On Complex Projective Hypersurfaces

which are Homology– P_n's

by

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Introduction. Taking into account the importance of the complex projective n-space $\mathbf{P}_n = \mathbf{P}_n(\mathbf{C})$ in algebraic geometry and topology, it is obvious that characterizing that space by algebro-geometric or topological properties always has been a matter of great interest. Therefore, it is quite natural to investigate spaces that share some of these properties. In this paper, we look for hypersurfaces in $\mathbf{P}_{n+1}$ with normal or even isolated singularities that have the integral homology of $\mathbf{P}_n$ (where $n \geq 2$). Such hypersurfaces will be called *homology-* $\mathbf{P}_n$*'s*. Our main results are as follows:

Theorem 1. ***(Cohomology-*** $\mathbf{P}_n$***'s are Hyperplanes.)*** *Let V be a closed subvariety of dimension $\dim V = n \geq 2$ in some projective space $\mathbf{P}_N$ which can be described by a system of at most $N-2$ homogeneous polynomials. If the cohomology rings $H^\bullet(V, \mathbf{Z})$ and $H^\bullet(\mathbf{P}_n, \mathbf{Z})$ are isomorphic, then V is a linear subspace of $\mathbf{P}_N$.*

Actually, in the precise statement (see section 1 below), the condition on the cohomology ring structure is slightly weakened. Note that the condition on the number of defining equations is always satisfied for complete intersection varieties — and only for these in the surface case $n = 2$. As the example of the Veronese surface $V \subset \mathbf{P}_5$ shows, that condition is sharp, as V can be described by 4 quadratic forms.

Theorem 2. ***(Examples of Homology-*** $\mathbf{P}_n$***'s with Isolated Singularities.)*** *For any dimension $n \geq 2$, degree $d \geq 3$, and integer a with $1 \leq a < d-1$, we consider the hypersurface $V := V^a_{n,d} : (f_{d,a} = 0)$ in $\mathbf{P}_{n+1}$ defined by*

$$f_{d,a}(x_0, x_1, \dots, x_n, x_{n+1}) := x_0^a x_1^{d-a} + x_1 x_2^{d-1} + \dots + x_{n-1} x_n^{d-1} + x_{n+1}^d \, .$$

This hypersurface has isolated singularites and satisfies
(i) $H_\bullet(V, \mathbf{Q}) \cong H_\bullet(\mathbf{P}_n, \mathbf{Q})$ *for* $(a, d) = 1$;
(ii) $H_\bullet(V, \mathbf{Z}) \cong H_\bullet(\mathbf{P}_n, \mathbf{Z})$ *for* $(a, d) = (a, d-1) = 1$.

The proof (in section 2) makes use of results from local singularity theory (monodromy arguments) and provides examples for the following phenomena that may be of interest in singularity theory and topology:

(i) Examples of hypersurface singularities with one-dimensional singular locus and having the monodromy operator equal to the identity (see section 2, Lemma 2 and Remark). This contrasts the situation for isolated hypersurface singularities, as described by A'Campo [–].

(ii) Examples of hypersurface singularity links in all dimensions ≥ 3 which are integral homology spheres (and hence topological spheres), but which are not associated to polynomials of Pham-Brieskorn type (see section 2, Corollary 1). This contrasts the situation in dimension 2 (see section 3, Appendix).

(iii) Examples of projective hypersurfaces in odd dimensions ≥ 3 with (at most two) isolated singularities which are topological manifolds (see section 2, Corollary 2). These varieties have the integral homology and the rational homotopy type of $\mathbf{P}_n$, but are not homotopy equivalent to $\mathbf{P}_n$ (e.g., by Theorem 1). Again, this contrasts the situation in dimension 2: By a famous result of Mumford [–], a surface with normal singularities (e.g., a two-dimensional hypersurface with isolated singularities) never is a topological manifold.

Note that the hypersurfaces $V^a_{n,d}$ admit a natural algebraic $\mathbf{C}^*$-action, as the affine equations at $(1:0:0:\ldots:0)$ and $(0:1:0:\ldots:0)$ are weighted homogeneous. In the case of surfaces (i.e., $n=2$) with such a $\mathbf{C}^*$-action, there are no other examples of homology planes (see section 3):

Theorem 3. *(Classification of Homology-* $\mathbf{P}_2$*'s with* $\mathbf{C}^*$*-Action.) Let V be hypersurface in $\mathbf{P}_3$ of degree $d \geq 3$ which has the integral homology of $\mathbf{P}_2$ and admits an algebraic $\mathbf{C}^*$-action. Then V is (isomorphic to) $V^a_{2,d} \subset \mathbf{P}_3$ for a unique integer a satisfying $1 \leq a < d-1$ and $(a, d-1) = (a, d) = 1$.*

Examples of homology- $\mathbf{P}_n$'s in dimensions $n \geq 3$ with singular locus of positive dimension can be obtained by more elementary methods than in the isolated singularity case. Such examples will be presented in section 4 (see Theorem 4).

We mention that Theorem 1 in the hypersurface case and some of the examples in Theorem 2 in the two-dimensional case (namely, the case $a = 1$, mildly disguised) have already appeared in [**ChDi**].

It is a great pleasure for us to thank Ludger Kaup for his stimulating interest. In particular, section 1 was strongly influenced by him through discussions with one of us. Moreover, in sections 3 and 4, we closely follow ideas of earlier joint papers of his and the first-named author. We think it quite appropriate to dedicate this paper to him on his 50^{th} birthday (with due delay).

Both authors enjoy(ed) the hospitality of the "Max Planck -Institut für Mathematik" in Bonn—the second one during the time when this was written, the first one at some earlier occasions. It is our pleasure to thank that institution, its members and staff, and in particular its director, F. Hirzebruch.

Notations and Conventions: Most of the varieties to be considered in the sequel are—in suitable affine coordinates—defined by *weighted homogeneous* (or quasihomogeneous) polynomials. Recall that by definition, such a polynomial $p(y_0, y_1, \ldots, y_m)$ satisfies an identity $p(t^{q_0}y_0, t^{q_1}y_1, \ldots, t^{q_m}y_m) = t^N \cdot p(y_0, y_1, \ldots, y_m)$ for a suitable vector $\boldsymbol{q} = (q_0, q_1, \ldots, q_m)$ of integers q_j and an integer N, the $\boldsymbol{q}$-degree $\boldsymbol{q}$-deg(p), so with respect to the grading of the polynomial algebra $\mathbf{C}[y_0, y_1, \ldots, y_m]$ given by $\boldsymbol{q}$-deg$(y_j) = q_j$, it is a homogeneous element of degree $\boldsymbol{q}$-deg$(p) = N$. Note that the q_j's are not necessarily positive. We adopt here the convention to call the q_j's the *weights*. They are just the weights of the $\mathbf{C}^*$-action on $\mathbf{C}^{m+1}$ given by $t\bullet(y_0, y_1, \ldots, y_m) = (t^{q_0}y_0, t^{q_1}y_1, \ldots, t^{q_m}y_m)$ that is associated to the grading. We always assume that the action is effective or, equivalently, that the weight vector $\boldsymbol{q}$ is primitive, i.e., $\gcd(q_0, q_1, \ldots, q_m) = 1$. We sometimes call p a $\boldsymbol{q}$-homogeneous polynomial. The pair $(\boldsymbol{q}, \boldsymbol{q}\text{-deg}(p))$ is called the *type* of p.

