A Characterization of Orbit Closure and Applications

by

Fritz Grunewald

and

Joyce O'Halloran

Fritz Grunewald
Mathematisches Institut
Universität Bonn
Bonn

BRD

Joyce O'Halloran
University of Wisconsin-Milwaukee
Milwaukee
WI 53201

USA

MPI 86-59

A Characterization of Orbit Closure and Applications

Fritz Grunewald
Mathematisches Institut, Universität Bonn, Bonn, BRD

Joyce O'Halloran*
University of Wisconsin-Milwaukee, Milwaukee, WI 53201, USA

In their book on representation varieties, Lubotsky and Magid give a useful characterization of orbit closure in representation varieties [2, 1.24]. Here we adapt this characterization to arbitrary categories. Applying this characterization of orbit closure to the category of Lie algebras and to the category of 2-cocycles for a fixed Lie algebra, we establish the following orbit space homeomorphism.

Let $\underset{\sim}{\mathcal{G}}$ be an n-dimensional Lie algebra over an algebraically closed field k. For each 2-cocycle B in $Z^2(\underset{\sim}{\mathcal{G}}, k^r)$, there is a central extension $\underset{\sim}{\mathcal{G}}(B)$ of $\underset{\sim}{\mathcal{G}}$ by k^r constructed as follows. On the vector space $\underset{\sim}{\mathcal{G}} \oplus k^r$, define a Lie product $[\ ,\ ]_B$ by:

$$[x+a, y+b]_B = [x,y]_{\underset{\sim}{\mathcal{G}}} + B(x,y),\ x,y \in \underset{\sim}{\mathcal{G}},\ a,b \in k^r.$$

Let $B^{\perp} = \{x \in \underset{\sim}{\mathcal{G}} \mid B(x, \underset{\sim}{\mathcal{G}}) = 0\}$. If $B^{\perp} \cap Z(\underset{\sim}{\mathcal{G}}) = 0$, then $Z(\underset{\sim}{\mathcal{G}}(B)) = k^r$. Let $\underset{\sim}{B} = \{B \in Z^2(\underset{\sim}{\mathcal{G}}, k^r) \mid B^{\perp} \cap Z(\underset{\sim}{\mathcal{G}}) = 0\}$ and let

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$\underline{L} = \{\underline{G}(B) \mid B \in \underline{B}\}$. Let G be the subgroup of $GL(\underline{G} \oplus k^r)$ consisting of elements of the form $\begin{bmatrix} \alpha & 0 \\ \theta & \psi \end{bmatrix}$, where $\alpha \in \mathrm{Aut}\,\underline{G}$ and $\psi \in GL(k^r)$. G acts on $\underline{B}$ by:

$$(g \cdot B)(x,y) = \psi B(\alpha^{-1}x, \alpha^{-1}y) + \theta[\alpha^{-1}x, \alpha^{-1}y]_{\underline{G}}$$

Skjelbred and Sund [5] established that the correspondence $B \to \underline{G}(B)$ induces a bijection between the G-orbits of $\underline{B}$ and the isomorphism classes of $\underline{L}$ (see [4] also for a discussion of this result). We show that this bijection is a homeomorphism of the orbit spaces. In Section 3 we apply this result to a specific example.

We would like to thank Andy Magid for pointing out that the orbit closure characterization for representations might carry over to the setting of Lie algebras.

§1. A characterization of orbit closure

Let k be an algebraically closed field. We consider functors on the category of k-algebras, i.e. commutative associative k-algebras with identity. The variety morphism defined by the k-algebra homomorphism $\eta: k[Y] \to k[X]$ is denoted $\bar{\eta}: X \to Y$.

Let $\mathcal{S}$ be a functor from the category of k-algebras to the category of sets; for a k-algebra homomorphism $f: A \to B$, let $f^*: \mathcal{S}(A) \to \mathcal{S}(B)$ be the corresponding map of sets. Assume that $\mathcal{S}$ satisfies the following conditions:

1) $\mathcal{S}(k)$ is a variety with coordinate ring S and there is an element $s_u \in \mathcal{S}(S)$ such that for $s_A \in \mathcal{S}(A)$, there is a homomorphism $\phi: S \to A$ such that $s_A = \phi^*(s_u)$.

2) For $v \in \mathcal{S}(k)$, $ev_v^*(s_u) = v$.

From this it follows that for any variety morphism $\bar{\phi}: X \to \mathcal{S}(k)$, given by $\phi: S \to k[X]$, we have $\bar{\phi}(x) = ev_{\bar{\phi}(x)}^*(s_u) = (ev_x \circ \phi)^*(s_u) = ev_x^*(\phi^* s_u)$.

3) If A is the coordinate ring of an affine set and t and u are elements of $\mathcal{S}(A)$ which satisfy $\phi^*(t) = \phi^*(u)$ for all homomorphisms $\phi: A \to k$, then $t = u$.

