Short closed curves on Riemannian manifolds with Ricci curvature bounded from below.

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Abstract. In [1] Michael Anderson proved the following remarkable theorem: Let M^n be a closed Riemannian manifold with a torsion-free fundamental group. Assume that Ricci curvature is bounded from below by $-(n-1)$. Assume that the volume of M^n is not less than $v > 0$ and the diameter of M^n does not exceed d. Then there exists an explicit $\epsilon = \epsilon(n, v, d) > 0$ such that every closed curve γ of length $\leq \epsilon$ on M^n is contractible. Moreover, one can drop the assumption that $\pi_1(M^n)$ is torsion-free, but in this case the theorem asserts only the existence of some positive integer $k(\gamma) \leq N(n, v, d)$ such that γ iterated $k(\gamma)$ times is contractible. Here $N(n, v, d)$ is an explicit function.

The purpose of the present paper is to derive several *effective* versions of this theorem. For example, we prove that for every positive r there exists an explicit positive $\epsilon = \epsilon(n, v, d, r)$ such that for every closed curve of length $\leq \epsilon$ one of its first $N(n, v, d)$ iterates is contractible via closed curves of length $\leq 2r$ inside a metric ball of radius r.

The proof by M. Anderson is based on an application of the Bishop volume comparison theorem to the universal covering space of a manifold with Ricci curvature bounded from below. Our main technical novelty is to replace this by an application of the Bishop volume comparison theorem to a tangent space of the manifold endowed with the (pseudo-)metric obtained as the pullback of the Riemannian metric on the manifold under the exponential map.

1. Introduction and main results.

To state our main results let $v_n(t)$ denote the volume of a metric ball of radius t is the n-dimensional hyperbolic space, $N(n, v, d)$ be the smallest odd integer number greater than $\frac{v_n(2d)}{v}$, and $\epsilon(n, v, d, c) = \frac{c}{N(n,v,d)-1}$. (Recall that $v_n(t) = \frac{\pi^{\frac{n}{2}} n}{\Gamma(\frac{n}{2}+1)} \int_0^t \sinh(\tau)^{n-1} d\tau$ (cf. [2]).) The ith iterate of a closed curve $\gamma : [0, 2\pi] \longrightarrow M^n$, $\gamma(0) = \gamma(2\pi)$, is, by definition, the curve γ^i defined by the formula $\gamma^i(\phi) = \gamma(i\phi)$.

Theorem 1. Let M^n be a closed Riemannian manifold with $Ric \geq -(n-1)$, volume bounded from below by $v > 0$ and the diameter bounded from above by d. Let $c \leq 2d$ be any positive real number. For every closed curve γ of length $\leq \epsilon(n, v, d, c)$ there exists a positive integer $k = k(\gamma) \leq N(n, v, d)$ such that the closed curve obtained by iterating γ k times can be contracted to a point via closed curves of length $\leq 2d + c$.

Example. One can take $c = 2d$, $\epsilon(n, v, d) = \epsilon(n, v, d, d) = \frac{2d}{N(n,v,d)-1}$ and to conclude that one of the first $N(n, v, d)$ iterates of every closed curve of length $\leq \epsilon(n, v, d)$ is contractible via closed curves of length $\leq 4d$.

Michael Anderson proved in [1] an assertion similar to the assertion of Theorem 1 but without the upper bound $2d + c$ for lengths of closed curves in the contracting homotopy. Here is a slighlty modified sketch of his proof that yields a slightly better estimate than the estimate in [1]: Assume that none of first the K iterates of γ is contractible. The same will be true for iterates of γ traversed in the opposite direction. Then there exists a segment S of the length $2K\ length(\gamma)$ in the universal covering of M^n that consists of $2K$ distinct copies of the lift of γ to the universal covering. Without any loss of generality we can assume that all $2K+1$ endpoints of these segments (that project to the basepoint) are in different fundamental domains. Therefore the d-neighborhood of S in the universal covering contains (at least) $2K+1$ fundamental domains, and therefore has volume $\geq (2K+1)v$. On the other had it is contained in a metric ball of radius $K\ length(\gamma)+d$. Assume that $K \leq \frac{d}{length(\gamma)}$. Then $K\ length(\gamma)+d \leq 2d$, and the Bishop volume comparison theorem implies that the volume of the d-neighborhood of S does not exceed $v_n(2d)$. Thus, $(2K+1)v \leq v_n(2d)$.

