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Abstract

We construct explicit quantization of closed conjugacy classes of the complex symplectic group $SP(2n)$ with non-Levi isotropy subgroups through an operator realization on highest weight modules over the quantum group $U_q\big(\mathfrak{sp}(2n)\big)$.

Mathematics Subject Classifications: 81R50, 81R60, 17B37.

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1 Introduction

We construct a quantization of conjugacy classes of the complex algebraic group $SP(2n)$ whose isotropy subgroup is not of Levi type. Such classes are not isomorphic to adjoint orbits in the Lie algebra $\mathfrak{sp}(2n)$, and their Poisson structure is not exactly $SP(2n)$-invariant. This quantization features a quantum group symmetry, which is a deformation of the conjugation action of $SP(2n)$ on itself.

The conjugacy classes of interest form a family that is as large as of Levi type: they involve diogonalizable symplectic matrices whose eigenvalues include simultaneously $+1$ and -1 (a Levi-type class may have at most one of them). Note that among the classical matrix groups only symplectic and orthogonal admit classes of this type: for the special linear group they are all isomorphic to adjoint orbits and have Levi isotropy subgroups.

The Poisson structure on the conjugacy classes comes from a Poisson structure on the group, which is analogous to the canonical invariant Poisson structure on the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n)$ (we assume the natural isomorphism of the adjoint and coadjoint representations of $\mathfrak{g}$). Quantization of this structure is analogous to quantization of the Kostant-Kirillov-Souriau bracket on the coadjoint orbits; with the difference that the former allows for quantum group symmetry rather than classical.

Conjugacy classes with Levi isotropy subgroups have been quantized in [1] using the representation theory of quantum groups. We should stress that the methods of [1] are inapplicable, as they are, for the non-Levi classes, whose quantization is still an open problem. In our recent paper [2] we have shown how to approach it on the simplest example of $SP(4)/SP(2) \times SP(2)$. In this work, we develop those ideas further and cover all non-Levi conjugacy classes of $SP(2n)$. Along with the Levi type treated in [1], this is closing the problem for all diagonalizable classes of $SP(2n)$.

Further we explain our methods. It is natural to seek for a quantization of an affine variety in terms of generators and relations, in other words, as a quotient of a free algebra. Supposedly this projection factors through a projection from a quantized coordinate ring $\mathbb{C}_\hbar[G]$ of the group $G = SP(2n)$, which is well studied and whose explicit description in generators and relations is available. To ensure that the deformation is flat, we seek to realize it in an algebra that is flat over the ring of formal series in the deformation parameter $\hbar$. Due to certain module properties of $\mathbb{C}_\hbar[G]$, this would also yield the defining relations, provided we have managed to find an ideal in the kernel turning to the defining ideal in the classical limit (such an ideal shall automatically coincide with the kernel). Since $\mathbb{C}_\hbar[G]$ can be realized as a subalgebra in the quantized universal enveloping algebra $U_\hbar(\mathfrak{g})$, one can try to construct the quantization through a representation of $\mathbb{C}_\hbar[G]$ in the algebra of endomorphisms of some $U_\hbar(\mathfrak{g})$-module.

Conjugacy classes with Levi isotropy subgroups have been quantized with the help of parabolic Verma modules. However, there is no immediate analog of parabolic Verma modules for non-Levi subalgebras in $\mathfrak{g}$, and the key step is to find their suitable replacement. We take for it a quotient of a special auxiliary parabolic Verma module, which is chosen as follows. Let $K \subset G$ denote the stabilizer of the initial point of the class. It contains a maximal Levi subgroup $L \subset K$ (there are actually two such subgroups). At the Lie algebra level, the isotropy subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is generated over the Levi subalgebra $\mathfrak{l}$ by a certain pair of root vectors e_δ, f_δ. We construct the parabolic Verma $U_\hbar(\mathfrak{g})$-module $\hat{M}_\lambda$ relative to $U_\hbar(\mathfrak{l})$, where highest weight λ is conditioned by the presence of a singular vector of weight $\lambda - \delta$.

The quotient M_λ of $\hat{M}_\lambda$ over the submodule generated by that singular vector is the module were we realize the quantization of $\mathbb{C}[G/K]$.

The subalgebra $\mathbb{C}_\hbar[G] \subset U_\hbar(\mathfrak{g})$ is generated by the entries of an invariant matrix $\mathcal{Q} \in \mathrm{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}_\hbar[G]$ canonically constructed of the universal R-matrix of $U_\hbar(\mathfrak{g})$. The problem then boils down to determining the minimal polynomial of $\mathcal{Q}$ regarded as an operator on $\mathbb{C}^{2n} \otimes M_\lambda$. The matrix $\mathcal{Q}$ is semi-simple on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$, and its eigenvalues are known. Therefore, $\mathcal{Q}$ satisfies the same polynomial equation on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$, which is however not necessarily minimal. We prove that the extra eigenvalue drops from the spectrum of $\mathcal{Q}$ in the transition from $\hat{M}_\lambda$ to M_λ, in this way producing the minimal polynomials on $\mathbb{C}^{2n} \otimes M_\lambda$ from that on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$.

The above described effect is analogous to the transition from to G/L to G/K, where the class G/L is obtained from G/K by splitting the eigenvalues -1 into the pairs of reciprocals μ, $\mu^{-1} \neq -1$. In the limit $\mu \to -1$ the eigenvalues μ and μ^{-1} glue up, and the isotropy subgroup jumps from L to K. The minimal polynomial of G/L acquires a non-simple factor $(x+1)^2$, which should be reduced in the minimal polynomial of G/K. Similarly, we check that the extra simple divisor of the minimal polynomial of $\mathcal{Q}$ is canceled on $\mathbb{C}^{2n} \otimes M_\lambda$, and the classical limit of $\mathcal{Q}$ yields the minimal polynomial of G/K. This implies the second essential step of our strategy being the analysis of the $U_\hbar(\mathfrak{g})$-module $\mathbb{C}^{2n} \otimes M_\lambda$ and the invariant operator $\mathcal{Q}$ on it.

Putting non-Levi conjugacy classes into a common quantization scheme with the classes of Levi type implies several far reaching consequences. First of all, recall that the latter (along with quantum semi-simple coadjoint orbits) gave rise to the theory of dynamical Yang-Baxter equation over general non-Abelian base, [3]. To a large extent, that theory was based on the properties of the parabolic $\mathcal{O}^{\mathfrak{l}}$-category featuring the structure of a module category over representations of the (quantum) group. We observe an analogous category $\mathcal{O}^{\mathfrak{k}}$ associated with the non-Levi type quantum classes, which are generated (as a module category) by M_λ with feasible weights λ. It is natural to expect that the category $\mathcal{O}^{\mathfrak{k}}$ should result in a proper generalization of the dynamical Yang-Baxter equation. The parabolic category $\mathcal{O}^{\mathfrak{l}}$ consists of $U_\hbar(\mathfrak{g})$-modules that are parabolically induced from $U_\hbar(\mathfrak{l})$-modules. At the same time, the algebra $U(\mathfrak{k})$ is not quantized as a Hopf subalgebra in $U_\hbar(\mathfrak{g})$. It is therefore interesting to understand its quantization, which will be a $\mathfrak{k}$-analog of the Levi subalgebra $U_\hbar(\mathfrak{l})$. This might help to understand the category $\mathcal{O}^{\mathfrak{k}}$.

2 Classical conjugacy classes

Throughout the paper, G designates the algebraic group $SP(2n)$ of symplectic matrices preserving a non-degenerate skew symmetric form $||C_{ij}||_{i,j=1}^{2n}$ in the complex vector space $\mathbb{C}^{2n}$; the Lie algebra of G will be denoted by $\mathfrak{g}$. We choose the realization corresponding to $C_{ij} = \epsilon_i \delta_{ij'}$, where δ_{ij} is the Kronecker symbol, $i' = 2n + 1 - i$, and $\epsilon_i = -\epsilon_{i'} = 1$ for $i = 1, \ldots, n$.

The polynomial ring $\mathbb{C}[G]$ is generated by the matrix coordinate functions $||A_{ij}||_{i,j=1}^{2n}$, modulo the set of $2n \times 2n$ relations written in the matrix form as

$$ACA^t = C. \tag{2.1}$$

The right conjugacy action of G on itself induces a left action on $\mathbb{C}[G]$ by duality; the matrix A is invariant as an element of $\mathrm{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}[G]$.

The group G is equipped with the Drinfeld-Sklyanin bivector field

$$\{A_1, A_2\} = \frac{1}{2}(A_2 A_1 r - r A_1 A_2), \tag{2.2}$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation, [4]. This equation is understood in $\mathrm{End}(\mathbb{C}^{2n}) \otimes \mathrm{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}[G]$, and the subscripts label the natural embeddings of $\mathrm{End}(\mathbb{C}^{2n})$ in $\mathrm{End}(\mathbb{C}^{2n}) \otimes \mathrm{End}(\mathbb{C}^{2n})$, as usual in the quantum groups literature.

The bivector field (2.2) is skew-symmetric when restricted to the functions on G and defines a Poisson bracket on $\mathbb{C}[G]$ making G a Poisson group. Of all possible solutions to the classical Yang-Baxter equation we choose

$$r = \sum_{i=1}^{2n} (e_{ii} \otimes e_{ii} - e_{ii} \otimes e_{i'i'}) + 2 \sum_{\substack{i,j=1 \\ i>j}}^{2n} (e_{ij} \otimes e_{ji} - \epsilon_i \epsilon_j e_{ij} \otimes e_{i'j'}), \tag{2.3}$$

which is the simplest factorizable r-matrix, [5]. At the end of the article, we lift this restriction.

We regard the group G as a G-space under the conjugation action. The object of our study is another Poisson structure on G,

$$\{A_1, A_2\} = \frac{1}{2}(A_2 r_{21} A_1 - A_1 r A_2 + A_2 A_1 r - r_{21} A_1 A_2), \tag{2.4}$$

in the matrix form. It is compatible with the conjugation action and makes G a Poisson space over the Poisson group G with the Drinfeld-Sklyanin bracket (2.2).

A closed conjugacy class $O \subset G$ consists of diagonalizable matrices and is determined by the set of eigenvalues $S_O = \{\mu_i, \mu_i^{-1}\}_{i=1}^n$ plus 1 if $N = 2n + 1$. Every eigenvalue μ

enters S_O with its reciprocal μ^{-1}, and, in particular, there may be $\mu = \mu^{-1} = \pm 1$. One should distinguish two situations: a) S_O contains either $+1$ or -1 or none and b) both $+1$ and -1 belong to S_O. In the first case, O is isomorphic to an orbit in $\mathfrak{g}$ via the Cayley transformation, and its isotropy subgroup is of Levi type. A conjugacy class of second type is not isomorphic to an adjoint orbit. In terms of Dynkin diagram, every Levi subgroup is obtained by scraping out a subset of nodes, while for non-Levi isotropy subgroups one should use the affine Dynkin diagram of $\mathfrak{g}$.

Informally, the non-Levi subgroup necessarily contains two symplectic blocks rotating the eigenspaces of eigenvalues ± 1.

We associate with a class O an integer valued vector $\boldsymbol{n} = (n_1, \dots, n_\ell, m, p)$ and a complex valued vector $\boldsymbol{\mu} = (\mu_1, \dots \mu_\ell, -1, 1)$ assuming μ_i, $i = 1, \dots, \ell$, all invertible, pairwise distinct and not a square root of 1. The initial point $o \subset O$ will be fixed to the diagonal matrix with entries

$$\underbrace{\mu_1, \dots, \mu_1}_{n_1}, \dots, \underbrace{\mu_\ell, \dots, \mu_\ell}_{n_\ell}, \underbrace{-1, \dots, -1}_{m}, \underbrace{1, \dots, 1}_{P}, \underbrace{-1, \dots, -1}_{m}, \underbrace{\mu_\ell^{-1}, \dots, \mu_\ell^{-1}}_{n_\ell}, \dots, \underbrace{\mu_1^{-1}, \dots, \mu_1^{-1}}_{n_1},$$

so that $\sum_{i=1}^{\ell} n_i + m + p = n$. We reserve the integers $m = n_{\ell+1}, p = n_{\ell+2}$ to denote respectively, ranks of the blocks corresponding to $-1 = \mu_{\ell+1}$ and $+1 = \mu_{\ell+2}$ (we view ± 1 as degeneration of the parameters $\mu_{\ell+1}$ and $\mu_{\ell+2}$). The specialization $n_1 = \ldots = n_\ell = 0$ is formally encoded by $\ell = 0$ and referred to as the *symmetric case*, because it corresponds to a symmetric conjugacy class.

The stabilizer subgroup of the initial point o is the direct product

$$K = GL(n_1) \times \ldots \times GL(n_\ell) \times SP(2m) \times SP(P) \tag{2.5}$$

and it is determined by the vector $\boldsymbol{n}$. The positive integer ℓ counts the number of GL-blocks in K. In the symmetric case, (2.5) reduces to $SP(2m) \times SP(P)$, and the class $O \simeq G/K$ to a symmetric space.