4) There is a group scheme action of $\mathcal{G}$ on $\mathcal{S}$, where $\mathcal{G}$ is an affine algebraic group scheme. In particular, the following diagram commutes for all homomorphisms $f: A \to B$:

$$\begin{array}{ccc} \mathcal{G}(A) \times \mathcal{S}(A) & \to & \mathcal{S}(A) \\ f^*\times f^* \downarrow & & \downarrow f^* \\ \mathcal{G}(B) \times \mathcal{S}(B) & \to & \mathcal{S}(B) \end{array}$$

5) Let K be the quotient field of $k[Z]$, Z an affine variety, so we have the inclusions $k \xrightarrow{i} k[Z] \xrightarrow{j} K$. Let $s_k \in \mathcal{S}(k)$, $s_z \in \mathcal{S}(k[Z])$, $g \in \mathcal{G}(K)$ satisfy $g\cdot((ji)^* s_k) = j^* s_z$. Then for all $x \in Z$ such that $g(x)$ is defined and has non-zero determinant, $g(x)\cdot s_k = ev_x^*(s_z)$.

<u>Lemma 1.1</u>. Let $\bar{\rho}: Y \to X$ be a dominant morphism from an affine set Y to a variety X, inducing the injective ring homomorphism $\rho: k[X] \to k[Y]$. Then there is a finite extension K of $k(X)$ and a homomorphism $q: k[Y] \to K$ such that $q\,\rho$ is the inclusion of $k[X]$ into K.

<u>Proof</u>: By considering $\bar{\rho}$ on irreducible components of Y, it suffices to establish the lemma for the case that Y is a variety.

By the Noether Normalization Lemma, there are algebraically independent elements $x_1,\dots,x_n$ in $k[X]$ such that $k[X]$ is an integral extension of $k[x_1,\dots,x_n]$. Let $y_i = \rho(x_i)$ and extend the set $\{y_1,\dots,y_n\}$ to a separating transcendence basis $\{y_1,\dots,y_r\}$ of $k(Y)$ in $k[Y]$. By the Noether Normalization Lemma, $k[Y]$ is an integral extension of $k[y_1,\dots,y_r]$. Let $\psi: k[y_1,\dots,y_r] \to k[x_1,\dots,x_n]$ be the homomorphism defined by

$$\psi(y_i) = x_i,\ 1 \le i \le n, \quad \text{and} \quad \psi(y_i) = 0,\ i > n.$$

The kernel of ψ, P_0, is a prime ideal (since the image of ψ is an integral domain). By the Going Up Theorem, there is a prime ideal P in $k[Y]$ with

$$P \cap k[y_1,\ldots,y_r] = P_0.$$

By construction, the quotient field K of $k[Y]/P$ is a finite extension of $k(X)$. The map $q\colon k[Y] \to K$ given by the quotient map $k[Y] \to k[Y]/p$ has the desired property. (For the commutative algebra theorems, see [3].)

The following characterization of orbit closure is a generalization of a theorem of Lubotsky and Magid [2, 1.24] for representation varieties.

<u>Theorem 1.2</u>. Let s_0, s_1 be elements of $\mathcal{S}(k)$. Then $s_0 \in \overline{\mathcal{O}(s_1)}$ if and only if there is a discrete valuation k-algebra A with residue field k, whose quotient field K is finitely generated over k of transcendence degree one, and an element s_A of $\mathcal{S}(A)$ such that

$$\tau^*(s_A) = g\cdot((\tau\eta)^* s_1) \quad \text{for some} \quad g \in \mathcal{G}(K)$$
$$\pi^*(s_A) = s_0$$

where η, τ, and π are the homomorphisms shown:

$$\begin{array}{ccccc} k & \xrightarrow{\eta} & A & \xrightarrow{\tau} & K \\ & & \downarrow{\scriptstyle \pi} & & \\ & & A/M = k & & \end{array}$$

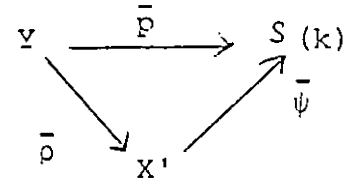
Proof: Suppose $s_0 \in \overline{O(s_1)}$. Choose an irreducible curve X in $\overline{O(s_1)}$ containing s_0 and s_1 and let $X' = X \cap O(s_1)$. X' is an affine open subset of X. Let Y be an affine subset of $G(k)$ whose image is Zariski dense in X' under the orbit map $\bar{p}$: $G(k) \to S(k)$ $(\bar{p}(g) = g \cdot s_1)$. Let $r: k \to k[Y]$ be the inclusion map and let $Y_{ij} \in k[Y]$ be the matrix coordinates on $G(k)$ restricted to Y. Then for $g \in Y$, we have

$$\begin{aligned} ev_g^*([Y_{ij}] \cdot r^*(s_1)) &= g \cdot s_1 && \text{by 4)} \\ &= ev_g^*(p^*(s_u)) && \text{by 2)} \end{aligned}$$

It follows from 3) that $[Y_{ij}] \cdot r^*(s_1) = p^*(s_u)$. Let $s_Y = p^*(s_u)$.