Now choose $N(n,v,d)$ as the minimal odd integer number greater than $\frac{v_n(2d)}{v}$, and let $K = \frac{N(n,v,d)-1}{2}$. Let $\epsilon = \epsilon(n,v,d) = \frac{2d}{N(n,v,d)-1}$. Then the assumption that a closed curve has length $\leq \epsilon$ but its first N iterates are non-contractible leads to an immediate contradiction.

Note that this argument cannot provide an upper bound for the length of loops during a contracting homotopy of an iterate of a short closed curve. Indeed, observe that Theorem 1 makes sense and is non-trivial even for simply connected Riemannian manifolds. In this case we obviously cannot use volume comparison in the universal covering space of M^n as this universal covering coincides with M^n and is useless for this purpose.

We became interested in finding an effective version of the result of Anderson in the course of our joint work with Shmuel Weinberger [4], as such a theorem could be helpful there. To prove such an effective version Shmuel Weinberger suggested to use some form of an "effective universal covering" instead of the universal covering. Here one possible idea could be to identify two paths leading from the base point to the same point if the loop formed by these two paths is not merely contractible, but is contractible with some control over geometry of the curves obtained in the process of contraction. Yet we were not able to construct any specific version of an "effective universal covering" that would be appropriate for a proof of Theorem 1. So, another possible reason to look for a proof of Theorem 1 is to find out what can replace the universal covering space when one is interested in an effective version of various arguments in comparison geometry that use the Bishop volume comparison theorem applied to domains in the universal covering.

Our answer for the last question is that one needs to apply the Bishop volume comparison theorem to a tangent space to the manifold endowed by the Riemannian pseudo-metric defined as the pullback of the Riemannian metric on the manifold under the exponential map. Our motivation for this idea will be explained in the last section. An obvious complication is that the tangent space with the pullback metric cannot be naturally subdivided into fundamental domains (unlike the universal covering space of M^n). This difficulty is circumvented by means of Lemma 2, which asserts that under certain conditions the inverse image of every point of the manifold under the exponential map will have many distinct points in a ball of controlled radius. (Or equivalently, every two points of the manifold can be connected by many distinct geodesics that are not too long.)

We are going to use essentially the same idea to prove an effective local version of the result of Anderson (Theorem 3 in section 4). Theorem 3 answers the following question: In Theorem 1 iterated short curves are contracted using all manifold and via closed curves of length bounded by $2d+c$. Does a smaller upper bound for the length of a curve guarantee that one of its iterates will be contractible within a small ball and via short curves? A possibility of such a local version of his theorem had been suggested by Anderson in Remark 2.2(1) in [1]. (But he was not interested in controlling the length of curves during a contracting homotopy). Anderson suggested to consider the δ-neighborhood $T(\delta)$ of a closed curve γ of interest for a sufficiently small δ. Then he suggested to replace M^n by this δ-neighborhood in his argument, obtaining as the result the inequality $N\ vol(T(\delta)) \leq v_n(N\ length(\gamma) + d)$ (in our notations) from which a local version of his theorem can be easily deduced. (However, there is a technical problem with this idea: In general, $T(\delta)$ and its universal covering have boundaries, and the Bishop volume comparison theorem is not applicable to manifolds with boundary. To appreciate the last assertion consider a thickened figure eight in R^2. Assume that its thickness is approximately equal to one. Its universal covering will be a thickened binary tree. The volume of the ball of radius R in the universal covering will grow exponentially with R, and will not be bounded from above by πR^2, as the Bishop volume comparison theorem would imply).

Note that all these results assert not the contractibility of a short closed curve but of one of its iterates. Our last results (Theorems 4 and 4.A) provide an explicit value of ϵ such that every closed curve of length $\leq \epsilon$ on M^n can be contracted to a point with only a controlled increase of its length as a loop based at one of its points. Yet this ϵ is defined not in terms of n, v, d. Instead, we use a distance x such that the volumes of all metric balls of radius $\leq x$ are close to the volumes of the balls of the same radius in R^n.

