ONE GROUP-THEORETIC PROPERTY OF THE RAMIFICATION FILTRATION

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ABSTRACT. Let $\Gamma(p)$ be the Galois group of a maximal p-extension of a complete discrete valuation field with perfect residue field of characteristic $p > 0$. If $v_0 > -1$ and $\Gamma(p)^{(v_0)}$ is the ramification subgroup of $\Gamma(p)$ in upper numbering, we prove that any closed but not open finitely generated subgroup of the quotient $\Gamma(p)/\Gamma(p)^{(v_0)}$ is a free pro-p-group. In particular, this quotient does not have non-trivial torsion and non-trivially commuting elements.

1. The statement of the main theorem.

Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$. Choose a separable closure K_{sep} of K and denote by $K(p)$ the maximal p-extension of K in K_{sep}.

If $\Gamma = \text{Gal}(K_{\text{sep}}/K)$ and $\{\Gamma^{(v)}\}_{v\geq 0}$ is the ramification filtration of Γ in upper numbering, cf. [Se, Ch.III], we have the induced filtration $\{\Gamma(p)^{(v)}\}_{v\geq 0}$ of the group $\Gamma(p) = \text{Gal}(K(p)/K)$. We note that for $-1 < v \leqslant 1$, $\Gamma(p)^{(v)} = I(p)$ is the inertia subgroup of $\Gamma(p)$, i.e. $K(p)^{I(p)}$ is the maximal unramified extension $K(p)_{\text{ur}}$ of K in $K(p)$.

Consider a real number $v_0 > -1$ and a closed subgroup H of $\Gamma(p)$ such that $H \supset \Gamma(p)^{(v_0)}$. If $\widetilde{H} = H/\Gamma(p)^{(v_0)}$, then $\widetilde{H}$ is a closed subgroup of $\Gamma(p)/\Gamma(p)^{(v_0)}$. We use the notation $d(\widetilde{H}) = \text{rk}_{\mathbf{Z}/p\mathbf{Z}}\left(\widetilde{H}/\widetilde{H}^p[\widetilde{H},\widetilde{H}]\right)$ for the minimal number of topological generators of the pro-p-group $\widetilde{H}$.

If $-1 < v_0 \leqslant 1$, then $\Gamma(p)/\Gamma(p)^{(v_0)}$ is a free pro-p-group, because it coincides with the Galois group of the maximal p-extension of the residue field k. So, in this case $\widetilde{H}$ is a free pro-p-group.

Suppose that $v_0 > 1$ and $\widetilde{H}$ is an open subgroup in $\Gamma(p)/\Gamma(p)^{(v_0)}$. Then H is an open subgroup in $\Gamma(p)$, $K_H := K(p)^H$ is a finite extension of K, $H = \Gamma_{K_H}(p)$ is the Galois group of the maximal p-extension $K(p)$ of K_H, and $\Gamma(p)^{(v_0)} = \Gamma_{K_H}(p)^{(v_{0H})}$ with $v_{0H} = \psi_{K_H/K}(v_0)$, where $\psi_{K_H/K}$ is the inverse to the Herbrandt's function of the extension K_H/K. So, in this case the study of the group $\widetilde{H}$ is equivalent to the study of the group $\Gamma(p)/\Gamma(p)^{(v_0)}$. This group is very far from to be a free pro-p-group: if k is finite then the number of its relations is infinite, cf. [Go] (but it has finitely many generators).

In this paper we consider almost the opposite situation. The main result can be stated as follows.

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Theorem. *If $v_0 > -1$ and $\widetilde{H}$ is a closed but not open subgroup of the pro-p-group $\Gamma(p)/\Gamma(p)^{(v_0)}$, then $\widetilde{H}$ is a free pro-p-group.*

We have noted already that for $v_0 \leqslant 1$ this theorem holds because in this case the group $\Gamma(p)/\Gamma(p)^{(v_0)}$ is itself a free pro-p-group. So, in the proof of the above theorem (cf. nn. 2 and 3 below) we can assume that $v_0 > 1$.

Corollary 1. a) *If $v_0 > -1$ and k is infinite, then any finitely generated closed pro-p-subgroup $\widetilde{H}$ of $\Gamma(p)/\Gamma(p)^{(v_0)}$ is a free pro-p-group.*
b) *Any finitely generated closed pro-p-subgroup of $I(p)/\Gamma(p)^{(v_0)}$, where $v_0 > 1$, is a free pro-p-group.*

Proof. The part a) follows from the above theorem, because here any open subgroup of $\Gamma(p)/\Gamma(p)^{(v_0)}$ has infinitely many generators and therefore, $\widetilde{H}$ is not open. The part b) is a special case of the part a), where K is replaced by the p-adic completion $\widehat{K(p)}_{\mathrm{ur}}$ of its maximal unramified p-extension, because the residue field of $\widehat{K(p)}_{\mathrm{ur}}$ is infinite.

Corollary 2. *The group $\Gamma(p)/\Gamma(p)^{(v_0)}$ does not have non-trivial torsion and non-trivially commuting elements.*

Proof. We can assume that $v_0 > 1$. Then for any open subgroup $\widetilde{H} \subset \Gamma(p)/\Gamma(p)^{(v_0)}$ we have $d(\widetilde{H}) \geq 2$. Therefore, if $\widetilde{H}$ is closed in $\Gamma(p)/\Gamma(p)^{(v_0)}$ and $d(\widetilde{H}) = 1$, then $\widetilde{H}$ is pro-p-free. Clearly, this is equivalent to the absence of non-trivial torsion.

The existence of non-trivially commuting elements is equivalent to the existence of a closed commutative subgroup $\widetilde{H} \subset \Gamma(p)/\Gamma(p)^{(v_0)}$ such that $d(\widetilde{H}) = 2$. Our theorem implies that $\widetilde{H}$ is an open subgroup, so we can assume that $\widetilde{H} = \Gamma(p)/\Gamma(p)^{(v_0)}$, where v_0 is still > 1 (cf. proposition 1 c) of n.3.1.1 below). Then $d(\widetilde{H}) = 2$ if and only if $k \simeq \mathbb{F}_p$ and $v_0 \leq 2$. Consider the set $\widetilde{H}^p = \{h^p \mid h \in \widetilde{H}\}$. Then $\widetilde{H}^p$ is a commutative subgroup of $\widetilde{H}$ (because $\widetilde{H}$ is commutative), $(\widetilde{H} : \widetilde{H}^p) = p^2$ and $d(\widetilde{H}^p) = 2$ (because $\widetilde{H}$ has no torsion). Therefore, $\widetilde{H}^p = \Gamma_{K_1}(p)/\Gamma_{K_1}(p)^{(v_1)}$, where K_1 is an extension of K of degree p^2 and $v_1 = \psi_{K_1/K}(v_0) > 1$. It is easy to see that $[k_1 : \mathbb{F}_p] = p$, where k_1 is the residue field of K_1. This gives the contradiction $2 = d(\widetilde{H}^p) \geq 2p$. The corollary is proved.

The above corollary gives that: a) if $\tau \notin \Gamma(p)^{(v_0)}$, then for any $n \in \mathbb{N}$, $\tau^{p^n} \notin \Gamma(p)^{(v_0)}$; b) if $\tau_1, \tau_2 \notin \Gamma(p)^{(v_0)}$, but the commutator $(\tau_1, \tau_2) \in \Gamma(p)^{(v_0)}$, then for some $a \in \mathbb{Z}_p$, we have either $\tau_1 = \tau_2^a$, or $\tau_2 = \tau_1^a$. These properties mean that the ramification filtration does not have any relation to the p-central filtration of the group $\Gamma(p)$. One can find indication to such phenomena in the paper of E.Mauss [Ma]. In fact our theorem means that the group $\Gamma(p)/\Gamma(p)^{(v_0)}$ does not have "simple" relations, e.g. there is no relations which can be expressed in terms of any proper subset of some minimal set of generators of the group $\Gamma(p)/\Gamma(p)^{(v_0)}$. In the characteristic p case these relations modulo the subgroup of commutators of order $\geq p$ were described in terms of generators of the group $\Gamma(p)$ in the papers [Ab1-3].

Let $I = \bigcup_{v>0} \Gamma^{(v)}$ be the higher ramification subgroup in Γ. The following analogue of the main theorem holds for the ramification filtration of the Galois group Γ.

Corollary 3. *If $v_0 > 0$ and a group $\widetilde{H} \subset I/\Gamma^{(v_0)}$ is a finitely generated pro-p-group, then $\widetilde{H}$ is pro-p-free (in particular, $I/\Gamma^{(v_0)}$ does not have torsion and non-trivial commuting elements).*

Proof. Let $K_{\rm tr}$ be the maximal tamely ramified extension of K in $K_{\rm sep}$. Then $K_{\rm tr} = \varinjlim_{\alpha\in\mathcal{A}} K_\alpha$, where $\{K_\alpha \mid \alpha \in \mathcal{A}\}$ is the set of all finite tamely ramified Galois extension of K in $K_{\rm sep}$. We shall provide the above notation with a lower index α, if the notation is related to the field K_α. Clearly, the family of groups $\{I_\alpha(p) \mid \alpha \in \mathcal{A}\}$ is a projective system induced by the projective system of the Galois groups $\Gamma_\alpha, \alpha \in \mathcal{A}$, and

$$I = \varprojlim_{\alpha\in\mathcal{A}} I_\alpha(p).$$

Using simplest functorial properties of the ramification filtration it is easy to see that we have a natural projective system $\{\Gamma_\alpha(p)^{(e_\alpha v_0)} \mid \alpha \in \mathcal{A}\}$, where $e_\alpha = e(K_\alpha/K)$ is the relative ramification index of the extension K_α/K, and we have

$$\Gamma^{(v_0)} = \varprojlim_{\alpha\in\mathcal{A}} \Gamma_\alpha(p)^{(e_\alpha v_0)}.$$

Therefore,

$$\widetilde{H} \subset \varprojlim_{\alpha\in\mathcal{A}} I_\alpha(p)/\Gamma_\alpha(p)^{(e_\alpha v_0)}.$$

If ${\rm pr}_\alpha$ is projection of the above projective limit to its component with the index α, then $\widetilde{H}_\alpha := {\rm pr}_\alpha(\widetilde{H})$ is a pro-p-free group by the above corollary 1 b). Clearly, there exists $\alpha_0 \in \mathcal{A}$ such that $d(\widetilde{H}) = d(\widetilde{H}_{\alpha_0})$ therefore, ${\rm pr}_{\alpha_0}|_{\widetilde{H}}$ is an isomorphism, and $\widetilde{H}$ is a pro-p-free group. The corollary is proved.

2. Proof of the theorem.

2.1. Let $\{K_\alpha \mid \alpha \in \mathcal{A}\}$ be the family of all finite Galois extensions of K in $K(p)$. $\mathcal{A}$ is a filtered set (for $\alpha_1, \alpha_2 \in \mathcal{A}$, $\alpha_1 \geq \alpha_2$ means that $K_{\alpha_1} \supset K_{\alpha_2}$), and $\Gamma(p) = \varprojlim_{\alpha\in\mathcal{A}} \Gamma_\alpha$, where $\Gamma_\alpha = {\rm Gal}(K_\alpha/K)$ for $\alpha \in \mathcal{A}$.

Consider the fields tower $K \subset K_\alpha^H \subset K_\alpha^{(v_0)} \subset K_\alpha$, where $\alpha \in \mathcal{A}$ and $K_\alpha^{(v_0)}$ is the subfield of K_α fixed by $\Gamma^{(v_0)}$. Then $\widetilde{H} = \varprojlim_{\alpha\in\mathcal{A}} \widetilde{H}_\alpha$, where $\widetilde{H}_\alpha = {\rm Gal}(K_\alpha^{(v_0)}/K_\alpha^H)$.