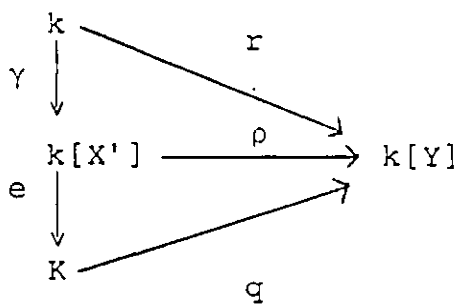
Because the image of $\bar{p}$ is contained in X', the diagram below commutes, where $\bar{\rho}(y) = \bar{p}(y)$ and $\bar{\psi}$ is the inclusion map. If we let $s_{X'} = \psi^*(s_u)$, then

$$\rho^*(s_{X'}) = s_Y = [Y_{ij}] \cdot r^*(s_1).$$

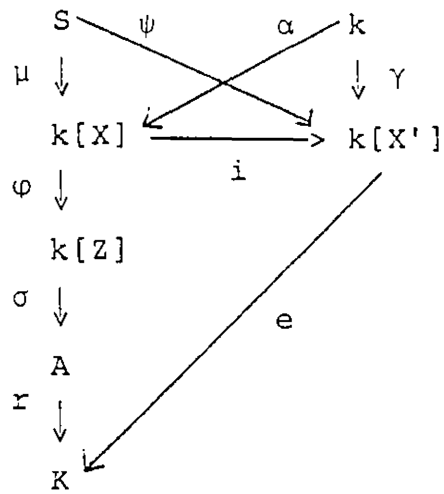


From Lemma 1.1, we have a finite extension K of $k(X')$ and a homomorphism $q: k[Y] \to K$ such that $q \circ \rho = e$, where e is the inclusion of $k[X']$ into K. From the diagram below, we have:

$$\begin{aligned} e^*(s_{X'}) &= (q\rho)^*(s_{X'}) \\ &= q^*([Y_{ij}]\cdot r^*s_1) \\ &= (q^*[Y_{ij}])\cdot((qr)^*s_1) \qquad \text{by 4)} \\ &= g\cdot((e\gamma)^*s_1) \quad \text{where} \quad g = q^*[Y_{ij}] \in G(K). \end{aligned}$$



Since $\bar{i}: X' \to X$ is a dominant morphism, $i: k[X] \to k[X']$ is an inclusion. Let $k[Z]$ be the integral closure of $ei(k[X])$ in K. Then the inclusion $\phi: k[X] \to k[Z]$ induces a surjection $\bar{\phi}: Z \to X$. (The morphism $\bar{\phi}$ is onto because $k[Z]$ is the integral closure of $k[X]$.) Because $\bar{\phi}$ is surjective, there is an element z_0 in Z such that $\bar{\phi}(z_0) = s_0$. Let A be the local ring of Z at z_0, and let $\sigma: k[Z] \to A$ and $\tau: A \to K$ be the inclusion maps.

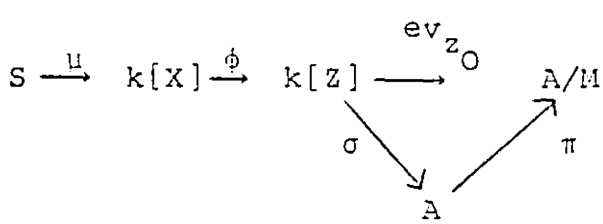


Let $s_X = \mu^*(s_u)$ and let $s_A = (\sigma\phi)^*(s_X)$. Then $s_{X'} = i^*(s_X)$ and we have:

$$\tau^*(s_A) = (\tau\sigma\phi)^*(s_X)$$
$$= e^*(i^* s_X)$$
$$= e^*(s_{X'})$$
$$= g\cdot((e\gamma)^* s_1) \quad \text{from above.}$$

Therefore $\tau^*(s_A) = g\cdot((\tau\eta)^* s_1)$, where η is the inclusion $\sigma\phi\alpha$.

From the inclusion $\{z_0\} \subset Z$, we have the homomorphism $ev_{z_0} : k[Z] \to A/M$ and the diagram below:



Then

$$s_O = \bar{\phi}(z_O) = ev_{z_O}^*((\phi\mu)^* s_u) \qquad \text{by } 2)$$
$$= \pi^*(\sigma\phi\mu)^* s_u$$
$$= \pi^* s_A$$

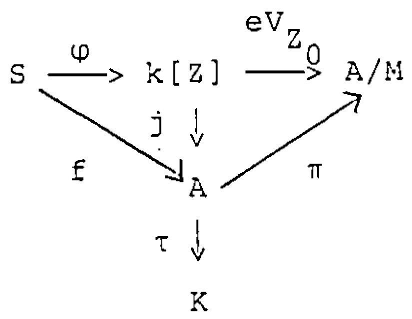
Therefore $\pi^* s_A = s_O$.