Let $\mathcal{M}_K$ denote the moduli space of conjugacy classes with the fixed isotropy subgroup (2.5), regarded as Poisson spaces. The set of all $\ell + 2$-tuples $\boldsymbol{\mu}$ with invertible components such that $\mu_{\ell+1}^2 = \mu_{\ell+2}^2 = 1$ and $\mu_i \neq \mu_j, \mu_j^{-1}$ for distinct i, j parameterize $\mathcal{M}_K$ albeit not uniquely. Multiplication by the scalar matrix $-1 \in G$ preserves this set and swaps $\mu_{\ell+1}$ with

$\mu_{\ell+2}$. This transformation is an automorphism of G as the adjoint G-space and preserves the Poisson structure (2.4). Therefore, the subset $\hat{\mathcal{M}}_K$ of $\boldsymbol{\mu}$ with fixed $\mu_{\ell+1} = -1$ and $\mu_{\ell+2} = 1$ can also be used for parameterization of $\mathcal{M}_K$. Its residual ambiguity is related to permutations of components $\mu_i \neq \pm 1$ with equal multiplicities.

The class O associated with $\boldsymbol{\mu}$ and $\boldsymbol{n}$ is determined by the set of polynomial equations

$$(A-\mu_1)\dots(A-\mu_\ell)(A+1)(A-1)(A-\mu_1^{-1})\dots(A-\mu_\ell^{-1}) = 0, \tag{2.6}$$

$$\mathrm{Tr}(A^k) = \sum_{i=1}^{\ell} n_i(\mu_i^k + \mu_i^{-k}) + 2m(-1)^k + 2p, \quad k = 1,\dots,2n, \tag{2.7}$$

on the entries of the matrix A. In fact, the ideal in $\mathbb{C}[G]$ generated by this set of relations is radical and therefore coincides with the defining ideal of $\mathbb{C}[O]$ in $\mathbb{C}[G]$. This is a consequence of the following general fact.

Consider a smooth variety X in affine space Y of dimension $\dim(Y)$. Suppose that X is defined by a system of polynomial equations $F_i(x) = 0$, $i \in I$, where I is a finite set of indices. The ideal $J' = (F_i)_{i\in I}$ is contained in the defining ideal J of X, i.e. the ideal of all polynomial functions vanishing on X. In general, J' might be less that J, and then the quotient $\mathbb{C}[Y]/J'$ cannot be regarded as a ring of functions on X, as containing nilpotent elements. It is essential for our approach to quantization to make sure that the ideal J' is exactly J, because the latter obeys certain maximality requirements. We will use the following criterion of radicality of J'.

Proposition 2.1. *Suppose that at every point $x \in X$ the rank of the differential $\{dF_i\}_{i\in I}$ is equal to* $\dim(Y) - \dim(X)$. *Then the ideal J' in $\mathbb{C}[Y]$ generated by $\{F_i\}_{i\in I}$ coincides with the defining ideal J of X.*

Proof. Denote by $A' = \mathbb{C}[Y]/J'$ and $A = \mathbb{C}[Y]/J$ the quotient algebras and consider their affine schemes with the structure sheafs $\mathcal{O}'$ and $\mathcal{O}$, respectively. Since J is the radical of J', the natural embedding $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is an isomorphism making $\mathcal{O}$ a subsheaf in $\mathcal{O}'$. The condition on the rank of $\{dF_i\}_{i\in I}$ implies, by the Jacobian criterion of smoothness, [6], that $\mathcal{O}_x$ and $\mathcal{O}'_x$ are regular local rings at every point $x \in \mathrm{Spec}(A')$, and $\mathcal{O}_x = \mathcal{O}'_x$. As the two sheafs coincide locally, they coincide globally. Hence $A' \simeq A$, and $J' = J$. $\square$

Proposition 2.1 provides a convenient test for verifying if a particular system of equations gives rise to the defining ideal. That is especially so for homogeneous varieties, as it suffices to look at the initial point only. Remark that the condition on the rank can be replaced by a more practical condition on the kernel: $\dim(\cap_i \ker dF_i) = \dim(X)$.

Theorem 2.2. *Let $\tilde{G}$ be the general linear group of the vector space $\mathbb{C}^N$ and let $\tilde{O} \subset \tilde{G}$ be the conjugacy class of matrices with distinct eigenvalues $(\mu_1, \ldots, \mu_l)$ of multiplicities $(n_1, \ldots, n_l)$. The system of polynomial equations*

$$(A - \mu_1) \ldots (A - \mu_l) = 0, \quad \mathrm{Tr}(A^k) = \sum_{i=1}^{l} n_i \mu_i^k, \quad k = 1, \ldots, N \tag{2.8}$$

has rank $N^2 - \dim(\tilde{O})$ everywhere on $\tilde{O}$, hence it generates the defining ideal of $\tilde{O}$.

Proof. Take $o = \mathrm{diag}(\mu_1, \ldots, \mu_1, \ldots, \mu_l, \ldots, \mu_l) \in \tilde{O}$ as the initial point in $\tilde{O}$. Denote by S_O the set of eigenvalues of o and let $P_i \colon \mathbb{C}^N \to \mathbb{C}^{n_i}$ be the projector to the μ_i-eigenspace.

The matrix o can be then written as $o = \sum_{i=1}^{l} \mu_i P_i$. Denote by E_{ij}, $i, j = 1, \ldots, l$, the subspace of matrices $P_i \mathrm{End}(\mathbb{C}^N) P_j$. We have $\mathrm{End}(\mathbb{C}^N) = \oplus_{i,j=1}^{l} E_{ij}$ for the matrix algebra, and $\tilde{\mathfrak{k}} = \oplus_{i=1}^{l} E_{ii}$ for the Lie algebra $\tilde{\mathfrak{k}}$ of the stabilizer of o. The tangent space $T_o(\tilde{G})$ is naturally identified with $\tilde{\mathfrak{m}} = \oplus_{\substack{i,j=1 \\ i \neq j}}^{l} E_{ij}$. The class $\tilde{O}$ is the zero locus of equations (2.8). To prove the statement, it is sufficient to check the rank of the system (2.8) at the point o.

Denote by F the matrix polynomial $\prod_{i=1}^{l} (A - \mu_i)$ and by ϑ_k the trace $\mathrm{Tr}(A^k)$, $k = 1, \ldots, N$. The system of relations involves $N \times N$ functions F_{ij} and N differences $\vartheta_k - \sum_{i=1}^{l} n_i \mu_i^k$. It is easy to check that

$$dF(\xi) = 0, \quad d\vartheta_k(\xi) = 0, \quad k = 1, \ldots, N,$$

for all $\xi \in E_{ij}$ with $i \neq j$ and

$$dF(\xi) = \prod_{\substack{i=1 \\ i \neq j}}^{l} (\mu_j - \mu_i)\, \xi, \quad d\vartheta_k(\xi) = k \mu_j^k \mathrm{Tr}(\xi), \quad k = 1, \ldots, N,$$

for all $\xi \in E_{jj}$. Note that the right equations are redundant as $\ker(dF) \subset \ker d\vartheta_k$: to see this, one should differentiate the trace of $F(\xi)$. The left equation tells us that $\mathrm{im}\, dF = \tilde{\mathfrak{k}}$, as the numerical coefficient before ξ does not vanish. This proves the assertion. □

Based on Theorem 2.2, we use Proposition 2.1 to describe the defining ideals of closed conjugacy classes of the symplectic group.

Theorem 2.3. *The system of polynomial relations (2.6) and (2.7) along with the defining relations of the group (2.1) generate the defining ideal of class $O \subset SP(2n)$.*

Proof. As shown in the proof of Theorem 2.2, the differential of the trace functions is lineally dependent of the differential of the minimal polynomials. Therefore, the essential part of the

Jacobian comes from the minimal polynomial and the equation of the group. The tangent space $T_o(G)$ is the set of fixed points of the liner endomorphism $\sigma_o\colon \xi \mapsto -o\sigma(\xi)o$, where σ is the involutive anti-automorphism $\xi \mapsto -C\xi^t C$. Clearly it can be presented as $T_o(G) = o\mathfrak{g}$.

The map σ_o is a linear involution, so the tangent $T_o(G)$ space is the image of the projector $\frac{\mathrm{id}+\sigma_o}{2}$. Using the same notation as in the proof of Theorem 2.2, we present $T_o(G)$ as a direct sum $o\mathfrak{g} = \mathfrak{k}_o \oplus \mathfrak{m}_o$, where $\mathfrak{k}_o = o\mathfrak{g} \cap \tilde{\mathfrak{k}}$ and $\mathfrak{m}_o = o\mathfrak{g} \cap \tilde{\mathfrak{m}}$. This is possible because the spaces $\tilde{\mathfrak{k}}$ and $\tilde{\mathfrak{m}}$ are stable under multiplication by o.

We need to find the rank at o of the mapping $\mathrm{End}(\mathbb{C}^{2n}) \to \mathrm{End}(\mathbb{C}^{2n}) \oplus \mathrm{End}(\mathbb{C}^{2n})$, $A \mapsto H(A) \oplus F(A)$, were $H(A) = ACA^tC + 1$. Equivalently we can find its kernel, which is the intersection $\ker dH_o \cap \ker dF_o$. The tangent space $T_o\bigl(\mathrm{End}(\mathbb{C}^{2n})\bigr)$ splits into the direct sum $o\mathfrak{g}^\perp \oplus o\mathfrak{g}$. The kernel of dH_o is exactly $o\mathfrak{g}$ so $\ker(dH_o \oplus dF_o)$ is just $\ker dF_o|_{o\mathfrak{g}}$. In the course of the proof of Theorem 2.2 we saw that $\mathfrak{m}_o \subset \ker dF_o|_{o\mathfrak{g}}$. This inclusion is, in fact, an equality. Indeed, $\mathfrak{k}_o \subset \tilde{\mathfrak{k}}$, and dF_o is injective on $\tilde{\mathfrak{k}}$. Hence it is injective on $\mathfrak{k}_o$. Thus, the kernel of the differential $dH_o \oplus dF_o$ is exactly $\mathfrak{m}_o$. But $\mathfrak{m}_o \simeq T_o(O)$, and the rank of the map $dH_o \oplus dF_o\colon \mathrm{End}(\mathbb{C}^{2n}) \to \mathrm{End}(\mathbb{C}^{2n}) \oplus \mathrm{End}(\mathbb{C}^{2n})$ is equal to the codimension of G. This completes the proof. $\square$

Although non-Levi conjugacy classes are of our main concern, Theorem 2.3 holds true for any semi-simple conjugacy class. It generalizes in the obvious way for the orthogonal groups, with the only stipulation for the D-series: the traces of matrix powers are not enough to fix a class, and one needs one more condition on the invariants of G, see e.g. [1].

3 Quantum group $U_\hbar\bigl(\mathfrak{sp}(2n)\bigr)$

Recall the definition of the quantum group $U_\hbar\bigl(\mathfrak{sp}(2n)\bigr)$, which is a deformation of the universal enveloping algebra $U\bigl(\mathfrak{sp}(2n)\bigr)$ along the formal parameter $\hbar$ in the class of Hopf algebras, [4]. Let R and R^+ denote respectively the root system and the set of positive roots of the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n)$. Let $\Pi_+ = (\alpha_1, \alpha_1, \ldots, \alpha_n)$ be the set of simple positive roots. We also reserve special notation β for the long root α_n. By $(\,.\,,\,.\,)$ we designate the canonical inner form on the linear span of Π^+. The set Π^+ can be conveniently expressed through an orthogonal basis $(\varepsilon_i)_{i=1}^n$ by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \ldots, n-1$, $\beta = 2\varepsilon_n$.

The inner product establishes a linear isomorphism between the linear span $\mathbb{C}\Pi^+$ and its dual $\mathfrak{h}$. We define $h_\lambda \in \mathfrak{h}$ for every $\lambda \in \mathfrak{h}^* = \mathbb{C}\Pi^+$ as its image under this isomorphism: $\mu(h_\lambda) = (\lambda, \mu)$ for all $h \in \mathfrak{h}$. In particular, we set h_ρ for the half-sum of all positive roots $\rho = \frac{1}{2}\sum_{\alpha\in \mathrm{R}_+} \alpha$.

The quantum group $U_\hbar(\mathfrak{g})$ is a $\mathbb{C}[[\hbar]]$-algebra generated by simple root vectors (Chevalley generators) e_μ, f_μ, and the Cartan generators $h_\mu \in \mathfrak{h}$, $\mu \in \Pi^+$. The vector space $\mathfrak{h}$ generate the Cartan subalgebra $U_\hbar(\mathfrak{h})$ in $U_\hbar(\mathfrak{g})$, which commutes with the simple root vectors by the rule

$$[h_\mu, e_\nu] = (\mu,\nu)e_\nu, \quad [h_\mu, f_\nu] = -(\mu,\nu)f_\nu.$$

Note that only the following inner products here do not vanish:

$$(\alpha_i,\alpha_i) = 2, \quad (\alpha_{i-1},\alpha_i) = -1, \quad (\beta,\beta) = 4, \quad (\beta,\alpha_{n-1}) = -2,$$

where i takes all admissible values in the range $1,\ldots,n-1$.