If $\alpha \in \mathcal{A}$, then the natural projection ${\rm pr}_\alpha : \widetilde{H} \longrightarrow \widetilde{H}_\alpha$ is a group epimorphism. If $\alpha_1, \alpha_2 \in \mathcal{A}$ and $\alpha_1 \geq \alpha_2$, then the connecting morphism

$${\rm pr}_{\alpha_1\alpha_2} : \widetilde{H}_{\alpha_1} \longrightarrow \widetilde{H}_{\alpha_2}$$

is uniquely defined by the relation ${\rm pr}_{\alpha_2} = {\rm pr}_{\alpha_1} \circ {\rm pr}_{\alpha_1\alpha_2}$.

Consider a free pro-p-group $\mathcal{G}$ with an epimorphic map of pro-p-groups

$$j : \mathcal{G} \longrightarrow \widetilde{H}$$

such that the induced morphism $\bar{j} : \mathcal{G}/\mathcal{G}^p[\mathcal{G},\mathcal{G}] \longrightarrow \widetilde{H}/\widetilde{H}^p[\widetilde{H},\widetilde{H}]$ is an isomorphism.

Let $\mathcal{G} = \varprojlim_{\beta\in\mathcal{B}} \mathcal{G}_\beta$, where $\{\mathcal{G}_\beta\}_{\beta\in\mathcal{B}}$ is a projective system of finite p-groups and all projections $\mathrm{pr}_\beta : \mathcal{G} \longrightarrow \mathcal{G}_\beta$ are group epimorphisms.

The morphism of pro-p-groups j can be given by the following data:

(j1) a map $\iota : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\iota(\alpha_1) \geq \iota(\alpha_2)$, where $\alpha_1, \alpha_2 \in \mathcal{A}$ and $\alpha_1 \geq \alpha_2$;

(j2) group epimorphisms $j_\alpha : \mathcal{G}_{\iota(\alpha)} \longrightarrow \widetilde{H}_\alpha$, where $\alpha \in \mathcal{A}$;

(j3) if $\alpha_1, \alpha_2 \in \mathcal{A}$ and $\alpha_1 \geq \alpha_2$, then the following diagram is commutative

$$\begin{array}{ccc} \mathcal{G}_{\iota(\alpha_1)} & \xrightarrow{j_{\alpha_1}} & \widetilde{H}_{\alpha_1} \\ {\scriptstyle \mathrm{pr}_{\iota(\alpha_1)\iota(\alpha_2)}}\downarrow & & {\scriptstyle \mathrm{pr}_{\alpha_1\alpha_2}}\downarrow \\ \mathcal{G}_{\iota(\alpha_2)} & \xrightarrow{j_{\alpha_2}} & \widetilde{H}_{\alpha_2} \end{array}$$

If $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ is such that $\beta \geq \iota(\alpha)$, define $j_{\beta\alpha} \in \mathrm{Hom}(\mathcal{G}_\beta, \widetilde{H}_\alpha)$ as the composition $j_{\beta\alpha} = \mathrm{pr}_{\beta\iota(\alpha)} \circ j_\alpha$. Then the property (j3) can be stated in the following form:

(j3′) if $\alpha_1, \alpha_2 \in \mathcal{A}$ and $\beta_1, \beta_2 \in \mathcal{B}$ are such that $\alpha_1 \geq \alpha_2$, $\beta_1 \geq \iota(\alpha_1)$, $\beta_2 \geq \iota(\alpha_2)$ and $\beta_1 \geq \beta_2$, then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G}_{\beta_1} & \xrightarrow{j_{\beta_1\alpha_1}} & \widetilde{H}_{\alpha_1} \\ {\scriptstyle \mathrm{pr}_{\beta_1\beta_2}}\downarrow & & {\scriptstyle \mathrm{pr}_{\alpha_1\alpha_2}}\downarrow \\ \mathcal{G}_{\beta_2} & \xrightarrow{j_{\beta_2\alpha_2}} & \widetilde{H}_{\alpha_2} \end{array}$$

2.2. Let $\mathcal{A}_1$ be the subset of $\mathcal{A}$ consisting of $\alpha \in \mathcal{A}$ such that

$$\mathrm{rk}_{\mathbf{Z}/p\mathbf{Z}} \left(\widetilde{H}_\alpha / \widetilde{H}_\alpha^p [\widetilde{H}_\alpha, \widetilde{H}_\alpha] \right) = d(\widetilde{H}),$$

i.e. the projection pr_α induces the isomorphism

$$\bar{\mathrm{pr}}_\alpha : \widetilde{H}/\widetilde{H}^p[\widetilde{H}, \widetilde{H}] \longrightarrow \widetilde{H}_\alpha / \widetilde{H}_\alpha^p [\widetilde{H}_\alpha, \widetilde{H}_\alpha].$$

Clearly, $\mathcal{A}_1$ is a cofinal subset in $\mathcal{A}$.

For $\alpha \in \mathcal{A}_1$, consider the fields tower from n.2.1

$$K \subset K_\alpha^H \subset K_\alpha^{(v_0)} \subset K_\alpha.$$

The following lemma will be proved in n.3 below. We use all notation from n.2.1.

The main lemma.

If $\alpha \in \mathcal{A}_1$ and $\beta \geq \iota(\alpha)$, then there exist finite extensions $E_{\beta\alpha}$ of K_α^H and $F_{\beta\alpha}$ of $E'_{\beta\alpha} := E_{\beta\alpha} K_\alpha^{(v_0)}$ such that

(a) $E_{\beta\alpha} \subset K(p)^H$ and therefore, we have the natural group isomorphism

$$f_{\beta\alpha} : \widetilde{H}_\alpha \longrightarrow \mathrm{Gal}(E'_{\beta\alpha}/E_{\beta\alpha});$$

(b) *$F_{\beta\alpha}$ is a Galois extension over $E_{\beta\alpha}$ and there exists a group isomorphism $g_{\beta\alpha}$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{G}_\beta & \xrightarrow{g_{\beta\alpha}} & \mathrm{Gal}(F_{\beta\alpha}/E_{\beta\alpha}) \\ {\scriptstyle j_{\beta\alpha}}\downarrow & & \downarrow \\ \widetilde{H}_\alpha & \xrightarrow{f_{\beta\alpha}} & \mathrm{Gal}(E'_{\beta\alpha}/E_{\beta\alpha}) \end{array}$$

(here the right vertical arrow is the natural projection);
(c) *$F_{\beta\alpha}$ is contained in the subfield $K(p)^{(v_0)}$ of $K(p)$ fixed by the group $\Gamma(p)^{(v_0)}$.*

2.3. For $\alpha \in \mathcal{A}_1$ and $\beta \geq \iota(\alpha)$, consider the fields $E_{\beta\alpha}$, $E'_{\beta\alpha}$ and $F_{\beta\alpha}$ from the above lemma. Denote by $D_{\beta\alpha}$ the normal closure of $F_{\beta\alpha}$ over K (in $K(p)$). Then there exists $\gamma \in \mathcal{A}$ such that $K_\gamma = D_{\beta\alpha}$, and we have the following commutative diagram in the category of finite extensions of the field K:

$$\begin{array}{ccccccc} K_\alpha^H & \subset & K_\alpha^{(v_0)} & \subset & K_\alpha & & \\ \cap & & \cap & & & & \\ E_{\beta\alpha} & \subset & E'_{\beta\alpha} & \subset & F_{\beta\alpha} & & \\ \cap & & & & \cap & & \\ K_\gamma^H & \subset & K_\gamma^{(v_0)} & = & K_\gamma^{(v_0)} & = K_\gamma = D_{\beta\alpha} & \end{array}$$

Note that $F_{\beta\alpha}$ is a Galois extension of $E_{\beta\alpha}$, $E_{\beta\alpha} \subset K(p)^H$, $F_{\beta\alpha} \subset K(p)^{(v_0)}$ and therefore, we have the natural group homomorphism

$$h_{\beta\alpha} : \widetilde{H} \longrightarrow \mathrm{Gal}(F_{\beta\alpha}/E_{\beta\alpha})$$

such that $h_{\beta\alpha} \circ g_{\beta\alpha}^{-1} \circ j_{\beta\alpha} = \mathrm{pr}_\alpha$. Because $\alpha \in \mathcal{A}_1$ and $\beta \geq \iota(\alpha)$, the minimal numbers of generators for the groups $\widetilde{H}$, $\widetilde{H}_\alpha$ and $\mathcal{G}_\beta$ coincide. Therefore, $h_{\beta\alpha} \circ g_{\beta\alpha}^{-1}$ is epimorphic, and we obtain that

$$h_{\beta\alpha}(\widetilde{H}) = g_{\beta\alpha}(\mathcal{G}_\beta) = \mathrm{Gal}(F_{\beta\alpha}/E_{\beta\alpha}).$$

This gives $F_{\beta\alpha}^H = E_{\beta\alpha}$ and thus, the fields $F_{\beta\alpha}$ and K_γ^H are linearly disjoint over $E_{\beta\alpha}$ and we have the group epimorphism

$$i_{\gamma\beta\alpha} : \widetilde{H}_\gamma \longrightarrow \mathrm{Gal}(F_{\beta\alpha}/E_{\beta\alpha}) \xrightarrow{g_{\beta\alpha}^{-1}} \mathcal{G}_\beta,$$

such that $\mathrm{pr}_{\gamma\alpha} = i_{\gamma\beta\alpha} \circ j_{\beta\alpha}$.

2.4. Consider the set

$$\mathcal{C} = \{(\beta, \alpha) \in \mathcal{B} \times \mathcal{A}_1 \mid \alpha \in \mathcal{A}_1, \beta \geq \iota(\alpha)\}.$$

Clearly, $\mathcal{C}$ is a filtered set.

If $(\beta,\alpha)\in\mathcal{C}$, consider the set

$$I_{(\beta,\alpha)}=\{i\in\mathrm{Hom}_{\mathrm{cont}}(\widetilde{H},\mathcal{G}_\beta)\mid \mathrm{pr}_\alpha=i\circ j_{\beta\alpha}\}.$$

This set is finite (because $\widetilde{H}$ is finitely generated) and non-empty (cf. n.2.3). The property (j3′) of n.2.1 gives that $\{I_{(\beta,\alpha)}\}_{(\beta,\alpha)\in\mathcal{C}}$ is a projective system and therefore, its projective limit $I\neq\emptyset$.

Take $i\in I$.

For any $\alpha\in\mathcal{A}_1$, the set

$$\mathcal{B}_\alpha=\{\beta\in\mathcal{B}\mid(\beta,\alpha)\in\mathcal{C}\}=\{\beta\in\mathcal{B}\mid\beta\geq\iota(\alpha)\}$$

is cofinal in $\mathcal{B}$. Therefore, for any $\alpha\in\mathcal{A}_1$, the collection

$$\{\mathrm{pr}_{(\beta,\alpha)}(i)\mid\beta\in\mathcal{B}_\alpha\}$$

gives a morphism of pro-p-groups $i_\alpha:\widetilde{H}\longrightarrow\mathcal{G}$ such that $\mathrm{pr}_\alpha=i_\alpha\circ j\circ\mathrm{pr}_\alpha$.

The property (j3′) gives that i_α does not depend on $\alpha\in\mathcal{A}_1$. So, $i=\varprojlim\limits_{\alpha\in\mathcal{A}_1}i_\alpha\in\mathrm{Hom}_{\mathrm{cont}}(\widetilde{H},\mathcal{G})$ and satisfies the identity $i\circ j=\mathrm{id}_{\widetilde{H}}$.

