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THE FREIHEITSSATZ FOR GENERIC POISSON ALGEBRAS

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Dedicated to the memory of Myung Hyo Chul (1937–2010)

1. Introduction

One of the classical achievements of the combinatorial group theory is the decidability of the word problem in a finitely generated group with one defining relation [1]. This result was a corollary of a fundamental statement called Freiheitssatz: Every equation over a free group is solvable in some extension. For solvable and nilpotent groups, this complex of problems was studied in [2].

In the context of Lie algebras, similar statements were proved [3]. For associative algebras, the problem turns to be surprisingly difficult: Over a field of characteristic zero, the Freiheitssatz was proved in [5], but the question about decidability of the word problem for an associative algebra with one defining relation remains open. One of the reasons for this is that the variety of associative algebras, contrary to those of groups or Lie algebras, is not a Schreier one, that is, a subalgebra of a free associative algebra is not necessary free. And as a matter of fact, the free algebras of Schreier varieties are usually more easy to deal with.

In [6], the Freiheitssatz was proved for right-symmetric (pre-Lie) algebras, and in [7]—for Poisson algebras. In this paper, we consider a modified approach to the proof in [7] which allows to prove the Freiheitssatz also for generic Poisson algebras.

There is plenty of varieties for which the Freiheitssatz is not true, e.g., so is the variety of Poisson algebras over a field of positive characteristic. One may find more examples of this kind in [8], e.g., for Leibniz algebras the Freiheitssatz does not hold (as well as for every variety of di-algebras in the sense of [9]).

Throughout the paper $\Bbbk$ denotes a field of characteristic zero.

A generic Poisson algebra (GP-algebra) is a linear space with two operations and one constant:

(1) associative and commutative product $x \cdot y = xy$;
(2) anti-commutative bracket $\{x, y\}$;
(3) multiplicative identity 1, $x \cdot 1 = 1 \cdot x = x$,

satisfying the Leibniz identity

$$\{x, yz\} = \{x, y\}z + \{x, z\}y.$$

These algebras were introduced in [10] in the study of speciality and deformations of Malcev–Poisson algebras.

Let $AC(X)$ be the free anti-commutative algebra (AC-algebra) generated by a set X with respect to operation denoted by $\{\cdot, \cdot\}$, and let $GP(X)$ be the free GP-algebra with a set of generators X. As a linear space, $GP(X)$ is isomorphic to the symmetric algebra $S(AC(X))$ [10].

2. Conditionally closed algebras and the Freiheitssatz

Suppose $\mathfrak{M}$ is a variety of algebras over a field $\Bbbk$. Denote by $\mathfrak{M}(X)$ the free algebra in $\mathfrak{M}$ generated by a set X. For $A, B \in \mathfrak{M}$, the notation $A *_{\mathfrak{M}} B$ stands for the free product of A and B in $\mathfrak{M}$.

If $A \in \mathfrak{M}$ then every $\Psi \in A *_{\mathfrak{M}} \mathfrak{M}(x)$ may be interpreted as an A-valued function on A. Moreover, for every extension $\bar{A}$ of A, $\bar{A} \in \mathfrak{M}$, $\Psi(x)$ is an $\bar{A}$-valued function on $\bar{A}$. An equation of the form $\Psi(x) = 0$ is *solvable over* A if there exists an extension $\bar{A} \in \mathfrak{M}$ of A such that the equation has a solution in $\bar{A}$. If such a solution can be found in A itself then $\Psi(x) = 0$ is said to be *solvable in* A.

Recall the common definition (see, e.g., [11, 12]): An algebra A is *(existentially) algebraically closed* if every system of equations which is solvable over A is solvable in A. Let us restrict this definition to a particular case of one equation: We will say $A \in \mathfrak{M}$ to be existentially closed in $\mathfrak{M}$ if every equation $\Psi(x) \in A *_{\mathfrak{M}} \mathfrak{M}(x)$ which is solvable in an appropriate extension $\bar{A}$ of A, $\bar{A} \in \mathfrak{M}$, has a solution in A. This definition is important for model theory, and it can be an efficient tool for studying algebras provided the principal question on the solvability of a particular equation is solved.

A stronger property (see [13]) can be stated as follows: An algebra $A \in \mathfrak{M}$ is called *algebraically closed in* $\mathfrak{M}$ if for every $\Psi \in A *_{\mathfrak{M}} \mathfrak{M}(x)$, $\Psi \notin A$, the equation $\Psi(x) = 0$ is solvable in A. We are going to propose an intermediate definition which is sufficient for our purpose.

Definition 1. *An algebra $A \in \mathfrak{M}$ is called* conditionally closed in $\mathfrak{M}$ *if for every $\Psi(x) \in A *_{\mathfrak{M}} \mathfrak{M}(x)$ which is not a constant function on A the equation $\Psi(x) = 0$ is solvable in A.*

Every algebraically closed in $\mathfrak{M}$ algebra is conditionally closed in $\mathfrak{M}$. However, there is plenty of conditionally closed systems that are not algebraically closed in $\mathfrak{M}$. For example, an algebraically closed field is conditionally closed but not algebraically closed in the variety of all associative algebras. Similarly, such a field may be considered as a Poisson algebra with respect to trivial bracket, and the Poisson algebra obtained is conditionally closed but not algebraically closed in the variety of all Poisson algebras.

It is also interesting to compare conditionally closed and existentially closed algebras. Neither of these notions is a formal generalization of another. For example, let $\mathfrak{M} = As$, the variety of associative algebras, and let $A \in As$ be the algebraic closure of the field $\Bbbk(t)$. As an algebraically closed field, A is conditionally closed in As, but the equation $[t, x] = 1$ has no solution in A although it is solvable in an appropriate extension (e.g., in the Makar-Limanov's skew field [5]). On the other hand, for the same variety As, the existential algebraic closure (see, e.g., [14, Ch. III]) of quaternions $\mathbb{H}$ is not conditionally closed: Equation $ix - xi = 1$ has no solution in any extension of $\mathbb{H}$ in As.

