Limit Theorems for Translation Flows.

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Abstract

The aim of this paper is to obtain an asymptotic expansion for ergodic integrals of translation flows on flat surfaces of higher genus (Theorem 1) and to give a limit theorem for these flows (Theorem 2).

Contents

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1 Introduction.

1.1 Outline of the results.

A compact Riemann surface endowed with an abelian differential admits two natural flows, called, respectively, *horizontal* and *vertical.* One of the main objects of this paper is the space $\mathfrak{B}^+$ of Hölder cocycles over the vertical flow, invariant under the holonomy by the horizontal flow. Equivalently, cocycles in $\mathfrak{B}^+$ can be viewed, in the spirit of F. Bonahon [5], [6], as finitely-additive transverse invariant measures for the horizontal foliation of our abelian differential. Cocycles in $\mathfrak{B}^+$ are closely connected to the invariant distributions for translation flows in the sense of G.Forni [12].

The space $\mathfrak{B}^+$ is finite-dimensional, and for a generic abelian differential the dimension of $\mathfrak{B}^+$ is equal to the genus of the underlying surface. Theorem 1, which extends earlier work of A.Zorich [33] and G. Forni [12], states that the time integral of a Lipschitz function under the vertical flow can be uniformly approximated by a suitably chosen cocycle from $\mathfrak{B}^+$ up to an error that grows more slowly than any power of time. The renormalizing action of the Teichmüller flow on the space of Hölder cocycles now allows one to obtain limit theorems for translation flows on flat surfaces (Theorem 2).

The statement of Theorem 2 can be informally summarized as follows. Taking the leading term in the asymptotic expansion of Theorem 1, to a generic abelian differential one assigns a compactly supported probability measure on the space of continuous functions on the unit interval. The normalized distribution of the time integral of a Lipschitz function converges, with respect to weak topology, to the trajectory of the corresponding "asymptotic distribution" under the action of the Teichmüller flow. Convergence is exponential with respect to the Lévy-Prohorov metric.

The cocycles in $\mathfrak{B}^+$ are constructed explicitly using a symbolic representation for translation flows as suspension flows over Vershik's automorphisms. Vershik's Theorem [30] states that every ergodic automorphism of a Lebesgue probability space can be represented as a Vershik's automorphism of a Markov compactum. For interval exchange transformations an explicit representation can be obtained using Rohlin towers given by Rauzy-Veech induction. Passing to Veech's zippered rectangles and their bi-infinite Rauzy-Veech expansions, one represents a minimal translation flow as a flow along the leaves of the asymptotic foliation of a bi-infinite Markov compactum. In this representation, cocycles in $\mathfrak{B}^+$ become finitely-invariant measures on the asymptotic foliations of a Markov compactum.

Thus, after passage to a finite cover (namely, the Veech space of zippered rectangles), the moduli space of abelian differentials is represented as a space of Markov compacta. The Teichmüller flow and the Kontsevich-Zorich cocycle admit a simple description in terms of this symbolic representation, and the cocycles in $\mathfrak{B}^+$ are constructed explicitly. Theorems 1, 2 are then derived from their symbolic counterparts, Theorems 4, 5.

1.2 Hölder cocycles over translation flows.

Let $\rho \geq 2$ be an integer, let M be a compact orientable surface of genus ρ, and let ω be a holomorphic one-form on M. Denote by $\mathbf{m} = (\omega \wedge \overline{\omega})/2i$ the area form induced by ω and assume that $\mathbf{m}(M) = 1$.

Let h_t^+ be the *vertical* flow on M (i.e., the flow corresponding to $\Re(\omega)$); let h_t^- be the *horizontal* flow on M (i.e., the flow corresponding to $\Im(\omega)$). The flows h_t^+, h_t^- preserve the area $\mathbf{m}$.

Take $x \in M$, $t_1, t_2 \in \mathbb{R}_+$ and assume that the closure of the set

$$\{h^+_{\tau_1} h^-_{\tau_2} x, 0 \leq \tau_1 < t_1, 0 \leq \tau_2 < t_2\} \tag{1}$$

does not contain zeros of the form ω. The set (1) is then called *an admissible rectangle* and denoted $\Pi(x, t_1, t_2)$. Let $\overline{\mathfrak{C}}$ be the semi-ring of admissible rectangles.

Consider the linear space $\mathfrak{B}^+$ of Hölder cocyles $\Phi^+(x, t)$ over the vertical flow h_t^+ which are invariant under horizontal holonomy. More precisely, a function $\Phi^+(x, t) : M \times \mathbb{R} \to \mathbb{R}$ belongs to the space $\mathfrak{B}^+$ if it satisfies:

Assumption 1. *1.* $\Phi^+(x, t + s) = \Phi^+(x, t) + \Phi^+(h_t^+ x, s)$*;*

2. There exists $t_0 > 0$*,* $\theta > 0$ *such that* $|\Phi^+(x, t)| \leq t^\theta$ *for all* $x \in M$ *and all* $t \in \mathbb{R}$ *satisfying* $|t| < t_0$*;*

3. If $\Pi(x, t_1, t_2)$ *is an admissible rectangle, then* $\Phi^+(x, t_1) = \Phi^+(h^-_{t_2} x, t_1)$.

For example, a cocycle Φ_1^+ defined by $\Phi_1^+(x, t) = t$ belongs to $\mathfrak{B}^+$.

In the same way define the space $\mathfrak{B}^-$ of Hölder cocyles $\Phi^-(x, t)$ over the horizontal flow h_t^- which are invariant under vertical holonomy, and set $\Phi_1^-(x, t) = t$.

Given $\Phi^+ \in \mathfrak{B}^+$, $\Phi^- \in \mathfrak{B}^-$, a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\overline{\mathfrak{C}}$ of admissible rectangles is introduced by the formula

$$\Phi^+ \times \Phi^-(\Pi(x, t_1, t_2)) = \Phi^+(x, t_1) \cdot \Phi^-(x, t_2). \tag{2}$$

In particular, for $\Phi^- \in \mathfrak{B}^-$, set $m_{\Phi^-} = \Phi_1^+ \times \Phi^-$:

$$m_{\Phi^-}(\Pi(x, t_1, t_2)) = t_1 \Phi^-(x, t_2). \tag{3}$$

For any $\Phi^- \in \mathfrak{B}^-$ the measure m_{Φ^-} satisfies $(h_t^+)_* m_{\Phi^-} = m_{\Phi^-}$ and is an invariant distribution in the sense of G. Forni [11], [12]. For instance, $m_{\Phi_1^-} = \mathbf{m}$.

An $\mathbb{R}$-linear pairing between $\mathfrak{B}^+$ and $\mathfrak{B}^-$ is given, for $\Phi^+ \in \mathfrak{B}^+$, $\Phi^- \in \mathfrak{B}^-$, by the formula

$$\langle \Phi^+, \Phi^- \rangle = \Phi^+ \times \Phi^-(M). \tag{4}$$

1.3 Characterization of cocycles.

Let $\mathfrak{B}_c^+(\mathbf{X})$ be the space of continuous holonomy-invariant cocycles: more precisely, a function $\Phi^+(x, t) : M \times \mathbb{R} \to \mathbb{R}$ belongs to the space $\mathfrak{B}_c^+(\mathbf{X})$ if it satisfies conditions 1 and 3 of Assumption 1, while condition 2 is replaced by the following weaker version:

For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\Phi^+(x,t)| \le \varepsilon$ for all $x \in M$ and all $t \in \mathbb{R}$ satisfying $|t| < \delta$.

Given an abelian differential $\mathbf{X} = (M, \omega)$, we construct an explicit mapping of $\mathfrak{B}_c^+(M,\omega)$ to $H^1(M,\mathbb{R})$ in the following way.

A continuous closed curve γ on M is called *rectangular* if

$$\gamma = \gamma_1^+ \sqcup \cdots \sqcup \gamma_{k_1}^+ \bigsqcup \gamma_1^- \sqcup \cdots \sqcup \gamma_{k_2}^-,$$

where γ_i^+ are arcs of the flow h_t^+, γ_i^- are arcs of the flow h_t^-.