Clearly, i is injective. But $\bar{i}=i\bmod\widetilde{H}^p[\widetilde{H},\widetilde{H}]=\bar{j}^{-1}$ is an isomorphism. So, i is surjective and $\widetilde{H}$ is a pro-p-free group (isomorphic to $\mathcal{G}$).

The theorem is proved.

3. Proof of the main lemma.

As we have noted in n.1 we can assume that $v_0>1$.

3.1. *Preliminaries.*

3.1.1. *The largest ramification numbers.* Let L be a complete discrete valuation field with perfect residue field of characteristic $p>0$. We recall some general facts from the higher ramification theory, cf. [Se, Ch.III].

If E/L is a finite Galois extension, $\Gamma_{E/L}=\mathrm{Gal}(E/L)$, and O_E is the valuation ring of the field E then for any $x>-1$, we have the ramification subgroup

$$\Gamma_{E/L,x}=\{\ \tau\in\Gamma_{E/L}\mid v_E(\tau a-a)\geq x+1\ \forall a\in O_E\ \},$$

where v_E is the valuation of E such that $v_E(E^*)=\mathbb{Z}$. This gives the ramification filtration $\{\Gamma_{E/L,x}\}_{x\geq0}$ of the group $\Gamma_{E/L}$ in lower numbering. This filtration is a decreasing left-continuous filtration of normal subgroups; for $-1<x\leq0$, $\Gamma_{E/L,x}$ is the ramification subgroup; and for $0<x\leq1$, $\Gamma_{E/L,x}$ is the higher ramification subgroup of the group $\Gamma_{E/L}$.

The Herbrandt's function of the extension E/L is defined for all $x\geq0$ by the expression

$$\varphi_{E/L}(x)=\int_0^x\left(\Gamma_{E/L,0}:\Gamma_{E/L,t}\right)^{-1}\mathrm{d}\,t.$$

For $-1<x<0$, $\varphi_{E/L}(x)=x$ by definition. Then $\varphi_{E/L}(x)$ is an increasing continuous piece-linear function, $\varphi_{E/L}(0)=0$, and for a sufficiently large x, one has that $\varphi'(x) = e(E/L)^{-1}$, where $e(E/L)$ is the ramification index of the extension E/L.

Set $\Gamma_{E/L}^{(v)} = \Gamma_{E/L,x}$, if $x > -1$ and $v = \varphi_{E/L}(x)$. This gives the ramification filtration of $\Gamma_{E/L}$ in upper numbering. If E_1 is a Galois extension of L and $E \subset E_1$ then the natural projection $\Gamma_{E_1/L} \longrightarrow \Gamma_{E/L}$ induces for any $v \geq 0$ the group epimorphism $\Gamma_{E_1/L}^{(v)} \longrightarrow \Gamma_{E/L}^{(v)}$. Taking projective limit with respect to these epimorphisms we obtain the ramification filtration $\{\Gamma_L^{(v)}\}_{v\geq 0}$ of the group $\Gamma_L = \mathrm{Gal}(L_{\mathrm{sep}}/L)$ in upper numbering.

The Herbrandt's function satisfies the composition property:

if E/L and E_1/L are finite Galois extensions such that $E \subset E_1$, then for any $x > -1$, one has

$$\varphi_{E_1/L}(x) = \varphi_{E/L}(\varphi_{E_1/E}(x)).$$

The definition of the Herbrandt's function $\varphi_{E/L}$ can be uniquely extended to the case of arbitrary finite separable extensions E/L under the requirement that the composition property should hold for arbitrary tower of finite extensions $L \subset E \subset E_1$, cf. [De].

Let $\psi_{E/L}$ be the inverse function for $\varphi_{E/L}$. This function is also an increasing piece-linear function satisfying the composition property:

if $L \subset E \subset E_1$ is a fields tower of finite extensions, then for any $x > -1$, one has

$$\psi_{E_1/L}(x) = \psi_{E_1/E}(\psi_{E/L}(x)).$$

If E/L is a finite extension such that $e(E/L) > 1$, then the set of edge points of the graph of the function $\varphi_{E/L}(x)$ is not empty and we denote by $(x(E/L), v(E/L))$ the coordinates of the last edge point. If $e(E/L) = 1$, we set $(x(E/L), v(E/L)) = (-1, -1)$. We have the following properties.

Proposition 1. *If $L \subset E \subset E_1$ is a tower of finite extensions, then:*
a) *the group $\Gamma_L^{(v)}$, where $v > -1$, acts trivially on E, if and only if $v > v(E/L)$;*
b) $v(E_1/L) = \max\{v(E/L), \varphi_{E/L}(v(E_1/E))\}$;
c) *if $v \geq v(E/L)$, then $\Gamma_L^{(v)} = \Gamma_E^{(\psi_{E/L}(v))}$.*

The above property a) follows directly from definitions, the property b) follows from the composition property. To prove c) let us consider an arbitrary finite Galois extension E_2 of L such that $E_2 \supset E$. It is sufficient to verify that

$$\Gamma_{E_2/L}^{(v)} = \Gamma_{E_2/E}^{(\psi_{E/L}(v))}.$$

For any $x \geqslant 0$, the equality $\Gamma_{E_2/E} \cap \Gamma_{E_2/L,x} = \Gamma_{E_2/E,x}$ follows directly from the definition of the lower numbering of the ramification filtration. Take $x = \psi_{E_2/L}(v)$ then $\Gamma_{E_2/L,x} \subset \Gamma_{E_2/E}$ (cf. n. a)) and therefore, $\Gamma_{E_2/L,x} = \Gamma_{E_2/E,x}$. It remains only to note that by the composition property we have $\varphi_{E_2/E}(x) = \psi_{E/L}(v)$, i.e. $\Gamma_{E_2/E,x} = \Gamma_{E_2/E}^{(\psi_{E/L}(v))}$. The proposition is proved.

We note that, if E is contained in the maximal p-extension $L(p)$ of L, then either $v(E/L) \geq 1$, or E/L is an unramified extension. So, if $\Gamma_L(p) = \mathrm{Gal}(L(p)/L)$, then for $-1 < v \leq 1$, $\Gamma_L(p)^{(v)} = \Gamma_L(p)^{(1)}$ is the ramification subgroup of $\Gamma_L(p)$.

Proposition 2. *Let E and L_1 be finite extensions of L in $L(p)$. Then*

$$\varphi_{E/L}(v(L_1E/E)) \leq v(L_1/L).$$

Proof. Let $[E:L] = p^{n_E}$ and $[L_1:L] = p^{n_1}$, where $n_E, n_1 \in \mathbb{Z}_{\geq 0}$.

The cases $n_E = 0$ or $n_1 = 0$ can be easily considered, so we can assume that $n_E, n_1 \in \mathbb{N}$. Let $v_E = v(E/L)$ and $v_1 = v(L_1/L)$.

Assume that $n_E = n_1 = 1$. Clearly, $v(L_1E/L) = \max\{v_E, v_1\}$.

If $v_1 \geq v_E$, we have by the proposition 1 b) that

$$v_1 = v(L_1E/L) = \max\{v_E, \varphi_{E/L}(v(L_1E/E))\}$$

therefore, $v_1 \geq \varphi_{E/L}(v(L_1E/E))$ and we obtain the formula of our proposition.

Consider the case $v_1 < v_E$. The equality $\varphi_{L_1E/L} = \varphi_{L_1E/E} \circ \varphi_{E/L}$ gives that the values of the function $\varphi_{L_1E/L}$ in its edge points equal $\varphi_{E/L}(v(L_1E/L))$ and v_E. The equality $\varphi_{L_1E/L} = \varphi_{L_1E/L_1} \circ \varphi_{L_1/L}$ gives that the values of this function in its edge points equal v_1 and $\varphi_{L_1/L}(v(L_1E/L_1))$. Now the inequality $v_1 < v_E$ implies that $v_1 = \varphi_{E/L}(v(L_1E/E))$. So, the case $n_E = n_1 = 1$ is completely considered.

Let $n_1 = 1$ and $n_E > 1$.

In this case there exists a field E' such that $L \subsetneq E' \subsetneq E$. By induction we can assume that our proposition is proved for the triples of fiields (E', L_1, L) and (E, E', L_1E'). Then

$$\varphi_{E/L}(v(L_1E/E)) = \varphi_{E'/L}(\varphi_{E/E'}(v(L_1E/E))) \leq \varphi_{E'/L}(v(L_1E'/E')) \leq v(L_1/L)$$

and the case $n_1 = 1$ and $n_E > 1$ is considered.

Assume that $n_1 > 1$ and n_E is an arbitrary natural number.

Consider the field L_2 such that $L \subsetneq L_2 \subsetneq L_1$. By induction we can assume that our proposition is proved for the triples (E, L_2, L) and (E, L_1, L_2). Applying also the composition property of the Herbrandt's function and the above proposition 1 we obtain that

$$\varphi_{E/L}(v(L_1E/E)) = \max\{\varphi_{E/L}(v(L_2E/E)), \varphi_{L_2E/E}(v(L_1E/L_2E))\} \leq$$

$$\max\{v(L_2/L), \varphi_{L_2/L}(v(L_1/L_2))\} = v(L_1/L).$$

The proposition is proved.

3.1.2. *A property of the field of norms functor.*

We use basic properties of the field of norms functor, cf. [Wtb].

Let E be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$ and absolute ramification index $e(E)$. Choose an algebraic closure $\bar{E}$ of E, a uniformizing element $\pi \in E$, and a sequence $\{\pi_n\}_{n\geq 0}$ of elements of $\bar{E}$ such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$ for all $n \geq 0$.

If $E_n = E(\pi_n)$ for $n \geq 0$, and $\widetilde{E} = \varinjlim E_n$, then $\widetilde{E}$ is an arithmetically profinite extension of E. Consider its field of norms $\mathcal{X}_E(\widetilde{E}) = \mathcal{E}$. Then $\mathcal{E}$ is a complete discrete valuation field of characteristic p and its residue field can be canonically identified with k.

If $\widetilde{L}$ is a finite extension of $\widetilde{E}$ in $\bar{E}$, then $\widetilde{L}$ is an arithmetically profinite extension of E and its field of norms $\mathcal{X}_E(\widetilde{L})$ is a separable finite extension of $\mathcal{E}$. The correspondence $\widetilde{L} \mapsto \mathcal{X}_E(\widetilde{L})$ induces an equivalence of the category of algebraic extensions of $\widetilde{E}$ and the category of separable extensions of its field of norms $\mathcal{E}$. Therefore, we can choose a separable closure of $\mathcal{E}$ in the form $\mathcal{E}_{\text{sep}} = \mathcal{X}_E(\bar{E})$ and obtain the following identification:

$$\mathcal{G}_{\mathcal{E}} := \mathrm{Gal}(\mathcal{E}_{\text{sep}}/\mathcal{E}) = \mathrm{Gal}(\bar{E}/\widetilde{E}) \subset \Gamma_E = \mathrm{Gal}(\bar{E}/E).$$

The (infinite) extension $\widetilde{E}/E$ has the Herbrandt's function

$$\varphi_{\widetilde{E}/E} = \lim_{n\to\infty} (\varphi_n \circ \cdots \circ \varphi_1),$$

where

$$\varphi_n(x) = \begin{cases} x, & \text{for } 0 \le x \le e^*(E_n) \\ e^*(E_n) + (x - e^*(E_n))/p, & \text{for } x \ge e^*(E_n), \end{cases}$$

and $e^*(E_n) = p^n e^*(E)$ with $e^*(E) = pe(E)/(p-1)$. The above identification $\mathcal{G}_{\mathcal{E}} \subset \Gamma_E$ is compatible with ramification filtrations: for any $v > -1$,

$$\mathcal{G}_{\mathcal{E}}^{(v)} = \mathcal{G}_{\mathcal{E}} \cap \Gamma_E^{(\varphi_{\widetilde{E}/E}(v))}.$$

One can verify that $\mathcal{G}_{\mathcal{E}}\Gamma_E^{(e^*(E))} = \Gamma_E$ and therefore, the embedding $\mathcal{G}_{\mathcal{E}} \subset \Gamma_E$ induces an isomorphism

$$\mathcal{G}_{\mathcal{E}}/\mathcal{G}_{\mathcal{E}}^{(e^*(E))} \simeq \Gamma_E/\Gamma_E^{(e^*(E))}.$$

This gives the following proposition.