Suppose $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two varieties of algebras over a field $\Bbbk$, and let $\omega : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a functor which acts as follows: Given $A \in \mathfrak{M}_1$, $A^{(\omega)} \in \mathfrak{M}_2$ is the same linear space equipped with new operations expressed in terms of initial operations. For example, one may consider the classical functor from the variety of associative algebras into the variety of Lie algebras defined by $[x, y] = xy - yx$.

Another important example comes from the following settings. Let $\mathfrak{M}_1 = Dif_{2n}$ be the variety of commutative associative algebras with $2n$ pairwise commuting derivations ∂_i, ∂_i', $i = 1, \dots, n$. Then, given $A \in Dif_{2n}$, the same space equipped with new binary operation

$$\{a, b\} = \sum_{i=1}^{n} \partial_i(a)\partial_i'(b) - \partial_i(b)\partial_i'(a), \quad a, b \in A, \tag{1}$$

is known to be a Poisson algebra denoted by $A^{(\partial)}$. If we allow the derivations ∂_i, ∂_i' to be non-commuting then (1) defines on a commutative algebra A a structure of a GP-algebra.

In general, ω may be a functor induced by a morphism of the governing operads. Functors of this kind were closely studied in [8].

Proposition 1. *If an algebra $A \in \mathfrak{M}_1$ is conditionally closed in $\mathfrak{M}_1$ then $A^{(\omega)}$ is conditionally closed in $\mathfrak{M}_2$.*

Note that for algebraically closed algebras this statement does not hold.

Proof. Since ω is a functor, the universal property of the free product implies the existence of a homomorphism $\varphi : A^{(\omega)} *_{\mathfrak{M}_2} \mathfrak{M}_2(x) \to (A *_{\mathfrak{M}_1} \mathfrak{M}_1(x))^{(\omega)}$ such that $f(a) = \varphi(f)(a)$ for all $f = f(x) \in A^{(\omega)} *_{\mathfrak{M}_2} \mathfrak{M}_2(x)$, $a \in A^{(\omega)}$.

Therefore, f is not a constant function on $A^{(\omega)}$ if and only if $\varphi(f)$ is not a constant function on A. If A is conditionally closed then there exists $a \in A$ such that $\varphi(f)(a) = 0$ and thus $f(a) = 0$. $\square$

In some cases, the converse statement is true: If $A^{(\omega)}$ is conditionally closed in $\mathfrak{M}_2$ then so is A in $\mathfrak{M}_1$.

Remark 1. *Let $\omega : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a functor with the following property: If $A \in \mathfrak{M}_1$ and $A^{(\omega)}$ is a subalgebra of $C \in \mathfrak{M}_2$ then there exists $B \in \mathfrak{M}_1$ such that $C = B^{(\omega)}$ and A is a subalgebra of B. Then A is conditionally closed in $\mathfrak{M}_1$ provided that $A^{(\omega)}$ is conditionally closed in $\mathfrak{M}_2$.*

Indeed, $A^{(\omega)} *_{\mathfrak{M}_2} \mathfrak{M}_2(x)$ is a $\mathfrak{M}_2$-algebra which contains $A^{(\omega)}$. Hence, $A^{(\omega)} *_{\mathfrak{M}_2} \mathfrak{M}_2(x) = B^{(\omega)}$ for a $\mathfrak{M}_1$-algebra B. Therefore, there exists a homomorphism of $\mathfrak{M}_1$-algebras $\psi : A *_{\mathfrak{M}_1} \mathfrak{M}_1(x) \to B$ such that $\psi(a) = a$ for $a \in A$, $\psi(x) = x$. Hence, $\varphi(\psi(f)) = f$ for all $f \in A *_{\mathfrak{M}_1} \mathfrak{M}_1(x)$, where φ is the homomorphism in the proof of Proposition 1.

Suppose $A^{(\omega)}$ is conditionally closed in $\mathfrak{M}_2$. If $f \in A *_{\mathfrak{M}_1} \mathfrak{M}_1(x)$ is not a constant function on A then so is $g = \psi(f)$ since $\varphi(g(a)) = \varphi(g)(a) = f(a)$ for all $a \in A$. Hence, there exists a solution of the equation $g(x) = 0$ in $A^{(\omega)}$ which is obviously a solution of $f(x) = 0$ in A.

The *Freiheitssatz problem* for a variety $\mathfrak{M}$ is to determine whether every nontrivial equation over the free algebra $\mathfrak{M}(X)$, $X = \{x_1, x_2, \dots\}$, is solvable over $\mathfrak{M}(X)$.

It is obviously equivalent to the following question about free algebras: Is the intersection of the ideal (f) generated by an element $f \in \mathfrak{M}(X \cup \{x\})$ and the subalgebra $\mathfrak{M}(X) \subset \mathfrak{M}(X \cup \{x\})$ trivial if $f \notin \mathfrak{M}(X)$ (i.e., depends on x)? If the answer is positive for all such f then we say that the Freiheitssatz holds for $\mathfrak{M}$.

Lemma 1. *Suppose $\mathfrak{M}$ is a variety of algebras with at least one binary operation $\cdot$ in the signature such that $\mathfrak{M}(X) = \mathfrak{M}(x_1, x_2, \dots)$ has no zero divisors with respect to $\cdot$. Then, if for every nonzero polynomial $h = h(x_1, \dots, x_n) \in \mathfrak{M}(X)$ there exists a conditionally closed algebra $A \in \mathfrak{M}$ which does not satisfy the polynomial identity $h(x_1, \dots, x_n) = 0$ then the Freiheitssatz holds for $\mathfrak{M}$.*

Proof. Suppose $X = \{x_1, x_2, \dots\}$, $x \notin X$, and let $f = f(x, x_1, \dots, x_n) \in \mathfrak{M}(X \cup \{x\}) \setminus \mathfrak{M}(X)$. Then $f = f_1 + f_0$, where f_1 belongs to the ideal generated by x, $f_0 \in \mathfrak{M}(X)$.