Positive and negative Chevalley generators commute to $U_\hbar(\mathfrak{h})$:

$$[e_\mu, f_\nu] = \delta_{\mu,\nu}\frac{q^{h_\mu} - q^{-h_\mu}}{q_\mu - q_\mu^{-1}}, \quad \mu \in \Pi^+,$$

where $q_\mu = q = e^\hbar$ for $\mu \neq \beta$ and $q_\beta = q^2$.

Non-adjacent positive Chevalley generators commute. Adjacent generators satisfy the Serre relations

$$e_\mu^2 e_\nu - (q+q^{-1})e_\mu e_\nu e_\mu + e_\nu e_\mu^2 = 0, \quad \text{for} \quad \nu \neq \beta,$$
$$e_\mu^3 e_\beta - (q^2+1+q^{-2})e_\mu^2 e_\beta e_{\alpha_{n-1}} + (q^2+1+q^{-2})e_\mu e_\beta e_\mu^2 - e_\beta e_\mu^3 = 0, \quad \text{for} \quad \mu = \alpha_{n-1}.$$

Similar relations holds for the negative Chevalley generators f_μ.

The Cartan involution $\omega\colon e_\mu \leftrightarrow f_\mu$ and $\omega(h_\mu) = -h_\mu$, $\mu \in \Pi^+$, extends to an algebra automorphism of $U_\hbar(\mathfrak{g})$

The comultiplication Δ and antipode γ are defined on the generators by

$$\Delta(h_\mu) = h_\mu \otimes 1 + 1 \otimes h_\mu, \quad \gamma(h_\mu) = -h_\mu,$$

$$\Delta(e_\mu) = e_\mu \otimes 1 + q^{h_\mu} \otimes e_\mu, \quad \gamma(e_\mu) = -q^{-h_\mu}e_\mu,$$
$$\Delta(f_\mu) = f_\mu \otimes q^{-h_\mu} + 1 \otimes f_\mu, \quad \gamma(f_\mu) = -f_\mu q^{h_\mu},$$

for all $\mu \in \Pi_+$. The counit homomorphism $\varepsilon\colon U_\hbar(\mathfrak{g}) \to \mathbb{C}[[\hbar]]$ annihilates e_μ, f_μ, h_μ.

Besides the Cartan subalgebra $U_\hbar(\mathfrak{h})$, the quantum group $U_\hbar(\mathfrak{g})$ contains the following Hopf subalgebras. The positive and negative Borel subalgebras $U_\hbar(\mathfrak{b}^\pm)$ are generated over $U_\hbar(\mathfrak{h})$ by $\{e_\mu\}_{\mu\in\Pi^+}$ and $\{f_\mu\}_{\mu\in\Pi^+}$, respectively. For any root subsystem in R the associated Levi subalgebra $U(\mathfrak{l})$ is quantized to a Hopf algebra $U_\hbar(\mathfrak{l})$, along with the parabolic subalgebras $U_\hbar(\mathfrak{p}^\pm)$ generated by $U_\hbar(\mathfrak{b}^\pm)$ over $U_\hbar(\mathfrak{l})$.

The quantum version of higher root vectors in $\mathfrak{g}$ reads:

$$e_\mu = e_\nu e_\sigma - q^{(\nu,\sigma)} e_\nu e_\sigma, \quad f_\mu = e_\sigma e_\nu - q^{-(\nu,\sigma)} e_\sigma e_\nu, \quad \nu, \sigma, \mu = \nu + \sigma \in R^+.$$

The ordered sets $(e_\mu)_{\mu\in R^+} \subset U_\hbar(\mathfrak{b}^+)$ and $(f_\mu)_{\mu\in R^+} \subset U_\hbar(\mathfrak{b}^-)$ generate a Poincare-Birkgoff-Witt basis over $U_\hbar(\mathfrak{h})$. Further on we redefine some root vectors to adapt them for our needs.

The triangular decomposition $\mathfrak{g} = \mathfrak{n}_\mathfrak{l}^- \oplus \mathfrak{l} \oplus \mathfrak{n}_\mathfrak{l}^+$ gives rise to the triangular factorization

$$U_\hbar(\mathfrak{g}) = U_\hbar(\mathfrak{n}_\mathfrak{l}^-) U_\hbar(\mathfrak{l}) U_\hbar(\mathfrak{n}_\mathfrak{l}^+), \tag{3.9}$$

where $U_\hbar(\mathfrak{n}_\mathfrak{l}^\pm)$ are subalgebras in $U_\hbar(\mathfrak{b}^\pm)$ generated by the positive or negative root vectors. This factorization makes $U_\hbar(\mathfrak{g})$ a free $U_\hbar(\mathfrak{n}_\mathfrak{l}^-) - U_\hbar(\mathfrak{n}_\mathfrak{l}^+)$-bimodule generated by $U_\hbar(\mathfrak{l})$. A special case of this decomposition is relative to $\mathfrak{l} = \mathfrak{h}$, in which case we use the notation $U_\hbar(\mathfrak{n}^\pm) = U_\hbar(\mathfrak{n}_\mathfrak{h}^\pm)$. Note that, contrary to the classical case, $U_\hbar(\mathfrak{n}_\mathfrak{l}^\pm)$ are not Hopf subalgebras in $U_\hbar(\mathfrak{g})$.

We shall also deal with the Hopf subalgebra $U_q(\mathfrak{g}) \subset U_\hbar(\mathfrak{g})$ generated by the Chevalley generators and the exponentials $t_{\alpha_i}^\pm = q^{\pm h_{\alpha_i}}$, $\alpha_i \in \Pi_+$. This algebra can be considered over the ring $\mathbb{C}[q, q^{-1}]$ and its extensions by fractions over the multiplicative system $\{q^l - 1\}_{l\in\mathbb{Z}}$. The other mentioned subalgebras of $U_\hbar(\mathfrak{g})$ have their counterparts in $U_q(\mathfrak{g})$, and we use the subscript q for their notation. The roles of quantum groups $U_\hbar(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are different in what follows. While $U_q(\mathfrak{g})$ is a source of non-commutative functions on quantum geometric spaces, $U_\hbar(\mathfrak{g})$ is a measure of their symmetry. This difference is somewhat camouflaged in the classical geometry but becomes more distinctive in quantum.

4 Quantum subgroup $U_\hbar\big(\mathfrak{gl}(n)\big)$

The quantum group $U_\hbar\big(\mathfrak{sp}(2n)\big)$ contains quantum subgroup $U_\hbar\big(\mathfrak{gl}(n)\big)$ with positive simple roots $(\alpha_1, \ldots \alpha_{n-1}) \subset \Pi^+$. We need a few technical facts about this subalgebra, which we use in the sequel.

Fix a pair of integers $i < j < n$ and put $\mu = \alpha_i + \ldots + \alpha_j \in R^+$. Define the root vectors

$$f_\mu = [f_{\alpha_i}, \ldots [f_{\alpha_{j-1}}, f_{\alpha_j}]_q]_q \ldots]_q, \quad \tilde{f}_\mu = [f_{\alpha_i}, \ldots [f_{\alpha_{j-1}}, f_{\alpha_j}]_{q^{-1}}]_{q^{-1}} \ldots]_{q^{-1}}.$$

Here are some commutation relations involving these vectors.

Lemma 4.1. *Suppose positive integer i, j, and k are such that $i<k<j$. Then*

$$[e_{\alpha_k}, f_\mu] = 0, \quad [e_{\alpha_k}, \tilde f_\mu] = 0.$$

Further,

$$[e_{\alpha_i}, f_\mu] = f_{\mu'} q^{-h_{\alpha_i}}, \quad [e_{\alpha_j}, f_\mu] = -q f_{\mu''} q^{h_{\alpha_j}}, \quad [e_{\alpha_i}, \tilde f_\mu] = \tilde f_{\mu'} q^{h_{\alpha_i}}, \quad [e_{\alpha_j}, \tilde f_\mu] = -q^{-1} \tilde f_{\mu''} q^{-h_{\alpha_j}},$$

where $\mu' = \alpha_{i+1} + \ldots + \alpha_j$ and $\mu'' = \alpha_i + \ldots + \alpha_{j-1}$.

Proof. It is sufficient to check only the group of equalities involving f_μ, as the other equalities can be obtained by formal replacement $q \to q^{-1}$. Let us start with the special case $j = i+2$, $k = i+1$:

$$[e_{\alpha_{i+1}}, f_\mu] \sim [f_{\alpha_i}, [q^{h_{\alpha_{i+1}}} - q^{-h_{\alpha_{i+1}}}, f_{\alpha_{i+2}}]_q]_q \sim [f_{\alpha_i}, f_{\alpha_{i+2}} q^{-h_{\alpha_{i+1}}}]_q = [f_{\alpha_i}, f_{\alpha_{i+2}}] q^{-h_{\alpha_{i+1}}} = 0.$$

The general case is verified in a similar way based on the formula $f_\mu = [f_{\mu_1}, [f_{\alpha_k}, f_{\mu_2}]_q]_q$, where the roots μ_1, μ_2 are determined by the decomposition $\mu = \mu_1 + \alpha_k + \mu_2$. This formula is an elementary corollary of the definition of f_μ. Further,

$$\begin{aligned}
[e_{\alpha_i}, f_\mu] &= [\frac{q^{h_{\alpha_i}} - q^{-h_{\alpha_i}}}{q - q^{-1}}, f_{\mu'}]_q = -\frac{1}{q-q^{-1}} [q^{-h_{\alpha_i}}, f_{\mu'}]_q = f_{\mu'} q^{-h_{\alpha_i}}, \\
[e_{\alpha_j}, f_\mu] &= [f_{\mu''}, \frac{q^{h_{\alpha_i}} - q^{-h_{\alpha_i}}}{q - q^{-1}}]_q = \frac{1}{q-q^{-1}} [f_{\mu''}, q^{h_{\alpha_j}}]_q = \frac{(1-q^2)}{q-q^{-1}} f_{\mu''} q^{h_{\alpha_j}} = -q f_{\mu''} q^{h_{\alpha_j}},
\end{aligned}$$

as required. □

Lemma 4.2. *Suppose $\mu = \alpha_i + \ldots + \alpha_j$ and g is a monomial (word) in the simple root vectors $\{f_{\alpha_k}\}_{k=i}^{j}$ that contains f_{α_i} and f_{α_j} at most once. Then*

1. *$[g, f_\mu] = 0$ if both f_{α_i} and f_{α_j} enter g or none,*
2. *$[g, f_\mu]_{q^{-1}} = 0$ if g contains only f_{α_i},*
3. *$[g, f_\mu]_q = 0$ if g contains only f_{α_j}.*

In particular, $[\tilde f_\mu, f_\mu] = 0$, $[\tilde f_{\mu'}, f_\mu]_q = 0$, and $[\tilde f_{\mu''}, f_\mu]_{q^{-1}} = 0$.

Proof. It is known that f_{α_k} commutes with $\tilde f_\gamma$ if $i<k<j$, see e.g. [7]. Further, the higher order Serre relations

$$\begin{aligned}
f_{\alpha_i} f_\mu &= f_{\alpha_i} [f_i, f_{\mu'}]_q = q^{-1} [f_i, f_{\mu'}]_q f_{\alpha_i} = q^{-1} f_\mu f_{\alpha_i}, \\
f_{\alpha_j} f_\mu &= f_{\alpha_j} [f''_\mu, f_j]_q = q [f''_\mu, f_j]_q f_{\alpha_j} = q f_\mu f_{\alpha_j}.
\end{aligned}$$

in $U_\hbar\big(\mathfrak{gl}(n)\big)$ readily imply the statement. □

5 Generalized Verma module M_λ

We need to set up a few conventions about representations of quantum groups. We assume that they are free modules over the ring of scalars and their rank will be referred to as dimension. We call a $U_q(\mathfrak{g})$-module irreducible if it is so for specialization at generic q. Similarly, a $U_\hbar(\mathfrak{g})$-module is called irreducible if it is irreducible over $U(\mathfrak{g})$ in the classical limit. As $U_q(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$ have different Cartan subalgebras, their sets of weights are different. Still we prefer to use additive language for $U_q(\mathfrak{g})$-weights, which are then parameterized by the assignment $\lambda \mapsto q^\lambda$.

We shall be dealing with weight modules of highest weights, i. e. containing a weight vector v annihilated by positive Chevalley generators. Under our convention, all weights in such modules belong to $-\mathbb{Z}\Pi^+ + \lambda$, where λ is the highest weight. To construct representations of $U_q(\mathfrak{g})$ as a $\mathbb{C}[q, q^{-1}]$-algebra, we should include weights from $\hbar^{-1}\mathfrak{h}^* \oplus \mathfrak{h}^*$ (the first summand is defined up to $\frac{2\pi\mathbb{Z}\sqrt{-1}}{\hbar}$, and the second summand can be restricted to the integral weight lattice). Of course, such modules need to be extended over Laurent series, in order to facilitate the extension from $U_q(\mathfrak{g})$ to $U_\hbar(\mathfrak{g})$. However, we are eventually interested in the algebra of endomorphisms of the regular parts, which can be shown to carry the action of $U_\hbar(\mathfrak{h})$. With all this said, we shall understand by weight an element of the vector space $\hbar^{-1}\mathfrak{h}^* \oplus \mathfrak{h}^*$.