Concerning the notion of "weight", there are different conventions used in the literature, especially in the case of a strictly positive grading (i.e., all $q_j > 0$, corresponding to a "good" $\mathbf{C}^*$-action). In addition to those discussed in [TRCS: Ch. 7, §1], we mention the one adopted by Milnor, Orlik, and some others, where the positive rational numbers $w_j := \boldsymbol{q}\text{-deg}(f)/q_j$ are called weights. Instead, we will call these w_j the *coweights* in the sequel. To emphasize that we are in the case of a strictly positive grading, we sometimes call a $\boldsymbol{q}$-homogeneous polynomial *positively* weighted homogeneous.

1. Projective Varieties which are Cohomology- $\mathbf{P}_n$'s. In this section, we prove the result mentioned in the introduction above: Cohomology- $\mathbf{P}_n$'s are hyperplanes. The actual—slightly more general—statement is as follows.

Theorem 1. *Let V be a closed subvariety of dimension $\dim V = n \geq 2$ in some projective space $\mathbf{P}_N$ which can be described by a system of at most $N-2$ homogeneous polynomials. If the cohomology group $H^2(V, \mathbf{Z})$ is generated (up to torsion) by a class u such that u^n generates $H^{2n}(V, \mathbf{Z})$, then V is a linear subspace of $\mathbf{P}_N$.*

Proof. Denote with $j : V \hookrightarrow \mathbf{P}_N$ the inclusion mapping and with $\omega \in H^2(\mathbf{P}_N)$ the canonical generator. Then there is an integer α (w.l.o.g. $\alpha > 0$) such that $j^*\omega = \alpha u$ and hence $j^*\omega^n = \alpha^n u^n$ holds in $H^2(V)$ (up to torsion) and in $H^{2n}(V)$, respectively. By a well known property of the degree (see, e.g., [**PAG**: pp. 171]), we have

$$\langle j^*\omega^n, [V]\rangle = \langle \omega^n, j_*[V]\rangle = \deg V = \alpha^n$$

(where $\langle \ldots, \ldots \rangle$ denotes the usual pairing, sometimes called "Kronecker product"). In order to show that $\alpha = 1$, look at the exact cohomology sequence of the pair $(\mathbf{P}_N, V)$:

$$H^2(\mathbf{P}_N) \to H^2(V) \to H^3(\mathbf{P}_N, V)\ .$$

By Lefschetz duality, the last group is isomorphic to $H_{2N-3}(\mathbf{P}_N \setminus V)$. As V can be defined by at most $N-2$ equations $(f_j = 0)$, the complex manifold $\mathbf{P}_N \setminus V$ is the union of at most $N-2$ affine open subsets $(f_j \neq 0)$ and hence is *topologically* $(N-3)$*-complete* (see [**FiKp₁**: §1 and 2.3] for the definition and for the result). It follows from the theorem stated in the introduction of [**FiKp₂**] that $H_{2N-3}(\mathbf{P}_N \setminus V)$ has no torsion. By the exactness of the sequence, $j^*\omega = \alpha u$ is a generator of $H^2(V)$. It follows that $\alpha = 1$ and hence $\deg V = 1$. •

The notion of a topologically q-complete space is modeled after the topological properties of *analytically* q-complete spaces. In fact, by a theorem of Hamm [–], an analytically q-complete complex space of dimension n is of the homotopy type of a CW-complex of (topological) dimension at most $n+q$. The topological completeness has a much nicer behaviour and better permanence properties with respect to standard operations; in particular, it is a homeomorphy invariant.

2. Projective Hypersurfaces with Isolated Singularities which are Homology-$\mathbf{P}_n$'s. In this section, we prove Theorem 2 as stated in the introduction. In order to show that those hypersurfaces $V = V^a_{n,d} : (f_{d,a} = 0)$ in $\mathbf{P}_{n+1}$ (with $n \geq 2$) of degree $d \geq 3$ with isolated singularities have the integral homology of $\mathbf{P}_n$, we first use duality and monodromy arguments to check that they are rational homology- $\mathbf{P}_n$'s (see paragraphs i)–iii)). In paragraph iv), we state conditions (in terms of Milnor lattices) for a rational homology- $\mathbf{P}_n$ with isolated singularities to be an integral homology- $\mathbf{P}_n$. Finally, using results of Milnor, Orlik, and Randell on the monodromy of certain weighted homogeneous singularities, we show in paragraphs v)–vii) that our examples satisfy these conditions.

i) We begin with a characterization of *rational* homology- $\mathbf{P}_n$'s in terms of the monodromy operator of the defining equation. Let $V : (f = 0)$ be a hypersurface of degree $d \geq 2$ in $\mathbf{P}_{n+1}$.

Lemma 1. *The following statements are equivalent:*
(α) $H_\bullet(V, \mathbf{Q}) = H_\bullet(\mathbf{P}_n, \mathbf{Q})$;

(β) *Let $F : (f - 1 = 0) \subset \mathbf{C}^{n+2}$ be the Milnor fibre associated to f, and let $h_f^* : \widetilde{H}^\bullet(F, \mathbf{Q}) \to \widetilde{H}^\bullet(F, \mathbf{Q})$ be the monodromy operator. Then all eigenvalues of h_f^* are different from 1.*

Proof. By a reasoning completely analoguous to that of section 1 (replacing Lefschetz by Alexander), statement (α) is equivalent to the vanishing of $\widetilde{H}^\bullet(\mathbf{P}_{n+1} \setminus V, \mathbf{Q})$. This affine variety $\mathbf{P}_{n+1} \setminus V$ is easily identified with the quotient $F/\langle h_f \rangle$ (where $h_f : F \to F$ is the geometric monodromy): By the homogeneity of f, the group μ_d of d-th roots of unity acts freely on F by multiplication, and the orbit space is $\mathbf{P}_{n+1} \setminus V$. The action of the standard generator $\zeta := \exp(2\pi i/d)$ of μ_d on $F \subset \mathbf{C}^{n+2}$ given by $(x_0, \ldots, x_{n+1}) \mapsto (\zeta x_0, \ldots, \zeta x_{n+1})$ is the geometric monodromy h_f. Hence, the cohomology under consideration is isomorphic to $\widetilde{H}^\bullet(F, \mathbf{Q})^{h_f^*}$, the fixed part under h_f^*, and the latter is $\ker(\mathrm{id} - h_f^*)$, the eigenspace of 1. •

ii) To obtain polynomials $f(x_0, x_1, \ldots, x_n, x_{n+1})$ satisfying property (β) above, we consider first the homogeneous polynomial

$$g = g_{d,a}(x_0, x_1, \ldots, x_n) := x_0^a x_1^{d-a} + x_1 x_2^{d-1} + \ldots + x_{n-1} x_n^{d-1}$$

of degree $d \geq 3$ with $n \geq 2$ and $1 \leq a < d - 1$.