Assume $g \in (f) \cap \mathfrak{M}(X)$, $g \neq 0$. Then $h = f_1 \cdot g \neq 0$, hence, there exist a conditionally closed $A \in \mathfrak{M}$ such that $h(x, x_1, \dots, x_n)$ is not a polynomial identity on A. Therefore, there exist $a, a_1, \dots, a_n \in A$ such that $f_1(a, a_1, \dots, a_n)g(a_1, \dots, a_n) \neq 0$ in A, so $f_1(a, a_1, \dots, a_n) \neq 0$. On the other hand, $f_1(0, a_1, \dots, a_n) = 0$. Therefore, $\Psi(x) = f_1(x, a_1, \dots, a_n)$ is a non-constant function on A. Since A is conditionally closed, there exists $a \in A$ such that $\Psi(a) = f_1(a, a_1, \dots, a_n) = -f_0(a_1, \dots, a_n)$. Thus, $f(a, a_1, \dots, a_n) = 0$ but $g(a_1, \dots, a_n) \neq 0$ which is impossible if $g \in (f) \triangleleft \mathfrak{M}(X \cup \{x\})$. □

There is a well-known functor ω from the variety As of associative algebras to the variety $Jord$ of Jordan algebras: Every associative algebra A turns into a Jordan algebra denoted by $A^{(+)}$ under new product $x \circ y = xy + yx$. A Jordan algebra is said to be *special* if it can be embedded into an algebra of the form $A^{(+)}$, $A \in As$. The class of all special Jordan algebras is not a variety since a homomorphic image of a special Jordan algebra may not be special. However, the class of all homomorphic images of all special Jordan algebras is a variety denoted by SJ. The free algebra $SJ(X)$ is obviously the subalgebra of $As(X)^{(+)}$ generated by X with respect to Jordan product.

Corollary 1. *The Freiheitssatz holds for the variety generated by special Jordan algebras.*

Proof. Consider the algebraically closed associative noncommutative algebra A from [5]. It is essential that A is a skew field and contains the first Weyl algebra W_1. Thus, A contains free associative algebra in any finite number of generators $x_1, \dots, x_n$ [4]. The special Jordan algebra $A^{(+)}$ is conditionally closed by Proposition 1 and contains free special Jordan algebra $SJ(x_1, \dots, x_n)$. Therefore, the variety SJ satisfies all conditions of Lemma 1. □

Note that for the entire variety $Jord$ the Freiheitssatz does not hold [8].

3. Jacobian polynomials in free anti-commutative algebras

In order to prove the Freiheitssatz for a variety $\mathfrak{M}$ by means of Lemma 1, we have to construct an algebra in $\mathfrak{M}$ which is conditionally closed algebra and does not satisfy a given polynomial identity.

In this section, we discuss technical questions that are used in subsequent sections for the study of polynomial identities on generic Poisson algebras.

3.1. **Preliminaries on $AC(X)$.** Let X be a set of generators, and let X^* stand for the set of all (nonempty) associative words u in the alphabet X. Denote by X^{**} the set of all non-associative words in X. Given a word $u \in X^*$, denote by (u) a non-associative word obtained from u by some bracketing. We will also use $[X^*]$ to denote the set of all associative and commutative words in X. Given $u \in X^*$, $[u]$ stands for the commutative image of u.

Suppose X^{**} is equipped with a linear order $\preceq$. A non-associative word $u \in X^{**}$ is *normal* if either $u = x \in X$ or $u = u_1 u_2$, where u_1 and u_2 are normal and $u_1 \prec u_2$. Obviously, normal words in X^{**} form a linear basis of the free anti-commutative algebra $AC(X)$ generated by X (see [15, 16]).

Let us call the elements of $AC(X)$ *AC-polynomials.* Given $u \in X^{**}$, define $\deg u$ to be the length of u. Thus, we have a well-defined degree function on $AC(X)$.

Choose a generator $x_i \in X = \{x_1, \dots, x_n\}$ and denote by V_i the subspace of $AC(X)$ spanned by all nonassociative words linear in x_i. Fix a linear order $\preceq$ on X^{**} such that any nonassociative word which contains x_i is greater than any word without x_i (there exist many linear orders with this property). With respect to such an order, the unique normal form of a monomial $w \in V_i$ is

$$w = \{u_1, \{u_2, \dots \{u_k, x_i\} \dots\}\}, \tag{2}$$

where u_j, $j = 1, \dots, k$, are normal words in the alphabet $X \setminus \{x_i\}$. The number k is called *x_i-height* ([17]) of w, let us denote it by $ht(w, x_i)$.

Let $V_0 = \bigcap_{i=1}^{n} V_i$ be the space of polylinear AC-polynomials. It is easy to compute x_i-height of a nonassociative word $w \in V_0$ just by the number of brackets in w to the left of x_i, assuming $\{$ is counted as 1 and $\}$ as -1. For example, the x_4-height of $\{\{x_1, \{\{x_2, x_3\}, x_4\}\}, \{x_5, x_6\}\}$ is equal to 3.

Denote by $M(AC(X))$ the algebra of left multiplications on $AC(X)$, i.e., the subalgebra of $\operatorname{End}_{\Bbbk} AC(X)$ generated by

$$\operatorname{ad} g : f \mapsto \{g, f\}, \quad f, g \in AC(X).$$

Since the variety AC of anti-commutative algebras is a Schreier one, $M(AC(X))$ is a free associative algebra (see [18]). Let U stand for the set of all normal words in X^{**}. It is easy to see that $M(AC(X)) \simeq As(U)$ provided that we identify $\operatorname{ad} u$ with $u \in U$.

Denote by U_i the set of normal words in the alphabet $X \setminus \{x_i\}$. Then V_i is a 1-generated free left module over $As(U_i)$: Every word of the form (2) may be presented as

$$w = W(x_i), \quad W = \operatorname{ad} u_1 \operatorname{ad} u_2 \dots \operatorname{ad} u_k,$$

where $u_1, \dots, u_k \in U_i$.

Denote by $*$ the involution of $As(U_i)$ given by $(u_1 \dots u_k)^* = (-1)^k u_k \dots u_1$, $u_j \in U_i$.

Definition 2. *A linear transformation of V_i defined by the rule*

$$F_i : W(x_i) \mapsto -W^*(x_i)$$

*is called an x_i-*flip. *Obviously, $(F_i)^{-1} = F_i$.*

The set of all flips $\{F_1, \dots, F_n\}$ acts on the space V_0 and thus generates a group $\mathcal{F} \subseteq \mathrm{GL}(V_0)$. Given a normal word $u \in V_0$, the orbit $\mathcal{F}u$ consists of AC-monomials (polynomials of the form εv, v is a nonassociative word, $\varepsilon = \pm 1$).