Suppose there is a discrete valuation k-algebra A with residue field k, whose quotient field K is finitely generated over k of transcedence degree one, and an element s_A of $S(A)$ such that $\tau^* s_A = g\cdot((\tau\eta)^* s_1)$ for some $g \in G(K)$ and $\pi^* s_A = s_O$.

By property 1), there is a homomorphism $f\colon S \to A$ such that $f^*(s_u) = s_A$. Choose an affine curve Z with $f(S) \subset k[Z] \subset A$ such that A is the local ring of $k[Z]$ at z_O. (Then $k(Z) = K$.)

Let $\bar{\phi}\colon Z \to S(k)$ be the variety morphism defined by $f\colon S \to k[Z]$, and let $j\colon k[Z] \to A$ and $\tau\colon A \to K$ be the inclusion maps. From the diagram below, we have:

$$\bar{\phi}(z_O) = ev_{z_O}^*(\phi^* s_u) \qquad \text{by } 2)$$
$$= \pi^*(j\phi)^* s_u$$
$$= \pi^* f\ s_u$$
$$= \pi^* s_A$$
$$= s_O$$



Let $s_z = \phi^*(s_u)$. Then $s_A = j^*(s_z)$. By hypothesis, there is some g in $G(K)$ such that $\tau^* s_A = g\cdot((\tau\eta)^* s_1)$. Then

$$g\cdot(\tau\eta)^* s_1 = \tau^* s_A = (\tau j)^* s_z.$$

Let Z' be the dense subset of Z consisting of points x such that $g(x)$ is defined and $\det g(x)$ is not zero. Then for $x \in Z'$, we have:

$$\begin{aligned} g(x)\cdot s_1 &= ev_x^* s_z && \text{by 5)} \\ &= ev_x^*(\phi^* s_u) \\ &= \bar{\phi}(x) && \text{by 2)} \end{aligned}$$

Since Z' is dense in Z and $\bar{\phi}(Z') \subset O(s_1)$, it follows that s_0 is in $\overline{O(s_1)}$.

□

§2. Equivalence of orbit closure for 2-cocycles and Lie algebras

In the introduction, we described the construction of a central extension $\underline{G}(b)$ of an n-dimensional Lie algebra $\underline{G}$ by k^r defined by a 2-cocycle b. We also described the action of a group G on a subset $\underline{B}$ of $Z^2(\underline{G}, k^r)$. For the convenience of the reader, we present here the proof of the theorem which appears in [5].

__Theorem 2.1__. The correspondence $b \to \underline{G}(b)$ induces a bijection between G-orbits of $\underline{B}$ and isomorphism classes of Lie algebras without direct abelian factor which are central extensions of $\underline{G}$ by k^r and have r-dimensional center.

__Proof__: From the construction of $\underline{G}(b)$, it is easy to see that for $b \in \underline{B}$, $\underline{G}(b)$ has no direct abelian factor and the center of $\underline{G}(b)$ has dimension r.

Suppose $b_1 = \phi \cdot b_2$ for some $\phi \in G$,

$$\phi = \begin{bmatrix} \alpha & 0 \\ \theta & \psi \end{bmatrix} .$$

Then we see that $\underline{G}(b_1)$ is isomorphic to $\underline{G}(b_2)$ via ϕ:

$$[x+a, y+c]_{\phi \cdot \underline{G}(b_2)} = \phi[\phi^{-1}(x+a), \phi^{-1}(y+c)]_{\underline{G}(b_1)} \quad \text{for } x,y \in \underline{G},\ a,c \in k^r$$

$$= \phi([\alpha^{-1}x, \alpha^{-1}y]_{\underline{G}} + b_2(\alpha^{-1}x, \alpha^{-1}y))$$

$$= \alpha[\alpha^{-1}x, \alpha^{-1}y]_{\underline{G}} + \theta[\alpha^{-1}x, \alpha^{-1}y]_{\underline{G}} + \psi b_2(\alpha^{-1}x, \alpha^{-1}y)$$

$$= [x,y]_{\underline{G}} + \theta[\alpha^{-1}x,\alpha^{-1}y]_{\underline{G}} + \psi b_2(\alpha^{-1}x,\alpha^{-1}y)$$

since $\alpha \in \text{Aut}\,\underline{G}$

$$= [x,y]_{\underline{G}(b_1)}$$

Suppose $\underline{G}(b_1)$ is isomorphic to $\underline{G}(b_2)$ via an isomorphism ϕ. Since b_1 and b_2 are in $\underline{B}$, the centers of $\underline{G}(b_1)$ and $\underline{G}(b_2)$ have dimension r, so ϕ induces

$$\bar{\phi}: \underline{G}(b_1)/Z(\underline{G}(b_1)) \to \underline{G}(b_2)/Z(\underline{G}b_2)).$$