Let L denote one of the two maximal Levi subgroups in K, which is

$$L = GL(n_1) \times \ldots \times GL(n_\ell) \times GL(m) \times SP(2p)$$

for K as in (2.5). The difference between L and K is only one Cartesian factor $GL(m) \subset SP(2m)$. By $\mathfrak{l}$ we denote the Lie algebra of L. It is a reductive subalgebra in $\mathfrak{g}$ of maximal rank n.

By $\mathfrak{c}_\mathfrak{l} \subset \mathfrak{h}$ we denote the center of $\mathfrak{l}$. In the presence of inner product, we identify its dual $\mathfrak{c}_\mathfrak{l}^*$ with a subspace in $\mathfrak{h}^*$. Any element $\lambda \in \hbar^{-1}\mathfrak{c}_\mathfrak{l}^* \oplus \mathfrak{c}_\mathfrak{l}^*$ defines a one-dimensional representation of $U_q(\mathfrak{l})$ denoted by $\mathbb{C}_\lambda$. This representation extends to $U_q(\mathfrak{p}^+)$ by setting it trivial on the subalgebra $U_q(\mathfrak{n}_\mathfrak{l}^+)$. Denote by $\hat{M}_\lambda = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}^+)} \mathbb{C}_\lambda$ the parabolic Verma $U_q(\mathfrak{g})$-module induced from $\mathbb{C}_\lambda$, [8]. We are interested in λ for which $\hat{M}_\lambda$ admits a singular vector of weight $-\delta + \lambda$, where $\delta = 2\alpha_{n-p} + \ldots + 2\alpha_{n-1} + \beta \in R^+$ (in the classical limit, the root vectors e_δ and f_δ generate the isotropy subalgebra $\mathfrak{k}$, over $\mathfrak{l}$).

For the sake of technical convenience, we assume that $\ell = 0$, $m = 1$, $n = 1 + p$. This restriction will be relaxed later on. In this setting, the root α_1 is distinguished, as f_{α_1} is the only negative Chevalley generator which does not kill v.

Lemma 5.1. *Put $\delta' = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_p + \beta$. The subspace of weight $-\delta' + \lambda$ in $\hat{M}_\lambda$ is spanned by the vector $f_{\alpha_2} \overset{<}{\ldots} f_{\alpha_p} f_\beta f_{\alpha_p} \overset{>}{\ldots} f_{\alpha_2} f_{\alpha_1} v$.*

Proof. As all negative Chevalley generators but f_{α_1} kill the highest weight vector, f_{α_1} must be next to v. Further, we conclude that all non-zero vectors should be combinations of $\phi_\sigma = g_\sigma f_{\alpha_p} \overset{>}{\ldots} f_{\alpha_2} f_{\alpha_1} v$ with $g_\sigma = \sigma(f_{\alpha_2} \overset{<}{\ldots} f_{\alpha_p} f_\beta)$, where σ is a permutation of the factors. Suppose that f_β is not rightmost in g_σ. Then ϕ_σ reads

$$\ldots f_{\alpha_k} f_{\alpha_p} \overset{>}{\ldots} f_{\alpha_2} f_{\alpha_1} v = \ldots f_{\alpha_k} f_{\alpha_{k+1}} f_{\alpha_k} \overset{>}{\ldots} f_{\alpha_1} v \sim \ldots (f^2_{\alpha_k} f_{\alpha_{k+1}} + f_{\alpha_{k+1}} f^2_{\alpha_k}) f_{\alpha_{k-1}} \overset{>}{\ldots} f_{\alpha_1} v,$$

for some $k > 1$. In the last transition we have used the Serre relations. The first term in the brackets disappears, because $f_{\alpha_{k+1}}$ goes freely to the right, where it kills v. In the second term, one factor from $f^2_{\alpha_k}$ percolates through $f_{\alpha_{k-1}}$ to the right due to the Serre relations, where it reaches v and annihilates it.

Thus, the permutation σ leaves f_β in the rightmost position in g_σ. Suppose $g_\sigma = \ldots f_{\alpha_i} f_{\alpha_k} \overset{<}{\ldots} f_{\alpha_p} f_\beta$ for some $k = 2, \ldots, p+1$ and $i < k-1$ (we assume formally that f_{α_i} stands next to f_β if $k = p+1$). Then f_{α_i} can be pushed through to the right of f_β, and this situation has been already treated above. Thus, the factors in g_σ are all ordered, and the permutation σ is identical.

Finally, the vector of concern is not zero. Indeed, the subspace of weight $-\delta' + \lambda$ in $\hat{M}_\lambda$ has the same dimension as the subspace of weight $-\delta'$ in $U_q(\mathfrak{n}_\mathfrak{l}^-)$, which is exactly 1, by virtue of the Poincaré-Birkgoff-Witt theorem. □

Put $\gamma = \alpha_1 + \ldots + \alpha_p$, $\delta = 2\gamma + \beta$ and introduce the vector

$$f_\delta = [f_\gamma, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2} = [\tilde{f}_\gamma, [f_\gamma, f_\beta]_{q^2}]_{q^{-2}},$$

were we use the standard notation $[x, y]_a$ for the combination $xy - ayx$ in any associative algebra and any scalar a. Remark that the right equality holds by virtue of Lemma 4.2.

Lemma 5.2. *The vector f_δ is presentable in the form*

$$[f_{\alpha_1}, [f_{\alpha_2}, \ldots [f_{\alpha_p}, [f_{\alpha_1}, \ldots [f_{\alpha_{p-1}}, [f_{\alpha_p}, f_\beta]_{q^2}]_q \ldots]_q]_{q^{-1}} \ldots]_{q^{-1}}]_{q^{-2}}.$$

Proof. First of al, remark that p internal commutators amount to $[f_\gamma, f_\beta]_{q^2}$. Further, fix $i = 2, \ldots, p+1$ and define the root ν from the equality $\gamma = \nu + \alpha_i + \ldots \alpha_p$ (if $i = p+1$ then ν is simply γ.) Suppose we have proved that f_δ is presentable in the form $[[f_\nu, f_{\alpha_i}]_{q^{-1}}, z]_{q^{-2}}$, where $z = [f_{\alpha_{i+1}}, \ldots [f_{\alpha_p}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}} \ldots]_{q^{-1}}$. In particular, this is true for $i = p+1$. The vector f_ν commutes with everything in z but f_γ. All Chevalley generators constituting f_ν excepting f_{α_1} commute with f_γ, and f_{α_1} enters f_ν exactly once. Applying Lemma 4.2, we conclude that $[f_\nu, z]_{q^{-1}} = 0$. Using the "Jacobi identity"

$$[x,[y,z]_a]_b = [[x,y]_c, z]_{\frac{ab}{c}} + c[y,[x,z]_{\frac{b}{c}}]_{\frac{a}{c}}, \tag{5.10}$$

which holds true in any associative algebra for any a, b and invertible c, we write

$$[f_\nu, [f_{\alpha_i}, z]_{q^{-1}},]_{q^{-2}} = [[f_\nu, f_{\alpha_i}]_{q^{-1}}, z]_{q^{-2}} + q^{-1}[[f_\nu, f_{\alpha_i}]_{q^{-1}}, z]$$

for $a = c = q^{-1}$, $b = q^{-2}$. The second term vanishes, and we come to the equality $[[f_\nu, f_{\alpha_i}]_{q^{-1}}, z]_{q^{-2}} = [f_\nu, [f_{\alpha_i}, z]_{q^{-1}},]_{q^{-2}}$. Descending induction on $i = p+1, \ldots, 2$ completes the proof. □

Now we lift the assumption $\ell = 0$, $m = 1$ and work out the case of general $\mathfrak{k}$ and $\mathfrak{l}$:

$$\mathfrak{k} = \mathfrak{gl}(n_1) \oplus \ldots \oplus \mathfrak{gl}(n_\ell) \oplus \mathfrak{sp}(2m) \oplus \mathfrak{sp}(2p), \quad \mathfrak{l} = \mathfrak{gl}(n_1) \oplus \ldots \oplus \mathfrak{gl}(n_\ell) \oplus \mathfrak{gl}(m) \oplus \mathfrak{sp}(2p).$$

Consider the subalgebra $U_q\bigl(\mathfrak{sp}(2+2p)\bigr) \subset U_q(\mathfrak{g})$ with the positive simple roots $(\alpha_{n-p}, \ldots \alpha_n)$. The root vectors $f_\gamma, \tilde{f}_\gamma, f_\delta \in U_q\bigl(\mathfrak{sp}(2+2p)\bigr)$ are carried to $U_q(\mathfrak{g})$, where we use the same notation for them. This relates the case $\ell = 0$, $m = 1$ to the general setting. The root α_{n-p} plays the same role as α_1 in the symmetric case with $m = 1$. We will denote it by α when we wish to emphasize the global meaning of formulas with it.

Proposition 5.3. *Suppose that $q^{2(\lambda,\alpha)} = -q^{-2p}$. Then $f_\delta v$ is a singular vector in $\hat{M}_\lambda$.*

Proof. To begin with, return to the symmetric case $\ell = 0$ with $m = 1$. Further, as the case $p = 1$ have been studied in [2], we assume $p > 1$.

Applying e_β to $f_\delta v$ we obtain, up a non-zero scalar factor,

$$[f_\gamma, [\tilde{f}_\gamma, q^{h_\beta} - q^{-h_\beta}]_{q^{-2}}]_{q^2} \sim [f_\gamma, [\tilde{f}_\gamma, q^{-h_\beta}]_{q^{-2}}]_{q^2} \sim [f_\gamma, \tilde{f}_\gamma q^{-h_\beta}]_{q^2} = [f_\gamma, \tilde{f}_\gamma q^{-h_\beta}]_{q^2} = [f_\gamma, \tilde{f}_\gamma] q^{-h_\beta}.$$

The last commutator is zero, by Lemma 4.2. If $1 < i < n-1$, then e_{α_i} commutes with f_γ and $\tilde{f}_\gamma$ by Lemma 4.1, and hence with $[f_\gamma, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2}$. It is therefore annihilates $f_\delta v$. Thus, we only need to check that $f_\delta v$ is annihilated by $e_\alpha = e_{\alpha_1}$ and $e_{\alpha_p} = e_{\alpha_{n-1}}$.

The action of e_{α_p} is considered in the following two cases: $p = 2$ and $p > 2$. Using

$$[e_{\alpha_p}, f_\gamma] = -q f_{\gamma''} q^{h_{\alpha_p}}, \quad [e_{\alpha_p}, \tilde{f}_\gamma] = -q^{-1} \tilde{f}_{\gamma''} q^{-h_{\alpha_p}}$$

we get

$$q[f_{\gamma''} q^{h_{\alpha_p}}, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2} + q^{-1}[f_\gamma, [\tilde{f}_{\gamma''} q^{-h_{\alpha_p}}, f_\beta]_{q^{-2}}]_{q^2}.$$

The second term disappears, because $[\tilde f_{\gamma''}q^{-h_{\alpha_p}}, f_\beta]_{q^{-2}} \sim [\tilde f_{\gamma''}, f_\beta]q^{-h_{\alpha_p}}$=0. Let us check that the first term vanishes too.

Suppose first that $p=2$. Then the first term is proportional to

$$[f_{\alpha_{p-1}}q^{h_{\alpha_p}}, [\tilde f_\gamma, f_\beta]_{q^{-2}}]_{q^2} = q[f_{\alpha_{p-1}}, [\tilde f_\gamma, f_\beta]_{q^{-2}}]_q q^{h_{\alpha_p}} = q[[f_{\alpha_{p-1}}, \tilde f_\gamma]_q, f_\beta]_{q^{-2}}q^{h_{\alpha_p}} = 0,$$

as $[f_{\alpha_{p-1}}, \tilde f_\gamma]_q = 0$.

If $p>2$, we present f_δ as $f_\delta = [\tilde f_\nu, [f_{\alpha_{p-1}}, [f_{\alpha_p}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}}]_{q^{-1}}]_{q^{-2}}$, where the $\nu = \gamma - \alpha_{p-1} - \alpha_p$. Then $[e_{\alpha_p}, f_\delta]$ (as we saw, we can focus only on commutation with $\tilde f_\gamma$) is proportional to

$$[\tilde f_\nu, [f_{\alpha_{p-1}}, [q^{h_{\alpha_p}} - q^{-h_{\alpha_p}}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}}]_{q^{-1}}]_{q^{-2}} = [\tilde f_\nu, [f_{\alpha_{p-1}}, [q^{h_{\alpha_p}}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}}]_{q^{-1}}]_{q^{-2}} = 0,$$

because

$$[f_{\alpha_{p-1}}, [q^{h_{\alpha_p}}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}}]_{q^{-1}} \sim [f_{\alpha_{p-1}}, [f_\gamma, f_\beta]_{q^2}q^{h_{\alpha_p}}]_{q^{-1}} = [f_{\alpha_{p-1}}, [f_\gamma, f_\beta]_{q^2}]q^{h_{\alpha_p}} = 0.$$

Thus we have shown that $f_\delta v$ is annihilated by $e_{\alpha_i} \in U_q\big(\mathfrak{sp}(2p)\big) \subset U_q(\mathfrak{g})$. Next we check that it is killed by e_{α_1}.