Lemma 2. *The monodromy operator h_g^* associated to $g = g_{d,a}$ is the identity operator if a and d are coprime (i.e., if $\gcd(a, d) = 1$).*

Proof. We denote with $G : (g - 1 = 0)$ the Milnor fibre of g in $\mathbf{C}^{n+1}$. We will define a $\mathbf{C}^*$-action on $\mathbf{C}^{n+1}$ such that G is invariant and the geometric monodromy $h_g : G \to G$ is given by "multiplication" (with respect to that action) by some element $\lambda \in \mathbf{C}^*$. Since $\mathbf{C}^*$ is connected, this implies that h_g is homotopy equivalent to the identity, thus proving the lemma.

As g is homogeneous of degree d, the geometric monodromy takes the same nice form $h_g(x_0, \ldots, x_n) = (\zeta x_0, \ldots, \zeta x_n)$ with $\zeta := \exp(2\pi i/d)$ as above. The $\mathbf{C}^*$-action will be given by a vector $\boldsymbol{q} = (q_0, \ldots, q_n)$ of integral weights $q_j = \boldsymbol{q}\text{-deg}(x_j)$. As G is invariant under that action, $\boldsymbol{q}$ has to be chosen such that g is $\boldsymbol{q}$-homogeneous with $\boldsymbol{q}\text{-deg}(g) = 0$. Hence, we have the condition

$$aq_0 + (d-a)q_1 = q_1 + (d-1)q_2 = \ldots = q_{n-1} + (d-1)q_n = 0$$

which is clearly satisfied by taking $q_n = a$, $q_{n-1} = (1-d)a$, $\ldots$, $q_1 = (1-d)^{n-1}a$ and $q_0 = (1-d)^{n-1}(a-d)$. As a and d are coprime by assumption, we can find an integer b with $ab \equiv 1 \,(\mathrm{mod}\, d)$. Since all weights satisfy $q_i \equiv a \,(\mathrm{mod}\, d)$, the element $\lambda := \exp(2\pi i b/d)$ has the required property that $\lambda\bullet(x_0, \ldots, x_n) = h_g(x_0, \ldots, x_n)$ as claimed at the beginning. •

Remark. Note that the affine hypersurface $(g = 0)$ in $\mathbf{C}^{n+1}$ has a one-dimensional singular locus. For *isolated* hypersurface singularities, the monodromy operator is the identity only in the case of an odd-dimensional A_1-singularity, as follows from the results of A'Campo [–: Thme. 2].

iii) Using the polynomials $g = g_{d,a}$ from above, we now introduce the homogeneous polynomials $f := f_{d,a}$ that define our hypersurfaces.

Lemma 3. *Denote with $f := f_{d,a}$ the polynomial*

$$f_{d,a}(x_0, x_1, \ldots, x_n, x_{n+1}) := g_{d,a}(x_0, x_1, \ldots, x_n) + x_{n+1}^d$$

and with $V := V_{n,d}^a$ the projective hypersurface $(f = 0)$ in $\mathbf{P}_{n+1}$. Then V is a rational homology- $\mathbf{P}_n$, i.e., we have $H_\bullet(V_{n,d}^a, \mathbf{Q}) \cong H_\bullet(\mathbf{P}_n, \mathbf{Q})$, if a and d are coprime.

Proof. By Lemma 1, it suffices to show that all eigenvalues of the monodromy operator h_f^* are different from 1 (condition (β)). That follows from Lemma 2: By results of Oka [–: Thm.1, Cor.2], the Milnor fibre F of f is homotopy equivalent to the join $G * \mu_d$ of the Milnor fibres of g and x_{n+1}^d, and the monodromy operator h_f^* on $\widetilde{H}^\bullet(F, \mathbf{Q}) \cong (\widetilde{H}^\bullet(G, \mathbf{Q}) \otimes \widetilde{H}^\bullet(\mu_d, \mathbf{Q}))$ is induced from the join of the geometric monodromies. Hence, we have the equality $h_f^* = h_g^* \otimes h_{(x_{n+1}^d)}^*$ (generalized Thom-Sebastiani Theorem). As $h_g^* = \mathrm{id}_{\widetilde{H}^\bullet(G,\mathbf{Q})}$ by Lemma 2 and all eigenvalues of $h_{(x_{n+1}^d)}^*$ on $\widetilde{H}^\bullet(\mu_d, \mathbf{Q})$ are different from 1, we are done. •

Remark. Any hypersurface $V \subset \mathbf{P}_{n+1}$ that is a rational homology- $\mathbf{P}_n$ also has the same rational cohomology ring as $\mathbf{P}_n$, so in particular, rational Poincaré duality holds. If V has isolated singularities, then it follows from L. Kaup's long exact Poincaré duality sequence (see the introduction in [$\mathbf{Ka_1}$]) that V is a rational homology manifold, i.e., all the singularities of V have links that are rational homology spheres (see also [$\mathbf{Di_2}$: Cor.(2.9)]). Moreover, a rational homology- $\mathbf{P}_n$ has the same rational homotopy type as $\mathbf{P}_n$, as the latter is determined by the rational cohomology ring (see [**Bab**: §2]).

iv) To show that we can actually obtain *integral* homology- $\mathbf{P}_n$'s among these varieties $V_{n,d}^a$, we use results of [$\mathbf{Di_1}$]. For an arbitrary hypersurface $V \subset \mathbf{P}_{n+1}$ with isolated singularities, [–: Thm. 2.1] says that we get isomorphisms $H_j(V, \mathbf{Z}) \cong H_j(\mathbf{P}_n, \mathbf{Z}) \oplus K_j$ for $j = n, n+1$, where K_n denotes the cokernel and K_{n+1} the kernel of a natural lattice homomorphism $\varphi_V : \bigoplus_i L_i \to \overline{L}$ associated to $V \subset \mathbf{P}_{n+1}$. The source of this homomorphism is the (orthogonal) direct sum of the Milnor lattices L_i at the singular points of V; the target is the *reduced* Milnor lattice $\overline{L} := L/\mathrm{Rad}\, L$ of X, the affine cone associated to a smooth hyperplane section of V, at the origin. Recall that the Milnor lattice of an isolated affine hypersurface singularity is the integral homology of the corresponding Milnor fibre, endowed with the intersection form. It is symmetric if the dimension n is even, and skew-symmetric if n is odd. Note that $H_{n+1}(V, \mathbf{Z})$ is torsion free (see [–: Cor. 2.3]). In the case where V is a rational homology- $\mathbf{P}_n$, it clearly follows that φ_V is a monomorphism and that its cokernel K_n is a finite torsion group of order $(\prod_i \det L_i) / \det \overline{L}$. Note that the target lattice $\overline{L}$ is unimodular if n is odd; if n is even, it has determinant $\pm d$ (see [–: Rem. 2.4 and Cor. 1.4, 1.5]). Hence, we state the following