For $\Phi^+ \in \mathfrak{B}_c^+$ define

$$\Phi^+(\gamma) = \sum_{i=1}^{k_1} \Phi^+(\gamma_i^+);$$

similarly, for $\Phi^- \in \mathfrak{B}_c^-$ write

$$\Phi^-(\gamma) = \sum_{i=1}^{k_2} \Phi^-(\gamma_i^-).$$

Thus, a cocycle $\Phi^+ \in \mathfrak{B}_c$ assigns a number $\Phi^+(\gamma)$ to every closed rectangular curve γ. It is shown in Proposition 39 below that if γ is homologous to γ', then $\Phi^+(\gamma) = \Phi^+(\gamma')$. For an abelian differential $\mathbf{X} = (M,\omega)$, we thus obtain maps

$$\check{\mathcal{I}}_{\mathbf{X}}^+ : \mathfrak{B}_c^+(\mathbf{X}) \to H^1(M,\mathbb{R}),\ \check{\mathcal{I}}_{\mathbf{X}}^- : \mathfrak{B}_c^-(\mathbf{X}) \to H^1(M,\mathbb{R}). \tag{5}$$

For a generic abelian differential, the image of $\mathfrak{B}^+$ under the map $\check{\mathcal{I}}_{\mathbf{X}}^+$ is the strictly unstable space of the Konstevich-Zorich cocycle over the Teichmüller flow.

More precisely, let $\kappa = (\kappa_1, \ldots, \kappa_\sigma)$ be a nonnegative integer vector such that $\kappa_1 + \cdots + \kappa_\sigma = 2\rho - 2$. Denote by $\mathcal{M}_\kappa$ the moduli space of pairs (M,ω), where M is a Riemann surface of genus ρ and ω is a holomorphic differential of area 1 with singularities of orders $k_1, \ldots, k_\sigma$. The space $\mathcal{M}_\kappa$ is often called the *stratum* in the moduli space of abelian differentials.

The Teichmüller flow $\mathbf{g}_s$ on $\mathcal{M}_\kappa$ sends the modulus of a pair (M,ω) to the modulus of the pair (M,ω'), where $\omega' = e^s\Re(\omega) + ie^{-s}\Im(\omega)$; the new complex structure on M is uniquely determined by the requirement that the form ω' be holomorphic. As shown by Veech, the space $\mathcal{M}_\kappa$ need not be connected; let $\mathcal{H}$ be a connected component of $\mathcal{M}_\kappa$.

Let $\mathbb{H}^1(\mathcal{H})$ be the fibre bundle over $\mathcal{H}$ whose fibre at a point (M,ω) is the cohomology group $H^1(M,\mathbb{R})$. The bundle $\mathbb{H}^1(\mathcal{H})$ carries the *Gauss-Manin connection* which declares continuous integer-valued sections of our bundle to be flat and is uniquely defined by that requirement. Parallel transport with respect to the Gauss-Manin connection along the orbits of the Teichmüller flow yields a cocycle over the Teichmüller flow, called the *Kontsevich-Zorich cocycle* and denoted $\mathbf{A} = \mathbf{A}_{KZ}$.

Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$. For $\mathbf{X} \in \mathcal{H}$, $\mathbf{X} = (M, \omega)$, let $\mathfrak{B}^+_{\mathbf{X}}$, $\mathfrak{B}^-_{\mathbf{X}}$ be the corresponding spaces of Hölder cocycles.

Denote by $E^u_{\mathbf{X}} \subset H^1(M, \mathbb{R})$ the space spanned by vectors corresponding to the positive Lyapunov exponents of the Kontsevich-Zorich cocycle, by $E^s_{\mathbf{X}} \subset H^1(M, \mathbb{R})$ the space spanned by vectors corresponding to the negative exponents of the Kontsevich-Zorich cocycle.

Proposition 1. *For $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$ the map $\check{\mathcal{I}}^+_{\mathbf{X}}$ takes $\mathfrak{B}^+_{\mathbf{X}}$ isomorphically onto $E^u_{\mathbf{X}}$, the map $\check{\mathcal{I}}^-_{\mathbf{X}}$ takes $\mathfrak{B}^-_{\mathbf{X}}$ isomorphically onto $E^s_{\mathbf{X}}$.*

The pairing $\langle , \rangle$ is nondegenerate and is taken by the isomorphisms $\mathcal{I}^+_{\mathbf{X}}$, $\mathcal{I}^-_{\mathbf{X}}$ to the cup-product in the cohomology $H^1(M, \mathbb{R})$.

Remark. In particular, if $\mathbb{P}$ is the Masur-Veech "smooth" measure [21, 24], then $\dim \mathfrak{B}^+_{\mathbf{X}} = \dim \mathfrak{B}^-_{\mathbf{X}} = \rho$.

Remark. The isomorphisms $\check{\mathcal{I}}^+_{\mathbf{X}}$, $\check{\mathcal{I}}^-_{\mathbf{X}}$ are analogues of G. Forni's isomorphism [12] between his space of invariant distributions and the unstable space of the Kontsevich-Zorich cocycle.

Consider the inverse isomorphisms

$$\mathcal{I}^+_{\mathbf{X}} = \left(\check{\mathcal{I}}^+_{\mathbf{X}}\right)^{-1}; \ \mathcal{I}^-_{\mathbf{X}} = \left(\check{\mathcal{I}}^-_{\mathbf{X}}\right)^{-1}.$$

Let $1 = \theta_1 > \theta_2 > \cdots > \theta_l > 0$ be the distinct positive Lyapunov exponents of the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$, and let

$$E^u_{\mathbf{X}} = \bigoplus_{i=1}^{l} E^u_{\mathbf{X},\theta_i}$$

be the corresponding Oseledets decomposition at $\mathbf{X}$.

Proposition 2. *Let $v \in E^u_{\mathbf{X},\theta_i}$, $v \neq 0$, and denote $\Phi^+ = \mathcal{I}^+_{\mathbf{X}}(v)$. Then for any $\varepsilon > 0$ the cocycle Φ^+ satisfies the Hölder condition with exponent $\theta_i - \varepsilon$ and for any $x \in M(\mathbf{X})$ we have*

$$\limsup_{T \to \infty} \frac{\log |\Phi^+(x, T)|}{\log T} = \theta_i.$$

Proposition 3. *If the Kontsevich-Zorich cocycle does not have zero Lyapunov exponent with respect to $\mathbb{P}$, then $\mathfrak{B}^+_c(\mathbf{X}) = \mathfrak{B}^+(\mathbf{X})$.*

Remark. The condition of the absence of zero Lyapunov exponents can be weakened: it suffices to require that the Kontsevich-Zorich cocycle act isometrically on the neutral Oseledets subspace corresponding to the Lyapunov exponent zero. Isometric action means here that there exists an inner product which depends measurably on the point in the stratum and which is invariant under the Kontsevich-Zorich cocycle.

Question. Does there exist a $\mathbf{g}_s$-invariant ergodic probability measure $\mathbb{P}'$ on $\mathcal{H}$ such that the inclusion $\mathfrak{B}^+ \subset \mathfrak{B}^+_c$ is proper almost surely with respect to $\mathbb{P}'$?

1.4 Approximation of weakly Lipschitz functions.

1.4.1 The space of weakly Lipschitz functions.

The space of Lipschitz functions is not invariant under h_t^+, and a larger function space $Lip_w^+(M,\omega)$ of weakly Lipschitz functions is introduced as follows. A bounded measurable function f belongs to $Lip_w^+(M,\omega)$ if there exists a constant C, depending only on f, such that for any admissible rectangle $\Pi(x,t_1,t_2)$ we have

$$\left|\int_0^{t_1} f(h_t^+x)dt - \int_0^{t_1} f(h_t^+(h_{t_2}^-x)dt\right| \le C. \tag{6}$$

Let C_f be the infimum of all C satisfying (6). We norm $Lip_w^+(M,\omega)$ by setting

$$||f||_{Lip_w^+} = \sup_M f + C_f.$$

By definition, the space $Lip_w^+(M,\omega)$ contains all Lipschitz functions on M and is invariant under h_t^+. We denote by $Lip_{w,0}^+(M,\omega)$ the subspace of $Lip_w^+(M,\omega)$ of functions whose integral with respect to $\mathbf{m}$ is 0.

For any $f \in Lip_w^+(M,\omega)$ and any $\Phi^- \in \mathfrak{B}^-$ the integral $\int_M f dm_{\Phi^-}$ can be defined as the limit of Riemann sums.

1.4.2 The cocycle corresponding to a weakly Lipschitz function.

If the pairing $\langle,\rangle$ induces an isomorphism between $\mathfrak{B}^+$ and the dual $(\mathfrak{B}^-)^*$, then one can assign to a function $f \in Lip_w^+(M,\omega)$ the functional Φ_f^+ by the formula

$$\langle \Phi_f^+, \Phi^- \rangle = \int\limits_M f dm_{\Phi^-}, \Phi^- \in \mathfrak{B}^-. \tag{7}$$

By definition, $\Phi^+_{f\circ h_t^+} = \Phi_f^+$.