By the construction of $\underline{G}(b_i)$, we see that $\underline{G}(b_i)/Z(\underline{G}(b_i))$ is isomorphic to $\underline{G}$. Thus $\bar{\phi}$ is an automorphism of $\underline{G}$. Fix a basis $\{e_1,\ldots,e_n\}$ for $\underline{G}$ and a basis $\{e_{n+1},\ldots,e_{n+r}\}$ for k^r. Then the matrix of ϕ relative to this basis is

$$\begin{bmatrix} \alpha & 0 \\ \theta & \psi \end{bmatrix}, \quad \alpha \in \text{Aut}\,\underline{G}, \ \psi \in GL_r(k), \quad \theta \in \text{Hom}(\underline{G},k^r).$$

Let $[\ ,\ ]_i$ denote the Lie product $\underline{G}(b_i)$ on $\underline{G} \oplus k^r$. For $x,y \in \underline{G}$, we have

$$\phi[x,y]_1 = [\phi x,\phi y]_2$$

$$= [\alpha x,\alpha y]_{\underline{G}} + b_2(\alpha x,\alpha y)$$

Also we have

$$\phi[x,y]_1 = \phi([x,y]_{\underset{\sim}{G}} + b_1(x,y))$$

$$= \alpha[x,y]_{\underset{\sim}{G}} + \theta[x,y]_{\underset{\sim}{G}} + \psi b_1(x,y)$$

From these two equations, it follows that

$$b_2(x,y) = \theta[\alpha^{-1}x,\alpha^{-1}y]_{\underset{\sim}{G}} + \psi b_1(\alpha^{-1}x,\alpha^{-1}y).$$

Therefore b_1 and b_2 are in the same G-orbit. □

In order to show that the bijection between orbits of 2-cocycles and isomorphism classes of Lie algebras preserves orbit closure, we introduce two functors with group scheme actions and apply Theorem 1.2.

Let $\underset{\sim}{G}$ be an n-dimensional Lie algebra over an algebraically closed field k. Define the functor $\mathcal{B}^r_{\underset{\sim}{G}}$ from the category of k-algebras to the category of sets by

$$\mathcal{B}^r_{\underset{\sim}{G}}(A) = \{b \in Z^2(\underset{\sim}{G} \otimes_k A, A^r) \mid b^\perp \cap Z(\underset{\sim}{G} \otimes_k A) = 0\}.$$

Fix a basis $\{e_i\}_{i=1}^n$ for $\underset{\sim}{G}$. Then a 2-cocycle b in $\mathcal{B}^r_{\underset{\sim}{G}}(A)$ is given by its values on the pairs of basis elements $(e_i \otimes 1_A, e_j \otimes 1_A)$; $b(e_i \otimes 1_A, e_j \otimes 1_A) = (a^s_{ij})^r_{s=1}$. For a homomorphism $f\colon A \to C$, let $f^*(b) = b \otimes_A C$. That is, for b in $Z^2(\underset{\sim}{G} \otimes A, A^r)$ given by $([a^s_{ij}])^r_{=1}$ define $f^*(b)$ in $Z^2(\underset{\sim}{G} \otimes C, C^r)$ by

$$([a^s_{ij} \otimes {}_A 1_B])^r_{s=1} = ([f(a^s_{ij})])^r_{s=1}.$$

$\underline{B}^r_{\underline{G}} = \mathcal{B}^r_{\underline{G}}(k)$ is a closed subvariety of $\mathrm{Hom}(\Lambda^2\underline{G}, k^r)$; let B denote its coordinate ring and let $\{X^s_{ij}\}_{1\leq i,j\leq n, 1\leq s\leq r}$ be the matrix coordinates. Let b_u in $\mathcal{B}^r_{\underline{G}}(B)$ be the 2-cocycle given by $([X^s_{ij}])^r_{s=1}$. If A is a k-algebra and b in $\mathcal{B}^r_{\underline{G}}(A)$ is given by $([a^s_{ij}])^r_{s=1}$, define $\phi: B \to A$ by $\phi(X^s_{ij}) = a^s_{ij}$. Then $b = \phi^*(b_u)$.

Let G be the closed sub-group scheme of $G\ell_{n+r}$ given by

$$G(A) = \{\begin{bmatrix}\alpha & 0\\ \theta & \psi\end{bmatrix} \mid \alpha \in \mathrm{Aut}(\underline{G}\otimes A),\ \psi \in G\ell(A^r),\ \theta \in \mathrm{Hom}(A^n, A^r)\}.$$

$G(A)$ acts on $\mathcal{B}^r_{\underline{G}}(A)$ by

$$\begin{bmatrix}\alpha & 0\\ \theta & \psi\end{bmatrix}\cdot b(x,y) = \psi b(\alpha^{-1}x, \alpha^{-1}y) + \theta[\alpha^{-1}x, \alpha^{-1}y]_{\underline{G}} .$$

It is easy to see that the functor $\mathcal{B}^r_{\underline{G}}$ with this action of the group scheme G satisfies the conditions listed in Section 1.