Based on Lemma 4.1, we find $e_\alpha f_\delta v$ to be equal to

$$[f_{\gamma'}q^{-h_{\alpha_1}}, [\tilde f_\gamma, f_\beta]_{q^{-2}}]_{q^2}v + [f_\gamma, [\tilde f_{\gamma'}q^{h_{\alpha_1}}, f_\beta]_{q^{-2}}]_{q^2}v =$$

$$= q^{1-(\alpha,\lambda)}[f_{\gamma'}, [\tilde f_\gamma, f_\beta]_{q^{-2}}]_q v + q^{(\alpha,\lambda)}[f_\gamma, [\tilde f_{\gamma'}, f_\beta]_{q^{-2}}]_q v.$$

With the help of Lemma 5.1, we develop the commutators in f_δ and find $e_\alpha f_\delta v$ proportional to

$$(q^{1-(\alpha,\lambda)}q^{-2-p+1} + q^{(\alpha,\lambda)+p})f_{\alpha_2} \overset{<}{\ldots} f_{\alpha_p} f_\beta f_{\alpha_p} \overset{>}{\ldots} f_{\alpha_2} f_\alpha v.$$

It turns zero if and only if $\quad q^{2(\alpha,\lambda)} = -q^{-2p}$. This completes the proof for $\ell=0$, $m=1$.

The vector $f_\delta v \in \hat M_\lambda$ has been shown to be singular with respect to the cental block subalgebra $U_q\big(\mathfrak{sp}(2+2p)\big)$. Therefore it is singular with respect to entire $U_q(\mathfrak{g})$, as the negative root vectors participating in f_δ all commute with the complementary positive root vectors. $\square$

Proposition 5.4. *The singular vector $f_\delta v$ is a linear combination of vectors*

$$f_{\alpha_i} \overset{<}{\ldots} f_{\alpha_{n-1}} f_\beta f_{\alpha_{i-1}} \overset{>}{\ldots} f_\alpha f_{\alpha_{n-1}} \overset{>}{\ldots} f_\alpha v, \quad i = n-p+1, \ldots, n.$$

Proof. Directly follows from the definition $f_\delta = [f_\gamma, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2}$. $\square$

We need to define certain weight subspaces in order to formalize further presentation. Let $\mathfrak{c}^*_{\mathfrak{l},reg}$ denote the set of all weights in $\mathfrak{c}^*_{\mathfrak{l}}$ that cannot be extended to characters of any reductive subalgebra in $\mathfrak{g}$ containing $\mathfrak{l}$. We denote by $\mathfrak{c}^*_{\mathfrak{k}}$ the subset in $\mathfrak{c}^*_{\mathfrak{l}}$ such that $q^{2(\alpha,\lambda)} = -1$ for $\lambda \in \hbar^{-1}\mathfrak{c}^*_{\mathfrak{k}}$; its intersection with $\mathfrak{c}^*_{\mathfrak{l},reg}$ is designated by $\mathfrak{c}^*_{\mathfrak{k},reg}$. It follows that $\mathfrak{c}^*_{\mathfrak{k}}$ is an affine space whose associated vector space is $\mathfrak{c}^*_{\hat{\mathfrak{l}}}$, where $\hat{\mathfrak{l}} \supset \mathfrak{l}$ is the Levi subalgebra $\oplus_{i=1}^{\ell}\mathfrak{gl}(n_i) \oplus \mathfrak{sp}(2m+2p)$. Finally, we denote by $\mathfrak{C}^*_{\mathfrak{k}}$ and $\mathfrak{C}^*_{\mathfrak{k},reg}$ the vector subspace in respectively $\hbar^{-1}\mathfrak{c}^*_{\mathfrak{k}} \oplus \mathfrak{c}^*_{\mathfrak{l}}$ and $\hbar^{-1}\mathfrak{c}^*_{\mathfrak{k},reg} \oplus \mathfrak{c}^*_{\mathfrak{l}}$ of weights λ satisfying $q^{2(\alpha,\lambda)} = -q^{-2p}$.

Definition 5.5. *Assuming $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$, we denote by M_λ the quotient of the parabolic Verma module $\hat{M}_\lambda$ by the submodule generated by $f_\delta v$.*

The weight subspaces introduced above can be explicitly described as follows. Introduce $\ell+2$ weights $\mathcal{E}_i \in \mathfrak{h}^*$ by setting

$$\mathcal{E}_1 = \varepsilon_1 + \ldots + \varepsilon_{n_1}, \quad \mathcal{E}_2 = \varepsilon_{n_1+1} + \ldots + \varepsilon_{n_1+n_2}, \quad \ldots, \quad \mathcal{E}_{\ell+2} = \varepsilon_{n-p+1} + \ldots + \varepsilon_n.$$

Then $\mathfrak{c}^*_{\mathfrak{l}}$ is formed by $\sum_{i=1}^{\ell+1}\Lambda_i\mathcal{E}_i$ with complex coefficients Λ_i, whilst $\mathfrak{c}^*_{\hat{\mathfrak{l}}}$ assumes zero $\Lambda_{\ell+1}$. The subset $\mathfrak{c}^*_{\mathfrak{l},reg}$ consists of combinations with pairwise distinct $\Lambda_i \neq 0$. The subset $\mathfrak{c}^*_{\mathfrak{k}}$ is characterized in $\mathfrak{c}^*_{\mathfrak{l}}$ by the condition $\Lambda_{\ell+1} = \frac{\sqrt{-1}\,\pi}{2}$. Then we can write $\mathfrak{C}^*_{\mathfrak{k}} = \hbar^{-1}\mathfrak{c}^*_{\mathfrak{k}} + \mathfrak{c}^*_{\hat{\mathfrak{l}}} - p\mathcal{E}_{\ell+1}$ and $\mathfrak{C}^*_{\mathfrak{k},reg} = \hbar^{-1}\mathfrak{c}^*_{\mathfrak{k},reg} + \mathfrak{c}^*_{\hat{\mathfrak{l}}} - p\mathcal{E}_{\ell+1}$.

6 Module $\mathbb{C}^{2n} \otimes M_\lambda$: the symmetric case

In this section we set $\ell = 0$ and work with the Levi subalgebra $U_q(\mathfrak{gl}(m)) \otimes U_q(\mathfrak{sp}(2p))$, $m + p = n$. In this setting, the distinguished root α is $\alpha_{n-p} = \alpha_m$. It is the scraped root of the Dynkin diagram of $\mathfrak{g}$, which complement is the Dynkin diagram of (semi-simple part of) $\mathfrak{l}$.

Consider the defining vector representation of $U_q(\mathfrak{sp}(2n))$ in $\mathbb{C}^{2n}$ and denote by $(w_i)_{i=1}^{2n} \subset \mathbb{C}^{2n}$, the standard basis carrying weights $(\varepsilon_1, \ldots \varepsilon_n, -\varepsilon_n, \ldots, -\varepsilon_1)$. In this basis, the matrices assigned to the generators of $U_q(\mathfrak{sp}(2n))$ are constant (independent of q) and coincide with the representation of $U(\mathfrak{sp}(2n))$ in the classical limit $q \to 1$.

For generic weight $\lambda \in \mathfrak{C}^*_{\mathfrak{l},reg}$, the tensor product $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$ is the direct sum of three submodules of weights $\nu_1 = \varepsilon_1 + \lambda$, $\nu_2 = \varepsilon_{m+1} + \lambda$, $\nu_3 = \varepsilon_{n+p+1} + \lambda$. We aim at proving the direct sum decomposition $\mathbb{C}^{2n} \otimes M_\lambda = M_1 \oplus M_2$, where M_i are submodules in $\mathbb{C}^{2n} \otimes M_\lambda$ of highest weights ν_i. We begin with finding singular vectors u_{ν_i} generating submodules

$\hat{M}_i \subset \mathbb{C}^{2n} \otimes \hat{M}_\lambda$ of weights ν_i, $i = 1, 2$. Obviously $u_{\nu_1} = w_1 \otimes v$. The singular vector u_{ν_3} is not interesting for us, as it drops from $\mathbb{C}^{2n} \otimes M_\lambda$. This effect is studied in detail in [2] for the simplest case $\mathfrak{g} = \mathfrak{sp}(4)$.

Lemma 6.1. *The vector*

$$\frac{q^{(\alpha,\lambda)} - q^{-(\alpha,\lambda)}}{q - q^{-1}} w_{m+1} \otimes v + (-q)^{-1} w_m \otimes f_{\alpha_m} v + \ldots + (-q)^{-m} w_1 \otimes f_{\alpha_1} \overset{<}{\ldots} f_{\alpha_m} v, \qquad (6.11)$$

is singular of weight $\varepsilon_{m+1} + \lambda$.

Proof. Straightforward. $\square$

Further we develop a diagram technique which will help us study the module $\mathbb{C}^{2n} \otimes M_\lambda$. Introduce the monomials by $\psi_i = f_{\alpha_i} \overset{<}{\ldots} f_{\alpha_m} \in U_q(\mathfrak{n}^-)$, $i = 1, \ldots, m$. In these terms, the singular vector u_{ν_2} reads

$$\frac{q^{(\alpha,\lambda)} - q^{-(\alpha,\lambda)}}{q - q^{-1}} w_{m+1} \otimes v + \sum_{i=1}^{m} (-q)^{i-m-1} w_i \otimes \psi_i v.$$

The defining representation restricted to the Levi subalgebra splits $\mathbb{C}^{2n}$ into three irreducible sub-representations, $\mathbb{C}^{2n} = \mathbb{C}^m \oplus \mathbb{C}^{2p} \oplus \mathbb{C}^m$. The action of $f_{\alpha_1}, \ldots, f_{\alpha_m}$ on the highest block $\mathbb{C}^m \otimes M_\lambda$ can be conveniently illustrated by the triangular diagram

D_0

$$\begin{array}{ccccccccccc|c}
 & f_{\alpha_1} & & f_{\alpha_2} & & f_{\alpha_3} & & f_{\alpha_{m-1}} & & f_{\alpha_m} & & \\
\hline
w_1\otimes\psi_1 v & \leftarrow & w_1\otimes\psi_2 v & \leftarrow & w_1\otimes\psi_3 v & \leftarrow & \ldots & \leftarrow & w_1\otimes\psi_m v & \leftarrow & w_1\otimes v & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & f_{\alpha_1} \\
 & & w_2\otimes\psi_2 v & \leftarrow & w_2\otimes\psi_3 v & \leftarrow & \ldots & \leftarrow & w_2\otimes\psi_m v & \leftarrow & w_2\otimes v & \\
 & & & & \downarrow & & & & \downarrow & & \downarrow & f_{\alpha_2} \\
 & & & & w_3\otimes\psi_3 v & \leftarrow & \ldots & & \ldots & & \ldots & \\
 & & & & & \ddots & \downarrow & & \downarrow & & \downarrow & f_{\alpha_{m-2}} \\
 & & & & & & w_{m-1}\otimes\psi_{m-1} v & \leftarrow & w_{m-1}\otimes\psi_m v & \leftarrow & w_{m-1}\otimes v & \\
 & & & & & & & & \downarrow & & \downarrow & f_{\alpha_{m-1}} \\
 & & & & & & & & w_m\otimes\psi_m v & \leftarrow & w_m\otimes v & \\
 & & & & & & & & & & \downarrow & f_{\alpha_m} \\
 & & & & & & & & & & w_{m+1}\otimes v &
\end{array}$$

The nodes on the diagram designate one dimensional subspaces in $\mathbb{C}^m \otimes M_\lambda$ spanned by the corresponding vectors. The horizontal arrows symbolize the action of the Chevalley generators on the M_λ-tensor factor while vertical arrows on the $\mathbb{C}^m$-tensor factor. The horizonal give the action on the whole $\mathbb{C}^m\otimes M_\lambda$ when it is distinct from the vertical arrow. When both arrows applied to a node coincide, the corresponding generator sends the node to the two dimensional space spanned by the nodes down and on the left. In the given diagram, this is exactly the m-th diagonal (above the principal). Note that rightmost arrows but f_{α_m} amount to the action on $\mathbb{C}^m\otimes M_\lambda$, as they kill v.

It is easy to see that the sub-triangle above the principal diagonal belongs to M_1. It is so for the rightmost column, which elements are obtained from $w_1\otimes v$ through the sequence of vertical arrows. The following induction on the column number proves it for the entire sub-triangle. Suppose it is checked for some column. Then all nodes in it except for the lowest are sent by the horizontal arrow leftward plus maybe to a node downward, which lies in M_1 by induction assumption. For the illustration, see the diagram in the proof of Theorem 6.3.

Applying f_{α_i} to $w_i\otimes\psi_{i+1}v\in M_1$ (which lies on the diagonal of the sub-triangle) gives

$$w_i\otimes\psi_i v+w_{i+1}\otimes q^{-1}\psi_{i+1}v,\quad i<m,\quad w_m\otimes\psi_m v+w_{m+1}\otimes q^{-(\alpha_m,\lambda)}v,\quad i=m. \tag{6.12}$$

From this we derive

Lemma 6.2. *The singular vector (6.11) is equal to* $q^{-m}\frac{q^{(\alpha,\lambda)+m}-q^{-(\alpha,\lambda)-m}}{q-q^{-1}}w_{m+1}\otimes v$ *modulo* M_1.