2. Proof of Theorem 1.

First, note that the general case of Theorem 1 follows from the special case when the Riemannian metric on M^n is analytic. Indeed, we can δ-approximate any given smooth Riemannian manifold M^n by an analytic Riemannian manifold N^n, where δ is much smaller than the convexity radius of M^n. (Here we consider an approximation in the Gromov-Hausdorff metric.) Then it becomes possible to "transfer" any closed curve γ in M^n to N^n, fill its appropriate iterate by a 2-disc in N^n made of curves of length $2d + c + O(\delta)$, and to "transfer" this 2-disc back to M^n. The idea that one can canonically fill "gaps" between points or between closed curves of size less than the convexity radius of M^n. The construction of such transfers is well-known in comparison geometry (cf. [3] for a detailed description of such transfers back and forth in a different setting).

Let p be a point of M^n. Consider the exponential map $\exp_p : TM^n_p \longrightarrow M^n$ of the tangent space of M^n at p to M^n. For each r let $B_r(TM^n_p)$ denote the open ball of radius r centered at the origin in TM^n_p. Consider the pullback pseudo-Riemannian metric on TM^n_p: For $v, w \in T(TM^n_p)_q$ we define $< v, w >$ as $g(d\exp_p(q)v, d\exp_p(q)(w))$. If $\exp_p(q)$ is not a conjugate point, this formula yields a Riemannian metric in an open neighborhood of q, and the exponential map will be a local isometry. If M^n is an analytic Riemannian manifold, then the set of points $X\ q \in TM^n_p$ such that $d\exp_p(q)$ is a singular linear map is a triangulable subset of TM^n_p of codimension ≥ 1. Note that one can also consider the pullback of the volume measure on M^n to TM^n_p. Denote the resulting pseudo-Riemannian manifold with the pullback measure by $(TM^n_p)_*$. Note that one can regard $B_r(TM^n_p)$ as a subset of $(TM^n_p)_*$. Classical proofs of the Bishop volume comparison inequality (cf. section 9.1 of [5]) also imply that:

Proposition. Let M^n be a complete Riemannian manifold satisfying $Ric \geq -(n-1)$.

Then for every r the volume of $B_r(TM_p^n) \subset (TM_p^n)_*$ does not exceed the volume $v_n(r)$ of a metric ball of radius r in the hyperbolic n-space.

Let γ be a closed curve on M^n of length $\leq \epsilon = \epsilon(n, v, d, c)$.

Example. Assume that M^n is the round n-dimensional sphere of radius 1. Let p be the South pole of the sphere. In order to understand the geometry of the pseudo-metric space $(TM_p^n)_*$ note that for every integer k $\exp_p$ maps every $(n-1)$-dimensional sphere of radius $2\pi k$ centered at the origin into the South pole and every sphere of radius $(2k+1)\pi$ centered at the origin into the North pole. Therefore, for every integer m the distance between each pair of points in the $(n-1)$-dimensional sphere of radius πm is equal to zero. In order to better understand the geometry of the pseudo-metric space $(TM_p^n)_*$ one can turn it into a metric space by identifying every pair of points such that pseudo-distance between them is equal to zero. After passing to the quotient $(TM_p^n)_*$ will become the infinite join of round n-spheres of radius one indexed by number $1, 2, \ldots$ and attached to each other so that for every i the North pole of the ith sphere is glued to the South pole of the $(i+1)$-sphere.

Choose $p = \gamma(0)$. Our idea is to carry out the proof by M. Anderson using $(TM_p^n)_*$ instead of the universal covering space of M^n. Instead of considering a metric ball of radius $2d$ in the universal covering space of a Riemannian manifold we are going to consider $B_{2d}(TM_p^n) \subset (TM_p^n)_*$. As it had been already noted, the volume of $B_{2d}(TM_p^n)$ regarded as a subset of $(TM_p^n)_*$ does not exceed $v_n(2d)$.

