spaces:

$$\bar{\partial}_t : \Gamma(X_t, \Omega^{0,1}_t \otimes \Theta_t) \to \Gamma(X_t, \Omega^{0,1}_t \otimes \Theta_t \otimes \Omega^{0,1}_t)$$

Let

$$\bar{\partial}^*_t \bar{\partial}_t = \nabla_t$$

where $\bar{\partial}^*_t$ is the conjugate of $\bar{\partial}_t$ with respect to the metric g(t).

Then the following THEOREM is true:

THEOREM (QUILLEN).

$\partial\bar{\partial}\log(\exp(-\zeta'_{\nabla_t}(0))) = \partial\bar{\partial}\log\|\det\bar{\partial}_t\|_Q$ is just the WEIL-PETRSSON METRIC on U.

REMARK. This THEOREM was also proved by FUJIKI and SCHUMACHER.

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