By "n-dimensional Lie algebra over A" we mean a Lie product on the free A-module on n generators $\{e^A_i\}^n_{i=1}$. Then the Lie algebra is uniquely determined by its structure constants (a^t_{ij}) relative to the generators $\{e^A_i\}^n_{i=1}$ $(L(e^A_i, e^A_j) = \sum_{t=1}^n a^t_{ij}e^A_t)$. If $f: A \to B$ is a k-algebra homomorphism, identify e^B_i with $e^A_i \otimes_A 1_B$.

Let L_n be the functor from the category of k-algebras to the category of sets defined by

$$L_n(A) = \{\text{n-dimensional Lie algebras over } A\}.$$

If $f: A \to B$ is a k-algebra homomorphism, let $f^*(L) = L \otimes_A B$. If the structure constants for L relative to $\{e_i^A\}_{i=1}^n$ are (a_{ij}^t), then, via the identification above, the structure constants for $f^*(L)$ relative to $\{e_i^B\}_{i=1}^n$ are $(f(a_{ij}^t))$.

$\underline{L}_n = L_n(k)$ is a closed subvariety of $\mathrm{Hom}(\Lambda^2 k^n, k^n)$; let L denote its coordinate ring and let L_u be the element of $L_n(L)$ with structure constants (X_{ij}^t). As above, for each $M \in L_n(A)$, there is a homomorphism $\phi: L \to A$ such that $M = \phi^*(L_u)$.

$G\ell_n(A)$ acts on $L_n(A)$ by change of basis:

$$(g \cdot L)(x,y) = g(L(g^{-1}x, g^{-1}y)).$$

The functor L_n with this action of $G\ell_n$ satisfies the conditions listed in Section 1.

Let $\{e_i\}_{i=1}^n$ be a basis for $\underline{G}$ and let $\{e_i\}_{i=n+1}^{n+r}$ be a basis for k^r. For any k-algebra A, let $e_i^A = e_i \otimes_k 1_A$ be generators for the free A-modules $\underline{G}_A = \underline{G} \otimes_k A$ and $A^r = k^r \otimes_k A$. The A-module $\underline{G}_A$ has the Lie product given by $\underline{G}$. For $b_A \in B_{\underline{G}}^r(A)$, let $\underline{G}_A(b_A)$ be the Lie product $[\ ,\ ]_{b_A}$ on $A^{n+r} = \underline{G}_A \oplus A^r$ defined by

$$[e_i^A, e_j^A]_{b_A} = [e_i^A, e_j^A]_{\underline{G}_A} + b_A(e_i^A, e_j^A) \quad \text{if } 1 \le i, j \le n$$

$$[e_i^A, e_j^A]_{b_A} = 0 \qquad \text{otherwise.}$$

For a k-algebra homomorphism $f: A \to C$, it is easy to see that

$$f^*(\underline{G}_A(b_A)) = \underline{G}_A(b_A) \otimes_A C = \underline{G}_C(f^*(b_A)).$$

As we have seen in Theorem 2.1, the correspondence from $\underline{B}_{\underline{G}}^r$ to $\underline{L}_{n+r}$ given by $b \to \underline{G}(b)$ induces a bijection between orbits in $\underline{B}_{\underline{G}}^r$ and isomorphism classes of central extensions of $\underline{G}$ by k^r with no direct abelian factor and r-dimensional center. Using Theorem 1.2, we show that orbit closure is preserved under this correspondence.

<u>Theorem 2.2</u>.

For b_0 and b_1 in $\underline{B}_{\underline{G}}^r$, $b_0 \in \overline{O(b_1)}$ if and only if $\underline{G}(b_0) \in \overline{O(\underline{G}(b_1))}$.

<u>Proof</u>: If $b_0 \in \overline{O(b_1)}$, then from Theorem 1.2, we know that there is a discrete valuation k-algebra A with residue field k and quotient field K and an element b_A of $\underline{B}_{\underline{G}}^r(A)$ such that

$$b_A \otimes_A A/M = b_0 \text{ and } b_A \otimes_A K = g\cdot(b_1 \otimes_k K) \quad \text{for some} \quad g \in G(K).$$

From the remarks above, we have

$$\underline{G}_A(b_A) \otimes_A K = \underline{G}_K(b_A \otimes_A K) = \underline{G}_K(g\cdot(b_1 \otimes_k K)).$$