Theorem 1. *Let $\mathbb{P}$ be an ergodic probability $\mathbf{g}_s$-invariant measure on $\mathcal{H}$. For any $\varepsilon > 0$ there exists a constant C_ε depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, any $f \in Lip_w^+(\mathbf{X})$, any $x \in M$ and any $T > 0$ we have*

$$\left|\int_0^T f\circ h_t^+(x)dt - \Phi_f^+(x,T)\right| \le C_\varepsilon ||f||_{Lip_w^+}(1+T^\varepsilon).$$

1.4.3 Invariant measures with simple Lyapunov spectrum.

Consider the case in which the Lyapunov spectrum of the Kontsevich-Zorich cocycle is simple in restriction to the space E^u (as, by the Avila-Viana theorem [3], is the case with the Masur-Veech smooth measure). Let $l_0 = \dim E^u$ and let

$$1 = \theta_1 > \theta_2 > \cdots > \theta_{l_0} \tag{8}$$

be the corresponding simple expanding Lyapunov exponents.

Let Φ_1^+ be given by the formula $\Phi_1^+(x,t) = t$ and introduce a basis

$$\Phi_1^+, \Phi_2^+, \ldots, \Phi_{l_0}^+ \tag{9}$$

in $\mathfrak{B}_{\mathbf{X}}^+$ in such a way that $\check{\mathcal{I}}_{\mathbf{X}}^+(\Phi_i^+)$ lies in the Lyapunov subspace with exponent θ_i. By Proposition 2, for any $\varepsilon > 0$ the cocycle Φ_i^+ satisfies the Hölder condition with exponent $\theta_i - \varepsilon$, and for any $x \in M(\mathbf{X})$ we have

$$\limsup_{T\to\infty} \frac{\log|\Phi_i^+(x,T)|}{\log T} = \theta_i.$$

Let $\Phi_1^-, \ldots, \Phi_{l_0}^-$ be the dual basis in $\mathfrak{B}_{\mathbf{X}}^-$. Clearly, $\Phi_1^-(x,t) = t$.
By definition, we have

$$\Phi_f^+ = \sum_{i=1}^{l_0} m_{\Phi_i^-}(f)\Phi_i^+. \tag{10}$$

Noting that by definition we have

$$m_{\Phi_1^-} = \mathbf{m},$$

we derive from Theorem 1 the following corollary.

Corollary 1. *Let $\mathbb{P}$ be an invariant ergodic probability measure for the Teichmüller flow such that with respect to $\mathbb{P}$ the Lyapunov spectrum of the Kontsevich-Zorich cocycle is simple in restriction to its strictly expanding subspace.*

Then for any $\varepsilon > 0$ there exists a constant C_ε depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, any $f \in Lip_w^+(\mathbf{X})$, any $x \in \mathbf{X}$ and any $T > 0$ we have

$$\left| \int_0^T f \circ h_t^+(x)dt - T(\int_M f d\mathbf{m}) - \sum_{i=2}^{l_0} m_{\Phi_i^-}(f)\Phi_i^+(x,T) \right| \leq C_\varepsilon ||f||_{Lip_w^+}(1+T^\varepsilon).$$

Remark. If $\mathbb{P}$ is the Masur-Veech smooth measure on $\mathcal{H}$, then it follows from the work of G.Forni [11], [12], [13] and S. Marmi, P. Moussa, J.-C. Yoccoz [19] that the left-hand side is bounded for any $f \in C^{1+\varepsilon}(M)$ (in fact, for any f in the Sobolev space $H^{1+\varepsilon}$). In particular, if $f \in C^{1+\varepsilon}(M)$ and $\Phi_f^+ = 0$, then f is a coboundary.

1.5 Holonomy invariant transverse finitely-additive measures for oriented measured foliations.

Holonomy-invariant cocycles assigned to an abelian differential can be interpreted as transverse invariant measures for its foliations in the spirit of Bonahon [5], [6].

Let M be a compact oriented surface of genus at least two, and let $\mathcal{F}$ be a minimal oriented measured foliation on M. Denote by $\mathbf{m}_{\mathcal{F}}$ the transverse invariant measure of $\mathcal{F}$. If $\gamma = \gamma(t), t \in [0,T]$ is a smooth curve on M, and s_1, s_2 satisfy $0 \le s_1 < s_2 \le T$, then we denote by $\mathrm{res}_{[s_1,s_2]}\gamma$ the curve $\gamma(t), t \in [s_1, s_2]$.

Let $\mathfrak{B}_c(\mathcal{F})$ be the space of uniformly continuous finitely-additive transverse invariant measures for $\mathcal{F}$. In other words, a map Φ which to every smooth arc γ transverse to $\mathcal{F}$ assigns a real number $\Phi(\gamma)$ belongs to the space $\mathfrak{B}_c(\mathcal{F})$ if it satisfies the following:

Assumption 2. *1. (finite additivity) For $\gamma = \gamma(t), t \in [0,T]$ and any $s \in (0,T)$, we have*

$$\Phi(\gamma) = \Phi(\mathrm{res}_{[0,s]}\gamma) + \Phi(\mathrm{res}_{[s,T]}\gamma);$$

2. *(uniform continuity) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any transverse arc γ satisfying $\mathbf{m}_{\mathcal{F}}(\gamma) < \delta$ we have $|\Phi(\gamma)| < \varepsilon$;*

3. *(holonomy invariance) the value $\Phi(\gamma)$ does not change if γ is deformed in such a way that it stays transverse to $\mathcal{F}$ while the endpoints of γ stay on their respective leaves.*

A measure $\Phi \in \mathfrak{B}_c(\mathcal{F})$ is called Hölder with exponent θ if there exists $\varepsilon_0 > 0$ such that for any transverse arc γ satisfying $\mathbf{m}_{\mathcal{F}}(\gamma) < \varepsilon_0$ we have

$$|\Phi(\gamma)| \le (\mathbf{m}_{\mathcal{F}}(\gamma))^{\theta}.$$

Let $\mathfrak{B}(\mathcal{F}) \subset \mathfrak{B}_c(\mathcal{F})$ be the subspace of Hölder transverse measures.

As before, we have a natural map

$$\mathcal{I}_{\mathcal{F}} : \mathfrak{B}_c(\mathcal{F}) \to H^1(M, \mathbb{R})$$

defined as follows. For a smooth closed curve γ on M, and a measure $\Phi \in \mathfrak{B}_c(\mathcal{F})$ the integral $\int_\gamma d\Phi$ is well-defined as the limit of Riemann sums; by holonomy-invariance and continuity of Φ, this operation descends to homology and assigns to Φ an element of $H^1(M, \mathbb{R})$.

Now take an abelian differential $\mathbf{X} = (M, \omega)$ and let $\mathcal{F}^-_{\mathbf{X}}$ be its horizontal foliation. We have a "tautological" isomorphism between $\mathfrak{B}_c(\mathcal{F}^-_{\mathbf{X}})$ and $\mathfrak{B}^+_c(\mathbf{X})$: every transverse measure for the horizontal foliation induces a cocycle for the vertical foliation and vice versa; to a Hölder measure corresponds a Hölder cocycle. For brevity, write $\mathcal{I}_{\mathbf{X}} = \mathcal{I}_{\mathcal{F}^-_{\mathbf{X}}}$. Denote by $E^u_{\mathbf{X}} \subset H^1(M, \mathbb{R})$ the unstable subspace of the Kontsevich-Zorich cocycle of the abelian differential $\mathbf{X} = (M, \omega)$.

Theorem 1 and Proposition 3 yield the following

Corollary 2. *Let $\mathbb{P}$ be an ergodic probability measure for the Teichmüller flow $\mathbf{g}_t$ on $\mathcal{H}$. Then for almost every abelian differential $\mathbf{X} \in \mathcal{H}$ the map $\mathcal{I}_{\mathbf{X}}$ takes $\mathfrak{B}(\mathcal{F}^-_{\mathbf{X}})$ isomorphically onto $E^u_{\mathbf{X}}$.*

If all the Lyapunov exponents of the Kontsevich-Zorich cocycle are nonzero with respect to $\mathbb{P}$, then for almost all $\mathbf{X} \in \mathcal{H}$ we have $\mathfrak{B}_c(\mathcal{F}_{\mathbf{X}}) = \mathfrak{B}(\mathcal{F}_{\mathbf{X}})$.

In other words, in the absence of zero Lyapunov exponents all continuous transverse invariant measures are in fact Hölder.

Remark. As before, the condition of the absence of zero Lyapunov exponents can be weakened: it suffices to require that the Kontsevich-Zorich cocycle act isometrically on the Oseledets subspace corresponding to the Lyapunov exponent zero.

By definition, the space $\mathfrak{B}(\mathcal{F}_{\mathbf{X}}^{-})$ only depends on the horizontal foliation of our abelian differential; so does $E_{\mathbf{X}}^{u}$.