Lemma 2. *Let $w = \{x_1, \{x_2, \dots \{x_{n-1}, x_n\} \dots\}\} \in V_0$. Then*

$$(-1)^\sigma \{x_{1\sigma}, \{x_{2\sigma}, \dots \{x_{(n-1)\sigma}, x_{n\sigma}\} \dots\}\} \in \mathcal{F}w$$

for every $\sigma \in S_n$ (here $(-1)^\sigma$ stands for the parity of a permutation σ.)

Proof. It is straightforward to compute that

$$F_1(F_i w) = -\{x_i, \{x_2, \dots x_{i-1}, \{x_1, \{x_{i+1}, \dots \{x_{n-1}, x_n\} \dots\}\}\}\},$$

$i = 2, \dots, n$. Since transpositions of the form $(1i)$ generate the entire symmetric group S_n, the lemma is proved. $\square$

3.2. **Jacobian AC-polynomials.** In this section, we describe polylinear AC-polynomials that have a specific property if considered as elements of the free GP-algebra.

Suppose $\Psi(x_1,\dots,x_n)$ is an element of the free GP-algebra $GP(X)$, $X=\{x_1,\dots,x_n\}$ which is linear with respect to x_n. We will say that Ψ is a derivation with respect to x_n if

$$\Psi(x_1,\dots,x_{n-1},yz)=y\Psi(x_1,\dots,x_{n-1},z)+z\Psi(x_1,\dots,x_{n-1},y)$$

in the free GP-algebra $GP(x_1,\dots,x_{n-1},y,z)$.

Definition 3. *A polylinear AC-polynomial $\Psi=\Psi(x_1,\dots,x_n)\in V_0$ is said to be a* jacobian *if Ψ is a derivation with respect to each variable x_i, $i=1,\dots,n$. A polylinear element of $GP(X)$ with the same property is called a jacobian GP-polynomial.*

For free Lie algebra considered as a part of the free Poisson algebra, a similar notion was considered in [17]. Obviously, if $n=2$ then $C_2=\{x_1,x_2\}$ is a jacobian AC-polynomial. It was shown in [17] that there are no other jacobian Lie polynomials (up to a multiplicative constant). However, there exists a jacobian AC-polynomial of degree 3:

$$J_3=\{\{x_1,x_2\},x_3\}+\{\{x_2,x_3\},x_1\}+\{\{x_3,x_1\},x_2\}.$$

The main purpose of this section is to show that C_2 and J_3 exhaust all jacobian AC-polynomials.

For a generic Poisson algebra A, $a\in A$, consider the linear map $\operatorname{ad}a : x\mapsto\{a,x\}$, $x\in A$. The set of all such transformations $\{\operatorname{ad}a \mid a\in A\}\subset \operatorname{End}_{\Bbbk}(A)$ generates a Lie subalgebra $L(A)\subset \mathrm{gl}\,(A)=\operatorname{End}_{\Bbbk}(A)^{(-)}$.

Given $L\in L(GP(x_1,\dots,x_{n-1}))$, one may easily note that $L(x_n)\in GP(x_1,\dots,x_n)$ is a derivation with respect to x_n. Indeed, the Leibniz identity implies that $\operatorname{ad}u$, $u\in GP(x_1,\dots,x_{n-1})$, is a derivation with respect to x_n, and the commutator of derivations is a derivation itself.

Lemma 3. *Let $\Psi(x_1,\dots,x_n)\in AC(X)\subset GP(X)$ be a polylinear element such that Ψ is a derivation with respect to x_n. Then there exists $L\in L(AC(X))$ such that $\Psi=L(x_n)$.*

Proof. The algebra of multiplications $M(AC(x_1,\dots,x_{n-1}))\simeq As(U_n)$ contains free Lie subalgebra $\mathcal{L}=Lie(U_n)\subset As(U_n)^{(-)}$ generated by $\operatorname{ad}u$ for all normal words $u\in U_n$.

As $\mathcal{L}$ acts on $V=AC(x_1,\dots,x_{n-1},y,z)$, we have the standard $\mathcal{L}$-module structure on $V\otimes V$, given by

$$a(u\otimes v)=au\otimes v+u\otimes av,\quad a\in\mathcal{L},\ u,v\in V.$$

Since $As(U_n)=U(\mathcal{L})$ is the universal enveloping algebra of $\mathcal{L}$, $V\otimes V$ is also an $U(\mathcal{L})$-module given by

$$a(u\otimes v)=\sum_{(a)}a_{(1)}u\otimes a_{(2)}v,\quad a\in U(\mathcal{L}),\ u,v\in V,$$

where $\Delta : a\mapsto\sum_{(a)}a_{(1)}\otimes a_{(2)}$ is the standard coproduct in $U(\mathcal{L})$.

An AC-polynomial $\Psi(x_1,\dots,x_n)$ may be presented as $L(x_n)$ for some $L\in As(U_n)$. By definition, Ψ is a derivation with respect to x_n if and only if

$$L(u\otimes v)=L(u)\otimes v+u\otimes L(v)\in V\otimes V$$

for all $u,v\in V$. Since $V\otimes V$ is a faithful $U(\mathcal{L})$-module, we obtain $\Delta(L)=L\otimes 1+1\otimes L$, thus the Friedrichs Criterion for the Lie elements in $As(U_n)$ implies L to be an element of $\mathcal{L}$, which proves the claim. □

Define a linear map

$$D(\cdot, x_n; y, z) : V_0 \to GP(x_1, \dots, x_{n-1}, y, z)$$

as follows: Given $w = W(x_n)$, $W \in M(AC(x_1, \dots, x_{n-1}))$, set

$$D(w, x_n; y, z) = W(yz) - yW(z) - zW(y).$$

A polylinear AC-polynomial $\Psi(x_1, \dots, x_n) \in V_0$ is a derivation with respect to x_n if and only if $D(\Psi, x_n; y, z) = 0$, or, as we have noticed above, W is primitive ($\Delta(W) = 1 \otimes W + W \otimes 1$). This property of W is homogeneous in $As(U_n)$, i.e., W splits into groups of homogeneous summands $W = W_1 + \dots + W_l$, where $W_i(x_n)$ is again a derivation with respect to x_n.