Observation: *Let V be a rational homology- $\mathbf{P}_n$. The following conditions are equivalent:*
(α) *V is an integral homology- $\mathbf{P}_n$;*
(β) *the cokernel K_n of the lattice homomorphism φ_V is trivial;*
(γ) $\prod_i \det L_i = \det \overline{L} = \begin{cases} \pm d & \text{if } n \text{ is even,} \\ \pm 1 & \text{if } n \text{ is odd.} \end{cases}$

To check that condition (γ) holds in suitable cases, we have to investigate the singularities of our hypersurfaces $V = V^a_{n,d}$ more in detail. Denote with o_i (for $i = 0, \ldots, n+1$) the origin of the standard affine coordinate system ($x_i = 1$) on $\mathbf{P}_{n+1}$. The affine equation for V at o_0 is $f_0 = x_1^{d-a} + x_1 x_2^{d-1} + \ldots + x_{n-1} x_n^{d-1} + x_{n+1}^d$, so o_0 is always an isolated singular point. At o_1, we have the affine equation $f_1 = x_0^a + x_2^{d-1} + x_2 x_3^{d-1} + \ldots + x_{n-1} x_n^{d-1} + x_{n+1}^d$, so o_1 is a singular point if (and only if) $a > 1$. It is easy to see that there are no other singularities. Hence, condition (γ) takes the following form:

Condition: *The product of the determinants of the Milnor lattices L_i at o_i is*

$$\det L_0 \cdot \det L_1 = \begin{cases} \pm d & \text{if } n \text{ is even,} \\ \pm 1 & \text{if } n \text{ is odd.} \end{cases}$$

Note that both f_0 and f_1 are (positively) weighted homogeneous. The explicit weights can be computed using the formulae given in the next paragraph.

v) Let $p(y_0, y_1, \ldots, y_m)$ be a positively weighted homogeneous polynomial that has an isolated singularity at the origin. Essentially following Milnor and Orlik [–: §4], we define the integers

$$\kappa(p) := \dim \ker(I - h_p^*) , \quad \text{and} \quad \rho(p) := \det(I - h_p^*) .$$

(Actually, our ρ is the $\Delta(1)$ of Milnor and Orlik, so it agrees with their definition of ρ if it is non-zero, the only case of interest.) Obviously, $\kappa(p)$ is the multiplicity of 1 as an eigenvalue of the monodromy operator, so we have $\kappa = 0 \iff \rho \neq 0$. Moreover, $\kappa(p)$ is the Betti number $b_{m-1}(K)$ of the singularity link $K := (p = 0) \cap S^{2m+1}$, and by one of Milnor's classical results [**SPCH**: Thm. 8.5], the latter is an integral homology sphere if (and only if) $\rho(p) = \pm 1$. Let $L(p)$ denote the Milnor lattice. It follows from the relation between the intersection form and the "variation operator" (or Seifert form—see Lamotke's paper [-: §6, Hauptsatz] or [**SDM II**: 2.5] for that relation) that $\pm\rho(p)$ equals the determinant $\det L(p)$ of the intersection form, so $L(p)$ is nondegenerate iff κ vanishes.

Now let p' be another positively weighted homogeneous polynomial with an isolated singularity in a new set of variables. Then the sum $p+p'$ (sometimes denoted with $p \oplus p'$) is again weighted homogeneous. (If p has type (q, N) and p' has type (q', N'), then in the case $\gcd(N, N') = 1$ to be considered below, the type of $p \oplus p'$ is $(N'q \oplus Nq', N \cdot N')$.) According to Milnor and Orlik [-: §4, Lemma 3], we have the following formula:

$$(1) \qquad \kappa(p+p') = \kappa(p) \cdot \kappa(p') \quad \text{and} \quad \rho(p+p') = \rho(p)^{\kappa(p')} \cdot \rho(p')^{\kappa(p)} \quad \text{if} \quad (N, N') = 1$$

(with $0^0 := 1$).

vi) To apply these formulae to the affine equations f_0 and f_1 in our case, we make use of the following decomposition. For $b \geq 2$ and $m \geq 1$, denote with $p := p_b$ the polynomial

$$(2) \qquad p_b(y_0, y_1, \ldots, y_m) := y_0^b + y_0 y_1^{d-1} + \ldots + y_{m-1} y_m^{d-1} .$$

Then we have

$$(3) \qquad f_0 = p_{d-a}(x_1, \ldots, x_n) + x_{n+1}^d , \quad \text{and} \quad f_1 = p_{d-1}(x_2, \ldots, x_n) + x_0^a + x_{n+1}^d .$$

The polynomial p_b introduced above belongs to the class of the weighted homogeneous polynomials $p = p_{a_0, a_1, \ldots, a_m}(y_0, y_1, \ldots, y_m) := y_0^{a_0} + y_0 y_1^{a_1 - 1} + \ldots + y_{m-1} y_m^{a_m - 1}$ (with $a_0 \geq 2$ and $m \geq 1$) investigated by Orlik and Randell in [-: 2]. The type (q, N) of $p_{a_0, a_1, \ldots, a_m}$ is easily expressed by means of the integers $r_k := \prod_{j=0}^k a_j$ defined in that paper [-: p. 203]: With the alternating sum $s_k := \sum_{j=-1}^k (-1)^{k-j} r_j$ $(= r_k - r_{k-1} + r_{k-2} \pm \ldots \pm 1 = (-1)^{k+1}(1 - a_0(1 - a_1(\ldots(1 - a_{k-1}(1 - a_k))\ldots)))$ for $k \geq 0$ and $s_{-1} = r_{-1} = 1)$, we have

$$N = r_m \quad \text{and} \quad q_k = \frac{s_{k-1} \cdot r_m}{r_k} = s_{k-1} \cdot \prod_{j=k+1}^{m} a_j \quad (\text{for } k = 0, \ldots, m) .$$