In order to complete the proof of Theorem 1 we need only to prove that if none of the first $N = N(n, v, d)$ iterates of γ is contractible in M^n via closed curves of length $\leq 2d + c$ then $B_{2d}(TM_p^n)$ contains at least N "copies" of M^n and has, therefore, volume $\geq Nv$. (Juxtaposing the upper and lower bounds for the volume of $B_{2d}(TM_p^n) \subset (TM_p^n)_*$ we would obtain the inequality $Nv \leq v_n(2d)$ contradicting the definition of $N = N(n, v, d)$.) More precisely, we would like to establish that M^n minus a set of measure zero can be partitioned into open domains such that each of these open domains is the image under $\exp_p$ of at least N disjoint open domains in $B_{2d}(TM_p^N) \subset (TM_p^n)_*$, and the restriction of $\exp_p$ on each of these open domains in $B_{2d}(TM_p^n)$ is an isometry. Since $(TM_p^n)_*$ is locally isometric to M^n outside of the inverse image of the set of conjugate points of p, it is sufficient to prove that for every point x of M^n the cardinality of the set $\exp_p^{-1}(x) \bigcap B_{2d}(TM_p^n)$ is at least N. In simple terms this means that there exists at least N geodesics of length $\leq 2d$ between p and x.

Thus, Theorem 1 follows the following lemma:

Lemma 2. Let M^n be a Riemannian manifold, and x a point of M^n. Let K be a positive integer number, and ϵ a positive real number. Finally, let γ is a closed curve of length $\leq \epsilon$ on M^n such that neither of its first K iterates can be contracted to a point via loops based at $\gamma(0)$ of length $\leq 2K\epsilon + 2dist(\gamma(0), x)$. Then there exist at least $2K + 1$ distinct geodesics of length $\leq K\epsilon + dist(\gamma(0), x)$ between $p = \gamma(0)$ and x.

Indeed, we can apply Lemma 2 with $K = \frac{N(n,v,d)-1}{2}$, $\epsilon = \epsilon(n, v, d, c)$ and to observe that for every point $x \in M^n$ $dist(x, \gamma(0)) \leq d$ and that $2K\epsilon = c$. Therefore, $2K\epsilon + 2dist(\gamma(0), x) \leq 2d + c$, and $K\epsilon + dist(\gamma(0), x) \leq \frac{c}{2} + d \leq d + d = 2d$, as needed. QED.

3. Proof of Lemma 2.

Proof: Let σ be a minimizing geodesic between p and x. Consider $2K+1$ paths $\tau_i, -(N-1)\leq i\leq N-1$ between p and x that first go along γ i times and then along σ. Apply a version of the Birkhoff curve-shortening process for curves with fixed endpoints to each of paths τ_i. As the result, for every i we will obtain a length non-increasing path homotopy H_i between τ_i and a geodesic between p and x that we are going to denote σ_i. (Recall that path homotopy is a homotopy of paths with fixed endpoints.) Of course, $\sigma_0=\sigma$.

We claim that, if $i>j$, then $\sigma_i\neq\sigma_j$. (This assertion immediately implies the lemma.) Indeed, assume that $i-j=k>0$, and $\sigma_i=\sigma_j$. Then there is a path homotopy between $\tau_i=\gamma^i*\sigma$ and $\tau_j=\gamma^j*\sigma$ that passes through paths of length not exceeding $K\epsilon+dist(x,p)$. This path homotopy H is obtained by combining path homotopy H_i with a path homotopy obtained from H_j by reversing the direction of time.

Now one can contract γ^k via closed curves of length $\leq 2d+2K\epsilon$ thereby obtaining a contradiction with the assumptions of the lemma as follows: First, create j new copies of γ followed by j copies of γ traversed in the opposite direction. This step can be described by the formula $\gamma^k\longrightarrow\gamma^k*\gamma^j*\gamma^{-j}=\gamma^i*\gamma^{-j}$. Now insert $\sigma*\sigma^{-1}$ in between of γ^i and γ^{-j}. We obtain $\gamma^i*\sigma*\sigma^{-1}*\gamma^{-j}=\tau_i*\sigma^{-1}*\gamma^{-j}$. Apply H to τ_i. We will obtain $\tau_j*\sigma^{-1}*\gamma^{-j}=\gamma^j*\sigma*\sigma^{-1}*\gamma^{-j}$, which can be contracted to a point over itself by a length decreasing homotopy. QED.