Proof. All entries above the main diagonal of the diagram D_0 belong to M_1. Formulas (6.12) imply that $w_i\otimes\psi_i v=-q^{-1}w_{i+1}\otimes\psi_{i+1}v$ modulo M_1, for $i\leqslant m$, if we set $\psi_{m+1}=1$. Therefore $w_i\otimes\psi_i v=-(-q)^{i-m}q^{-(\alpha,\lambda)}w_{m+1}\otimes v$ modulo M_1, for $i=1,\ldots,m$. Hence the singular vector equals

$$\Bigl(\frac{q^{(\alpha,\lambda)}-q^{-(\alpha,\lambda)}}{q-q^{-1}}+q^{-(\alpha,\lambda)-1}\sum_{i=0}^{m-1}q^{2(i-m)}\Bigr)w_{m+1}\otimes v\quad \mathrm{mod}\ M_1.$$

This implies the statement immediately. □

We have studied the action of the Chevalley generators on the highest block $\mathbb{C}^m\otimes M_\lambda$ of the tensor product $\mathbb{C}^{2n}\otimes M_\lambda$. Next we move further down through the block $\mathbb{C}^{2p}\otimes M_\lambda$ in order to reach the vector $w_{n+p+1}\otimes v$ of weight ν_3. For generic λ, it is proportional to the singular vector u_{ν_3}, modulo $\hat M_1+\hat M_2$. We will show that $w_{n+p+1}\otimes v\in M_1+M_2$ for $\lambda\in\mathfrak{C}^*_{\mathfrak{k}}$. This is the key step in proving the direct decomposition $\mathbb{C}^{2n}\otimes M_\lambda=M_1\oplus M_2$.

To that end, we need to develop further the diagrammatic technique used above. Introduce monomials $\phi_i\in U_q(\mathfrak{n}^-)$, $i=1,\ldots,p+1$, of degree $2p+1$ by the formulas (recall that

$m = n - p$ in this section)

$$\phi_i := (f_{\alpha_{m+i-1}} \overset{<}{\ldots} f_{\alpha_{n-1}} f_\beta f_{\alpha_{m+i-2}} \overset{>}{\ldots} f_{\alpha_m})(f_{\alpha_{n-1}} \overset{>}{\ldots} f_{\alpha_m}), \quad i = 1, \ldots, p+1.$$

According to Proposition 5.4, the root vector f_δ can be written as a linear combination of ϕ_i.

Denote by f_i^l, $l = 1, \ldots, 2p+1$, the l-th factor in ϕ_i counting from the right. If follows that $f_i^l = f_{\alpha_{l+m-1}}$ for $1 \leqslant l \leqslant p$ and all i. Thus, every i corresponds to a reordering of the leftmost $n-m$ terms of the sequence

$$f_{\alpha_m}, \ldots, f_{\alpha_{n-1}}, f_\beta, f_{\alpha_{n-1}}, \ldots, f_{\alpha_m}. \tag{6.13}$$

Denote by ϕ_i^l the product $f_i^l \overset{>}{\ldots} f_i^1$ for all $l = 1, \ldots, 2p+1$. In particular, $\phi_i^l = f_{\alpha_{l-1+m}} \overset{>}{\ldots} f_{\alpha_m}$ for all $1 \leqslant l \leqslant p$, and $\phi_i^{2p-1} = \phi_i$. It is also convenient to put $\phi_i^0 := 1$ for all i.

With every $i = 1, \ldots, p+1$, we associate a diagram D_i of $p+1$ rows if $i > 1$ and $2p+2$ rows if $i = 1$. The lengths of the rows vary from $2p+2$ to 1 in D_1 and to $p+2$ if $i > 1$, from top to bottom. The rows are leveled on the right, so D_1 is a full triangle and D_i are trapezoids for $i > 1$. In fact, D_i can be extended further down to triangular diagrams as D_1, but we need only first $p+1$ rows in them.

The rightmost column is formed by the vectors $w_{m+l-1} \otimes v$, where l runs from 1 to $p+1$ if $i > 1$ and to $2p+2$ in D_1. The intersection of l-th row and j-th column is the vector $w_{m+l-1} \otimes \phi_i^{j-1} v$. As before, the nodes of the diagrams span one-dimensional subspaces in $\mathbb{C}^{2n} \otimes M_\lambda$ and the arrows designate the action of negative Chevalley generators: horizontal on the M_λ-factors and vertical on the $\mathbb{C}^{2n}$-factors. When the vertical and horizontal arrows are distinct, the horizontal arrow gives the action on entire $\mathbb{C}^{2n} \otimes M_\lambda$, otherwise the node is sent to the span of the two nodes: next down and next to the left.

In all diagrams the vertical arrow applied to the j-th row is labeled with f_1^j, i.e. the j-th term of the sequence (6.13) from the right. The horizontal arrows form the reordered sequence (6.13) making up ϕ_i.

$$D_i, \quad i > 1$$

$$\begin{array}{ccccccccccc|c}
 & f_i^{2p+1} & & f_i^{2p} & \ldots & & f_i^{p} & \ldots & f_i^{1} & & \\
\hline
w_m \otimes \phi_i^{2p+1} v & \leftarrow & w_m \otimes \phi_i^{2p} v & & & & & & \leftarrow & w_m \otimes v & \\
 & & \downarrow & & & & & & & \downarrow & f_1^1 \\
 & & w_{m+1} \otimes \phi_i^{2p} v & \leftarrow & & & & & & & \\
 & & & & \ddots & & & & & & \vdots \\
 & & & & & \downarrow & & & & \downarrow & f_1^p \\
 & & & & & w_{m+p} \otimes \phi_i^{m+p+1} v & \leftarrow & \ldots & \leftarrow & w_{m+p} \otimes v &
\end{array}$$

$$D_1$$

$$
\begin{array}{ccccccccc|c}
f_1^{2p+1} & & f_1^{2p} & & \dots & & f_1^2 & & f_1^1 & \\
\hline
w_m\otimes\phi_1^{2p+1}v & \leftarrow & w_m\otimes\phi_1^{2p}v & & & & & & & \\
 & & \downarrow & & & & & & & f_1^1 \\
 & & w_{m+1}\otimes\phi_1^{2p}v & \leftarrow & & & & & & \\
 & & & & & & & & & \vdots \\
 & & & & \downarrow & & & & & f_1^{2p-1} \\
 & & & & w_{n+p-1}\otimes\phi_1^{2}v & \leftarrow & w_{n+p-1}\otimes\phi_1^{1}v & & & \\
 & & & & & & \downarrow & & & f_1^{2p} \\
 & & & & & & w_{n+p}\otimes\phi_1^{1}v & \leftarrow & w_{n+p}\otimes v & \\
 & & & & & & & & \downarrow & f_1^{2p+1} \\
 & & & & & & & & w_{n+p+1}\otimes v &
\end{array}
$$

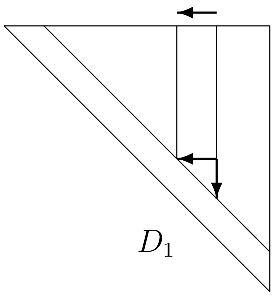
We present the diagrams D_1, D_2, D_3 in Appendix, in order to illustrate the formalism in the case $m = 1$, $p = 2$, $n = 3$.

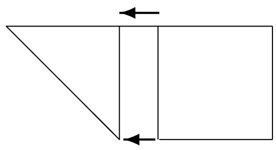
Theorem 6.3. *Suppose that $q^{-2p+2m} \neq -1$. Then the $U_q(\mathfrak{g})$-module $\mathbb{C}^{2n} \otimes M_\lambda$ is isomorphic to the direct sum $M_1 \oplus M_2$.*

Proof. Obviously, the modules M_1 and M_2 have zero intersection, as they carry different eigenvalues of the invariant matrix $\mathcal{Q}$, see Section 8. We must show that the sum $M = M_1 \oplus M_2$ exhausts all $\mathbb{C}^{2n} \otimes M_\lambda$. To that end, it is sufficient to show that $\mathbb{C}^{2n} \otimes v$ lies in M. The symbol $\equiv$ below will mean $\equiv \mod M$.

First of all, it follows from the proof of Lemma 6.1 that $w_i \otimes v \in M_1$ for $i = 1, \dots m$. By Lemma 6.2, the vector $w_{m+1} \otimes v$ belongs to M if $q^{2(\alpha,\lambda)+2m} = -q^{-2p+2m} \neq 1$. This implies $w_l \otimes v \in M$ for $l = m+2, \dots n+p$, by the analysis of the diagram D_1, see below.

The crucial step is to show that $w_{n+p+1} \otimes v \in M$. In the diagram D_1 the triangle above the principal diagonal lies in $M_1 + M_2$. This is checked by induction on column number as for the diagram D_0 above. The left diagram below displays schematically the induction transition.





The only nodes in question are on the main diagonal, which are $w_{m+l} \otimes \phi_1^{2p+1-l}v$, $l = 0, \ldots, 2p+1$. Applying the arrows to the diagonal of sub-triangle we get

$$a_1 w_m \otimes \phi_1^{2p+1}v + w_{m+1} \otimes \phi_1^{2p}v \equiv 0, \quad \ldots, \quad a_{2p+1}w_{n+p} \otimes \phi_1^1 v + w_{n+p+1} \otimes \phi_1^0 v \equiv 0,$$

where $a_i \neq 0$ are non-zero scalars. Thus, all the diagonal terms are proportional, modulo M.

Now turn to the diagram D_i, $i = 2, \ldots, p+1$, see the right diagram above. As was mentioned, the square of size $p+1$ on the right is the same as in D_1 and therefore belongs to M. If we extend the diagram further down, we shall have $f_1^{p+1} \neq f_i^{p+1}$, therefore the operator f_i^{p+1} is sending the bottom node in $p+1$-the column to bottom node in the $p+2$-th. This implies that the entire rectangle supported on the bottom line of D_i belongs to M. A simple induction proves that the $p+1 \times p+1$-triangular left part of D_i also belongs to M. In particular, the vertex node $w_m \otimes \phi_i^{2p+1}$ lies in M_1.

Summing up the equality $a_1 w_m \otimes \phi_1^{2p+1}v + w_{m+1} \otimes \phi_1^{2p}v \equiv 0$ and the equalities $w_m \otimes \phi_i^{2p+1}v \equiv 0$ for $i = 2, \ldots, 2p+1$ with appropriate coefficients, we replace $w_m \otimes \phi_1^{2p+1}v$ with $w_m \otimes f_\delta v$, which belongs to M. Hence $w_{m+1} \otimes \phi_1^{2p}v \equiv 0$, and moving down the diagonal we eventually find $w_{n+p+1} \otimes v \equiv 0$.

To complete the proof, we must check that $w_j \otimes v \in M$ for $j > n+p+1$. This readily follows from the action $f_{\alpha_{m-i}}(w_{2n-m+i} \otimes v) = -w_{2n-m+i+1} \otimes v$, $i = 1, \ldots, m-1$. Thus, $\mathbb{C}^{2n} \otimes v$ is contained in M, and therefore $M = \mathbb{C}^{2n} \otimes M_\lambda$. $\square$

The direct sum decomposition is a strong property and hard to prove for general $\mathfrak{k}$. For our purposes, it is possible to relax it and consider an increasing filtration, which construction is easier. Now we rephrase the above result for the symmetric case in this milder setting, which will be a part of construction for general $\mathfrak{k}$ further on.

Set $V_1 = M_1$ to be the $U_q(\mathfrak{g})$-module generated by $w_1 \otimes v$ and denote by V_2 the $U_q(\mathfrak{g})$-module generated by $\{w_1 \otimes v, w_{m+1} \otimes v\}$. Thus, $V_1 \subset V_2$. While $w_{m+1} \otimes v$ is not a singular vector, it is so modulo V_1. Identified with its projection to $V_2/V_1 \simeq M_2$, it is a highest weight vector in the quotient V_2/V_1.

Proposition 6.4. *The module V_2 coincides with $\mathbb{C}^{2n} \otimes M_\lambda$, and $V_2/V_1 \simeq M_2$.*

Proof. Using similar reasoning as in the proof of Theorem 6.3, we show that $\mathbb{C}^{2n} \otimes v$ and hence $\mathbb{C}^{2n} \otimes M_\lambda$ lies in V_2. The only difference is that the inclusion $w_{m+1} \otimes v \subset V_2$ holds now by the very construction, and this is a simplification. $\square$

7 Module $\mathbb{C}^{2n} \otimes M_\lambda$: general case

For general Levi subalgebra $\mathfrak{l}$, the vector space $\mathbb{C}^{2n}$ is decomposed in the direct sum of irreducible $\mathfrak{l}$-submodules

$$\mathbb{C}^{2n} = W_1 \oplus \ldots \oplus W_{\ell+1} \oplus W_{\ell+2} \oplus W_{\ell+3} \oplus \ldots \oplus W_{2\ell+3}$$

of dimensions $n_1, \ldots, n_\ell, p, 2m, p, n_\ell, \ldots, n_1$. The highest weights ν_i of the blocks are

$$\varepsilon_1,\ \varepsilon_{n_1+1}, \ldots, \varepsilon_{n_1+\ldots+n_\ell+1},\ \varepsilon_{n_1+\ldots+n_\ell+m+1},\ -\varepsilon_{n_1+\ldots+n_\ell+m},\ -\varepsilon_{n_1+\ldots+n_\ell}, \ldots, -\varepsilon_{n_1}. \tag{7.14}$$

The highest weight vectors w_{ν_i}, $i = 1, \ldots, 2\ell + 3$, belong to the standard basis in $\mathbb{C}^{2n}$.