In the special case $p = p_b$ to consider (with $a_0 = b$ and $a_1 = \ldots = a_m = d-1$), the explicit values are $r_k = b(d-1)^k$ and $s_k = b \cdot \frac{(d-1)^{k+1} - (-1)^{k+1}}{d} - (-1)^k$ (for $k = 0, \ldots, m$), so the type (q, N) of p_b is given by $q_k = s_{k-1} \cdot (d-1)^{m-k}$ and $N = b(d-1)^m$. Next, we can apply the formula

$$\det(t \cdot I - h_p^*) = \delta(t) = (t^{r_m} - 1) \cdot (t^{r_{m-1}} - 1)^{-1} \cdot \ \ldots \ \cdot (t^{r_0} - 1)^{(-1)^m} \cdot (t^{r_{-1}} - 1)^{(-1)^{m+1}}$$

for the characteristic polynomial of the integral monodromy operator given by Orlik and Randall [–: (2.12)]. That yields immediately the multiplicity $\kappa(p)$ of 1 as eigenvalue of the monodromy operator, namely,

$$\kappa(p) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases} \tag{4}$$

vi) In order to apply formula (1) from above to the decomposition (3) of f_0 and f_1, we need the values of κ and ρ for the "remainder". Simple direct computation yields

$$\kappa(x_{n+1}^d) = 0 \quad \text{and} \quad \rho(x_{n+1}^d) = d \tag{5}$$

as well as

$$\kappa(x_0^a + x_{n+1}^d) = 0 \quad \text{and} \quad \rho(x_0^a + x_{n+1}^d) = 1 \quad \text{if} \quad \gcd(a, d) = 1\ . \tag{6}$$

We now assume that the condition $(a, d-1) = (a, d) = 1$ holds. This allows to apply formula (1), as the respective degrees N and N' are coprime. It follows immediately that we have $\kappa(f_0) = \kappa(f_1) = 0$, so both local equations have nondegenerate Milnor lattices L_j. Using (6), the computation of ρ yields $\rho(f_1) = 1$, so L_1 is always unimodular. By (4) and (5), we get two different values for $\rho(f_0) = \pm\det(L_0)$, according to the parity of n, namely

$$\rho(f_0) = \begin{cases} d & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

It follows that the condition of paragraph iv) is satisfied, so $V = V_{n,d}^a$ has the integral homology of $\mathbf{P}_n$.

That completes the proof of Theorem 2 as stated in the introduction. • •

We mention explicitely the following consequences of the proof, as announced in the introduction.

Corollary 1. *For $n \geq 3$, $d \geq 3$, and $b \geq 2$, consider the weighted homogeneous polynomials $\hat{h} := \hat{h}_{d,a}$ and $\check{h} := \check{h}_{d,b}$ given by*

$$\hat{h}_{d,a}(x_0, \ldots, x_n) := x_0^a + x_1^{d-1} + x_1 x_2^{d-1} + \ldots + x_{n-2} x_{n-1}^{d-1} + x_n^d\ ;$$
$$\check{h}_{d,b}(x_0, \ldots, x_n) := x_0^b + x_0 x_1^{d-1} + x_1 x_2^{d-1} + \ldots + x_{n-2} x_{n-1}^{d-1} + x_n^d$$

with an isolated singularity at the origin. The corresponding singularity links $\hat{K}$ and $\check{K}$ are $(2n-1)$-dimensional homology spheres (and hence actually topological spheres) if the following conditions hold:
for $\hat{h}_{d,a}$: n is arbitrary, and a, $d-1$, and d are pairwise coprime;
for $\check{h}_{d,b}$: n is odd, and b and d are coprime.

Proof. With the notations of (2), the polynomials are $\hat{h}_{d,a} = p_{d-1}(x_1, \dots, x_{n-1}) + x_0^a + x_n^d$ and $\check{h}_{d,b} = p_b(x_0, \dots, x_{n-1}) + x_n^d$. The claim now follows from our computation of ρ, as $\rho = \pm 1$ implies that the link is a homology sphere (see [**SPCH**: Lemma 8.3]). — Note that in general, these equations have fractional coweights, so they are not equivalent to polynomials of Pham-Brieskorn type. Of course, the construction can be generalized to yield more examples.

Corollary 2. *If the dimension $n \geq 3$ is odd and if the hypersurface $V := V_{n,d}^a : (f_{d,a} = 0)$ of the theorem is an integral homology- $\mathbf{P}_n$, then it is a topological manifold. For $n = 3$, that manifold even admits a smooth structure.*

Proof. By the previous corollary, the singularities are integral homology manifold points and hence even topological manifold points. The smooth structure for $n = 3$ comes from the non-existence of exotic spheres of (real) dimension 5: if the link K is a homology sphere, than it is h-cobordant to S^5 (see [KeMi:§1]). — Note that these topological manifolds are not of the same homotopy type as $\mathbf{P}_n$, though they have the integral homology and the rational homotopy type of $\mathbf{P}_n$.

3. Normal Homology Planes in $\mathbf{P}_3$ with $\mathbf{C}^*$-action. Our examples $V_{n,d}^a$ of homology-$\mathbf{P}_n$'s with isolated singularities constructed in section 2 admit a natural algebraic $\mathbf{C}^*$-action, as their affine equations f_0 and f_1 are both weighted homogeneous. Assuming the existence of such an action, we can give a classification in the case of homology *planes*, i.e., in the two-dimensional case. It turns out that for degree $d \geq 3$, the only such surfaces are our examples $V = V_{2,d}^a$. For simplicity, we omit the dimension index $n = 2$ in this section and just write V_d^a.

Theorem 3. *Let V be a normal surface of degree d in $\mathbf{P}_3$ which has the same $\mathbf{Z}$-homology groups as $\mathbf{P}_2$ (i.e., a homology plane) and which admits an algebraic $\mathbf{C}^*$-action. Then V is one of the following surfaces.*
($d = 1$) $V = V_1$ is the projective plane $\mathbf{P}_2$;
($d = 2$) $V = V_2$ is the quadratic cone $(x^2 + y^2 + z^2 = 0)$;
($d \geq 3$) $V = V_d^a$ for a positive integer $a < d-1$ relatively prime to $d-1$ and d.
These surfaces are pairwise non homeomorphic.

Proof. As the quadratic cone is the only surface of degree $d = 2$ in $\mathbf{P}_3$ satisfying the assumptions, we may restrict to the case $d \geq 3$. The $\mathbf{C}^*$-action on V is induced from an action on the ambient space, so in a suitable system $(x_0 : x_1 : x_2 : x_3)$ of homogeneous coordinates, it is of the form $t{\bullet}x := (x_0 : t^{q_1}x_1 : t^{q_2}x_2 : t^{q_3}x_3)$ for a triple $q := (q_1, q_2, q_3)$ of integral weights with $q_1 \geq q_2 \geq q_3 \geq 0$ and $\gcd(q_1, q_2, q_3) = 1$ (e.g., see [$\mathbf{Bar_3}$: 1.1]). The corresponding affine equation $f(1, x_1, x_2, x_3)$ defining $V \cap (x_0 = 1)$ is q-(quasi-)homogeneous of some q-degree N. It is easy to see (e.g., in [$\mathbf{Bar_3}$: 1.4]) that, up to the only exeption of the smooth quadric, every normal $\mathbf{C}^*$-surface in $\mathbf{P}_3$ has an *elliptic* fixed point, i.e., a fixed point that lies in the closure of every orbit passing through a suitable neighbourhood. By taking that point as centre of the invariant affine chart $(x_0 = 1)$, we may assume $q_3 > 0$ (up to reversing the action, i.e., replacing t by t^{-1}). Then the affine equation f is (positively) weighted homogeneous of type (q, N) (recall our conventions from the beginning), and $N \geq d$.