If $g = \begin{bmatrix} \alpha & 0 \\ \theta & \psi \end{bmatrix}$, then $g\cdot \underline{G}_K(b_1 \otimes_k K)$ has Lie product

$$g([g^{-1}(x+s), g^{-1}(y+t)]_{\underline{G}_K(b_1 \otimes_k K)}) \text{ for } x,y \in \underline{G}_K;\ s,t \in K^r$$

$$= g([\alpha^{-1}x, \alpha^{-1}y]_{\underline{G}_K} + (b_1 \otimes_k K)(\alpha^{-1}x, \alpha^{-1}y))$$

$$= [x,y]_{\underline{G}_K} + \theta[\alpha^{-1}x, \alpha^{-1}y]_{\underline{G}_K} + \psi(b_1 \otimes_k K)(\alpha^{-1}x, \alpha^{-1}y)$$

because $\alpha \in \mathrm{Aut}\ \underline{G}_K$

$$= [x+s, y+t]_{\underline{G}_K(g\cdot(b_1 \otimes_k K))}$$

Therefore $\underline{G}_A(b_A) \otimes_A K = g\cdot(\underline{G}(b_1) \otimes_k K)$.

Because $b_0 = b_A \otimes_A A/M$, we have

$$\underline{G}(b_0) = \underline{G}(b_A \otimes_A A/M) = \underline{G}_A(b_A) \otimes_A A/M$$

It follows from Theorem 1.2 that $\underline{G}(b_0) \in \overline{O(\underline{G}(b_1))}$.

Suppose that $\underline{G}(b_0) \quad \overline{O(\underline{G}(b_1))}$. By Theorem 1.2, there is a discrete valuation k-algebra A with residue field k and quotient field K and $L \in L_n(A)$ such that

$$L \otimes_A K \approx \underline{G}(b_1) \otimes_k K \quad \text{and} \quad L \otimes_A A/M = \underline{G}(b_0).$$

Let $\phi: L \otimes_A K \to \underline{G}(b_1) \otimes_k K$ be the isomorphism and let $\{x_i\}_{i=1}^{n+r}$ be a basis for L. Let M be the A-submodule of $\underline{G}(b_1) \otimes K$ generated by $\{\phi x_i\}_{i=1}^{n+r}$. Because ϕ is an isomorphism of Lie algebras, we have

$$[\phi x_i, \phi x_j] = \phi[x_i, x_j] = \phi(\sum_s c_{ij}^s x\ ) = \sum_s \ c_{ij}^s \phi(x_s),$$

where $c_{ij}^s \in A$. Thus M is a Lie algebra over A which is isomorphic to L. Since K is the quotient field of A, there is an element a in A such that $\{a\phi(x_i)\}_{i=1}^{n+r}$ is a basis for $\underline{G}_A(b_1 \otimes {}_k A) \otimes_A 1_K$. Therefore $\underline{G}_A(b_1 \otimes {}_k A) \otimes_A 1_A \subset M$. But $\dim_A M = \dim_A \underline{G}_A(b_1 \otimes {}_k A)$, so $M = \underline{G}_A(b_1 \otimes {}_k A) \otimes_A 1_K$. It follows that ϕ induces an isomorphism between L and $\underline{G}_A(b_1 \otimes {}_k A)$.

Because L is isomorphic to $\underline{G}_A(b_1 \otimes {}_k A)$, the dimension of $Z(L)$ is r. Let $\underline{G}_A^L = L/Z(L)$ and define $b'\colon \underline{G}_A^L \times \underline{G}_A^L \to A^r$ as follows. Let L' be a subspace of L complementary to $Z(L)$ and let $\pi\colon L' \oplus Z(L) \to Z(L)$ be the projection. For $x,y \in L$, define b' by

$$b'(\bar{x},\bar{y}) = \pi[x,y]_L.$$

Then $L = \underline{G}_A^L(b')$.

Because ϕ is a Lie algebra isomorphism from $\underline{G}_A^L(b')$ to $\underline{G}_A(b_1 \otimes {}_k A)$, $\phi(Z(\underline{G}_A^L(b')) = Z(\underline{G}_A(b_1 \otimes {}_k A))$ and so ϕ induces an isomorphism $\bar{\phi}\colon \underline{G}_A^L \to \underline{G}_A$. Identify $\underline{G}_A^L$ with $\underline{G}_A$ via $\bar{\phi}$, so that $b\colon \underline{G}_A \times \underline{G}_A \to A^r$ is given by

$$b(x,y) = b'(\bar{\phi}^{-1}x, \bar{\phi}^{-1}y) \quad \text{and} \quad L = \underline{G}_A(b).$$

Then $\underline{G}_A(b)$ is isomorphic to $\underline{G}_A(b_1 \otimes {}_k A)$. Since ϕ preserves the center, the matrix of ϕ is of the form

$$\phi = \begin{bmatrix} \alpha & 0 \\ \theta & \psi \end{bmatrix}$$

relative to a basis $\{e_i^A\}_{i=1}^{n+r}$ where $\{e_i^A\}_{i=1}^{n}$ is a basis for $\underline{G}_A$ and $\{e_i^A\}_{i=n+1}^{n+r}$ is a basis for A^r. For $x,y \in \underline{G}_A$, we have

$$\phi([x,y]_{\underline{G}_A(b)}) = [\phi x,\phi y]_{\underline{G}_A(b_1 \otimes_k A)}$$

So $\alpha[x,y]_{\underline{G}_A} + (\phi[x,y]_{\underline{G}_A} + \psi b(x,y)) = [\alpha x,\alpha y]_{\underline{G}_A} + (b_1 \otimes_k A)(\alpha x,\alpha y)$.