For generic weight $\lambda \in \mathfrak{c}^*_{\mathfrak{l},reg}$ this decomposition induces decomposition

$$\mathbb{C}^{2n} \otimes \hat{M}_\lambda = \oplus_{i=1}^{2\ell+3} \hat{M}_i \tag{7.15}$$

of $U_q(\mathfrak{g})$-submodules. The blocks are generated by singular vectors of weights $\nu_i + \lambda$, where ν_i are given by (7.14).

Under the transition to the subalgebra $\mathfrak{k} \subset \mathfrak{l}$, the $\mathfrak{l}$-modules $W_{\ell+1}$ and $W_{\ell+3}$ are merged into a single irreducible $\mathfrak{k}$-module. The other $\mathfrak{l}$-modules W_i remain so with respect to $\mathfrak{k}$.

Denote by M_i the images of $\hat{M}_i$ under the projection $\mathbb{C}^{2n} \otimes \hat{M}_\lambda \to \mathbb{C}^{2n} \otimes M_\lambda$. One should expect that $\hat{M}_{\ell+3}$ is annihilated by the projection, and decomposition (7.15) turns into

$$\mathbb{C}^{2n} \otimes M_\lambda = M_1 \oplus \ldots \oplus M_{\ell+1} \oplus M_{\ell+2} \oplus M_{\ell+4} \oplus \ldots \oplus M_{2\ell+3}.$$

However, this is not an easy thing to prove in the general case. On the other hand, it is sufficient for our purposes to replace the direct sum with a suitable filtration, which is much easier.

Denote by V_j the submodule in $\mathbb{C}^{2n} \otimes M_\lambda$ generated by $\{w_{\nu_i} \otimes v\}_{i=1,\ldots,j}$ assuming $j = 1, \ldots, 2\ell + 3$. We have the obvious inclusion $V_{i-1} \subset V_i$. It is convenient to set $V_0 = \{0\}$.

Proposition 7.1. *The submodules* $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_{2\ell+3}$ *form an ascending filtration of* $\mathbb{C}^{2n} \otimes M_\lambda$. *For each* $k = 1, \ldots 2\ell + 3$, *the graded component* V_k/V_{k-1} *is either* $\{0\}$ *or it is generated by (the image of)* $w_{\nu_k} \otimes v$, *which is the highest weight vector in* V_k/V_{k-1}. *In particular,* $V_{\ell+2} = V_{\ell+3}$.

Proof. We will show that $\oplus_{i=1}^{k} W_i \otimes v \subset V_k$ meaning $\mathbb{C}^{2n} \otimes v \subset V_{2\ell+3}$ for $k = 2\ell + 3$. This will imply that $e_\alpha(w_{\nu_k} \otimes v) = 0 \mod V_{k-1}$, i.e. $w_{\nu_j} \otimes v$ is a singular vector in V_{k-1}/V_k if not zero. Since V_{k-1}/V_k is generated by $w_{\nu_k} \otimes v$, it is the highest weight vector. This will imply $\mathbb{C}^{2n} \otimes v \subset V_{2\ell+3}$ and $V_{2\ell+3} = \mathbb{C}^{2n} \otimes v \subset M_\lambda$.

Thus, our next goal is to prove that $W_j \otimes v \subset V_j$. This is true for $j = 0$ if we set $W_0 = \{0\}$. Suppose we have done this for some $j \geqslant 0$. By construction, $w_{\nu_{j+1}} \otimes v \in V_{j+1}$. Consecutively applying the negative Chevalley generators from the appropriate block of $U_q(\mathfrak{l})$ we conclude that $W_{\nu_{j+1}} \otimes v \subset V_{j+1}$. Induction on j proves $W_j \otimes v \subset V_j$ for all j.

Finally, the equality $V_{l+2} = V_{l+3}$ follows from $W_{l+3} \otimes v \subset V_{l+2}$, and this boils down to the symmetric case studied in Proposition 6.4: it is sufficient to apply the negative Chevalley generators corresponding to the centered block $\mathfrak{sp}(2 + 2p) \subset \mathfrak{sp}(2n)$ to $w_{n-p} \otimes v \in V_{l+2}$ in order to get $w_{n+p+1} \otimes v$. The latter generates V_{l+3} modulo V_{l+2} This completes the proof. $\square$

Remark that for our purposes we need not prove that $V_i \neq V_{i+1}$ for i other than $\ell + 2$.

8 The matrix of quantum coordinate functions

The classical description of semi-simple conjugacy classes is formulated in terms of operations (multiplication, transposition, trace functional) with the matrix A of coordinate functions on $\mathrm{End}(\mathbb{C}^{2n})$. The matrix A is G-invariant, and its entries generate the polynomial algebra of the class. A similar description of the quantum conjugacy classes involves a matrix A with non-commutative entries or its image $\mathcal{Q} \in \mathrm{End}(\mathbb{C}^{2n}) \otimes U_q(\mathfrak{g})$, which should be regarded as "restriction" of A to the quantum group G_q. In this section we study algebraic properties of $\mathcal{Q}$.

The operator $\mathcal{Q}$ is defined through the universal R-matrix of $U_\hbar(\mathfrak{g})$, which is an invertible element of (completed) tensor square of $U_\hbar(\mathfrak{g})$, conventionally denoted by $\mathcal{R}$:

$$\mathcal{Q} = (\pi \otimes \mathrm{id})(\mathcal{R}_{12}\mathcal{R}) \in \mathrm{End}(\mathbb{C}^{2n}) \otimes U_q(\mathfrak{g}).$$

Regarded as an operator on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$, it satisfies a polynomial equation with the roots

$$q^{2(\lambda+\rho,\nu)-2(\rho,\varepsilon_1)} = q^{2(\lambda,\nu)+2(\rho,\nu-\varepsilon_1)},$$

where ν are the highest weights of the irreducible $\mathfrak{l}$-submodules in $\mathbb{C}^{2n}$.

Assuming $\lambda \in \mathfrak{C}^*_{\mathfrak{l},reg}$, put $\Lambda_i = (\lambda, \varepsilon_{n_1+\ldots+n_{i-1}+1}) = (\lambda, \varepsilon_{n_1+\ldots+n_i})$ for $i = 1, \ldots \ell+2$ (recall that $n_{\ell+1} = m$ and $n_{\ell+2} = p$). The weight λ depends on the parameters (Λ_i) with $\Lambda_{\ell+2} = 0$. Define the vector $\boldsymbol{\mu}$ by

$$\mu_i = q^{2\Lambda_i - 2(n_1+\ldots+n_{i-1})}, \quad i = 1, \ldots, \ell+2. \tag{8.16}$$

The eigenvalues of $\mathcal{Q}$ on $\mathrm{End}(\mathbb{C}^{2n} \otimes \hat{M}_\lambda)$ are expressed through $\boldsymbol{\mu}$ by

$$\mu_i, \quad \mu_i^{-1} q^{-4n+2(n_i-1)}, \quad i = 1, \ldots \ell+1, \quad \mu_{\ell+2}. \tag{8.17}$$

We call a weight $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg}$ *admissible* if the vector $\boldsymbol{\mu}$ belongs to $\hat{\mathcal{M}}_K$ modulo $\hbar$. Recall that $\hat{\mathcal{M}}_K$ parameterizes the moduli space $\mathcal{M}_K$ of classes with given K. By definition, this property is determined by the meromorphic part $\hbar^{-1}\mathfrak{c}^*_{\mathfrak{k},reg} \subset \mathfrak{C}^*_{\mathfrak{k},reg}$. Clearly admissible weights are dense in $\mathfrak{C}^*_{\mathfrak{k},reg}$.

Proposition 8.1. *For admissible $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg}$ the operator $\mathcal{Q}$ satisfies a polynomial equation of degree $2\ell+2$ on $\mathbb{C}^{2n} \otimes M_\lambda$ with the roots*

$$\mu_i, \quad \mu_i^{-1} q^{-4n+2(n_i-1)}, \quad i = 1, \ldots \ell, \quad \mu_{\ell+1}, \quad \mu_{\ell+2}. \tag{8.18}$$

Proof. The proof is based on the following fact: a linear operator in a complex vector space is semi-simple if and only if it satisfies a polynomial equation with simple roots.

It is known that $\mathcal{Q}$ satisfies on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$ a polynomial equation of degree $2\ell+3$ with $2\ell+3$ roots (8.17), see [1]. Its eigenvalues are pairwise distinct in the classical level, apart from $\lim_{q\to 1} \mu_{\ell+1} = \lim_{q\to 1} \mu_{\ell+1}^{-1} q^{-4n+2(m-1)}$, which are equal to -1. However, for $q \neq 1$ this coincidence is no longer the case, and the eigenvalues (8.16) become pairwise distinct for q in a neighborhood of 1: "quantization eliminates degeneration". This implies that $\mathcal{Q}$ is semi-simple on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$ for all q close to 1 and hence for generic q. Therefore it is semi-simple on the quotient $\mathbb{C}^{2n} \otimes M_\lambda$, where $\mu_{\ell+1}^{-1} q^{-4n+2(m-1)}$ is no longer its eigenvalue, by Proposition 7.1. This proves the proposition for generic q and therefore for all q. □

The matrix $\mathcal{Q}$ produces the center of $U_\hbar(\mathfrak{g})$ via the q-trace construction. Recall that for any invariant matrix $X \in \mathrm{End}(\mathbb{C}^{2n}) \otimes A$ with the entries in a $U_\hbar(\mathfrak{g})$-module A one can define an invariant element

$$\mathrm{Tr}_q(X) := \mathrm{Tr}\big(q^{2h_\rho} X\big) \in A. \tag{8.19}$$

This construction, when applied $X = \mathcal{Q}^k$, $k \in \mathbb{Z}_+$, gives a series of elements of $U_\hbar(\mathfrak{g})$ that are invariant under adjoint representation and hence central. First $2n$ traces generate the center, which is isomorphic to a polynomial algebra in n elements. We will use the shortcut notation τ_k for $\mathrm{Tr}_q(\mathcal{Q}^k)$, $k \in \mathbb{Z}_+$.

A module M of highest weight λ defines a one dimensional representation χ_λ of the center of $U_\hbar(\mathfrak{g})$, which assigns a scalar to each τ_ℓ:

$$\chi^\lambda(\tau_k) = \sum_\nu q^{2k(\lambda+\rho,\nu)-2k(\rho,\nu_1)} \prod_{\alpha\in \mathrm{R}_+} \frac{q^{(\lambda+\nu+\rho,\alpha)} - q^{-(\lambda+\nu+\rho,\alpha)}}{q^{(\lambda+\rho,\alpha)} - q^{-(\lambda+\rho,\alpha)}}, \tag{8.20}$$

where the summation is taken over weights $\nu \in \{\pm\varepsilon_j\}_{j=1}^n$ of the module $\mathbb{C}^{2n}$. Restriction of λ to $\mathfrak{C}^*_{\mathfrak{k},reg}$ makes the right hand side a function of $\boldsymbol{\mu}$ defined in (8.16). We denote this function by $\vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu})$, where $\boldsymbol{n} = (n_1, \ldots, n_\ell, m, p)$ is the integer valued vector of multiplicities. In the limit $\hbar \to 0$ the function $\vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu})$ goes over into the right hand side of (2.7), where $\mu_i = \lim_{\hbar\to 0} q^{2(\lambda,\nu_i)}$ with ν_i being the highest weights of the $\mathfrak{k}$-submodules in the upper part $\mathbb{C}^{n+p} \subset \mathbb{C}^{2n}$.

9 Quantum conjugacy classes of non-Levi type

By quantization of a commutative $\mathbb{C}$-algebra A we understand a $\mathbb{C}[[\hbar]]$-algebra $A_\hbar$, which is free as a $\mathbb{C}[[\hbar]]$-module and $A_\hbar/\hbar A_\hbar \simeq A$ as a $\mathbb{C}$-algebra. Note that we do not require $\hbar$-adic completion because algebras of our interest are direct sums of $U_\hbar(\mathfrak{g})$-submodules, which we prefer to preserve under quantization. Below we describe the quantization of $\mathbb{C}[G]$ along the Poisson bracket (2.4).

Recall from [9] that the image of the universal R-matrix of the quantum group $U_\hbar(\mathfrak{g})$ in the defining representation is equal, up to a scalar factor, to

$$R = \sum_{i,j=1}^{2n} q^{\delta_{ij}-\delta_{ij'}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1\\ i>j}}^{2n} (e_{ij} \otimes e_{ji} - q^{\rho_i-\rho_j} \epsilon_i\epsilon_j e_{ij} \otimes e_{i'j'}),$$

where $\rho_i = -\rho_{i'} = (\rho, \varepsilon_i) = n + 1 - i$ for $i = 1, \ldots n$.