1.6 Finitely-additive invariant measures for interval exchange transformations.

Let $m \in \mathbb{N}$. Let Δ_{m-1} be the standard unit simplex

$$\Delta_{m-1} = \{\lambda \in \mathbb{R}_{+}^{m},\ \lambda = (\lambda_1, \ldots, \lambda_m), \lambda_i > 0, \sum_{i=1}^{m} \lambda_i = 1\}.$$

Let π be a permutation of $\{1, \ldots, m\}$ satisfying the *irreducibility* condition: we have $\pi\{1, \ldots, k\} = \{1, \ldots, k\}$ if and only if $k = m$.

On the half-open interval $I = [0, 1)$ consider the points $\beta_1 = 0$, $\beta_i = \sum_{j<i} \lambda_j$, $\beta_1^{\pi} = 0$, $\beta_i^{\pi} = \sum_{j<i} \lambda_{\pi^{-1}j}$ and denote $I_i = [\beta_i, \beta_{i+1})$, $I_i^{\pi} = [\beta_i^{\pi}, \beta_{i+1}^{\pi})$. The length of I_i is λ_i, while the length of I_i^{π} is $\lambda_{\pi^{-1}i}$. Set

$$\mathbf{T}_{(\lambda,\pi)}(x) = x + \beta_{\pi i}^{\pi} - \beta_i \text{ for } x \in I_i.$$

The map $\mathbf{T}_{(\lambda,\pi)}$ is called *an interval exchange transformation* corresponding to (λ, π). By definition, the map $\mathbf{T}_{(\lambda,\pi)}$ is invertible and preserves the Lebesgue measure on I. By the theorem of Masur [21] and Veech [24], for any irreducible permutation π and for Lebesgue-almost all $\lambda \in \Delta_{m-1}$, the corresponding interval exchange transformation $\mathbf{T}_{(\lambda,\pi)}$ is uniquely ergodic: the Lebesgue measure is the only invariant probability measure for $\mathbf{T}_{(\lambda,\pi)}$.

Consider the space of complex-valued continuous finitely-additive invariant measures for $\mathbf{T}_{(\lambda,\pi)}$.

More precisely, let $\mathfrak{B}(\lambda, \pi)$ the space of all continuous functions $\Phi : [0, 1] \to \mathbb{R}$ satisfying

1. $\Phi(0) = 0$;
2. if $0 \le t_1 \le t_2 < 1$ and $\mathbf{T}_{(\lambda,\pi)}$ is continuous on $[t_1, t_2]$, then $\Phi(t_1) - \Phi(t_2) = \Phi(\mathbf{T}_{(\lambda,\pi)}(t_1)) - \Phi(\mathbf{T}_{(\lambda,\pi)}(t_2))$.

Each function Φ induces a finitely-additive measure on $[0, 1]$ defined on the semi-ring of subintervals (for instance, the function $\Phi_1(t) = t$ yields the Lebesgue measure on $[0, 1]$).

Let $\mathcal{R}$ be the Rauzy class of the permutation π.

Proposition 4. *For any irreducible permutation π and for Lebesgue-almost all λ all functions in the space $\mathfrak{B}(\lambda, \pi)$ are Hölder. For any irreducible permutation π there exists a natural number $\rho = \rho(\mathcal{R})$ depending only on the Rauzy class of π and such that for Lebesgue-almost all λ we have* $\dim \mathfrak{B}(\lambda, \pi) = \rho$.

If f is a Lipschitz function on $[0,1]$, and Φ a Hölder function on $[0,1]$, then the Riemann-Stieltjes integral $\int_0^1 f d\Phi$ is well-defined. The following proposition extends the Theorem of Zorich [33].

Proposition 5. *For any Rauzy class $\mathcal{R}$ of irreducible permutations there exist constants*

$$1 = \theta_1(\mathcal{R}) > \theta_2(\mathcal{R}) > \cdots > \theta_{\rho(\mathcal{R})} > 0 = \theta_{\rho(\mathcal{R})+1}$$

such that for every $\pi \in \mathcal{R}$, and Lebesgue-almost every λ the space $\mathfrak{B}(\lambda, \pi)$ admits a basis $\Phi_1, \ldots, \Phi_{\rho(\mathcal{R})}$ such that the following holds:

1. *$\Phi_1(t) = t$, and for any $\varepsilon > 0$ the function Φ_i is Hölder with exponent $\theta_i - \varepsilon$*
2. *for every $x \in [0,1]$ and every Lipschitz f on $[0,1]$ we have*

$$\limsup_{N\to\infty} \frac{\log \left| \sum\limits_{k=0}^{N-1} f \circ \mathbf{T}^k_{(\lambda,\pi)}(x) \right|}{\log N} = \theta_{i(f)+1},$$

where $i(f) = \max\{i : \int_0^1 f d\Phi_j = 0 \text{ for all } j \le i\}$.

The numbers $\rho(\mathcal{R})$, $\theta_1(\mathcal{R}), \theta_2(\mathcal{R}), \ldots, \theta_{\rho(\mathcal{R})}$ admit an explicit characterization. To every Rauzy class $\mathcal{R}$ there corresponds [18] a connected component $\mathcal{H}(\mathcal{R})$ of the moduli space of abelian differentials with prescribed singularities; $\rho(\mathcal{R})$ is then the genus of the underlying surface, while $1 = \theta_1 > \theta_2(\mathcal{R}) > \cdots > \theta_\rho(\mathcal{R})$ are the positive Lyapunov exponents of the Kontsevich-Zorich cocycle on $\mathcal{H}(\mathcal{R})$ with respect to the Masur-Veech smooth measure (the simplicity of the Lyapunov spectrum was proved by Avila and Viana in [3]).

Remark. Objects related to finitely-additive measures for interval exchange transformations have been studied by X. Bressaud, P. Hubert and A. Maass in [7] and by S. Marmi, P. Moussa and J.-C. Yoccoz in [20].

1.7 Limit Theorems for Translation Flows.

1.7.1 Time integrals as random variables.

As before, (M, ω) is an abelian differential, and h_t^+, h_t^- are, respectively, its vertical and horizontal flows. Take $\tau \in [0,1]$, $s \in \mathbb{R}$, a real-valued $f \in Lip^+_{w,0}(M, \omega)$ and introduce the function

$$\mathfrak{S}[f, s; \tau, x] = \int_0^{\tau \exp(s)} f \circ h_t^+(x) dt. \tag{11}$$

For fixed f, s and x the quantity $\mathfrak{S}[f, s; \tau, x]$ is a continuous function of $\tau \in [0,1]$; therefore, as x varies in the probability space $(M, \mathbf{m})$, we obtain a random element of $C[0,1]$. In other words, we have a random variable

$$\mathfrak{S}[f, s] : (M, \mathbf{m}) \to C[0,1] \tag{12}$$

defined by the formula (11).

For any fixed $\tau \in [0,1]$ the formula (11) yields a real-valued random variable

$$\mathfrak{S}[f, s; \tau] : (M, \mathbf{m}) \to \mathbb{R}, \tag{13}$$

whose expectation, by definition, is zero.

Our first aim is to estimate the growth of its variance as $s \to \infty$. Without losing generality, one may take $\tau = 1$.

1.7.2 The growth rate of the variance.

Let $\mathbb{P}$ be an invariant ergodic probability measure for the Teichmüller flow such that with respect to $\mathbb{P}$ the second Lyapunov exponent θ_2 of the Kontsevich-Zorich cocycle is positive and simple (recall that, as Veech and Forni showed, the first one, $\theta_1 = 1$, is always simple [28, 12] and that, by the Avila-Viana theorem [3], the second one is simple for the Masur-Veech smooth measure).

For an abelian differential $\mathbf{X} = (M, \omega)$, denote by $E^+_{2,\mathbf{X}}$ the one-dimensional subspace in $H^1(M, \mathbb{R})$ corresponding to the second Lyapunov exponent θ_2, and let $\mathfrak{B}^+_{2,\mathbf{X}} = \mathcal{I}^+_{\mathbf{X}}(E^+_{2,\mathbf{X}})$. Similarly, denote by $E^-_{2,\mathbf{X}}$ the one-dimensional subspace in $H^1(M, \mathbb{R})$ corresponding to the Lyapunov exponent $-\theta_2$, and let $\mathfrak{B}^-_{2,\mathbf{X}} = \mathcal{I}^-_{\mathbf{X}}(E^-_{2,\mathbf{X}})$.

Recall that the space $H^1(M, \mathbb{R})$ is endowed with the Hodge norm $|\cdot|_H$; the isomorpshims $\mathcal{I}^\pm_{\mathbf{X}}$ take the Hodge norm to a norm on $\mathfrak{B}^\pm_{\mathbf{X}}$; slightly abusing notation, we denote the latter norm by the same symbol.