Remark 2. *Suppose $\Psi = L(x_n) \in AC(X)$ is a polynomial as in Lemma 3, then $L \in Lie(\operatorname{ad} u_1, \dots, \operatorname{ad} u_k)$ for some normal words $u_1, \dots, u_k \in U_n$ (assume k is minimal). Then for every $i = 1, \dots, k$ Ψ must contain a term*

$$\{u_{i_1}, \{u_{i_2} \dots \{u_i, x_n\} \dots\}\},$$

in which the word u_i appears as the last entry.

Without loss of generality, assume L is polylinear with respect to $\operatorname{ad} u_1, \dots, \operatorname{ad} u_k$. As an element of the free Lie algebra it may be uniquely written as a linear combination of $w_{i_1,\dots,i_{k-1}} = [\operatorname{ad} u_{i_1}, [\operatorname{ad} u_{i_2}, \dots, [\operatorname{ad} u_{i_{k-1}}, \operatorname{ad} u_i] \dots]]$. The expansion of such a monomial in the free associative algebra $As(U_n)$ contains unique term $\operatorname{ad} u_{i_1} \operatorname{ad} u_{i_2} \dots \operatorname{ad} u_{i_{k-1}} \operatorname{ad} u_i \in As(U_n)$ ending with $\operatorname{ad} u_i$. These terms for different $w_{i_1,\dots,i_{k-1}}$ do not cancel.

Corollary 2. *Let $\Psi(x_1, \dots, x_n) \in AC(X)$ be a polylinear AC-polynomial of degree n such that Ψ is a derivation with respect to x_n. Suppose $\Psi = \sum_w \alpha_w w$, $w \in X^{**}$ are normal words, and*

$$\max_{w:\alpha_w \neq 0} ht(w, x_n) \geq \max_{w:\alpha_w \neq 0} ht(w, x_i), \quad i = 1, \dots, n.$$

(x_n has maximal height in Ψ). Then

$$\max_{w:\alpha_w \neq 0} ht(w, x_n) = n - 1$$

and thus Ψ contains a monomial of the form

$$w = \{x_{1s}, \dots \{x_{(n-1)s}, x_n\} \dots\}$$

for some $s \in S_{n-1}$.

Proof. Assume $k < n - 1$ is the maximal height of x_n in Ψ, i.e., Ψ contains a summand of the form $\{u_1, \dots \{u_k, x_n\} \dots\}$, $k < n - 1$. Then there exists at least one u_i whose degree is greater than 1. Remark 2 implies that Ψ contains a summand $\alpha_w w$, where $w = \{u_{j_1}, \dots \{u_{j_k}, x_n\} \dots\}$, $j_k = i$, $\alpha_w \neq 0$. Since $ht(u_i, x_j) > 1$ for some x_j, we have $ht(w, x_j) > k$, which contradicts to the condition $ht(w, x_j) \leq k$. Hence, $k = n - 1$. □

Lemma 4. *Suppose $\Psi = \Psi(x_1, \dots, x_n)$ is a jacobian AC-polynomial. Then Ψ is invariant with respect to the action of the group $\mathcal{F}$ generated by all x_i-flips, $i = 1, \dots, n$.*

Proof. Let us fix $i \in \{1, \dots, n\}$. Without loss of generality we may assume $i = n$. By Lemma 3, $\Psi = L(x_n)$, where L is a linear operator constructed by commutators of operators $\operatorname{ad} u$, $u \in U_n$. The set of all such $\operatorname{ad} u$ generates an associative subalgebra $\mathcal{U} \subset \operatorname{End}_{\Bbbk} V_n$, $\mathcal{U} \simeq As(U_n)$.

Since Ψ is polylinear (with respect to X), L naturally splits into a sum of operators presented by polylinear (with respect to U_n) elements of $\mathcal{U}$.

Consider the linear transformation τ of $\mathcal{U}$ given by

$$\tau : W \mapsto -W^*, \quad W \in \mathcal{U}.$$

The map τ acts as an identity on $Lie(U_n) \subset \mathcal{U}^{(-)}$, it follows from the obvious observation $\tau([W_1, W_2]) = [\tau(W_1), \tau(W_2)]$ for $W_1, W_2 \in \mathcal{U}$.

By Definition 2,

$$F_n(\Psi) = F_n(L(x_n)) = \tau(L)(x_n) = L(x_n) = \Psi.$$

As Ψ is invariant with respect to all flips, we have $\mathcal{F}(\Psi) = \{\Psi\}$. □

Lemma 5. *Suppose $U = \{u_1, \dots, u_m\}$ is a set, $As(U)$ is the free associative algebra generated by U, $Lie(U)$ is the free Lie algebra generated by U, $Lie(U) \subset As(U)^{(-)}$. Consider*

$$A_m = \sum_{s \in S_m} (-1)^s u_{1s} \dots u_{ms}.$$

Then $A_m \in Lie(U)$ if and only if $m = 1$ or $m = 2$.

Proof. For $m = 1, 2$ it is obvious that $A_m \in Lie(U)$.

Assume $m \geq 3$ and $A_m \in Lie(U)$. Consider the homomorphism $\Phi : As(U) \to \wedge(\Bbbk U)$ given by $u \mapsto u$, where $\wedge(\Bbbk U)$ is the exterior algebra of the linear space spanned by U. Note that $\Phi(A_m) = m! u_1 \dots u_m \neq 0$ in $\wedge(\Bbbk U)$. However, $\Phi(Lie(U)) \subset \wedge(\Bbbk U)^{(-)}$ is a Lie subalgebra generated by U. It is easy to see that $\wedge(\Bbbk U)^{(-)}$ is a 3-nilpotent Lie algebra, so $\Phi(Lie(U))$ does not contain elements of degree $m \geq 3$. □

Theorem 1. *Let $X = \{x_1, \dots, x_n\}$, and let $\Psi = \Psi(x_1, \dots, x_n) \in AC(X)$ be a jacobian AC-polynomial. Then either $n = 2$ and $\Psi = \alpha C_2$, or $n = 3$ and $\Psi = \alpha J_3$, where $\alpha \in \Bbbk^*$.*