Proposition. *If L is a finite extension of E and $\widetilde{L} = L\widetilde{E}$, then the correspondence*

$$L \mapsto \widetilde{\mathcal{X}}_{\widetilde{E}}(L) := \mathcal{X}_E(\widetilde{L})$$

induces an equivalence of the category of algebraic extensions L/E such that $v(L/E) < e^*(E)$ and the category of separable extensions $\mathcal{L}/\mathcal{E}$ such that $v(\mathcal{L}/\mathcal{E}) < e^*(E)$.

Remark. If $[L : E] < \infty$, $v(L/E) < e^*(E)$ and $\mathcal{L} = \mathcal{X}_{\widetilde{E}}(L)$, then we have the equality of Herbrandt's functions $\varphi_{L/E} = \varphi_{\mathcal{L}/\mathcal{E}}$.

3.1.3. *An application of the Artin-Schreier theory.*

Let K_1 be a complete discrete valuation field with perfect residue field k_1 of characteristic $p > 0$. Choose a maximal p-extension $K_1(p)$ of K_1 and denote by $F(K_1)$ the category of separable extensions of the field K_1 in $K_1(p)$ (if $L_1, L_2 \in F(K_1)$ and $L_1 \subset L_2$, then $\mathrm{Hom}_{F(K_1)}(L_1, L_2)$ contains only one element — the embedding of L_1 into L_2; if $L_1 \not\subset L_2$, then $\mathrm{Hom}_{F(K_1)}(L_1, L_2) = \emptyset$).

We use the notation $e(K_1)$ for the absolute ramification index of the field K_1 if it has characteristic 0 and define $e(K_1) = \infty$ if $\operatorname{char} K_1 = p$. In the both cases we set $e^*(K_1) = pe(K_1)/(p-1)$.

Let $v_1, v_1' \in \mathbb{R}$ be such that $1 \le v_1' < v_1 \le e^*(K_1)$. Consider the $\mathbb{F}_p$-linear space

$$V(k_1, v_1', v_1) = \bigoplus_{a \in [v_1', v_1)_p} (k_1)_a,$$

where $[v_1', v_1)_p = \{n \in \mathbb{N} \mid v_1' \le n < v_1, (n,p) = 1\}$.

Denote by $S(k_1, v_1', v_1)$ the category of finite dimensional linear subspaces of $V(k_1, v_1', v_1)$ (here we have also for any 2 objects V_1 and V_2 of this category, that $\operatorname{Hom}_{S(k_1,v_1',v_1)}(V_1, V_2) = \emptyset$ if $V_1 \not\subset V_2$, otherwise the set $\operatorname{Hom}_{S(k_1,v_1',v_1)}(V_1, V_2)$ consists only of one element — the embedding $V_1 \subset V_2$).

Proposition 1. *There exists a fully faithful functor*

$$\mathcal{F} : S(k_1, v_1', v_1) \longrightarrow F(K_1)$$

such that for any $L \in S(k_1, v_1', v_1)$ one has
a) $\mathcal{F}(L)$ is a finite Galois extension of K_1 and there exists a natural identification $\operatorname{Gal}(\mathcal{F}(L)/K_1) = \widehat{L} := \operatorname{Hom}(L, \mathbb{F}_p)$;
b) If $L \ne 0$, then $v_1' \le v(\mathcal{F}(L)/K_1) < v_1$.

Proof. Consider first the case $\operatorname{char} K_1 = p$.

If σ is Frobenius and $\Gamma_{K_1}(p) = \operatorname{Gal}(K_1(p)/K_1)$, then one has the natural identification of the Artin-Schreier theory $K_1/(\sigma - \mathrm{id})K_1 = \operatorname{Hom}(\Gamma_{K_1}(p), \mathbb{F}_p)$.

Choose a uniformizer t_1 of K_1 and consider the identification of $V(k_1, v_1', v_1)$ with a linear subspace of $K_1/(\sigma - \mathrm{id})K_1$ induced by the correspondence

$$\{\alpha_a\}_{a \in [v_1', v_1)_p} \mapsto \Big(\sum_{a \in [v_1', v_1)_p} \alpha_a t_1^{-a}\Big) \bmod (\sigma - \mathrm{id})K_1.$$

If $L \in S(k_1, v_1', v_1)$ is an $\mathbb{F}_p$-linear subspace of $V(k_1, v_1', v_1)$, then we set $\mathcal{F}(L) = K_1(p)^{H(L)}$, where

$$H(L) = \bigcap_{l \in L} \operatorname{Ker} l \subset \Gamma_{K_1}(p),$$

and elements $l \in L$ are considered as elements of the group $\operatorname{Hom}(\Gamma_{K_1}(p), \mathbb{F}_p)$ by the use of the above identifications

$$V(k_1, v_1', v_1) \subset K_1/(\sigma - \mathrm{id})K_1 = \operatorname{Hom}(\Gamma_{K_1}(p), \mathbb{F}_p).$$

It is easy to see that the correspondence $L \mapsto \mathcal{F}(L)$ determines a functor which satisfies the properties of our proposition. We note that this functor depends only on the choice of a uniformizer t_1 in the field K_1.

If $\operatorname{char} K_1 = 0$, we choose a uniformizer $\pi_0 \in K_1$, a sequence $\pi_n \in K_{1\,\mathrm{sep}} \supset K_1(p)$, such that $\pi_{n+1}^p = \pi_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$, and construct the functor $\widetilde{\mathcal{X}}_{\widetilde{K}_1}$ from n.3.1.2. If t_1 is the uniformizing element of the field $\mathcal{K}_1 = \widetilde{\mathcal{X}}_{\widetilde{K}_1}(K_1)$, which corresponds to the sequence $\{\pi_n\}_{n \geqslant 0}$, $\mathcal{F}'$ is the above constructed functor for the field $\mathcal{K}_1$ and its uniformizer t_1, and $\widetilde{\mathcal{X}}_{\widetilde{K}_1}$ is the functor from n.3.1.2, then the functor $\mathcal{F} = \mathcal{F}' \circ \widetilde{\mathcal{X}}_{\widetilde{K}_1}^{-1}$ satisfies the properties of our proposition. The proposition is proved.

We shall apply the above proposition in the following situation.

Suppose that $1 < v_1 \leq e^*(K_1)$, H_1 is a closed subgroup of $\Gamma_{K_1}(p)$ containing the ramification subgroup $\Gamma_{K_1}(p)^{(v_1)}$, $\widetilde{H}_1 = H_1/\Gamma_{K_1}(p)^{(v_1)}$ and $d(\widetilde{H}_1)$ is the minimal number of topological generators of the group $\widetilde{H}_1$.

Proposition 2. *If $1 \leq v_1' < v_1 (\leqslant e^*(K_1))$ and $\dim_{\mathbb{F}_p} V(k_1, v_1', v_1) > d(\widetilde{H}_1)$, then there exists an extension K_2 of K_1 of degree p such that $K_2 \subset K_1(p)^{H_1}$ and $v(K_2/K_1) \geq v_1'$.*

Proof. In the notation of the above proposition 1 consider the field $E_1 = \mathcal{F}(L)$, where $L \in S(k_1, v_1', v_1)$ is such that $\dim_{\mathbb{F}_p} L > d(\widetilde{H}_1)$. Then $v_1' \leq v(E_1/K_1) < v_1$. So, if

$$f : \Gamma_{K_1}(p) \longrightarrow \mathrm{Gal}(E_1/K_1) = \widehat{L}$$

is the natural projection, then $\Gamma_{K_1}(p)^{(v_1)} \subset \mathrm{Ker}\, f$ and $d(f(H_1)) \leq d(\widetilde{H}_1)$. Therefore, $f(H_1)$ is a proper subgroup of $\widehat{L}$ and there exists a subextension K_2 of $E_1^{f(H_1)}$ over K_1 such that $[K_2 : K_1] = p$. The proposition is proved.

3.2. *The field $K_{\alpha 1}$.*

As earlier, consider for $\alpha \in \mathcal{A}_1$, the fields tower

$$K \subset K_\alpha^H \subset K_\alpha^{(v_0)} \subset K_\alpha.$$

Denote by $C(\widetilde{H})$ a positive real number such that for any $r > 0$ the interval $(r, r + C(\widetilde{H}))$ contains at least $d(\widetilde{H}) + 1$ prime to p integers.

Proposition. *There exists a finite extension $K_{\alpha 1}$ of K_α^H in $K(p)^H$ such that*

$$\min\{\psi_{K_{\alpha 1}/K}(v_0), e^*(K_{\alpha 1})\} > C(\widetilde{H}) + 1.$$

Proof.

3.2.1. Prove first that there exists an infinite fields tower

$$K_\alpha^H = K_{\alpha 0,0} \subset K_{\alpha 0,1} \subset \cdots \subset K_{\alpha 0,n} \subset \ldots$$

such that for any $n \in \mathbb{N}$, we have $[K_{\alpha 0,n} : K_{\alpha 0,n-1}] = p$ and $K_{\alpha 0,n} \subset K(p)^H$.

Indeed, let $n_0 \in \mathbb{Z}_{\geq 0}$ and assume that we have constructed such fields $K_{\alpha 0,n}$ for $n \leqslant n_0$. Because $K_{\alpha 0,n_0} \subset K(p)^H$, we have $H \subset \Gamma_{n_0} = \mathrm{Gal}(K(p)/K_{\alpha 0,n_0})$. Because $[K_{\alpha 0,n_0} : K] < \infty$ and H is not open subgroup of $\Gamma(p)$, we have $H \neq \Gamma_{n_0}$ and therefore, $H_{n_0} := H\Gamma_{n_0}^p[\Gamma_{n_0}, \Gamma_{n_0}] \subsetneq \Gamma_{n_0}$. Let $E_{n_0} = K(p)^{H_{n_0}}$. Then E_{n_0} is a non-trivial abelian extension of $K_{\alpha 0,n_0}$ in $K(p)^H$. Clearly, there exists the field $K_{\alpha 0,n_0+1}$ such that $K_{\alpha 0,n_0} \subset K_{\alpha 0,n_0+1} \subset E_{n_0}$ and $[K_{\alpha 0,n_0+1} : K_{\alpha 0,n_0}] = p$.

3.2.2. We want to prove here that the fields tower $K_{\alpha 0,n}$, $n \geqslant 0$, from n.3.2.1 can be chosen in such a way that for almost all $n \in \mathbb{N}$, the field $K_{\alpha 0,n}$ is totally ramified over $K_{\alpha 0,n-1}$.

Denote by $K'_{\alpha 0,n}$, $n \geqslant 0$, the fields tower from n.3.2.1. If this tower does not satisfy the above condition, then there exists $n_0 \geqslant 0$ such that the residue field k_1 of $K'_{\alpha 0,n_0}$ contains more that $p^{d(\widetilde{H})}$ elements (recall that $d(\widetilde{H})$ is the minimal number of topological generators of the pro-p-group $\widetilde{H} = H/\Gamma(p)^{(v_0)}$).