Denote by S the $U_\hbar(\mathfrak{g})$-invariant quantum permutation $PR \in \mathrm{End}(\mathbb{C}^{2n}) \otimes \mathrm{End}(\mathbb{C}^{2n})$, where P is the ordinary flip of $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$. This matrix has three invariant projectors to its eigenspaces, among which there is a one-dimensional projector κ to the trivial $U_\hbar(\mathfrak{g})$-submodule, proportional to $\sum_{i,j=1}^{2n} q^{\rho_i-\rho_j} \epsilon_i\epsilon_j e_{i'j} \otimes e_{ij'}$.

Denote by $\mathbb{C}_\hbar[G]$ the associative algebra generated by the entries of a matrix $A = ||A_{ij}||_{i,j=1}^{2n} \in \mathrm{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}_\hbar[G]$ modulo the relations

$$S_{12}A_2S_{12}A_2 = A_2S_{12}A_2S_{12}, \quad A_2S_{12}A_2\kappa = -q^{-2n-1}\kappa = \kappa A_2S_{12}A_2. \tag{9.21}$$

These relations are understood in $\mathrm{End}(\mathbb{C}^{2n})\otimes\mathrm{End}(\mathbb{C}^{2n})\otimes\mathbb{C}_\hbar[G]$, and the indices distinguish the two copies of $\mathrm{End}(\mathbb{C}^{2n})$, in the usual way. Note that the factor $-q^{-2n-1}$ before κ is missing in [1]. Due to dilation symmetry of the left equation, this factor can be taken arbitrary for $\mathbb{C}_\hbar[G]$, however it is fixed by the correspondence $A_{ij}\mapsto\mathcal{Q}_{ij}$, below.

The algebra $\mathbb{C}_\hbar[G]$ is a quantization of $\mathbb{C}[G]$, [10], which is different from the RTT-quantization and not a Hopf algebra. It carries a $U_\hbar(\mathfrak{g})$-action being a deformation of the conjugation action of $U(\mathfrak{g})$ on $\mathbb{C}[G]$. This action can be characterized by the requirements that A commutes with $(\pi\otimes\mathrm{id})\circ\Delta U_\hbar(\mathfrak{g})$ in the tensor product $\mathrm{End}(\mathbb{C}^{2n})\otimes\mathbb{C}_\hbar[G]\rtimes U_\hbar(\mathfrak{g})$, where $\pi\colon U_\hbar(\mathfrak{g})\to\mathrm{End}(\mathbb{C}^{2n})$ is the defining representation. It is important that $\mathbb{C}_\hbar[G]$ can be realized as a $U_\hbar(\mathfrak{g})$-invariant subalgebra in $U_\hbar(\mathfrak{g})$ (and even in $U_q(\mathfrak{g})$), where the latter is regarded as the adjoint module. The embedding is implemented by the assignment

$$A\mapsto(\pi\otimes\mathrm{id})(\mathcal{R}_{21}\mathcal{R})=\mathcal{Q}\in\mathrm{End}(\mathbb{C}^{2n})\otimes U_\hbar(\mathfrak{g}).$$

The following properties of $\mathbb{C}_\hbar[G]$ will be of importance. Denote by $I_\hbar(G)\subset\mathbb{C}_\hbar[G]$ the subalgebra of $U_\hbar(\mathfrak{g})$-invariants. It coincides with the center of $\mathbb{C}_\hbar[G]$ and generated by the q-traces $\mathrm{Tr}_q(A^l)$, $l=1,\ldots 2n$, which go over to τ_l under the embedding to $U_\hbar(\mathfrak{g})$. Not all traces are independent, as $I_\hbar(G)$ is a polynomial algebra in n variables, but that is immaterial for our presentation.

The algebra $\mathbb{C}_\hbar[G]$ is freely generated over $I_\hbar(G)$ by a $U_\hbar(\mathfrak{g})$-module whose isotypic components are finite dimensional, [10]. This is a quantum version of the Kostant-Richardson theorem, see [10].

Our approach to quantization is based on the following strategy that is similar to [1]. Suppose we have constructed two $U_\hbar(\mathfrak{g})$-algebras $S_\hbar$ and $T_\hbar$ along with an equivariant homomorphism $\varphi\colon S_\hbar\to T_\hbar$ possessing the following properties. 1) all isotypic components in $S_\hbar$ are $\mathbb{C}[[\hbar]]$-finite, 2) $T_\hbar$ has no $\hbar$-torsion (multiplication by $\hbar$ has zero kernel), 3) there is an ideal $J_\hbar\subset\ker\varphi$ such that the image $J_0^\flat$ of $J_0=J_\hbar/\hbar J_\hbar$ in $S_0=S_\hbar/\hbar S_\hbar$ is a maximal $\mathfrak{g}$ invariant ideal in S_0, 4) S_0 is commutative. Then a) the kernel of φ coincides with $J_\hbar$, b) $\varphi(S_\hbar)$ is a quantization of the algebra $S_0/J_0^\flat$. Remark that if S_0 is the coordinate ring of a $\mathfrak{g}$-variety, maximal proper $\mathfrak{g}$-invariant ideals are exactly the radical ideals of orbits in it.

In our situation, $T_\hbar=\mathrm{End}(M_\lambda)$ is the algebra of linear endomorphisms of the $U_q(\mathfrak{g})$ module M_λ and $S_\hbar$ is the quotient of $\mathbb{C}_\hbar[G]$ by the central ideal annihilated in M_λ. Note that we cannot take simply $\mathbb{C}_\hbar[G]$ for $S_\hbar$, because isotopic components of $\mathbb{C}_\hbar[G]$ are not finite due to large center $I_\hbar(G)$. However, the quantum Richardson theorem allows us to use the aforementioned quotient. Explicitly the central ideal is generated by the relations

(8.20). The homomorphism φ is the composition of the embedding $\mathbb{C}_\hbar[G] \to U_q(\mathfrak{g})$ and the representation homomorphism $U_q(\mathfrak{g}) \to \mathrm{End}(M_\lambda)$. The defining ideal of a class in G is a maximal G-invariant proper ideal in $\mathbb{C}[G]$, therefore its projection to S_0 will be maximal G-invariant proper ideal too. Thus, to construct quantization, it is sufficient to check that the φ annihilates an ideal that turns into the defining ideal of the class in the classical limit. As the kernel of central character is factored out in $S_\hbar$, this reduces to checking the polynomial equation on $\mathcal{Q}$. That is already done in Proposition 8.1.

There is a subtle issue about the action of $U_\hbar(\mathfrak{g})$ as mentioned in Section 5. The quantum group $U_\hbar(\mathfrak{g})$ cannot act on the $U_q(\mathfrak{g})$-module M_λ because the operators from $\mathfrak{h}$ are irregular in $\hbar$ for admissible $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg}$. We have to extend M_λ by the Laurent series, to incorporate the action of $U_\hbar(\mathfrak{g})$. The subalgebra of endomorphisms of the regular part of this extended module is $U_\hbar(\mathfrak{g})$-invariant, and it is that subalgebra where we represent $\mathbb{C}_\hbar[G]$.

Theorem 9.1. *Suppose that $\lambda = \mathfrak{C}^*_{\mathfrak{k},reg}$ is admissible, and let $\boldsymbol{\mu}$ be as explained in (8.16). The quotient of $\mathbb{C}_\hbar[G]$ by the ideal of relations*

$$\prod_{i=1}^{\ell}(\mathcal{Q}-\mu_i) \times (\mathcal{Q}-\mu_{\ell+1})(\mathcal{Q}-\mu_{\ell+2}) \times \prod_{i=1}^{\ell}(\mathcal{Q}-\mu_i^{-1}q^{-4n+2(n_i-1)}) = 0, \tag{9.22}$$

$$\mathrm{Tr}_q(\mathcal{Q}^k) = \vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu}) \tag{9.23}$$

is an equivariant quantization of the class $\boldsymbol{\mu}^0 = \hat{\mathcal{M}}_K$, where $\boldsymbol{\mu}^0 = \lim_{\hbar\to 0}\boldsymbol{\mu}$. It is the image of $\mathbb{C}_\hbar[G]$ in the algebra of endomorphisms of the $U_q(\mathfrak{g})$-module M_λ.

Theorem 9.1 describes quantization in terms of the matrix $\mathcal{Q}$, which is the image of the matrix A. To obtain the description in terms of A, one should replace $\mathcal{Q}$ by A and add the relations (9.21).

The constructed quantization is equivariant with respect to the standard or Drinfeld-Jimbo quantum group $U_\hbar(\mathfrak{g})$. Other quantum groups are obtained from standard $U_\hbar(\mathfrak{g})$ by twist, [11]. Formulas (9.22) and (9.23) are valid for any quantum group $U_\hbar(\mathfrak{g})$ upon the following modifications. The matrix $\mathcal{Q}$ is expressed through the universal R-matrix as usual. The q-trace should be redefined as $\mathrm{Tr}_q(X) = q^{1+2\rho_1}\mathrm{Tr}\Big(\pi\big(\gamma^{-1}(\mathcal{R}_1)\mathcal{R}_2\big)X\Big) = q^{1+2n}\mathrm{Tr}\Big(\pi\big(\gamma^{-1}(\mathcal{R}_1)\mathcal{R}_2\big)X\Big)$. This can be verified along the lines of [12].

10 Appendix

Below we present the diagrams D_1, D_2, D_3 to illustrate the formalism of Section 6 for the case $m = 1$, $p = 2$, $n = 3$.

$$D_1$$

$$\begin{array}{ccccccccccc|c}
 & f_{\alpha_1} & & f_{\alpha_2} & & f_{\beta} & & f_{\alpha_2} & & f_{\alpha_1} & & \\
\hline
w_1\otimes\phi_1^5 v & \leftarrow & w_1\otimes\phi_1^4 v & \leftarrow & w_1\otimes\phi_1^3 v & \leftarrow & w_1\otimes\phi_1^2 v & \leftarrow & w_1\otimes\phi_1^1 v & \leftarrow & w_1\otimes v & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & f_{\alpha_1} \\
 & & w_2\otimes\phi_1^4 v & \leftarrow & w_2\otimes\phi_1^3 v & \leftarrow & w_2\otimes\phi_1^2 v & \leftarrow & w_2\otimes\phi_1^1 v & \leftarrow & w_2\otimes v & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & f_{\alpha_2} \\
 & & & & w_3\otimes\phi_1^3 v & \leftarrow & w_3\otimes\phi_1^2 v & \leftarrow & w_3\otimes\phi_1^1 v & \leftarrow & w_3\otimes v & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & f_{\beta} \\
 & & & & & & w_4\otimes\phi_1^2 v & \leftarrow & w_4\otimes\phi_1^1 v & \leftarrow & w_4\otimes v & \\
 & & & & & & & & \downarrow & & \downarrow & f_{\alpha_2} \\
 & & & & & & & & w_5\otimes\phi_1^1 & \leftarrow & w_5\otimes v & \\
 & & & & & & & & & & \downarrow & f_{\alpha_1} \\
 & & & & & & & & & & w_6\otimes v &
\end{array}$$

$$D_2$$

$$\begin{array}{ccccccccccc|c}
 & f_{\alpha_2} & & f_{\beta} & & f_{\alpha_1} & & f_{\alpha_2} & & f_{\alpha_1} & & \\
\hline
w_1\otimes\phi_2^5 v & \leftarrow & w_1\otimes\phi_2^4 v & \leftarrow & w_1\otimes\phi_2^3 v & \leftarrow & w_1\otimes\phi_2^2 v & \leftarrow & w_1\otimes\phi_2^1 v & \leftarrow & w_1\otimes v & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & f_{\alpha_1} \\
 & & w_2\otimes\phi_2^4 v & \leftarrow & w_2\otimes\phi_2^3 v & \leftarrow & w_2\otimes\phi_2^2 v & \leftarrow & w_2\otimes\phi_2^1 v & \leftarrow & w_2\otimes v & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & f_{\alpha_2} \\
 & & & & w_3\otimes\phi_2^3 v & \leftarrow & w_3\otimes\phi_2^2 v & \leftarrow & w_3\otimes\phi_2^1 v & \leftarrow & w_3\otimes v &
\end{array}$$

$$D_3$$

$$\begin{array}{ccccccccccc|c}
 & f_{\beta} & & f_{\alpha_2} & & f_{\alpha_1} & & f_{\alpha_2} & & f_{\alpha_1} & & \\
\hline
w_1\otimes\phi_3^5 v & \leftarrow & w_1\otimes\phi_3^4 v & \leftarrow & w_1\otimes\phi_3^3 v & \leftarrow & w_1\otimes\phi_3^2 v & \leftarrow & w_1\otimes\phi_3^1 v & \leftarrow & w_1\otimes v & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & f_{\alpha_1} \\
 & & w_2\otimes\phi_3^4 v & \leftarrow & w_2\otimes\phi_3^3 v & \leftarrow & w_2\otimes\phi_3^2 v & \leftarrow & w_2\otimes\phi_3^1 v & \leftarrow & w_2\otimes v & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & f_{\alpha_2} \\
 & & & & w_3\otimes\phi_3^3 v & \leftarrow & w_3\otimes\phi_3^2 v & \leftarrow & w_3\otimes\phi_3^1 v & \leftarrow & w_3\otimes v &
\end{array}$$

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