We note first that this affine equation f is not homogeneous: Otherwise, V would be the cone over the smooth plane curve $(x_0 = f = 0)$ of degree d and hence have the third Betti number $b_3(V) = (d-1)(d-2)$ strictly positive (as $d > 2$). The proof of the claim is an easy consequence of the following lemma and Theorem 3A in the appendix to this section.

Lemma. *For a surface V as in Theorem 3, there is a system of homogeneous coordinates $(w : x : y : z)$ with the following properties:*

a) *The origin $o := (1 : 0 : 0 : 0)$ of the affine chart $(w = 1)$ is an elliptic fixed point of the action;*

b) *the second integral local homology $\mathcal{H}_{2,o}$ at o is trivial;*

c) *the curve at infinity $V_\infty := V \cap (w = 0)$ is a projective line.*

Now the second local homology group $\mathcal{H}_{2,o}$ at o is isomorphic to the first homology of the corresponding singularity link K, so the latter is an integral homology sphere. Hence, by Theorem 3A below, the affine equation in the chart $(w = 1)$ of the lemma is $x^a + y^b + z^c = 0$, where the exponents a, b, c are pairwise coprime. To complete the proof of the theorem, we only have to observe that the affine surface defined by a polynomial of Pham-Brieskorn type has a normal projective closure (i.e., isolated singularities at infinity) if and only if the two highest exponents differ by at most 1. Assuming $a \leq b \leq c = d$ (w.l.o.g.) and $d > 1$, we must thus have $b = d - 1$. It follows that V has the equation $w^{d-a}x^a + wy^{d-1} + z^d = f_{d,a}(x, w, y, z) = 0$. $\bullet$

Proof of the Lemma. We may again restrict to the case $d \geq 3$. As the affine equation in the coordinate system $(x_0 = 1)$ chosen above is not homogeneous, we may apply the results of

[**Bar**$_1$]. First, by [–: (3.5.4)(i)], we have $b_2(V) = b_3(V) + b_2(A)$ and hence $b_2(A) = 1$ for the curve $A := V \cap (x_0 = 0)$, so A is irreducible. The argument preceeding [–:(3.5.5)] yields that A is not only homeomorphic to a projective line, but that, interchanging the roles of x_0 and x_1 if necessary, we may even assume that A actually is a projective line. (Note that the condition of [–: (2.3.1)] only concerns the affine singularities.) Then [–: (3.5.5)(i)] yields $H_2(V) = \mathbf{Z} \oplus \mathcal{H}_{2,O}$ and hence $\mathcal{H}_{2,O} = 0$ for the local homology at the affine origin, so we have proved our claim. •

Remark. It is easy to see that the surfaces V_d of [**ChDi**] and our V_d^1 are isomorphic. In fact, after renaming the coordinates for $\mathbf{P}_3$ so that V_d is defined by $w^{d-1}x + w^d + wy^{d-1} + z^d$, the linear transformation $(w : x : y : z) \mapsto (w : x - w : y : z)$ takes V_d into V_d^1. — The fact that the surfaces V_d^1 have the integral homology of the projective plane has been mentioned in [**Bar**$_2$: 2].)

Appendix: Weighted Homogeneous Surface Singularity Links that are Homology Spheres. In this appendix, we discuss a theorem from two-dimensional singularity theory. Though a more general statement can be found in the literature, the result is apparently not widely known. For that reason, the discussion has been included here.

Theorem 3A. *Let $p(x, y, z)$ be a positively weighted homogeneous polynomial with an isolated singularity at the origin. Assume that the link K of the singularity is an integral homology sphere. Then (up to scalar factors), we have $p(x, y, z) = x^a + y^b + z^c$, where the exponents a, b, c (which agree with the coweights in that case) are pairwise coprime.*

Proof. In the class of weighted homogeneous polynomials with integral coweights, the result is a special case of Brieskorn's characterization of homology spheres (see [**Bri**: 2, Satz 1]). For a thorough discussion of that class of surface singularity links, we refer to Milnor's article [**Mil**]. To exclude the various classes of weighted homogeneous polynomials with at least one fractional coweight, we give three arguments.

Maybe the simplest—but also the least illuminating—way is by checking that for polynomials in these classes, the group $H := H_1(K) \cong \mathcal{H}_{2,O}$ never vanishes. That group can be computed using a general formula given by Orlik (see [–: 2.6, 3.3, 3.4]), which is made explicit in our case as follows: Write the coweights w_j of $p(x_1, x_2, x_3)$ as reduced fractions u_j/v_j. Then the rank $\kappa := b_1(K) = b_{2,O}$ of H is

$$\kappa = \frac{w_1 w_2 w_3}{\operatorname{lcm}(u_1, u_2, u_3)} - \sum_{i \neq j} \frac{w_i w_j}{\operatorname{lcm}(u_i, u_j)} + \sum_k \frac{w_k}{u_k} - 1.$$

To compute the torsion subgroup T of H, introduce the numbers

$$\kappa_{ij} := \frac{w_i w_j}{\operatorname{lcm}(u_i, u_j)} - \frac{w_i}{u_i} - \frac{w_j}{u_j} + 1$$

and define integers c, c_i, and c_{ij} by the factorization of the denominators u_j as follows: Let

$$c := \gcd(u_1, u_2, u_3),\ c_i := \frac{\gcd(u_j, u_k)}{c},\ \text{ and }\ c_{ij} := \frac{u_k}{cc_ic_j}\ \text{ for }\ \{i,j,k\} = \{1,2,3\}$$

(so $u_i = cc_jc_kc_{jk}$). Finally, introduce the integers

$$t_l := \prod_{\kappa_{ij} \geq l} c_{ij}\ \text{ for }\ 1 \leq l \leq m := \max\{\kappa_{ij}\}.$$

Then the torsion subgroup is the direct sum of cyclic groups

$$T \cong \mathbf{Z}/(ct_1) \oplus \mathbf{Z}/(t_2) \oplus \ldots \oplus \mathbf{Z}/(t_m).$$

Explicit formulae in terms of the exponents of typical monomials for the different classes of weighted homogeneous polynomials are listed in [TSCC: pp. 285–286].