Proof. By Lemma 4 $F\Psi = \Psi$ for every $F \in \mathcal{F}$. Without loss of generality we may assume that x_n has the maximal height in Ψ (re-numerate variables if needed). Corollary 2 implies that Ψ contains a summand of the form αw, where $\alpha \in \Bbbk^*$, $w = \{x_{1s}, \dots \{x_{(n-1)s}, x_n\} \dots\}$ for some $s \in S_{n-1}$. Without loss of generality, $\alpha = 1$ and $s = \mathrm{id}$. By Lemma 2, Ψ contains all monomials obtained from w by all permutations of variables, i.e.,

$$\Psi = \sum_{s \in S_{n-1}} (-1)^s \{x_{1s}, \dots \{x_{(n-1)s}, x_n\} \dots\} + \Phi(x_1, \dots, x_n),$$

where the x_n-height of all monomials in Φ is smaller than $n-1$. Since all summands of Ψ with the same x_n-height form a derivation with respect to x_n, the AC-polynomial

$$\Psi_1 = \sum_{s \in S_{n-1}} (-1)^s \{x_{1s}, \dots \{x_{(n-1)s}, x_n\} \dots\}$$

must be a derivation with respect to x_n. But

$$\Psi_1 = A_{n-1}(u_1, \dots, u_{n-1})(x_n), \quad u_i = \operatorname{ad} x_i,$$

so $A_{n-1}(u_1, \dots, u_{n-1}) \in Lie(u_1, \dots, u_{n-1})$. By Lemma 5, $n - 1 \leq 2$, so $n \leq 3$. Obviously, C_2 and J_3 are the only jacobian AC-polynomials for $n = 2$ and $n = 3$, respectively. □

4. Identities of generic Poisson algebras

Let A be a GP-algebra, and let $f \in GP(x_1, \dots, x_n)$, $f \neq 0$. As usual, we say that f is a polynomial identity on A if for every homomorphism $\varphi : GP(x_1, \dots, x_n) \to A$ we have $\varphi(f) = 0$. In this case we also say that A satisfies the polynomial identity f.

Proposition 2. *Suppose a GP-algebra A satisfies a polynomial identity. Then there exists a polynomial identity Ψ on A which is a jacobian GP-polynomial.*

This statement, as well as its proof, is similar to the result by Farkas [17] on polynomial identities of Poisson algebras.

Proof. The standard linearization procedure (see, e.g., [19, Chapter 1]) allows to assume that A satisfies a polylinear polynomial identity $f \in GP(X)$, $X = \{x_1, \dots, x_n\}$.

As an element of $GP(X)$, f may be uniquely presented as a linear combination of GP-monomials $w = u_1 \dots u_k$, $u_j \in U$, where $U \subset AC(X)$ is the set of normal words. We may assume that u_j are of degree two or more (if an AC-monomial of degree one appears, e.g., $u_j = x_i$, then one may plug in $x_i = 1$ and obtain a polylinear polynomial identity without x_i). Denote by $FH_i(w)$ (the Farkas height) the degree of u_j in which the variable x_i occurs, and let $FH_i(f)$ be the maximal of $FH_i(w)$ among all GP-monomials w that appear in f with a nonzero coefficient. Finally, set

$$FH(f) = \sum_{i=1}^{n} 3^{FH_i(f)}$$

Observe that if f is not a derivation in x_i then the derivation difference $D(f, x_i; x_i, x_{n+1})$ is a nonzero polylinear element of $GP(X \cup \{x_{n+1}\})$ which has a smaller Farkas height. Indeed, for a GP-monomial w from f we have

$$FH_j(D(w, x_i; x_i, x_{n+1})) \leq FH_j(w),$$
$$FH_i(D(w, x_i; x_i, x_{n+1})) \leq FH_i(w) - 1,$$
$$FH_{n+1}(D(w, x_i; x_i, x_{n+1})) \leq FH_i(w) - 1,$$

which implies

$$FH(w) - FH(D(w, x_i; x_i, x_{n+1})) \leq 3^{FH_i(w)} - 2 \cdot 3^{FH_i(w)-1} > 0.$$

Obviously, $D(f, x_i; x_i, x_{n+1})$ is a polynomial identity on A.

Therefore, after a finite number of steps we obtain a nonzero polynomial identity on A which is a jacobian GP-polynomial in a larger set of variables $\widetilde{X} \supseteq X$. □

Let us recall the notion of fine grading [17]. First, given a set X, the free anti-commutative algebra $AC(X)$ carries $[X^*]$-grading such that $u \in X^{**}$ has weight $[u]$. Next, if $w = (u_1) \dots (u_n) \in GP(X)$, $u_i \in X^*$, then the weight of w is $[u_1] + \dots + [u_n] \in \Bbbk[X^*]$. As a result,

$$GP(X) = \bigoplus_{p \in \mathbb{Z}_+[X^*] \setminus \{0\}} GP_p(X),$$

where $\mathbb{Z}_+$ stands for the set of non-negative integers, $GP_p(X)$ is the space spanned by all generic Poisson monomials of degree p. An element $f \in GP_p(X)$ is said to be *finely homogeneous.*

Proposition 3. *A jacobian GP-polynomial Ψ can be presented as a linear combination of products of jacobian AC-polynomials (on the appropriate set of variables).*

Proof. Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of variables, and let $U = \{u_1, u_2, \dots\}$ be the set of normal nonassociative words in X (with respect to some ordering), then $GP(X) = \Bbbk[U]$. For $\Psi \in GP(X)$, denote by $\operatorname{supp}(\Psi)$ all variables from X that appear in Ψ and by $\operatorname{psupp}(\Psi)$ all elements from U that appear in Ψ.

Suppose $f \in GP(x_1, \dots, x_n) \subseteq GP(X)$ is a jacobian GP-polynomial. Without loss of generality we may assume f to be finely homogeneous and $f \notin AC(X)$. Proceed by induction on $|\operatorname{psupp}(f)|$.