It follows that $\alpha[x,y]_{\underline{G}_A} = [\alpha x,\alpha y]_{\underline{G}_A}$ for all $x,y \in \underline{G}_A$, i.e. $\alpha \in \mathrm{Aut}\ \underline{G}_A$, and $b_1 \otimes_k A = \phi \cdot b$. Therefore

$$b \otimes_A K = \phi^{-1} \cdot (b_1 \otimes_k K),$$ considering ϕ^{-1} as an element of $Gl_{n+r}(K)$.

Because $\underline{G}(b_0) = L \otimes_A A/M$, we have

$$\underline{G}(b_0) = L \otimes_A A/M = \underline{G}_A(b) \otimes_A A/M = \underline{G}(b \otimes_A A/M).$$

Therefore $b_0 = b \otimes_A A/M$.

It follows from Theorem 1.2 that $b_0 \in \overline{O(b_1)}$.

□

§3. An application of the orbit closure characterization for 2-cocycles and Lie algebras

By comparing invariants like the dimensions of the upper and lower central series, one can often establish that a Lie algebra L is not in the closure of the orbit of a Lie algebra M (see [1] for examples). One case where this method fails is the case of the two central extensions of $\underline{G} = \underline{G}_3 \times \underline{G}_1$ by k^2 whose structure is given in Table I. ($\underline{G}_3$ is the non-abelian 3-dimensional nilpotent Lie algebra and $\underline{G}_1$ is the 1-dimensional abelian Lie algebra.) In this case $L = \underline{G}(B_0)$ and $M = \underline{G}(B_1)$ where B_0 and B_1 are given in Table II.

Proposition 3.1. L is not in the closure of the orbit of M.

Proof: By Theorem 2.1, L is in the closure of the orbit of M if and only if B_0 is in the closure of the orbit of B_1.

Suppose B_0 is in the closure of the orbit of B_1. Then by Theorem 1.2, there is a coordinate ring $k[Z]$ for some affine set Z, an element g in $G(k(Z))$, and an element $x \in Z$ such that B_0 is the evaluation of $g \cdot B_1$ at x.

The element g is of the form

$$\begin{bmatrix} \alpha & 0 \\ \theta & \psi \end{bmatrix} \quad \text{where} \quad \alpha^{-1} = \begin{bmatrix} a & b & 0 & s \\ c & d & 0 & t \\ e & f & w & u \\ g & h & 0 & v \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} p & y \\ q & z \end{bmatrix}$$

We have:

$$g \cdot B_1(e_1, e_3) = \psi(cwx_1) = pcwx_1 + qcwx_2$$

$$g \cdot B_1(e_2, e_3) = \psi(dwx_1) = pdwx_1 + qdwx_2$$

If $g \cdot B_1$ evaluated at x is B_0, then we have:

$$1 = p(x)c(x)w(x) = q(x)d(x)w(x)$$

$$0 = p(x)d(x)w(x)$$

It follows that $0 = p(x)d(x)w(x) = d(x)/c(x)$, so $d(x) = 0$; but this contradicts the statement $1 = q(x)d(x)w(x)$.

Therefore L is not in the closure of the orbit of M.

□

Table I

$L(e_1,e_2) = e_4$	$M(e_1,e_2) = e_4$	$G_3(e_1,e_2) = e_3$
$L(e_1,e_3) = e_5$	$M(e_1,e_4) = e_5$	$G_3(e_i,e_j) = 0$
$L(e_2,e_4) = e_5$	$M(e_2,e_4) = e_6$	other $i \leq j$
$L(e_2,e_3) = e_6$	$M(e_2,e_3) = e_5$	
$L(e_i,e_j) = 0$	$M(e_i,e_j) = 0$	
other $i \leq j$	other $i \leq j$	

Table II

$B_0(e_1,e_3) = x_1$	$B_1(e_1,e_4) = x_1$
$B_0(e_2,e_3) = x_2$	$B_1(e_2,e_3) = x_1$
$B_0(e_2,e_4) = x_1$	$B_1(e_2,e_4) = x_2$
$B_0(e_i,e_j) = 0$	$B_1(e_i,e_j) = 0$
other $i \leq j$	other $i \leq j$

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