For $0 \leqslant n \leqslant n_0$ set $K_{\alpha 0,n} = K'_{\alpha 0,n}$. Let $n_1 \in \mathbb{Z}_{\geqslant 0}$ be such that $n_1 \geqslant n_0$ and assume that we have consructed for $n_0 < n \leqslant n_1$, the fields $K_{\alpha 0,n}$ such that for $0 < n \leqslant n_1$ we have $[K_{\alpha 0,n} : K_{\alpha 0,n-1}] = p$, $K_{\alpha 0,n} \subset K(p)^H$ and for $n_0 < n \leqslant n_1$, $v(K_{\alpha 0,n}/K_{\alpha 0,n-1}) \geqslant 1$ (i.e. $K_{\alpha 0,n}$ is totally ramified over $K_{\alpha 0,n-1}$).

Let $K_1 = K_{\alpha 0,n_1}$, $v_1 = \min\{\psi_{K_1/K}(v_0), e^*(K_1)\}$ and $H_1 = H\Gamma_1^{(v_1)}$, where $\Gamma_1 = \mathrm{Gal}(K(p)/K_1)$. We note that k_1 is the residue field of K_1, $v_1 > 1$ and $\dim_{\mathbb{F}_p} V(k_1, 1, v_1) \geqslant \dim_{\mathbb{F}_p} k_1 > d(\widetilde{H})$ (cf. the notation of n.3.1.3). Because $K_1 \subset K(p)^H$ and $H \supset \Gamma(p)^{(v_0)}$, we have $v(K_1/K) < v_0$ and therefore, $\Gamma(p)^{(v_0)} = \Gamma_1^{(\psi_{K_1/K}(v_0))}$ (cf. proposition 1 of n.3.1.1). Because $v_1 \leqslant \psi_{K_1/K}(v_0)$, we have $\Gamma_1^{(\psi_{K_1/K}(v_0))} \subset \Gamma_1^{(v_1)}$ and therefore, there exists the natural group epimorphism

$$\widetilde{H} = H/\Gamma(p)^{(v_0)} \longrightarrow H/H \cap \Gamma_1^{(v_1)} = H_1/\Gamma_1^{(v_1)} := \widetilde{H}_1.$$

This gives $d(\widetilde{H}) \geqslant d(\widetilde{H}_1)$. So, we can apply the proposition 2 of n.3.1.3 to obtain the field extension K_2/K_1 of relative degree p such that $K_2 \subset K(p)^{H_1}$ and $v(K_2/K_1) \geq 1$. Because $H_1 \supset H$ we can take $K_{\alpha 0,n_1+1} = K_2$.

3.2.3. By the above arguments, we can consider the fields tower $K_{\alpha 0,n}$, $n \geqslant 0$, from n.3.2.1 such that the set $\{n \in \mathbb{Z}_{\geqslant 0} \mid v(K_{\alpha 0,n+1}/K_{\alpha 0,n}) \geqslant 1\}$ is infinite. Clearly, the properties $\psi_{K_{\alpha 0,n}/K}(v_0) \to +\infty$ and $e^*(K_{\alpha 0,n}) \to +\infty$ for $n \to \infty$ will imply the statement of our proposition.

We note first that if $v_n^* := v(K_{\alpha 0,n+1}/K_{\alpha 0,n}) < 1$, then $v_n^* = -1$, i.e. $K_{\alpha 0,n+1}$ is unramified over $K_{\alpha 0,n}$. In this case we have $\psi_{K_{\alpha 0,n+1}/K} = \psi_{K_{\alpha 0,n}/K}$ and $e^*(K_{\alpha 0,n+1}) = e^*(K_{\alpha 0,n})$. If $v_n^* \geqslant 1$, then by the composition property of the Herbrandt function we have

$$\psi_{K_{\alpha 0,n+1}/K}(x) = \begin{cases} \psi_{K_{\alpha 0,n}/K}(x), & \text{for } \psi_{K_{\alpha 0,n}/K}(x) < v_n^* \\ v_n^* + p(\psi_{K_{\alpha 0,n}/K}(x) - v_n^*), & \text{for } \psi_{K_{\alpha 0,n}/K}(x) \geqslant v_n^*. \end{cases}$$

and $e^*(K_{\alpha 0,n+1}) = pe^*(K_{\alpha 0,n})$.

The property $e^*(K_{\alpha 0,n}) \to +\infty$ for $n \to \infty$, is obvious.

For any $n \in \mathbb{Z}_{\geqslant 0}$ by proposition 1 b) of n.3.1.1, we have

$$\varphi_{K_{\alpha 0,n}/K}(v_n^*) \leqslant v(K_{\alpha 0,n+1}/K) < v_0.$$

Therefore, $v_n^* < \psi_{K_{\alpha 0,n}/K}(v_0)$. So, if we set $\psi_n = \psi_{K_{\alpha 0,n}/K}(v_0)$, then

$$\psi_{n+1} = \begin{cases} \psi_n, & \text{if } v_n^* = -1 \\ v_n^* + p(\psi_n - v_n^*), & \text{if } v_n^* \geqslant 1. \end{cases}$$

Note that in the second case we have $v_n^* \in \mathbb{N}$, $v_n^* < \psi_n$ and $\psi_{n+1} > \psi_n$. Consider the strictly increasing sequence $\{\psi_m^*\}_{m \in \mathbb{N}}$ of all elements of the set $\{\psi_n \mid n \in \mathbb{Z}_{\geqslant 0}\}$.

This sequence is infinite by the choice of the fields tower $K_{\alpha 0,n}$, $n \geqslant 0$ (cf. n.3.2.2), and it is sufficient to prove that $\psi^*_m \to +\infty$ if $m \to \infty$.

For $x \in \mathbb{R}$, set $\{\{x\}\} = \min\{x - n \mid n \in \mathbb{Z}, n < x\}$.

For any $m \in \mathbb{N}$, we have

$$\psi^*_{m+1} - \psi^*_m = (p-1)(\psi^*_m - v^*_{n(m)}),$$

where $n(m) \in \mathbb{Z}_{\geqslant 0}$ is such that $\psi^*_m = \psi_{n(m)}$ and $\psi^*_{m+1} = \psi_{n(m)+1}$. Therefore,

$$\psi^*_{m+1} - \psi^*_m \geqslant (p-1)\{\{\psi^*_m\}\}$$

for any $m \in \mathbb{N}$. So,

$$\psi^*_m \geqslant (p-1) \sum_{1 \leqslant n < m} \{\{\psi^*_n\}\} = (p-1) \sum_{1 \leqslant n < m} \{\{p^{n-1}\psi^*_1\}\} \to +\infty,$$

when $m \to \infty$. The proposition is proved.

3.3. *The fields $K_{\alpha 1,n}$ and $L_{\alpha 1,n}$, $n \geq 0$.*

3.3.1. Consider the field $K_{\alpha 1}$ given by proposition of n.3.2. Let

$$v'_0 = \varphi_{K_{\alpha 1}/K}(\psi_{K_{\alpha 1}/K}(v_0) - 1)$$

and let

$$v^*_0 = \max\left\{v'_0, v(K^{(v_0)}_\alpha/K), v(K_{\alpha 1}/K)\right\}.$$

We note that $v^*_0 < v_0$ and $\psi_{K_{\alpha 1}/K}(v^*_0) > C(\widetilde{H})$.

Proposition. *There exists a fields tower*

$$K_{\alpha 1,0} := K_{\alpha 1} \subset K_{\alpha 1,1} \subset \dots \subset K_{\alpha 1,n} \subset \dots,$$

where for all $n \in \mathbb{N}$ it holds

a) $[K_{\alpha 1,n} : K_{\alpha 1,n-1}] = p$ and $K_{\alpha 1,n} \subset K(p)^H$;

b) if $v^{(n)} = v(K_{\alpha 1,n}/K_{\alpha 1,n-1})$ and $A_n = \min\left\{\psi_{K_{\alpha 1,n-1}/K}(v^*_0), e^*(K_{\alpha 1,n-1})\right\}$, then $v^{(n)} \in \left(A_n - C(\widetilde{H}), A_n\right)$.

Proof. We use induction on n. Let $n_0 \in \mathbb{N}$ and assume that such fields are constructed for $0 \leqslant n < n_0$. We note that for all $0 < n \leqslant n_0$, we have $A_n \geqslant A_1 > C(\widetilde{H})$ and therefore, the interval $\left(A_n - C(\widetilde{H}), A_n\right)$ contains at least $d(\widetilde{H}) + 1$ integers prime to p.

Set $K_1 = K_{\alpha 1,n_0-1}$ and $H_1 = H\Gamma_1^{(A_{n_0})}$, where $\Gamma_1 = \mathrm{Gal}(K(p)/K_1)$. If k_1 is the residue field of K_1, $v_1 = A_{n_0}$ and $v'_1 = \left[A_{n_0} - C(\widetilde{H})\right] + 1$, then $v'_1 \geqslant 1$ and $\dim_{\mathbb{F}_p} V(k_1, v'_1, v_1) > d(\widetilde{H})$. As in n.3.2.2 we obtain now that $v(K_1/K) < v_0$, $\Gamma(p)^{(v_0)} = \Gamma_1^{(\psi_{K_1/K}(v_0))} \subset \Gamma_1^{(A_{n_0})}$ (because $A_{n_0} \leqslant \psi_{K_{\alpha 1,n_0-1}/K}(v_0)$), we have the natural group epimorphism

$$\widetilde{H} = H/\Gamma(p)^{(v_0)} \longrightarrow H/H \cap \Gamma_1^{(A_{n_0})} = H_1/\Gamma_1^{(A_{n_0})} := \widetilde{H}_1$$

and therefore, $d(\widetilde{H}) \geqslant d(\widetilde{H}_1)$.

So, the proposition 2 of n.3.1.3 gives the extension K_2/K_1 such that $[K_2 : K_1] = p$, $A_{n_0} > v(K_2/K_1) \geqslant v_1' > A_{n_0} - C(\widetilde{H})$ and $K_2 \subset K(p)^{H_1} \subset K(p)^H$.

Clearly, we can take $K_2 = K_{\alpha 1, n_0}$. The proposition is proved.

3.3.2. Consider the fields tower $K_{\alpha 1,n}$, $n \geqslant 0$, from the above proposition. Set $L_{\alpha 1,n} = K_\alpha^{(v_0)} K_{\alpha 1,n}$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Because $K(p)^H$ and $K_\alpha^{(v_0)}$ are linearly disjoint over K_α^H, we have for all $n \geq 0$ the natural isomorphism

$$\widetilde{H}_\alpha = \mathrm{Gal}(K_\alpha^{(v_0)}/K_\alpha^H) \simeq \mathrm{Gal}(L_{\alpha 1,n}/K_{\alpha 1,n}).$$

Proposition. *If v_0^* is the real number from n.3.3.1, then for all $n \in \mathbb{Z}_{\geqslant 0}$, we have $v(L_{\alpha 1,n}/K) \leqslant v_0^*$.*

Proof. Because

$$v(L_{\alpha 1,n}/K) = \max\left\{v(K_\alpha^{(v_0)}/K), v(K_{\alpha 1,n}/K)\right\},$$

it is sufficient to prove that $v(K_{\alpha 1,n}/K) \leqslant v_0^*$.