Consider a GP-monomial w in f. Since $f \notin AC(X)$, there exist u_i and $w' \neq 1$ for which $w = u_i w'$. Write $f = u_i g + h$, $g, h \in GP(X)$, $g \neq 1$, where all GP-monomials of h are not divisible by u_i (in $\Bbbk[U]$). Since f is polylinear, $\operatorname{supp}(g) \cap \operatorname{supp}(u_i) = \emptyset$.

Denote by D_i a map $GP(X) \to GP(X \cup \{y, z\})$ defined as follows:

$$D_i(\Psi) = \begin{cases} D(\Psi, x_i; y, z), & x_i \in \operatorname{supp}(\Psi), \\ \Psi, & x_i \notin \operatorname{supp}(\Psi). \end{cases}$$

Then $D_j(f) = u_i D_j(g) + D_j(h) = 0$ if $x_j \in \operatorname{supp}(g)$.

Consider $GP(X \cup \{y, z\})$ as a polynomial algebra with a set $\widetilde{U}$ of generators including U. Then $u_i \notin \operatorname{psupp}(h)$ and $u_i \notin \operatorname{psupp}(D_j(h))$. Hence, $D_j(g) = 0$ and g is a jacobian GP-polynomial.

Let us now fix the deg-lex order on the set $[U^*]$, i.e., commutative monomials in U are first compared by their length and then lexicographically, assuming $u_1 < u_2 < \dots$. Recall that $f = u_i g + h$, where $\operatorname{psupp}(h) \not\ni u_i$, and presented h as $h = gp + r$, where all GP-monomials of r are not divisible (in $\Bbbk[U]$) by the leading GP-monomial $\bar{g}$ of g. Then $f = gq + r$, $q = u_i + p$, and $\operatorname{psupp}(r) \not\ni u_i$. In particular, $\operatorname{psupp}(r) \subset \operatorname{psupp}(f)$.

By definition, $D_j(f) = g D_j(q) + D_j(r) = 0$ if $x_j \in \operatorname{supp}(q)$. If $D_j(q) \neq 0$ then some of the monomials in $D_j(r)$ are divisible by $\bar{g}$. Consider a GP-monomial M of r. Since it is not divisible by $\bar{g}$ there is at least one variable u_a which appears in $\bar{g}$ and does not appear in M. Note that if $\operatorname{supp}(u_b) \not\ni x_i$ then $D_i(u_b) = u_b$, and if $\operatorname{supp} u_b \ni x_i$ then $D_i(u_b)$ is a GP-polynomial of degree two (in $\Bbbk[\widetilde{U}]$) in which none of variables belongs to U. Hence, $D_j(M)$ is not divisible by u_a and none of the GP-monomials of $D_j(r)$ is divisible by $\bar{g}$. Therefore, $D_j(q) = 0$ and q is a jacobian GP-polynomial.

Since a product of two jacobian GP-polynomials is also jacobian (with respect to the corresponding sets of variables), $r = f - gq$ is a jacobian GP-polynomial. By induction, the statement holds for r, as well as for g and q. □

Corollary 3. *Let $F(t_1, \dots, t_n) \in GP(t_1, t_2, \dots)$ be a finely homogeneous jacobian GP-polynomial. Then F contains a summand $\alpha u_1 \dots u_k$, where $\alpha \in \Bbbk^*$, $u_i \in AC(t_1, t_2, \dots)$ are of the form*

$$\{t_{i_1}, t_{i_2}\} \quad \text{or} \quad \{t_{i_1}, \{t_{i_2}, t_{i_3}\}\}$$

5. The Freiheitssatz for (generic) Poisson algebras

The following statement is well-known in the theory of differential fields [20, 21]. We will sketch a proof below in order to make the exposition more convenient for a reader. Recall that the characteristic of the base field $\Bbbk$ is assumed to be zero, and that Dif_n denotes the variety of commutative associative algebras with n pairwise commuting derivations.

Theorem 2. *Every algebra from Dif_n which is a field can be embedded into an algebraically closed algebra in Dif_n.*

Proof. Let F be a differential field of characteristic zero with a set $\Delta = \{\partial_i \mid i = 1, \dots, n\}$ of pairwise commuting derivations. Denote by $F[x;\Delta] = F *_{Dif_n} Dif_n(x)$ the set of all differential polynomials in one variable x over F. Suppose $f(x) \in F[x;\Delta]\setminus F$. Then there exists a differential field K which is an extension of the differential field F such that the equation $f(x) = 0$ has a solution in K.

Indeed, differential polynomials $F[x;\Delta]$ may be considered as ordinary polynomials in infinitely many variables

$$X = \{x^{(i_1,\dots,i_n)} \mid (i_1, \dots, i_n) \in \mathbb{Z}_+^n\},$$

where $x^{(i_1,\dots,i_n)}$ is identified with $\partial_1^{i_1} \dots \partial_n^{i_n}(x)$. Then the differential ideal $I(f;\Delta)$ generated by $f(x)$ in $F[x;\Delta]$ coincides with the ordinary ideal in $F[X]$ generated by f and all its derivatives $\partial_1^{i_1} \dots \partial_n^{i_n}(f)$.

Note that if $f \notin F$ then $I(f;\Delta)$ is proper: One may apply the notion of a characteristic set (see, e.g., [21, Ch. I.10]) or simply note that the set of all derivatives of f is a Gröbner basis provided that we choose an ordering of monomials in such a way that highest derivative (leader) is contained in the leading monomial (e.g., rank ordering in [21, Ch. I.8]). Indeed, if uy is the leading monomial of f ($y \in X$ is the leader of f, u is an ordered monomial in X) then $uy^{(i_1,\dots,i_n)}$ is the leading monomial of $\partial_1^{i_1} \dots \partial_n^{i_n}(f)$. It is easy to see that there are no compositions (we follow the terminology of Shirshov [3], see [22] for details) among f and its derivatives except for the case when $uy = y^k$, but in the latter case the only series of compositions of intersection of f with itself is obviously trivial.

Hence, if $f \notin F$ then $I = I(f;\Delta)$ is proper, and so is its radical $\sqrt{I}$. By the differential prime decomposition theorem (see, e.g., [20, Ch. 1]), $I = p_1 \cap \dots \cap p_k$, where p_i are prime differential ideals in $F[x;\Delta]$. In particular, $f \in p_1$, and $F[x;\Delta]/p_1$ is a differential domain containing a root $x + p_1$ of f. Finally, the quotient field of that domain $Q(F[x;\Delta]/p_1)$ is the desired differential field.