4. A local version of Theorem 1.

In this section we are going to prove the following theorem:

Theorem 3. Let M^n be a closed Riemannian manifold with $Ric\geq -(n-1)$, volume greater than or equal to v and the diameter less than or equal to d. Let N be the smallest odd number greater than $\frac{v_n(2d)}{v}$, and δ be any positive number. Assume that a positive ϵ satisfies the inequality

$$\frac{v_n((N-1)\frac{\epsilon}{2}+\delta)}{v_n(\delta)}<\frac{v_n(2d)}{v_n(d)}\ . \qquad (*)$$

Let $r=\frac{N-1}{2}\epsilon+\delta$. Then for every closed curve γ of length $\leq\epsilon$ in M^n there exists $i\leq N$ such that the ith iterate of this curve can be contracted to a point inside the metric ball of radius r centered at $\gamma(0)$ via loops based at $\gamma(0)$ of length $\leq 2r$.

Since $v_n(x)$ is continuous, the left hand side of (*) converges to 1, as $\epsilon\longrightarrow 0$, and so the inequality (*) holds for all sufficiently small positive ϵ. Thus, we obtain the following corollary of Theorem 3:

Corollary 3.A Let M^n be a closed Riemannian manifold with $Ric\geq-(n-1)$, volume not less than $v>0$ and diameter not exceeding d. Then for every positive r there exists an explicit $\epsilon(n,v,d,r)>0$ such that every closed curve of length $\leq\epsilon(n,v,d,r)$ on M^n can be contracted to a point within a metric ball of radius r via closed curves of length $\leq 2r$.

Proof of Theorem 3. We are going to prove Theorem 3 by contradiction. Assume that its conclusion is false. Then we can apply Lemma 2 with $K=\frac{N-1}{2}$. It implies that for every point $x\in M^n$ such that the distance between x and $\gamma(0)$ does not exceed δ there exist at least N distinct geodesics between $\gamma(0)$ and x of length not exceeding r. Therefore, as in the proof of Theorem 1 in section 2, we can assume w.l.o.g. that the Riemannian on M^n is analytic, and can conclude in this case that the volume of $B_r(TM^n_{\gamma(0)})$ regarded as a subset of $(TM^n_p)_*$ is not less than $Nv(\delta,\gamma(0))$, where $v(\delta,\gamma(0)$ denotes the volume of the metric ball in M^n of radius δ centered at $\gamma(0)$. An application of the Bishop-Gromov volume comparison inequality implies that $v(\delta,\gamma(0))\geq \frac{v\ v_n(\delta)}{v_n(d)}$. Therefore,

$$v_n(r)\geq \frac{Nv\ v_n(\delta)}{v_n(d)}\geq \frac{v_n(2d)}{v}\frac{v\ v_n(\delta)}{v_n(d)}=\frac{v_n(2d)}{v_n(d)}v_n(\delta).$$

Hence

$$\frac{v_n(r)}{v_n(\delta)}\geq \frac{v_n(2d)}{v_n(d)},$$

which contradicts (*). QED.

Our last result provides ϵ and δ such that every closed curve γ of length $\leq\epsilon$ can be contracted to a point via loops based at $\gamma(0)$ of length $\leq 2\epsilon+2\delta$. In this case one does not need to iterate γ to make the result contractible. But ϵ and δ are not defined in terms of n, v and d anymore. (It seems that a modification of the example constructed in the proof of Proposition 3.1 of [1] can be used to show that there is no positive lower bound for ϵ of the form $f(n,v,d)$ such that every closed curve of length $\leq\epsilon$ is contractible. One needs just to replace the Eguchi-Hanson metrics on TS^2 used in the proof of Proposition 3.1 of [1] by analogous Riemannian metrics on TRP^2. I learned this idea from Vitali Kapovitch and would like to thank him for pointing out to me this construction.) To state out result note that for small ϵ volumes of all balls of radius ϵ in M^n are very close to the volume of the ball of radius ϵ in R^n. The same is true for the volume of the ball of the radius ϵ in the n-dimensional hyperbolic space.

Theorem 4. Let M^n be a closed Riemannian manifold with $Ric\geq -(n-1)$. Define δ_0 as supremum of all positive numbers τ such that the volume of every metric ball of radius τ in M^n is not less than $\frac{1}{2}v_n(\tau)$. Assume that δ is in the open interval $(0,\delta_0)$. Let ϵ_δ be the solution of the equation $v_n(\epsilon_\delta+\delta)=\frac{3}{2}v_n(\delta)$. Then every closed curve γ of length $\epsilon<\epsilon_\delta$ can be contracted to a point via loops based at $\gamma(0)$ of length $\leq 2\epsilon+2\delta$.