We can assume by induction that $v(K_{\alpha 1,n-1}/K) \leqslant v_0^*$ for some $n \in \mathbb{N}$. Then

$$v(K_{\alpha 1,n}/K) = \max\left\{v(K_{\alpha 1,n-1}/K), \varphi_{K_{\alpha 1,n-1}/K}(v^{(n)})\right\},$$

where $v^{(n)} = v(K_{\alpha 1,n}/K_{\alpha 1,n-1}) < A_n \leqslant \psi_{K_{\alpha 1,n-1}/K}(v_0^*)$ and therefore,

$$\varphi_{K_{\alpha 1,n-1}/K}(v^{(n)}) < v_0^*.$$

The proposition is proved.

3.3.3. In the above notation we have the following proposition.

Proposition. *If $v_n = v(L_{\alpha 1,n}/K_{\alpha 1,n})$ and $e_n = e(K_{\alpha 1,n}/K_\alpha^H)$, then*

$$\lim_{n\to\infty} (v_n/e_n) = 0.$$

Proof. By the proposition 2 of n.3.1.1 we have

$$v(L_{\alpha 1,n}/K_{\alpha 1,n}) \leqslant \psi_{K_{\alpha 1,n}/K_{\alpha 1,0}}(v(L_{\alpha 1,0}/K_{\alpha 1,0}))$$

and by proposition 1 b) of n. 3.1.1 it holds

$$\varphi_{K_{\alpha 1,0}/K}(v(L_{\alpha 1,0}/K_{\alpha 1,0})) \leqslant v(L_{\alpha 1,0}/K) \leqslant v_0^*.$$

Therefore, $v(L_{\alpha 1,n}/K_{\alpha 1,n}) \leqslant \psi_n := \psi_{K_{\alpha 1,n}/K}(v_0^*)$ and it is sufficient to prove that $\lim_{n\to\infty}(\psi_n/e_n) = 0$.

Prove first that there exists $n_0 \in \mathbb{Z}_{\geqslant 0}$ such that $\psi_{n_0} < e^*(K_{\alpha 1,n_0})$.

If such n_0 does not exist we have for all $n \in \mathbb{Z}_{\geqslant 0}$, that $\psi_n \geqslant e^*(K_{\alpha 1,n}) = p^n e^*(K_{\alpha 1,0})$ and $A_{n+1} = e^*(K_{\alpha 1,n})$. Therefore,

$$\psi_{n+1} = \psi_{K_{\alpha 1,n+1}/K_{\alpha 1,n}}(\psi_n) = v^{(n+1)} + p(\psi_n - v^{(n+1)}),$$

because $\psi_n \geqslant e^*(K_{\alpha 1,n})$ and $v^{(n+1)} < A_{n+1}$. Using the inequalities $v^{(n+1)} > A_{n+1} - C(\widetilde{H})$ we obtain for any $n \in \mathbb{Z}_{\geqslant 0}$, that

$$\psi_{n+1} \leqslant p^{n+1}\psi_0 - (p^{n+1}-p^n)(e^*(K_{\alpha 1,0})-C(\widetilde{H})) - \cdots - (p-1)(e^*(K_{\alpha 1,n})-C(\widetilde{H})) =$$

$$p^{n+1}\left(\psi_0 + C(\widetilde{H})\left(1-p^{-n-1}\right) - (n+1)e^*(K_{\alpha 1,0})(1-p^{-1})\right).$$

This gives the contradiction, because the right-hand side of the above equality tends to $-\infty$, if $n \to \infty$.

Let $n_0 \in \mathbb{Z}_{\geqslant 0}$ be such that $\psi_{n_0} < e^*(K_{\alpha 1,n_0})$. Then for any $n \geqslant n_0$, we have also $\psi_n < e^*(K_{\alpha 1,n})$, because for any n, it holds

$$\psi_{n+1} = \max\{\psi_n, v^{(n+1)} + p(\psi_n - v^{(n+1)})\} \leqslant p\psi_n.$$

Prove that for any $n \geqslant n_0$,

$$\psi_{n+1} - \psi_n < (p-1)C(\widetilde{H}). \tag{1}$$

Indeed, if $\psi_n \leqslant v^{(n+1)}$, then

$$\psi_{n+1} = \psi_{K_{\alpha 1,n+1}/K_{\alpha 1,n}}(\psi_n) = \psi_n$$

and the inequality (1) holds. If $\psi_n > v^{(n+1)}$, then

$$\psi_{n+1} = v^{(n+1)} + p(\psi_n - v^{(n+1)}).$$

But $\psi_n = A_{n+1}$, $v^{(n+1)} > A_{n+1} - C(\widetilde{H})$ and

$$\psi_{n+1} - \psi_n = (p-1)\left(\psi_n - v^{(n+1)}\right) < (p-1)C(\widetilde{H}).$$

The inequality (1) is proved.

Therefore, for any $n \geqslant n_0$, we have

$$\psi_{n_0} \leqslant \psi_n < (p-1)C(\widetilde{H})(n-n_0) + \psi_{n_0},$$

and because $e_n = p^{n-n_0}e_{n_0}$, this implies obviously that $\psi_n/e_n \to 0$, if $n \to \infty$. The proposition is proved.

3.4. *The fields $K_{\alpha 2,n}$ and $L_{\alpha 2,n}$, $n \geq 0$.*

Proposition 1. *There exists a tower of finite extensions of the field K_α^H in $K(p)^H$ of relative degree p*

$$K_{\alpha 2,0} \subset K_{\alpha 2,1} \subset \dots \subset K_{\alpha 2,n} \subset \dots$$

such that if for $n \geq 0$ we set $L_{\alpha 2,n} = K_{\alpha 2,n} K_\alpha^{(v_0)}$, $v_{\alpha 2,n} = v(L_{\alpha 2,n}/K)$, $v_{\alpha,n} = v(L_{\alpha 2,n}/K_{\alpha 2,n})$, $v^{(n+1)} = v(K_{\alpha 2,n+1}/K_{\alpha 2,n})$, and $e_{\alpha 2,n} = e(K_{\alpha 2,n}/K)$, then

$$\text{(1)} \qquad v_{\alpha 2,n} + \frac{p}{(p-1)e_{\alpha 2,n}} \left(\frac{v_{\alpha,n}}{p-1} + C(\widetilde{H}) + 1 \right) < v_0;$$

$$\text{(2)} \qquad \frac{v_{\alpha,n}}{e_{\alpha 2,0}} + \frac{(p-1)(C(\widetilde{H})+1)}{p e_{\alpha 2,0}} \leqslant 1;$$

$$\text{(3)} \qquad v^{(n+1)} > p v_{\alpha,n}/(p-1).$$

Proof. Consider the fields tower $K_{\alpha 1,n}$, $n \geqslant 0$, from n.3.3.1. The propositions of nn. 3.3.2 and 3.3.3 imply the existence of a sufficiently large $N_1 \in \mathbb{N}$ such that if $K_{\alpha 2,0} := K_{\alpha 1,N_1}$ then the properties (1) and (2) hold for $n = 0$.

We use induction on $n \geq 0$. Assume that the fields $K_{\alpha 2,n}$ are constructed for all $n \leq N$, where $N \in \mathbb{Z}$, $N \geq 0$.

Lemma.

$$\varphi_{K_{\alpha 2,N}/K} \left(\frac{p v_{\alpha,N}}{p-1} + C(\widetilde{H}) + 1 \right) + \frac{1}{(p-1)e_{\alpha 2,N}} \left(\frac{v_{\alpha,N}}{p-1} + C(\widetilde{H}) + 1 \right) < v_0.$$

Proof. Let $x = \max\{x(K_{\alpha 2,N}/K), v_{\alpha,N}\}$, then $\varphi_{K_{\alpha 2,N}/K}(x) = v_{\alpha 2,N}$, cf. proposition 1 b) of n.3.1.1.

If $p v_{\alpha,N}/(p-1) + C(\widetilde{H}) + 1 \leqslant x$, then

$$\varphi_{K_{\alpha 2,N}/K} \left(p v_{\alpha,N}/(p-1) + C(\widetilde{H}) + 1 \right) \leq \varphi_{K_{\alpha 2,N}/K}(x) = v_{\alpha 2,N}.$$

If $p v_{\alpha,N}/(p-1) + C(\widetilde{H}) + 1 > x$, then

$$\varphi_{K_{\alpha 2,N}/K} \left(\frac{p v_{\alpha,N}}{p-1} + C(\widetilde{H}) + 1 \right) = \varphi_{K_{\alpha 2,N}/K}(x) + \frac{p v_{\alpha,N}/(p-1) + C(\widetilde{H}) + 1 - x}{e_{\alpha 2,N}}$$

(we use that if $x > x(K_{\alpha 2,N}/K)$, then $\varphi'_{K_{\alpha 2,N}/K}(x) = e_{\alpha 2,N}^{-1}$)

$$\leqslant v_{\alpha 2,N} + \frac{1}{e_{\alpha 2,N}} \left(\frac{v_{\alpha,N}}{p-1} + C(\widetilde{H}) + 1 \right)$$

(this follows from the inequality $x \geq v_{\alpha,N}$).

In the both cases the inequality of our lemma is implied now by the property (1) for $n = N$.

The lemma is proved.

Set $K_{\alpha 2,N} = K_1$, $\Gamma_1 = \mathrm{Gal}(K(p)/K_1)$ and $v_1 = pv_{\alpha,N}/(p-1) + C(\widetilde{H}) + 1$.

We note that the property (2) for $n = N$ gives that $v_1 \leqslant e^*(K_{\alpha 2,0}) \leqslant e^*(K_{\alpha 2,N})$. Because by the property (1) it holds $v(K_1/K) = v_{\alpha 2,N} < v_0$, we can apply the proposition 1 c) of n. 3.1.1 and the inequality $\varphi_{K_1/K}(v_1) < v_0$ (cf. the above lemma) to obtain

$$\Gamma(p)^{(v_0)} = \Gamma_1^{(\psi_{K_1/K}(v_0))} \subset \Gamma_1^{(v_1)}.$$

Therefore, if $H_1 = H\Gamma_1^{(v_1)}$, then we have the natural group epimorphism

$$\widetilde{H} = H/\Gamma(p)^{(v_0)} \longrightarrow H/H \cap \Gamma_1^{(v_1)} = H_1/\Gamma_1^{(v_1)} := \widetilde{H}_1,$$

and therefore, $d(\widetilde{H}_1) \leqslant d(\widetilde{H})$.

Now we can apply the proposition 2 of n.3.1.3 with the above chosen K_1, v_1, H_1 and $v_1' = pv_{\alpha,N}/(p-1) + 1$ to obtain the extension K_2 of degree p over K_1 such that $K_2 \subset K(p)^{H_1} \subset K(p)^H$ and $v(K_2/K_1) \geqslant pv_{\alpha,N}/(p-1) + 1$. If we set $K_2 = K_{\alpha 2,N+1}$, then the property (3) is satisfied for $n = N$.

By the proposition 2 of n.3.1.1 we have the inequality $v_{\alpha,N+1} \leqslant \psi_{K_2/K_1}(v_{\alpha,N})$. But $\psi_{K_2/K_1}(v_{\alpha,N}) = v_{\alpha,N}$, because $v_{\alpha,N} < pv_{\alpha,N}/(p-1) + 1 \leqslant v(K_2/K_1)$. Therefore, $v_{\alpha,N+1} \leqslant v_{\alpha,N}$ and the property (2) holds for $n = N + 1$.

Because $v(K_2/K_1) = v^{(N+1)} \geqslant pv_{\alpha,N}/(p-1)+1 \geqslant 1$, we have $e_{\alpha 2,N+1} = pe_{\alpha 2,N}$. By the above construction of the field $K_{\alpha 2,N+1}$ and the property 1 b) of n.3.1.1, we have

$$v_{\alpha 2,N+1} = \max\{v_{\alpha 2,N}, \varphi_{K_{\alpha 2,N}/K}(v^{(N+1)})\}.$$

Therefore, the property (1) for $n = N + 1$ follows from the inequality of the above lemma.