Introduce a multiplicative cocycle $H_2(s, \mathbf{X})$ over the Teichmüller flow $\mathbf{g}_s$ by taking $v \in E^+_{2,\mathbf{X}}$, $v \neq 0$, and setting

$$H_2(s, \mathbf{X}) = \frac{|\mathbf{A}(s, \mathbf{X})v|_H}{|v|_H}. \tag{14}$$

The Hodge norm is chosen only for concreteness in (14); any other norm can be used instead.

By definition, we have

$$\lim_{s \to \infty} \frac{\log H_2(s, \mathbf{X})}{s} = \theta_2. \tag{15}$$

Now take $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$ $\Phi_2^- \in \mathfrak{B}^-_{2,\mathbf{X}}$ in such a way that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$.

Proposition 6. *There exists positive measurable functions*

$$C : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+, \; V : \mathcal{H} \to \mathbb{R}_+, \; s_0 : \mathcal{H} \to \mathbb{R}_+$$

such that the following is true for $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$.

If $f \in Lip^+_{w,0}(\mathbf{X})$ satisfies $m_{\Phi_2^-}(f) \neq 0$, then for all $s \geq s_0(\mathbf{X})$ we have

$$\left| \frac{Var_{\mathbf{m}} \mathfrak{S}(f, x, e^s)}{V(\mathbf{g}_s \mathbf{X})(m_{\Phi_2^-}(f)|\Phi_2^+| H_2(\mathbf{X}, s))^2} - 1 \right| \leq C(\mathbf{X}, \mathbf{g}_s \mathbf{X}) \exp(-\alpha s). \tag{16}$$

Remark. Observe that the quantity $(m_{\Phi_2^-}(f)|\Phi^+|)^2$ does not depend on the specific choice of $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$, $\Phi_2^- \in \mathfrak{B}^-_{2,\mathbf{X}}$ such that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$.

Proposition 6 is based on

Proposition 7. *There exists a positive measurable function $V : \mathcal{H} \to \mathbb{R}_+$ such that for $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$, we have*

$$Var_{\mathbf{m}(\mathbf{X})}\Phi_2^+(x, e^s) = V(\mathbf{g}_s\mathbf{X})|\Phi_2^+|^2(H_2(\mathbf{X}, s))^2. \tag{17}$$

In particular $Var_{\mathbf{m}}\Phi_2^+(x, e^s) \neq 0$ for any $s \in \mathbb{R}$. The function $V(\mathbf{X})$ is given by

$$V(\mathbf{X}) = \frac{Var_{\mathbf{m}(\mathbf{X})}\Phi_2^+(x, 1)}{|\Phi_2^+|^2}.$$

Observe that the right-hand side does not depend on a particular choice of $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$, $\Phi_2^+ \neq 0$.

1.7.3 The limit theorem.

Go back to the $C[0,1]$-valued random variable $\mathfrak{S}[f, s]$ and denote by $\mathfrak{m}[f, s]$ the distribution of the normalized random variable

$$\frac{\mathfrak{S}[f, s]}{\sqrt{Var_{\mathbf{m}}\mathfrak{S}[f, s; 1]}}. \tag{18}$$

By definition, $\mathfrak{m}[f, s]$ is a Borel probability measure on $C[0, 1]$; furthermore, if $\xi = \xi(\tau) \in C[0, 1]$, then we have

1. $\xi(0) = 0$ almost surely with respect to $m[f, s]$;
2. $\mathbb{E}_{\mathfrak{m}[f,s]}\xi(\tau) = 0$ for all $\tau \in [0, 1]$;
3. $Var_{\mathfrak{m}[f,s]}\xi(1) = 1$.

We are interested in the weak accumulation points of $\mathfrak{m}[f, s]$ as $s \to \infty$.

Consider the space $\mathcal{H}'$ given by the formula

$$\mathcal{H}' = \{\mathbf{X}' = (M, \omega, v), v \in E_2^+(M, \omega), |v|_H = 1\}.$$

By definition, the space $\mathcal{H}'$ is a $\mathbb{P}$-almost surely defined two-to-one cover of the space $\mathcal{H}$. The skew-product flow of the Kontsevich-Zorich cocycle over the Teichmüller flow yields a flow $\mathbf{g}'_s$ on $\mathcal{H}'$ given by the formula

$$\mathbf{g}'_s(\mathbf{X}, v) = (\mathbf{g}_s\mathbf{X}, \frac{(\mathbf{A}(s, \mathbf{X})v}{|(\mathbf{A}(s, \mathbf{X})v|_H}).$$

Given $\mathbf{X}' \in \mathcal{H}'$, set

$$\Phi^+_{2,\mathbf{X}'} = \mathcal{I}^+(v).$$

Take $\tilde{v} \in E_2^-(M, \omega)$ such that $\langle v, \tilde{v} \rangle = 1$ and set

$$\Phi_{2,\mathbf{X}'}^- = \mathcal{I}^-(v), \ m_{2,\mathbf{X}'}^- = m_{\Phi_{2,\mathbf{X}'}}^-.$$

Let $\mathfrak{M}$ be the space of all probability distributions on $C[0,1]$ and introduce a $\mathbb{P}$-almost surely defined map $\mathcal{D}_2^+ : \mathcal{H}' \to \mathfrak{M}$ by setting $\mathcal{D}_2^+(\mathbf{X}')$ to be the distribution of the $C[0,1]$-valued normalized random variable

$$\frac{\Phi_{2,\mathbf{X}'}^+(x, \tau)}{\sqrt{Var_{\mathbf{m}} \Phi_{2,\mathbf{X}'}^+(x, 1)}}, \ \tau \in [0, 1].$$

By definition, $\mathcal{D}_2^+(\mathbf{X}')$ is a Borel probability measure on the space $C[0,1]$; it is, besides, a compactly supported measure as its support consists of equibounded Hölder functions with exponent $\theta_2/\theta_1 - \varepsilon$.

Consider the set $\mathfrak{M}_1$ of probability measures $\mathfrak{m}$ on $C[0,1]$ satisfying, for $\xi \in C[0,1]$, $\xi = \xi(t)$, the conditions:

1. the equality $\xi(0) = 0$ holds $\mathfrak{m}$-almost surely;
2. for all τ we have $\mathbb{E}_{\mathfrak{m}} \xi(\tau) = 0$:
3. we have $Var_{\mathfrak{m}} \xi(1) = 1$ and for any $\tau \neq 0$ we have $Var_{\mathfrak{m}} \xi(\tau) \neq 0$.

It will be proved in what follows that $\mathcal{D}_2^+(\mathcal{H}') \subset \mathfrak{M}_1$.

Consider a semi-flow J_s on the space $C[0,1]$ defined by the formula

$$J_s \xi(t) = \xi(e^{-s} t), \ s \geq 0.$$

Introduce a semi-flow G_s on $\mathfrak{M}_1$ by the formula

$$G_s \mathfrak{m} = \frac{(J_s)_* \mathfrak{m}}{Var_{\mathfrak{m}}(\xi(\exp(-s)))}, \mathfrak{m} \in \mathfrak{M}_1. \tag{19}$$

By definition, the diagram

$$\begin{array}{ccc} \mathcal{H}' & \xrightarrow{\mathcal{D}_2^+} & \mathfrak{M}_1 \\ \downarrow{\scriptstyle \mathbf{g}_s} & & \uparrow{\scriptstyle G_s} \\ \mathcal{H}' & \xrightarrow{\mathcal{D}_2^+} & \mathfrak{M}_1 \end{array}$$

is commutative.

Let d_{LP} be the Lévy-Prohorov metric on the space of probability measures on $C[0,1]$ (see, e.g., [4]).

We are now ready to formulate the main result.

Theorem 2. *Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$ such that the second Lyapunov exponent of the Kontsevich-Zorich cocycle is positive and simple with respect to $\mathbb{P}$.*

There exists a positive measurable function $C : \mathcal{H}' \times \mathcal{H}' \to \mathbb{R}_+$ and a positive constant α depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $\mathbf{X}' \in \mathcal{H}'$ and any $f \in Lip^+_{w,0}(\mathbf{X})$ satisfying $m^-_{2,\mathbf{X}'}(f) > 0$ we have

$$d_{LP}(\mathfrak{m}[f,s], \mathcal{D}_2^+(\mathbf{g}'_s\mathbf{X}')) \leq C(\mathbf{X}', \mathbf{g}'_s\mathbf{X}')\exp(-\alpha s). \tag{20}$$

Now fix $\tau \in \mathbb{R}$ and let $\mathfrak{m}_2(\mathbf{X}', \tau)$ be the distribution of the $\mathbb{R}$-valued random variable

$$\frac{\Phi^+_{2,\mathbf{X}'}(x,\tau)}{\sqrt{Var_{\mathbf{m}}\Phi^+_{2,\mathbf{X}'}(x,\tau)}}.$$

For brevity, write $\mathfrak{m}_2(\mathbf{X}', 1) = \mathfrak{m}_2(\mathbf{X}')$.