Proof. Assume that the conclusion of the theorem is false. We are going to bring this assumption to a contradiction. Denote the length of γ by ϵ. Apply Lemma 2 for $K=1$ and $x\in M^n$ such that the distance from $\gamma(0)$ and x does not exceed δ. Its conclusion is that there exists at least three distinct geodesics between $\gamma(0)$ and x of length $\leq\epsilon+\delta$. Therefore, the volume of $B_{\epsilon+\delta}(TM^n_{\gamma(0)})$ regarded as a subset of $(TM^n_{\gamma(0)})_*$ is not less than three times the volume of the metric ball of radius δ centered at $\gamma(0)$ in M^n. The definition of δ implies that this volume is not less than $\frac{3}{2}v_n(\delta)$. On the other hand, the Bishop volume comparison theorem (or, more precisely, Proposition in Section 2) implies that this volume does not exceed $v_n(\epsilon+\delta)$, which is less than $\frac{3}{2}v_n(\delta)$. (The last inequality follows from the fact that $\epsilon < \epsilon_\delta$.) Juxtaposing these two inequalities we obtain the desired contradiction. QED.

Note that we used the lower bound for the Ricci curvature in this proof only to majorize the volume of $B_{\epsilon+\delta}(TM^n_{\gamma(0)})$ regarded as a subset of $(TM^n_{\gamma(0)})_*$ by the volume of the ball of the same radius in the hyperbolic space. Therefore, essentially the same argument proves the following slightly more general theorem. Denote the volume of a metric ball of radius t in the simply-connected space of constant sectional curvature k by $v_{n,k}(t)$. It is well-known that $v_{n,k}(t) = \frac{\pi^{\frac{n}{2}} n}{\Gamma(\frac{n}{2}+1)} \int_0^t S_k(\tau)^{n-1} d\tau$, where $S_k(t) = \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}$, if $k < 0$, $S_k(t) = t$, if $k = 0$, and $S_k(t) = \frac{\sin(\sqrt{k}t)}{\sqrt{k}}$, if $k > 0$ (cf. [2]).

Theorem 4.A. Let M^n be a closed Riemannian manifold. Let k denote the infimum of the Ricci curvature of M^n divided by $n-1$. Define δ_0 as the supremum of all positive numbers τ such that the volume of every metric ball of radius τ in M^n is not less than $\frac{1}{2}v_{n,k}(\tau)$. Let δ be a positive number less than δ_0, and ϵ_δ be the solution of the equation $v_{n,k}(\epsilon_\delta + \delta) = \frac{3}{2}v_{n,k}(\delta)$. Then every closed curve γ of length $\epsilon < \epsilon_\delta$ can be contracted to a point via loops based at $\gamma(0)$ of length $\leq 2\epsilon + 2\delta$.

5. A concluding remark.

The main idea behind our improvements of Anderson's theorem is to use the tangent space of M^n with a (pseudo-) Riemannian metric defined as the pullback of the Riemannian metric on M^n under the exponential map. Our rationale can be explained as follows. The universal covering space of M^n is usually constructed by considering all paths emanating from a fixed base point $p \in M^n$. Two paths are being identified into the same point of the universal covering if they have the common endpoints and together form a contractible loop. An almost obvious observation is that here it is sufficient to consider only geodesics emanating from p. Indeed, every path emanating from p can be connected with a geodesic with the same endpoints by a (length non-increasing) homotopy. So, the path and the geodesic will correspond to the same point of the universal covering. Thus, we have a surjective map from the tangent space TM^n_p that parametrizes all geodesics on M^n emanating from p to the universal covering space of M^n. Informally, one can say that TM^n_p is "not smaller" than the universal covering space of M^n. Yet the Bishop volume comparison theorem is still true for TM^n_p with the pullback metric and measure (see Proposition in section 2). Therefore, this space seems to be a natural replacement for the universal covering space of M^n in arguments involving volume comparison on the latter, if by some reasons the latter cannot be used.

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