The proposition is proved.

We use the above construction to obtain the following proposition.

Proposition 2. *For any $\alpha \in \mathcal{A}_1$, there exists a commutative diagram in the category of finite extensions of the field K:*

$$\begin{array}{ccccccccc}
K_\alpha^H & \subset & K_{\alpha 2,0} & \subset & \dots & \subset & K_{\alpha 2,n} & \subset & \dots \\
\cap & & \cap & & & & \cap & & \\
K_\alpha^{(v_0)} & \subset & L_{\alpha 2,0} & \subset & \dots & \subset & L_{\alpha 2,n} & \subset & \dots
\end{array}$$

such that for any $n \in \mathbb{Z}_{\geqslant 0}$ it holds:

a) $K_{\alpha 2,n} \subset K(p)^H$, $L_{\alpha 2,n} = K_\alpha^{(v_0)} K_{\alpha 2,n}$ *and the natural map*

$$\mathrm{Gal}(L_{\alpha 2,n}/K_{\alpha 2,n}) \longrightarrow \widetilde{H}_\alpha = \mathrm{Gal}(K_\alpha^{(v_0)}/K_\alpha^H)$$

is an isomorphism;

b) $[K_{\alpha 2,n+1} : K_{\alpha 2,n}] = p$ *and* $v^{(n+1)} := v(K_{\alpha 2,n+1}/K_{\alpha 2,n}) > pv_{\alpha,n}/(p-1)$, *where* $v_{\alpha,n} = v(L_{\alpha 2,n}/K_{\alpha 2,n})$;

c) $v(L_{\alpha 2,n}/K_{\alpha 2,0}) < e^*(K_{\alpha 2,0})$;

d) $v^{(n+1)} \leqslant v^{(N^*)}$ *for some* $N^* \in \mathbb{N}$, *and* $\varphi_{K_{\alpha 2,N^*-1}/K}(v^{(N^*)}) < v_0$.

Proof. Consider the fields tower $K_{\alpha 2,n}$, $n \geqslant 0$, from the above proposition 1. Obviously, the statements a) and b) hold for this tower.

For any $n \in \mathbb{N}$, we have

$$v^{(n)} = v(K_{\alpha 2,n}/K_{\alpha 2,n-1}) < pv_{\alpha,n}/(p-1) + C(\widetilde{H}) + 1 \leqslant pe_{\alpha 2,0}/(p-1) \leqslant e^*(K_{\alpha 2,0}).$$

Applying the proposition 1 b) of n.3.1.1 we obtain that

$$v(K_{\alpha 2,n}/K_{\alpha 2,0}) \leqslant \max\left\{v^{(m)} \mid 1 \leqslant m \leqslant n\right\} < e^*(K_{\alpha 2,0}).$$

Because $v(L_{\alpha 2,0}/K_{\alpha 2,0}) = v_{\alpha,0} < pv_{\alpha,0}/(p-1) + C(\widetilde{H}) + 1 < e^*(K_{\alpha 2,0})$ and $L_{\alpha 2,n} = L_{\alpha 2,0}K_{\alpha 2,n}$ we obtain the statement c).

Because all $v^{(n)}$ belong to $\mathbb{N}$ and are less than $pe_{\alpha 2,0}/(p-1)$, the set $\{v^{(n)} \mid n \in \mathbb{N}\}$ has the maximal element $v^{(N^*)}$, where $N^* \in \mathbb{N}$. By the proposition 1 b) of n.3.1.1 we have also

$$\varphi_{K_{\alpha 2,N^*-1}/K}(v^{(N^*)}) \leqslant v(K_{\alpha 2,N^*}/K) \leqslant v_{\alpha 2,N^*} < v_0.$$

The proposition is proved.

3.5. In this section we prove an auxiliary proposition in the case $\operatorname{char} K = p$.

As earlier, $K \subset K_1 \subset L_1$ is a tower of finite extensions in $K(p)$ and $v_1 = v(L_1/K_1) \geq 1$. Consider an extension K_2 of degree p over K_1 in $K(p)$ such that $v^* = v(K_2/K_1) > pv_1/(p-1)$.

Obviously, K_2 and L_1 are linearly disjoint over K_1. Therefore, if $L_2 = L_1K_2$ then $[L_2 : K_2] = [L_1 : K_1]$. In fact, we have the following more strong statement:

Proposition. *With the above notation and assumptions there exist field isomorphisms $f : K_1 \longrightarrow K_2$ and $g : L_1 \longrightarrow L_2$ such that $g|_{K_1} = f$.*

Proof. If E is one of the fields K_1, K_2, L_1, L_2, denote by O_E its valuation ring.

Lemma 1. *There exist uniformizing elements t_1 in K_1 and t_2 in K_2 such that*

$$t_1 \equiv t_2^p \bmod t_2^{p+v^*(p-1)} O_{K_2}.$$

Proof of lemma. If k_1 is the residue field of K_1 and t is its uniformiser, then $K_1 = k_1((t))$. By the Artin-Schreier theory, $K_2 = K_1(T)$, where $T^p - T = a \in K_1$ and $a = \alpha t^{-v^*} +$ (higher terms), $\alpha \in k_1^*$.

Set $\alpha_1 = \sigma^{-1}(\alpha^{-1})$ (where σ is Frobenius) and $\alpha_1 T = T_1$, then

$$(*) \qquad T_1^p(1 - \alpha_1^{\sigma-1}T_1^{1-p}) = t^{-v^*}\varepsilon,$$

where $\varepsilon \in k_1[[t]]$ is a principal unit.

Clearly, there exists a uniformizer t_0 of K_1 such that $t^{-v^*}\varepsilon = t_0^{-v^*}$. Now the relation $(*)$ implies that $T_1 = t_2^{-v^*}$, where t_2 is a uniformizer of K_2, and can be rewritten in the following form

$$t_2^p\left(1 - \alpha_1^{\sigma-1}t_2^{v^*(p-1)}\right)^{-1/v^*} = \beta t_0,$$

where $\beta \in k_1$ is such that $\beta^{v^*} = 1$. This gives $t_2^p \equiv t_1 \bmod t_2^{p+v^*(p-1)}$, where $t_1 = \beta t_0$ is a uniformizer of K_1. The lemma is proved.

Let

$$P = P(X) = X^r + \sum_{1 \le i \le r} a_i X^{r-i} \in O_{K_1}[X]$$

be the characteristic polynomial of some generator of the valuation ring O_{L_1} of the field L_1 over the valuation ring O_{K_1} of the field K_1.

The proof of the following statement can be found in [De].

Lemma 2. *If $y \in K_{\rm sep}$ is such that $v_{K_1}(P(y)) > 1 + v_1$, then there exists $\theta \in K_{\rm sep}$ such that $P(\theta) = 0$ and $v_{K_1}(y - \theta) > v_{K_1}(y - \theta')$, where $\theta' \in K_{\rm sep}, \theta' \ne \theta$, and $P(\theta') = 0$.*

Remark. If $\tau \in \mathrm{Gal}(K_{\rm sep}/K_1) \subset \Gamma$ and $\tau y = y$, then $\tau\theta = \theta$. Therefore, $\theta \in K_1(y)$.

We use uniformizing elements t_1 and t_2 from the lemma 1 for identifications $K_1 = k_1((t_1))$ and $K_2 = k_1((t_2))$ and define the isomorphism $f : K_1 \longrightarrow K_2$ by the following conditions: $f(t_1) = t_2$ and $f|_{k_1} = \sigma^{-1}$, where σ is Frobenius.

Consider the extension $\widetilde{L}_2$ of K_2 in $K(p)$ generated by some root θ_2 of the polynomial

$$f_*P = X^r + \sum_{1 \le i \le r} f(a_i) X^{r-i} \in O_{K_2}[X].$$

If θ_2 is a root of f_*P in $K(p)$, then

$$P(\theta_2^p) = P(\theta_2^p) - \sigma\left(\left(f_*P\right)(\theta_2)\right) = \sum_{1 \le i \le r} \left(a_i(t_1) - a_i(t_2^p)\right) \theta_2^{p(r-i)}$$

and by lemma 1

$$v_{K_1}\left(P(\theta_2^p)\right) \geqslant \frac{1}{p}(p + v^*(p-1)) > 1 + v_1.$$

Now lemma 2 gives the existence of $\theta \in K_1(\theta_2^p) \subset \widetilde{L}_2$ such that $P(\theta) = 0$. Therefore, $L_1 \subset \widetilde{L}_2$, $L_1 K_2 = \widetilde{L}_2$ and the correspondence $\theta \mapsto \theta_2$ gives the extension of f to the isomorphism $g : L_1 \longrightarrow \widetilde{L}_2 = L_2$.

The proposition is proved.

3.6. *The fields $\mathcal{K}_{\alpha n}$ and $\mathcal{L}_{\alpha n}$, $n \ge 0$.*

Assume first that $\mathrm{char}\, K = 0$.

In this case one can apply considerations of n.3.3 to construct for the field $K_{\alpha 2,0}$ its infinite extension $\widetilde{K}_{\alpha 2,0}$ in $K_{\rm sep}$, consider the complete discrete valuation field $\mathcal{K}_{\alpha 0} = \mathcal{X}_{K_{\alpha 2,0}}(\widetilde{K}_{\alpha 2,0})$ of characteristic p, and the equivalence $\widetilde{\mathcal{X}}_0 := \widetilde{\mathcal{X}}_{\widetilde{K}_{\alpha 2,0}}$ of the category of algebraic extensions $L/K_{\alpha 2,0}$ such that $v(L/K_{\alpha 2,0}) < e^*(K_{\alpha 2,0})$ and the category of separable extensions $\mathcal{L}/\mathcal{K}_{\alpha 2,0}$ such that $v(\mathcal{L}/\mathcal{K}_{\alpha 0}) < e^*(K_{\alpha 2,0})$.

If $\widetilde{\mathcal{X}}_0(K_{\alpha 2,n}) = \mathcal{K}_{\alpha n}$ and $\widetilde{\mathcal{X}}_0(L_{\alpha 2,n}) = \mathcal{L}_{\alpha n}$ for $n \ge 0$, then we obtain the following commutative diagram of complete discrete valuation fields of characteristic p and their embeddings:

$$\begin{array}{ccccccc} \mathcal{K}_{\alpha 0} & \subset & \dots & \subset & \mathcal{K}_{\alpha n} & \subset & \dots \\ \cap & & & & \cap & & \\ \mathcal{L}_{\alpha 0} & \subset & \dots & \subset & \mathcal{L}_{\alpha n} & \subset & \dots \end{array}$$

such that for any $n \geq 0$, $\mathcal{L}_{\alpha n} = \mathcal{L}_{\alpha 0}\mathcal{K}_{\alpha n}$. Note that the functor $\widetilde{\mathcal{X}}_0$ induces the identifications

$$\mathrm{Gal}(\mathcal{L}_{\alpha n}/\mathcal{K}_{\alpha n}) = \mathrm{Gal}(L_{\alpha 2,n}/K_{\alpha 2,n}).$$

Because the equivalence $\widetilde{\mathcal{X}}_0$ is compatible with ramification filtrations we have also for $n \geq 1$, that

$$v(\mathcal{K}_{\alpha,n}/\mathcal{K}_{\alpha,n-1}) = v^{(n)} > pv_{\alpha,n-1}/(p-1),$$

where $v_{\alpha,n-1} = v(\mathcal{L}_{\alpha,n-1}/\mathcal{K}_{\alpha,n-1})$.