The next approach is somewhat more conceptual—in fact, it shows the background of the formulae above. For every weighted homogeneous surface V in $\mathbf{C}^3$ with an isolated singularity, the link $K = V \cap S^5_\epsilon \cong (V \setminus 0)/\mathbf{R}_{>0}$ is a closed oriented three-dimensional manifold with a fixed point free S^1-action. As such, it has the structure of a Seifert fiber space (see, e.g., Orlik's Lecture Notes [SM]). It follows from Seifert's computation of the fundamental group [SM: 5.3] that if $H_1(K)$ vanishes, then necessarily, the genus g of the "decomposition surface" $K/S^1 \cong (V \setminus 0)/\mathbf{C}^*$ vanishes, the number of exceptional orbits is at least three (unless $K \cong S^3$, i.e., $V \cong \mathbf{C}^2$), and their orders are pairwise coprime (see [Sei: §12, Satz 12], where such homology spheres are called "Poincarésche Räume", or [**OrWa₁**: p. 280]). Now all the exceptional orbits lie in the intersection of V with the coordinate hyperplanes ($x_i = 0$). Orlik and Wagreich show in [$-_2$: 3.5] how the orders of such orbits and the numbers of orbits of a given order can be expressed explicitely in terms of the type (q, N). From their results, it follows that for weighted homogeneous polynomials with fractional coweights, the necessary condition for K to be a homology sphere is never fulfilled.

The most satisfactory argument comes from a result of W. Neumann, and we are grateful to him for pointing this out to us. If the genus of the decomposition surface

K/S^1 vanishes, then the homology group H is a finite abelian group. There is a corresponding finite-sheeted unramified covering K' of K, the "universal abelian covering", having H as group of decktransformations. This can be extended to a covering $V' \to V$ of normal weighted homogeneous surfaces, ramified only at the fixed point. By Neumann's result [–: Thm.1], that universal abelian covering surface V' is always a complete intersection $V_{a_1,\ldots,a_N} : (\sum_{i=1}^{N} \lambda_{ij} x_i^{a_i} = 0)_{j=1,\ldots,N-2}$ defined by Pham-Brieskorn type polynomials, where the integers a_i are the orders and N is the number of the exceptional orbits of V. If K is a homology sphere, then, of course, the covering is trivial, i.e., $K = K'$ and $V = V'$. Hence, if in the case $V \subset \mathbf{C}^3$ under consideration, the link K is a homology sphere, then the defining polynomial is of Pham-Brieskorn type $x_1^{a_1} + x_2^{a_2} + x_3^{a_3} = 0$ and there are exactly three exceptional orbits, so the exponents a_i are pairwise coprime. — An explicit (and earlier) reference for the characterization of Seifert fibred homology spheres as complete intersections $V_{a_1,\ldots,a_N}$ with coprime exponents a_i is Theorem 4.1 in the article by Neumann and Raymond [–].

4. Homology of "Asymptotically Linear" Hypersurfaces in $\mathbf{P}_{n+1}$ with $\mathbf{C}^*$-Action and Examples of Homology-$\mathbf{P}_n$'s. In the two-dimensional case, the affine equation of our examples $V_{n,d}^a$ at o_1 is $x_0^a + x_2^{d-1} + x_3^d$, so the *leading form* (i.e., the homogeneous part of the highest degree d) is the d-th power of a coordinate function. Accordingly, the corresponding (reduced) hyperplane section $V \cap (x_1 = 0)$ "at infinity" is the linear subspace $(x_1 = x_3 = 0)$. By the natural good $\mathbf{C}^*$-action, the affine part $V \cap (x_1 = 1)$ is contractible. Using singular duality theory, this decomposition into topologically simple pieces allowed to reduce the homology computation in section 3 to the study of the singularity link at o_1.

This observation leads to a rather straightforward generalization to the higher-dimensional case. For $x = (x_1, \ldots, x_{n+1})$ (with $n \geq 2$), let $p(x)$ be a weighted homogeneous polynomial of degree $d \geq 2$ with an isolated singularity at the affine origin $o \in \mathbf{C}^{n+1}$ and assume that the leading form is $p_d = x_{n+1}^d$ (up to a non-zero scalar factor). Denote with $\hat{p}(x_0, \ldots, x_{n+1}) := x_0^d \cdot p(x/x_0)$ the (usual) homogenization of p and with $V = V(p) : (\hat{p} = 0) \subset \mathbf{P}_{n+1}$ the projective closure of the normal affine weighted homogeneous variety $U : (p = 0) \subset \mathbf{C}^{n+1}$. The (reduced) part at infinity $V_\infty := V \setminus U = V \cap (x_0 = 0)$ is the projective subvariety $(p_d = 0)$ of the hyperplane $(x_0 = 0)$ at infinity that is defined by the leading form, so it is the linear subspace $(x_0 = x_{n+1} = 0) \cong \mathbf{P}_{n-1}$. (Its points correspond to the asymptotic directions on U, hence the name "asymptotically linear".) The singularities at infinity are in general non-isolated. If the "sub-leading form" p_{d-1} is not divisible by x_{n+1}, however, they have codimension at least two and are thus normal.

The projective variety V is invariant under the $\mathbf{C}^*$-action on $\mathbf{P}_{n+1}$ given by $t_\bullet(x_0 : x_1 : \ldots : x_{n+1}) := (x_0 : t^{q_1}x_1 : \ldots : t^{q_{n+1}}x_{n+1})$ that extends the natural $\mathbf{C}^*$-action on $\mathbf{C}^{n+1} \cong (x_0 = 1)$ corresponding to the weight vector q of p. The (co-)homology of these projective hypersurfaces is determined by that of the the affine singularity link. The ring structure is very similar to that of a hyperplane, i.e., of $\mathbf{P}_n$.

Theorem 4. *For V as above, the integral cohomology is*

$$H^k(V) \cong \begin{cases} H^k(\mathbf{P}_n) & \text{for } k \neq n, n+1 \\ H^k(\mathbf{P}_n) \oplus H_{2n-k}(K) & \text{for } k = n, n+1. \end{cases}$$

Here, K denotes the singularity link $K := U \cap S_\varepsilon^{2n+1}$ at the affine origin o.

The cohomology ring structure is described by the homomorphism $j^k : H^k(\mathbf{P}_{n+1}) \to H^k(V)$ induced by inclusion:

- *For $k \neq 2n+2$, that mapping is injective.*
- *For $k \neq n, n+1, 2n, 2n+2$, it is an isomorphism.*
- *For $k = n, n+1$, the subgroup $j^k(H^k(\mathbf{P}_{n+1}))$ has a direct complement, namely, $\ker(i^k : H^k(V) \to H^k(V_\infty)) \cong H_{2n-k}(K)$, and all cup products with positive-dimensional classes vanish on that complement.*
- *For $k = 2n$, the canonical generator ω^n of $H^{2n}(\mathbf{P}_{n+1})$ is mapped onto $d \cdot u_n$, where $u_n \in H^{2n}(V)$ is the canonical generator (dual to the fundamental class).*

Complement. *If the link K is a rational homology sphere, then V has the rational cohomology ring and hence also the rational homotopy type of $\mathbf{P}_n$.*

That holds in particular for the homology- $\mathbf{P}_n$'s in our class, which are obtained in the obvious manner:

Corollary. *If the singularity link $K := U \cap S_\varepsilon^{2n+1}$ at the affine origin o is an integral homology sphere, the variety V has the integral homology of $\mathbf{P}_n$.*

To give two simple examples, note that K is an integral homology sphere if f is regular at o or has integral coweights w_j which are pairwise coprime. In the first case, the affine variety V is isomorphic to $\mathbf{C}^n$, and V is a singular compactification with $\mathbf{P}_{n-1}$ as part at infinity. In the second case, the polynomial p is of the Pham-Brieskorn type $p(x) = \sum x_j^{w_j}$. Of course, there are much more such examples, e.g., those obtained by Pham-Brieskorn type polynomials satisfying the conditions of [**Bri**: 2, Satz 1], or by modifying the results of section 2, Corollary 1.