Proposition 8. *For $\mathbb{P}$-almost any $\mathbf{X}' \in \mathcal{H}'$, the measure $\mathfrak{m}_2(\mathbf{X}', \tau)$ admits atoms for a dense set of times $\tau \in \mathbb{R}$.*

By definition, $\mathfrak{m}_2(\mathbf{X}')$ is always compactly supported; the following Proposition shows, however, that the family

$$\{\mathfrak{m}_2(\mathbf{X}'), \mathbf{X}' \in \mathcal{H}'\}$$

is in general not closed. Let δ_0 stand for the delta-measure at zero.

Proposition 9. *Let $\mathcal{H}$ be endowed with the Masur-Veech smooth measure. Then the measure δ_0 is an accumulation point for the set $\{\mathfrak{m}_2(\mathbf{X}'), \mathbf{X}' \in \mathcal{H}'\}$ in the weak topology.*

1.8 A symbolic coding for translation flows.

1.8.1 Interval exchange transformations as Vershik's automorphisms.

Recall that, by Vershik's Theorem [30], every ergodic automorphism of a Lebesgue probability space can be represented as a Vershik's automorphism. The proof of Vershik's Theorem [30] proceeds by constructing an increasing sequence of Rohlin towers which intersect "in a Markov way". In the case of interval exchange transformations, such a sequence of towers is given, for instance, by the Rauzy-Veech induction; as a result, one obtains an explicit representation of minimal interval exchange transformations as Vershik's automorphisms (see [14]). In the next subsection, a bi-infinite variant of this construction will give a symbolic representation for translation flows on flat surfaces.

Let π be an irreducible permutation on m symbols and let $\mathbf{T} : [0,1) \to [0,1)$ be a minimal interval exchange transformation of m intervals with permutation π.

One can find a sequence of intervals $I^{(n)} = [0, b^{(n)})$, $n = 0, \ldots,$ such that

1. $\lim\limits_{n\to\infty} b^{(n)} = 0$;
2. $I^{(n+1)} \subset I^{(n)}$;

3. the induced map of T on $I^{(n)}$ is again an interval exchange of m subintervals.

Denote by $\mathbf{T}_n$ the induced map of $\mathbf{T}$ on $I^{(n)}$, and let $I_1^{(n)}, \ldots, I_m^{(n)}$, $I_i^{(n)} = [a_i^{(n)}, b_i^{(n)})$ be the subintervals of the interval exchange $\mathbf{T}_n$. By definition, we have $a_1^{(n)} = 0$, $b_i^{(n)}) = a_i^{(n+1)}$.

Now represent $I^{(n)}$ as a union of Rohlin towers over $I^{(n+1)}$ with respect to the map $\mathbf{T}_n$:

$$I^{(n)} = \bigsqcup_{i=1}^{m} \bigsqcup_{k=0}^{N_i^{(n+1)}-1} \mathbf{T}_n^k I_i^{(n+1)}. \tag{21}$$

Here $N_i^{(n+1)}$, the height of the tower, is the time of the first return of $I_i^{(n+1)}$ into $I^{(n)}$ under the map $\mathbf{T}_n$.

Let

$$\mathcal{E}_{n+1} = \{(i,k) : i \in \{1, \ldots, m\}, k \in \{0, \ldots, N_i^{(n+1)} - 1\}\}.$$

and for $e \in \mathcal{E}_{n+1}$, $e = (i,k)$, denote

$$J_e^{(n+1)} = \mathbf{T}_{n+1}^k I_i^{(n+1)}.$$

For any $e \in \mathcal{E}_{n+1}$ there exists a unique $j \in \{1, \ldots, m\}$ such that

$$J_e^{(n+1)} \subset I_j^{(n)}.$$

We denote $j = F(e)$; we also write $i = I(e)$ if $e = (i,k)$.

We thus have

$$I_j^{(n)} = \bigsqcup_{e \in \mathcal{E}_{n+1} : F(e) = j} J_e^{(n+1)}. \tag{22}$$

Now represent $I = [0,1)$ as a union of Rohlin towers over $I^{(n)}$ with respect to $\mathbf{T}$:

$$I = \bigsqcup_{i=1}^{m} \bigsqcup_{k=0}^{L_i^{(n+1)}-1} \mathbf{T}^k I_i^{(n)}. \tag{23}$$

Substituting (22) into (23), write

$$I = \bigsqcup_{i=1}^{m} \bigsqcup_{k=0}^{L_i^{(n+1)}-1} \bigsqcup_{e \in \mathcal{E}_{n+1} : F(e) = j} \mathbf{T}^k J_e^{(n+1)}. \tag{24}$$

Denote the resulting partition of I into subintervals by π_{n+1}. By definition, the maximal length of an element of π_n tends to 0 as $n \to \infty$, so the increasing sequence of partitions π_n tends (in the sense of Rohlin) to the partition into points. As usual, for $x \in I$, let $\pi_n(x)$ be the element of π_n containing x.

Introduce a function $\mathtt{i}_n : I \to \mathcal{E}_n$ by setting $\mathtt{i}_n(x) = e$ if $\pi_n(x)$ has the form $\mathbf{T}^l J_e^{(n)}$. A finite string $(e_1, \ldots, e_n)$, $e_i \in \mathcal{E}_i$, satisfying $F(e_i) = I(e_{i-1})$, will be called *admissible*.

Proposition 10. *Let* $(e_1, \ldots, e_n)$*,* $e_i \in \mathcal{E}_i$*, be an admissible string. There exists a unique interval* $J = J(e_1, \ldots, e_n)$ *such that*

1. J *is an element of the partition* π_n
2. *for any* $x \in J$ *we have* $\mathfrak{i}_l(x) = e_l$*,* $l = 1, \ldots, n$*.*

Conversely, any element of the partition π_n *has the form* $J(e_1, \ldots, e_n)$ *for a unique admissible string* $(e_1, \ldots, e_n)$*.*

The proof is immediate by induction.

Introduce the Markov compactum

$$Y = \{y = y_1 \ldots y_n \cdots : y_i \in \mathcal{E}_i, F(y_i) = I(y_{i-1})\}. \tag{25}$$

We thus have a natural map $p : Y \to [0, 1]$ which sends $y \in Y$ to the point

$$\bigcap_{n=1}^{\infty} \overline{J}(y_1, \ldots, y_n). \tag{26}$$

(here $\overline{J}$ stands for the closure of J). The map p is surjective and is indeed a bijection except at the endpoints of the intervals $J = J(e_1, \ldots, e_n)$, all of which, except 0 and 1, have two preimages. In particular, the map p is almost surely bijective with respect to the Lebesgue measure on $[0, 1]$; by definition, the image ν_Y of the Lebesgue measure on $[0, 1]$ under p^{-1} is a Markov measure on Y.

The Markov compactum Y also has the following additional structure. Each set $\mathcal{E}_n$ is partially ordered: for $e_1, e_2 \in \mathcal{E}_n$, $e_1 = (i_1, k_1)$, $e_2 = (i_2, k_2)$, we write $(i_1, k_1) < (i_2, k_2)$ if $i_1 = i_2$, $k_1 < k_2$.

This ordering induces a partial ordering $\mathfrak{o}$ on Y: we write $y < \tilde{y}$ if there exists n_0 such that $y_n = \tilde{y}_n$ for $n > n_0$ while $y_{n_0} < \tilde{y}_{n_0}$.

The map $p^{-1} \circ \mathbf{T} \circ p : Y \to Y$ is a Vershik's automorphism on Y with respect to the partial ordering $\mathfrak{o}$ (see [30, 32, 23]).

1.8.2 Translation flows as symbolic flows.

Let (M, ω) be an abelian differential such that both corresponding flows h_t^+ and h_t^- are minimal. A rectangle $\Pi(x, t_1, t_2) = \{h_{\tau_1}^+ h_{\tau_2}^- x, 0 \le \tau_1 < t_1, 0 \le \tau_2 < t_2\}$ is called *weakly admissible* if for all sufficiently small $\varepsilon > 0$ the rectangle $\Pi(h_\varepsilon^+ h_\varepsilon^- x, t_1 - \varepsilon, t_2 - \varepsilon)$ is admissible (in other words, the boundary of Π may contain zeros of ω but the interior does not).