If the case $\mathrm{char}\, K = p$ we have the same result by setting $\mathcal{K}_{\alpha n} = K_{\alpha 2,n}$ and $\mathcal{L}_{\alpha n} = L_{\alpha 2,n}$ for all $n \geq 0$.

With the above notation we obtain from the above proposition of n.3.5 the following proposition.

Proposition. *For $n \geq 0$ there exist field isomorphisms $i_n : \mathcal{K}_{\alpha 0} \longrightarrow \mathcal{K}_{\alpha n}$ and $j_n : \mathcal{L}_{\alpha 0} \longrightarrow \mathcal{L}_{\alpha n}$ such that $j_n|_{\mathcal{K}_{\alpha 0}} = i_n$, i.e. j_n is a prolongation of i_n.*

3.7. Because the Galois group of a maximal p-extension of the field $\mathcal{K}_{\alpha 0}$ is pro-p-free and $d(\mathcal{G}_\beta) = d(\widetilde{H}_\alpha)$ (indeed, $d(\widetilde{H}) = d(\mathcal{G}) \geqslant d(\mathcal{G}_\beta) \geqslant d(\widetilde{H}_\alpha) = d(\widetilde{H})$), there exists a Galois extension $\mathcal{F}_{\beta\alpha 0}$ of $\mathcal{K}_{\alpha 0}$ such that $\mathcal{F}_{\beta\alpha 0} \supset \mathcal{L}_{\alpha 0}$ and there exists a group isomorphism $\tilde{g}_{\beta\alpha 0}$ such that the following diagram is commutative:

$$(*_0) \qquad \begin{array}{ccc} \mathcal{G}_\beta & \xrightarrow{\tilde{g}_{\beta\alpha 0}} & \mathrm{Gal}(\mathcal{F}_{\beta\alpha 0}/\mathcal{K}_{\alpha 0}) \\ {\scriptstyle j_{\beta\alpha}}\downarrow & & \downarrow \\ \widetilde{H}_\alpha & \xrightarrow{\tilde{f}_{\alpha 0}} & \mathrm{Gal}(\mathcal{L}_{\alpha 0}/\mathcal{K}_{\alpha 0}) \end{array}$$

where $\tilde{f}_{\alpha 0}$ is induced by identifications

$$\widetilde{H}_\alpha \longrightarrow \mathrm{Gal}(L_{\alpha 2,0}/K_{\alpha 2,0}) \longrightarrow \mathrm{Gal}(\mathcal{L}_{\alpha 0}/\mathcal{K}_{\alpha 0})$$

and the right vertical arrow is the natural projection.

If $n \geq 0$, consider a prolongation of the isomorphism $j_n : \mathcal{L}_{\alpha 0} \longrightarrow \mathcal{L}_{\alpha n}$ from n.3.6 to an isomorphism of separable closures

$$\bar{j}_n : \widetilde{\mathcal{X}}_0(\bar{K}) = \mathcal{K}_{\alpha 0,\mathrm{sep}} \longrightarrow \mathcal{K}_{\alpha n,\mathrm{sep}} = \mathcal{K}_{\alpha 0,\mathrm{sep}},$$

and let $\mathcal{F}_{\beta\alpha n} = \bar{j}_n(\mathcal{F}_{\beta\alpha 0})$.

Then $\mathcal{F}_{\beta\alpha n} \supset \mathcal{L}_{\alpha n} \supset \mathcal{K}_{\alpha n}$ and $v(\mathcal{F}_{\beta\alpha n}/\mathcal{K}_{\alpha n}) = v(\mathcal{F}_{\beta\alpha 0}/\mathcal{K}_{\alpha 0}) = v_{\beta\alpha 0}$ does not depend on n.

By the use of the above prolongations $\bar{j}_n$, $n \geqslant 0$, we obtain the following commutative diagrams:

$$(*_n) \qquad \begin{array}{ccc} \mathcal{G}_\beta & \xrightarrow{\tilde{g}_{\beta\alpha n}} & \mathrm{Gal}(\mathcal{F}_{\beta\alpha n}/\mathcal{K}_{\alpha n}) \\ {\scriptstyle j_{\beta\alpha}}\downarrow & & \downarrow \\ \widetilde{H}_\alpha & \xrightarrow{\tilde{f}_{\alpha n}} & \mathrm{Gal}(\mathcal{L}_{\alpha n}/\mathcal{K}_{\alpha n}) \end{array}$$

where the right vertical arrow is the natural projection, $\tilde{g}_{\beta\alpha n}$ and $\tilde{f}_{\alpha n}$ are group isomorphisms.

Lemma. *There exists $N_1 \geq 0$ such that for $n \geq N_1$, we have*

$$v(\mathcal{F}_{\beta\alpha n}/\mathcal{K}_{\alpha 0}) < e^*(K_{\alpha 2,0}).$$

Proof. If $n \geq 0$, then

$$v(\mathcal{F}_{\beta\alpha n}/\mathcal{K}_{\alpha 0}) = \max\left\{v(\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha 0}), \varphi_{\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha 0}}(v_{\beta\alpha 0})\right\}.$$

Clearly (cf. proposition 2 of n.3.4),

$$v(\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha 0}) = v(K_{\alpha 2,n}/K_{\alpha 2,0}) \leqslant v(L_{\alpha 2,n}/K_{\alpha 2,0}) < e^*(K_{\alpha 2,0}).$$

For all $n \in \mathbb{N}$, the natural number $N^* \in \mathbb{N}$ from n.3.4 and any $x \geqslant 0$, we have

$$\varphi_{\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha,n-1}}(x) \leqslant \varphi^*(x),$$

where

$$\varphi^*(x) = \varphi_{\mathcal{K}_{\alpha,N^*}/\mathcal{K}_{\alpha,N^*-1}}(x) = \begin{cases} x, & \text{for } 0 \leqslant x \leqslant v^{(N^*)} \\ v^{(N^*)} + (x - v^{(N^*)})/p, & \text{for } x > v^{(N^*)}. \end{cases}$$

By the composition property, we have $\varphi_{\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha,0}}(v_{\beta\alpha 0}) \leqslant \varphi^{*(n)}(v_{\beta\alpha 0})$, where $\varphi^{*(n)}$ is the n-th iteration of the function φ^*. It is easy to see that:

1) if $v_{\beta\alpha 0} \leqslant v^{(N^*)}$, then $\varphi^{*(n)}(v_{\beta\alpha 0}) \leqslant v^{(N^*)}$;
2) if $v_{\beta\alpha 0} > v^{(N^*)}$, then $\varphi^{*(n)}(v_{\beta\alpha 0}) \underset{n\to\infty}{\longrightarrow} v^{(N^*)}$.

Because $v^{(N^*)} < e^*(K_{\alpha 2,0})$ (cf. the beginning of the proof of proposition 2 of n.3.4), the above properties 1) and 2) imply the existence of $N_1 \in \mathbb{N}$ such that

$$\varphi^{*(n)}(v_{\beta\alpha 0}) < e^*(K_{\alpha 2,0})$$

for all $n \geqslant N_1$. Clearly, this gives the statement of our lemma.

3.8. If $\operatorname{char} K = 0$, then by the lemma of n.3.7, we can apply the inverse equivalence $\widetilde{\mathcal{X}}_0^{-1}$ to obtain the following commutative diagrams for all $n \geq N_1$ from the above diagrams $(*_n)$ of n.3.7:

$$(**_n) \qquad \begin{array}{ccc} \mathcal{G}_\beta & \xrightarrow{g_{\beta\alpha n}} & \mathrm{Gal}(F_{\beta\alpha n}/K_{\alpha 2,n}) \\ {\scriptstyle j_{\beta\alpha}}\downarrow & & \downarrow \\ \widetilde{H}_\alpha & \xrightarrow{f_{\alpha n}} & \mathrm{Gal}(L_{\alpha 2,n}/K_{\alpha 2,n}), \end{array}$$

where $\widetilde{\mathcal{X}}_0(F_{\beta\alpha n}) = \mathcal{F}_{\beta\alpha n}$, $f_{\alpha n}$ is the natural identification, $g_{\beta\alpha n}$ is a group isomorphism, and the vertical arrow is the natural projection.

The same result holds also in the characteristic p case, if we take identical functor instead of $\widetilde{\mathcal{X}}_0^{-1}$.

Lemma. *There exists* $N_2 \geq N_1$ *such that for all* $n \geq N_2$, *we have*

$$v(F_{\beta\alpha n}/K) < v_0.$$

Proof. Let $N_1^* = \max\{N_1, N^* - 1\}$.

If $n \geqslant N_1^*$, then

$$v(F_{\beta\alpha n}/K) = \max\left\{v(K_{\alpha 2,N_1^*}/K), \varphi_{K_{\alpha 2,N_1^*}/K}\left(v(F_{\beta\alpha n}/K_{\alpha 2,N_1^*})\right)\right\}.$$

We know that $v(K_{\alpha 2,N_1^*}/K) < v_0$.

By proposition 1 b) of n.3.1.1, we have

$$v(F_{\beta\alpha n}/K_{\alpha 2,N_1^*}) = v(\mathcal{F}_{\beta\alpha n}/\mathcal{K}_{\alpha N_1^*}) = \max\left\{v(\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha N_1^*}), \varphi_{\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha N_1^*}}(v_{\beta\alpha 0})\right\}.$$

Clearly, for all $n \geqslant N_1^*$, we have $v(\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha N_1^*}) \leqslant v^{(N^*)}$. As in n.3.7, we obtain either

$$\varphi_{\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha N_1^*}}(v_{\beta\alpha 0}) \leqslant v^{(N^*)}$$

for all $n \geqslant N_1^*$, or

$$\varphi_{\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha N_1^*}}(v_{\beta\alpha 0}) \underset{n\to\infty}{\longrightarrow} v^{(N^*)}.$$

By the proposition 2 d) of n.3.4, we have $v^{(N^*)} < \psi_{K_{\alpha 2,N^*-1}/K}(v_0)$ therefore, there exists $N_2 \geqslant N_1^*$ such that for any $n \geqslant N_2$ one has

$$\varphi_{\mathcal{K}_{\alpha n}/\mathcal{K}_{\alpha N_1^*}}(v_{\beta\alpha 0}) < \psi_{K_{\alpha 2,N^*-1}/K}(v_0).$$

Therefore, for $n \geqslant N_2$ it holds

$$v(F_{\beta\alpha n}/K_{\alpha 2,N_1^*}) < \psi_{K_{\alpha 2,N^*-1}/K}(v_0)$$

and

$$\varphi_{K_{\alpha 2,N_1^*}/K}(v(F_{\beta\alpha n}/K_{\alpha 2,N_1^*})) < \varphi_{K_{\alpha 2,N_1^*}/K}\left(\psi_{K_{\alpha 2,N^*-1}/K}(v_0)\right) \leqslant$$

$$\varphi_{K_{\alpha 2,N^*-1}/K}\left(\varphi_{K_{\alpha 2,N_1^*}/K_{\alpha 2,N^*-1}}\left(\psi_{K_{\alpha 2,N^*-1}/K}(v_0)\right)\right) \leqslant$$

$$\varphi_{K_{\alpha 2,N^*-1}/K}\left(\psi_{K_{\alpha 2,N^*-1}/K}(v_0)\right) = v_0.$$

The lemma is proved.

3.9. Finally, we note that the statement of the main lemma is satisfied with $E_{\beta\alpha} = K_{\alpha 2,N_2}$, $E'_{\beta\alpha} = L_{\alpha 2,N_2}$, $F_{\beta\alpha} = F_{\beta\alpha N_2}$ and the diagram $(**_{N_2})$ of n.3.8. The main lemma is completely proved.

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