Therefore, every nontrivial equation over an arbitrary differential field F has a solution in an extension K of F. If F is infinite then K has the same cardinality as F, so the standard transfinite induction arguments similar to those applied to ordinary fields show that F can be embedded into a differential field $\bar{F} \in Dif_n$ in which every nontrivial differential polynomial has a root. □

Corollary 4 ([7]). *The Freiheitssatz holds for the variety of Poisson algebras.*

Proof. Let $A_{2n} = \Bbbk(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ be the algebra of (commutative) rational functions over $\Bbbk$, $\partial_i = \partial_{x_i}$, $\partial_i' = \partial_{y_i}$ be ordinary partial derivatives with respect to x_i, y_i, respectively. As $A_{2n} \in Dif_{2n}$, there exists its algebraically closed extension $\bar{A}_{2n} \in Dif_{2n}$. Let $PS_n = A_{2n}^{(\partial)}$ be the Poisson algebra defined by (1). Then $PS_n \subseteq \bar{A}_{2n}^{(\partial)}$, where the latter is a conditionally closed Poisson algebra by Proposition 1.

It was shown in [17] that for every nonzero Poisson polynomial $h = h(x_1, \dots, x_m)$, $m \geq 1$, there exists a sufficiently large N such that PS_N (and thus $\bar{A}_{2N}^{(\partial)}$) does not satisfy the identity $h(x_1, \dots, x_m) = 0$. Lemma 1 implies the claim. □

Let us twist the functor $\partial : Dif_{2n} \to Pois$ in order to obtain a conditionally closed generic Poisson algebra that does not satisfy a fixed polynomial identity.

Consider the variety $CDif_n$ of commutative differential algebras with pairwise commuting derivations ∂_i and constants c_i, $i = 1, \dots, n$, such that $\partial_i(c_j) = \delta_{ij}$. Then there exists a natural forgetful functor $\omega : CDif_n \to Dif_n$ erasing the information about constants.

In particular, A_{2n} may be considered as an algebra from $CDif_{2n}$ with derivations $\partial_i = \partial_{x_i}$, $\partial'_i = \partial_{y_i}$, and constants $c_i = x_i$, $c'_i = y_i$, $i = 1, \dots, n$. Moreover, if $A_{2n} \subseteq A \in Dif_{2n}$ then $A = B^{(\omega)}$ for an appropriate $B \in CDif_{2n}$. Hence (see Remark 1), for every $n \geq 1$ there exists a conditionally closed algebra $\bar{B}_{2n}$ in $CDif_{2n}$, $\bar{B}_{2n}^{(\omega)} = \bar{A}_{2n}$.

Suppose $B \in CDif_{2n}$ with derivations ∂_i, ∂'_i and constants c_i, c'_i, $i = 1, \dots, n$. Let us consider the following functor τ from $CDif_{2n}$ to the variety $NDif_{2n}$ of commutative differential algebras with non-commuting derivations ξ_i, ξ'_i, $i = 1, \dots, n$. On the same space B, define new derivations by

$$\begin{gathered} \xi_i(a) = c'_{i+1}\partial_i, \quad i = 1, \dots, n-1, \\ \xi_n(a) = c'_1\partial_n, \\ \xi'_i(a) = \partial'_i(a), \quad i = 1, \dots, n, \end{gathered} \tag{3}$$

for $a \in B$. If B is conditionally closed in $CDif_{2n}$ then $B^{(\tau)}$ is conditionally closed in $NDif_{2n}$.

Finally, define a functor ξ from $NDif_{2n}$ to the variety GP of generic Poisson algebras by means of

$$\{a, b\} = \sum_{i \geq 1} \xi_i(a)\xi'_i(b) - \xi_i(b)\xi'_i(a). \tag{4}$$

Denote by GPS_n the GP-algebra $\left(A_{2n}^{(\tau)}\right)^{(\xi)}$.

Proposition 4. *For every $n \geq 1$ there exists $N \geq 1$ such that the GP-algebra GPS_m does not satisfy a polynomial identity of degree n for all $m \geq N$.*

Proof. Suppose $f \in GP(t_1, t_2, \dots)$ is a GP-polynomial of degree n which is an identity on GPS_m. By Proposition 2 there also exists a polylinear identity g on GPS_m which is a jacobian GP-polynomial.

Let us split g into finely homogeneous components:

$$g = g_1 + \dots + g_k,$$

each g_i is a jacobian GP-polynomial (but not an identity on GPS_m).

According to Corollary 3, g_1 contains a summand $\alpha u_1 \dots u_l$, $\alpha \in \Bbbk^*$,

$$u_i = \{t_{i_1}, \dots \{t_{i_{m_i}}, t_{i_{m_i+1}}\} \dots\}, \quad m_i = 1, 2.$$

Assume m is large enough (e.g., $m > 2l$), and evaluate the variables in such a way that

$$\begin{gathered} t_{i_{m_i+1}} = y_{k_i}, \\ t_{i_{m_i}} = x_{k_i},\ t_{i_{m_i-1}} = x_{k_i+1}, \dots, t_{i_1} = x_{k_i+m_i-1}, \\ k_{i+1} \geq k_i + m_i, \quad k_l + m_l < m. \end{gathered}$$

Then the only summand in $g_1(t_1, \dots, t_n)$ is nonzero, namely, the summand mentioned in Corollary 3: It turns into $\alpha y_{k_1+m_1} \dots y_{k_l+m_l} \neq 0$. Other $g'_i s$ turn into zero.

Hence, g cannot be a polynomial identity on GPS_m. □

Theorem 3. *The Freiheitssatz holds for the variety of generic Poisson algebras.*

Proof. Given $N \geq 1$, $G_N = \big(\bar{B}_{2N}^{(\tau)}\big)^{(\xi)}$ is a conditionally closed algebra in GP by Proposition 1, and $GPS_N \subseteq G_N$. The claim now follows from Proposition 4 and Lemma 1. □

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