Proof of Theorem 4. The result is a rather straightforward application of the "APL" (Alexander-Poincaré-Lefschetz) type duality theory for singular varieties, as developped by L. Kaup in his papers [$-_{1,2}$], and of J. Milnor's classical results on the topology of hypersurface singularities in his book [**SPCH**]. Essentially, the proof follows the lines of [BaKa: 3.5]. The affine part $U = V \setminus V_\infty$ is either smooth, or it has an isolated singularity at the origin o of $\mathbf{C}^{n+1}$. Hence, the pair (V, V_∞) is a "relative variety with isolated singularities", so relative (Lefschetz type) duality theory yields a long exact sequence

$$\begin{aligned} 0 \to H^1(V,V_\infty) \to H_{2n-1}(U) \to \mathcal{H}_{2n-1,o} \to H^2(V,V_\infty) \to H_{2n-2}(U) \to \dots \\ \dots \to H_{2n-k+1}(U) \to \mathcal{H}_{2n-k+1,o} \to H^k(V,V_\infty) \to H_{2n-k}(U) \to \dots \\ \dots \to H_1(U) \to \mathcal{H}_{1,o} \to H^{2n}(V,V_\infty) \to H_0(U) \to \mathcal{H}_{0,o} \to 0 \end{aligned}$$

(see [**Ka**$_1$: Bsp. 2.1, p. 14]), where $\mathcal{H}_{l,o}$ is the l-th integral local homology at the affine origin o. As U is contractible, that long exact sequence yields $H^1(V, V_\infty) = 0$ and breaks into isomorphisms $H^k(V,V_\infty) \cong \mathcal{H}_{2n-k+1,o}$ for $2 \le k \le 2n-1$. As the affine variety U is locally near o (and even globally) isomorphic to the open real cone over the singularity link K, there is an isomorphism $\mathcal{H}_{l,o} \cong \widetilde{H}_{l-1}(K)$. By [**SPCH**: Thm. 5.2, p. 45], an n-dimensional hypersurface singularity link is $(n-2)$-connected, so the homology groups $\mathcal{H}_{l,o} \cong \widetilde{H}_{l-1}(K)$ vanish for $l \ne n, n+1, 2n$. It follows that $H^k(V, V_\infty)$ vanishes for all k with $2 \le k \le 2n-1$ and $k \ne n, n+1$.

We now consider the long exact cohomology sequence

$$\dots \to H^k(V,V_\infty) \xrightarrow{r^*} H^k(V) \xrightarrow{i^*} H^k(V_\infty) \xrightarrow{\delta^*} H^{k+1}(V,V_\infty) \to \dots$$

of the pair (V, V_∞). The part at infinity $V_\infty := V \cap (x_0 = 0)$ is the linear subspace $(x_0 = x_{n+1} = 0) \cong \mathbf{P}_{n-1}$. It follows that the composed map $i^k j^k : H^k(\mathbf{P}_{n+1}) \to H^k(V) \to H^k(V_\infty)$ is an isomorphism of free cyclic groups in all even dimensions $k < 2n$, whereas $H^k(V_\infty)$ vanishes in all other cases. In particular, the homomorphism $i^* : H^k(V) \to H^k(V_\infty)$ is always split surjective, so the exact cohomology sequence breaks into split short exact sequences

$$0 \to H^k(V,V_\infty) \to H^k(V) \to H^k(V_\infty) \to 0.$$

Now the result on the cohomology group structure and on the homomorphism j^* follows easily.

To prove the vanishing of "higher" cup products on $\ker(i^k : H^k(V) \to H^k(V_\infty)) = \operatorname{im}(r^k : H^k(V,V_\infty) \to H^k(V))$ for $k = n, n+1$, we use again singular duality theory: By

[**Ka**$_1$: Thm. 2.1, p. 10], the long exact sequences of (V, V_∞) in cohomology and of (V, U) in homology are joined to a "ladder", i.e., there is a commutative diagram

$$\begin{array}{ccccccccc} \ldots \longrightarrow & H^k(V,V_\infty) & \longrightarrow & H^k(V) & \longrightarrow & H^k(V_\infty) & \longrightarrow \ldots \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \longrightarrow & H_{2n-k}(U) & \longrightarrow & H_{2n-k}(V) & \longrightarrow & H_{2n-k}(V,U) & \longrightarrow \ldots \end{array}$$

where the vertical arrows are singular duality homomorphisms. Hence, by the contractibility of U, the image of $H^k(V, V_\infty)$ lies in the kernel of the "absolute" Poincaré duality homomorphism $H^k(V) \to H_{2n-k}(V)$. That homomorphism is nothing but the cap product with the fundamental homology class (see [**BaKa**: 2.5]). With the standard relations between cup and cap product, that proves the result.

The complement is just a restatement of the remark in section 2, paragraph iii). •

Remarks. (i) Most of the results on the (co-)homology of the varieties V are already contained in [**Ka**$_2$: Kor. 3.6, 3.7, pp. 502/503] as special case: put $r = 0$ and interchange the roles of x_0 and x_{n+1}. Note that the variety ${}_0F_{n-1}$ occuring there is just $\mathbf{P}_{n-1}$.

(ii) It is immediate to see that for $0 < 2k < 2n$, the Poincaré duality homomorphism maps $j^{2k}\omega^k$ onto $d \cdot i_{2n-2k} l_{n-k}$, where $l_m \in H_{2m}(V_\infty)$ is the canonical generator represented by $[\mathbf{P}_m]$: With $m := n - k$, one has $j_{2m}(j^{2k}\omega^k \cap [V]) = \omega^k \cap j_{2n}[V] = \omega^k \cap d \cdot l_n = j_{2m} i_{2m}(d \cdot l_m)$. To see that the image of the Poincaré homomorphism lies in the image of i_{2m}—wherefore the restriction of j_{2m} to the image is injective—, one uses the analoguous "ladder" to the one above where the roles of U and V_∞ are changed.—In the general duality theory of [**Ka**$_1$], one has to be careful about supports. If no supports are explicitely noted, compact supports are understood in homology and closed supports in cohomology. As these two families agree on the compact varieties V and V_∞, there is no problem in our case.

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