There exists a sequence of partitions

$$M = \Pi_1^{(n)} \sqcup \ldots \Pi_m^{(n)}, \ n \in \mathbb{Z}, \tag{27}$$

where $\Pi_i^{(n)}$ are weakly admissible rectangles and for any $n_1, n_2 \in \mathbb{Z}$, $i_1, i_2 \in \{1, \ldots, m\}$, the rectangles $\Pi_{i_1}^{(n_1)}$ and $\Pi_{i_2}^{(n_2)}$ intersect in a Markov way in the following precise sense.

Take a weakly admissible rectangle $\Pi(x, t_1, t_2)$ and decompose its boundary into four parts:

$$\partial_h^1(\Pi) = \overline{\{}h_{t_1}^+ h_{\tau_2}^- x, 0 \le \tau_2 < t_2\};$$

$$\partial_h^0(\Pi) = \overline{\{}h_{\tau_2}^- x, 0 \le \tau_2 < t_2\};$$

$$\partial_v^1(\Pi) = \overline{\{}h_{t_2}^- h_{\tau_1}^+ x, 0 \le \tau_1 < t_1\};$$

$$\partial_v^0(\Pi) = \overline{\{}h_{\tau_1}^+ x, 0 \le \tau_1 < t_1\}.$$

The *Markov condition* is then the requirement that for any $n \in \mathbb{Z}$ and $i \in \{1, \dots, m\}$ there exist $i_1, i_2, i_3, i_4 \in \{1, \dots, m\}$ such that

$$\partial_h^1(\Pi_i^{(n)}) \subset \partial_h^1 \Pi_{i_1}^{n-1};$$

$$\partial_h^0(\Pi_i^{(n)}) \subset \partial_h^0 \Pi_{i_2}^{n-1};$$

$$\partial_v^1(\Pi_i^{(n)}) \subset \partial_v^1 \Pi_{i_3}^{n+1};$$

$$\partial_v^0(\Pi_i^{(n)}) \subset \partial_v^0 \Pi_{i_4}^{n+1}.$$

Furthermore, for a weakly admissible rectangle $\Pi = \Pi(x, t_1, t_2)$ we write $|\partial_h(\Pi) = t_2|$, $|\partial_v(\Pi)| = t_1$ and require

$$\lim_{n\to\infty} \max_{i=1,\dots,m} |\partial_v \Pi_i^{(n)}| = 0; \ \lim_{n\to\infty} \max_{i=1,\dots,m} |\partial_h \Pi_i^{(-n)}| = 0. \tag{28}$$

A sequence of partitions (27) satisfying the Markov condition and (28) exists by the minimality of the vertical and horizontal foliations; this sequence allows us to identify our surface M with the space of paths of a non-autonomous topological Markov chain.

Indeed, for $n \in \mathbb{Z}$, let $\mathcal{E}_n$ be the set of connected components of intersections $\Pi_i^{(n)} \cap \Pi_j^{(n-1)}$. If e is such a connected component , then we write $i = I(e)$, $j = F(e)$.

Now consider the set

$$X = \{x = \dots x_{-n} \dots x_n \dots, x_n \in \mathcal{E}_n, F(x_n) = I(x_{n-1}),\ n \in \mathbb{Z}\}.$$

Given a point $x \in X$, consider the intersection of closures of the corresponding connected components

$$\pi(x) = \bigcap \overline{x}_n. \tag{29}$$

Nonemptiness of this intersection follows from the Markov condition, while condition (28) implies that the intersection (29) is a point.

We therefore obtain a measurable map $\pi : X \to M$. It is immediate that $\mathbf{m}$-almost all points in M have exactly one preimage, and we thus obtain an almost sure identification of M and X. By construction, the measure $\mathbf{m}$ yields a Markov measure on X. Furthermore, by construction we immediately obtain the following

Proposition 11. *Let $n_0 \in \mathbb{Z}$. If x, x' are such that $x_n = x'_n$ for all $n \geq n_0$, then $\pi(x)$ and $\pi(x')$ lie on the same orbit of the flow h_t^+; if x, x' are such that $x_n = x'_n$ for all $n \leq n_0$, then $\pi(x)$ and $\pi(x')$ lie on the same orbit of the flow h_t^-.*

In other words, the horizontal and the vertical flows of the abelian differential ω correspond to flows along asymptotic foliations of a Markov compactum. It will develop that Hölder cocycles correspond to special finitely-additive measures on the asymptotic foliations.

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2 Finitely-additive Measures on the Asymptotic Foliations of a Markov Compactum.

2.1 Markov Compacta.

Let $m \in \mathbb{N}$ and let Γ be an oriented graph with m vertices $\{1, \ldots, m\}$ and possibly multiple edges. We assume that that for each vertex there is an edge starting from it and an edge ending in it.

Let $\mathcal{E}(\Gamma)$ be the set of edges of Γ. For $e \in \mathcal{E}(\Gamma)$ we denote by $I(e)$ its initial vertex and by $F(e)$ its terminal vertex. Denote by $A(\Gamma)$ the incidence matrix of Γ given by the formula:

$$A_{ij}(\Gamma) = \#\{e \in \mathcal{E}(\Gamma) : I(e) = i, F(e) = j\}.$$

Let $\mathfrak{G}$ be the set of all oriented graphs on m vertices such that there is an edge starting at every vertex and an edge ending at every vertex.

Assume we are given a sequence $\{\Gamma_n\}$, $n \in \mathbb{Z}$ of graphs belonging to $\mathfrak{G}$. To this sequence we assign the Markov compactum of paths in our sequence of graphs:

$$X = \{x = \dots x_{-n} \dots x_n \dots, x_n \in \mathcal{E}(\omega_n), F(x_{n+1}) = I(x_n)\}. \tag{30}$$

Write $A_n(X) = A(\Gamma_n)$.

2.2 Asymptotic foliations.

For $x \in X$, $n \in \mathbb{Z}$, introduce the sets

$$\gamma_n^+(x) = \{x' \in X(\omega) : x'_t = x_t, t \geq n\};\ \gamma_n^-(x) = \{x' \in X(\omega) : x'_t = x_t, t \leq n\};$$

$$\gamma_\infty^+(x) = \bigcup_{n\in\mathbb{Z}} \gamma_n^+(x);\ \gamma_\infty^-(x) = \bigcup_{n\in\mathbb{Z}} \gamma_n^-(x).$$

The sets $\gamma_\infty^+(x)$ are leaves of the asymptotic foliation $\mathcal{F}^+(X)$ on X corresponding to the infinite future; the sets $\gamma_\infty^-(x)$ are leaves of the asymptotic foliation $\mathcal{F}^-(X)$ on X corresponding to the infinite past.

For $n \in \mathbb{Z}$ let $\mathfrak{C}_n^+(X)$ be the collection of all subsets of X of the form $\gamma_n^+(x)$, $n \in \mathbb{Z}$, $x \in X$; similarly, $\mathfrak{C}_n^-(X)$ is the collection of all subsets of the form $\gamma_n^-(x)$.

By definition, the collections $\mathfrak{C}_n^+(X)$, $\mathfrak{C}_n^-(X)$ are semi-rings. Introduce the collections

$$\mathfrak{C}^+(X) = \bigcup_{n\in\mathbb{Z}} \mathfrak{C}_{n,\omega}^+; \mathfrak{C}_\omega^- = \bigcup_{n\in\mathbb{Z}} \mathfrak{C}_{n,\omega}^-. \tag{31}$$

Since every element of $\mathfrak{C}_n^+$ is a disjoint union of elements of $\mathfrak{C}_{n+1}^+$, the collection $\mathfrak{C}^+$ is a semi-ring as well. The same statements hold for $\mathfrak{C}_n^-$ and $\mathfrak{C}^-$.

Cylinders in X are subsets of the form $\{x : x_{n+1} = e_1, \dots, x_{n+k} = e_k\}$, where $n \in \mathbb{Z}$, $k \in \mathbb{N}$, $e_1 \in \mathcal{E}(\Gamma_{n+1}), \dots, e_k \in \mathcal{E}(\Gamma_{n+k})$ and $F(e_i) = I(e_{i+1})$. The family of all cylinders forms a semi-ring which we denote by $\mathfrak{C}$.

2.3 Finitely-additive measures.

Let $\mathfrak{V}^+(X)$ be the family of finitely-additive complex-valued measures Φ^+ on the semi-ring $\mathfrak{C}^+$ such that if $F(x_n) = F(x'_n)$, then $\Phi^+(\gamma_n^+(x)) = \Phi^+(\gamma_n^+(x'))$.

Let $\mathbf{v} = v^{(l)}$, $l \in \mathbb{Z}$, be a sequence of vectors, real or complex, satisfying

$$v^{(l+1)} = A_l v^{(l)}.$$

Such a sequence will be called *equivariant*.

To any equivariant sequence $\mathbf{v}$ we assign a finitely-additive measure $\Phi_{\mathbf{v}}^+ \in \mathfrak{V}^+(X)$ by the following formula, valid for any $n \in \mathbb{Z}$

$$\Phi_{\mathbf{v}}^+(\gamma_{n+1}^+(x)) = (v^{(n)})_{F(x_{n+1})}. \tag{32}$$

Similarly, let $\mathfrak{V}^-(X)$ be the family of finitely-additive complex-valued measures Φ^- on the semi-ring $\mathfrak{C}^-$ such that if $I(x_n) = I(x'_n)$, then $\Phi^-(\gamma_n^-(x)) = \Phi^-(\gamma_n^-(x'))$.

Again, if a sequence of vectors $\mathbf{v} = v^{(l)}$, $l \in \mathbb{Z}$ satisfies

$$v^{(l)} = A_l^t v^{(l+1)},$$

then such a sequence will be called *reverse equivariant* and to any reverse equivariant sequence $\mathbf{v}$ we assign a finitely additive measure $\Phi_{\mathbf{v}}^-$ on the semi-ring $\mathfrak{C}^-(X)$ by the following formula, valid for any $n \in \mathbb{Z}$:

$$\Phi_{\mathbf{v}}^-(\gamma_n^-(x)) = (v^{(n)})_{I(x_n)}. \tag{33}$$

Given a vector $v \in \mathbb{R}^m$, we introduce its norm by the formula

$$|v| = \sum_{i=1}^{m} |v_i|.$$

2.4 Unique Ergodicity.

Assume that each space $\mathfrak{V}^+(X)$, $\mathfrak{V}^-(X)$ contains a unique positive measure up to scaling. Assume furthermore that the positive measure $\nu^+ \in \mathfrak{V}^+(X)$ satisfies

$$\lim_{n\to\infty} \max_{x\in X} \nu^+(\gamma_{-n}^+(x)) = 0; \ \lim_{n\to\infty} \min_{x\in X} \nu^+(\gamma_n^+(x)) = \infty, \tag{34}$$

while the positive measure $\nu^- \in \mathfrak{V}^-(X)$ satisfies

$$\lim_{n\to\infty} \max_{x\in X} \nu^-(\gamma_n^-(x)) = 0; \ \lim_{n\to\infty} \min_{x\in X} \nu^-(\gamma_{-n}^-(x)) = \infty. \tag{35}$$

The Markov compactum X will then be called *uniquely ergodic*. Unique ergodicity can be equivalently reformulated as the following

Assumption 3. *1. For any $l \in \mathbb{Z}$, there exists a vector $\lambda^{(l)} = (\lambda_1^{(l)}, \dots, \lambda_m^{(l)})$, all whose coordinates are positive, such that $\lambda^{(l)} = A_l^t \lambda^{(l+1)}$ and*

$$\bigcap_{n\in\mathbb{N}} A_{l+1}^t \dots A_{l+n}^t \mathbb{R}_+^m = \mathbb{R}_+ \lambda^{(l)}.$$

2. For any $l \in \mathbb{Z}$, there exists a vector $h^{(l)} = (h_1^{(l)}, \dots, h_m^{(l)})$, all whose coordinates are positive, such that $h^{(l)} = A_l h^{(l-1)}$ and

$$\bigcap_{n\in\mathbb{N}} A_{l-1} \dots A_{l-n} \mathbb{R}_+^m = \mathbb{R}_+ h^{(l)}.$$

3. $|\lambda^{(l)}| \to 0$ as $l \to \infty$, $|h^{(l)}| \to 0$ as $l \to -\infty$.

4. $\min_i \lambda_i^{(l)} \to \infty$ as $l \to -\infty$, $\min_i h^{(l)} \to \infty$ as $l \to \infty$.

The sequences of vectors $\lambda^{(l)}$ and $h^{(l)}$ are defined up to a multiplicative constant (independent of l). The sequence $(h^{(l)})$ is equivariant while the sequence $(\lambda^{(l)})$ is reverse equivariant.

To normalize the vectors $(\lambda^{(l)})$, $(h^{(l)})$ given by Assumption 3, we write

$$|\lambda^{(0)}| = 1, \langle \lambda^{(0)}, h^{(0)} \rangle = 1. \tag{36}$$

By equivariance we have

$$\langle \lambda^{(l)}, h^{(l)} \rangle = 1 \text{ for all } l \in \mathbb{Z}. \tag{37}$$

Denote by ν_X^+ the measure corresponding to the equivariant sequence $h^{(l)}$; observe that it is a positive sigma-finite sigma-additive measure on the sigma-algebra generated by the semi-ring $\mathfrak{C}^+$. Similarly, denote by ν_X^- the measure corresponding to the equivariant sequence $\lambda^{(l)}$. Furthermore, we define a probability measure ν_X on X by the formula

$$\nu_X = \nu_X^+ \times \nu_X^-.$$

2.5 Finitely-additive measures with the Hölder property.

Let X be a uniquely ergodic Markov compactum. Let $\mathfrak{B}^+(X) \subset \mathfrak{V}^+(X)$ be the subspace of finitely-additive measures Φ^+ satisfying the following Hölder condition: there exist $\varepsilon > 0$ and $\theta > 0$ (depending on Φ^+) such that for any element $\gamma^+ \in \mathfrak{C}^+$ satisfying $\nu^+(\gamma^+) \leq \varepsilon$ we have

$$\Phi^+(\gamma^+) \leq \left(\nu^+(\gamma^+)\right)^\theta . \tag{38}$$

Similarly, let $\mathfrak{B}^-(X) \subset \mathfrak{V}^-(X)$ be the subspace of finitely-additive measures Φ^- for which there exist $\varepsilon > 0$ and $\theta > 0$ (depending on Φ^-) such that for any element $\gamma^- \in \mathfrak{C}^-$ satisfying $\nu^-(\gamma^-) \leq \varepsilon$ we have

$$\Phi^-(\gamma^-) \leq \left(\nu^-(\gamma^-)\right)^\theta . \tag{39}$$

2.6 Concatenation and Aggregation.

For two graphs $\Gamma, \Gamma' \in \mathfrak{G}$, their concatenation $\Gamma\Gamma' \in \mathfrak{G}$ is defined as follows. The set of edges $\mathcal{E}(\Gamma\Gamma')$ is given by the formula

$$\mathcal{E}(\Gamma\Gamma') = \{(e, e'),\ e \in \mathcal{E}(\Gamma), e \in \mathcal{E}(\Gamma'), I(e) = F(e')\},$$

and we define $F(e, e') = F(e), I(e, e') = I(e')$. We clearly have $A(\Gamma\Gamma') = A(\Gamma')A(\Gamma)$.

Denote by $W(\mathfrak{G})$ the set of all finite words over the alphabet $\mathfrak{G}$, and for a word $w \in W(\mathfrak{G})$, $w = w_0 \dots w_n$, $w_i \in \mathfrak{G}$, denote $\Gamma(w) = \Gamma(w_0) \dots \Gamma(w_n)$.

Now take a sequence $\Gamma_n \in \mathfrak{G}$, $n \in \mathbb{Z}$, and a strictly increasing sequence of indices $i_n \in \mathbb{Z}$, $n \in \mathbb{Z}$. Consider the concatenations

$$\check{\Gamma}_n = \Gamma_{i_n} \dots \Gamma_{i_{n+1}-1}.$$

The sequence $\check{\Gamma}_n$ will be called an *aggregation* of the sequence Γ_n, while the sequence Γ_n will be called a *refinement* of $\check{\Gamma}_n$.

Let X be the Markov compactum corresponding to the sequence Γ_n, $\check{X}$ the Markov compactum corresponding to the sequence $\check{\Gamma}_n$.

We have a natural "tautological" homeomorphism

$$\mathfrak{t}_{Ag(i_n)} : X \to \check{X}. \tag{40}$$

By definition, the homeomorphism $\mathfrak{t}_{Ag(i_n)}$ sends foliations $\mathcal{F}^{\pm}_X$ to the respective foliations $\mathcal{F}^{\pm}_{\check{X}}$ and identifies the spaces $\mathfrak{V}^{\pm}(X)$ and $\mathfrak{V}^{\pm}(\check{X})$.

The Markov compactum X is uniquely ergodic if and only if $\check{X}$ is, and if $\Phi^+ \in \mathfrak{B}^+(X)$, then $\left(\mathfrak{t}_{Ag(i_n)}\right)_* \Phi^+ \in \mathfrak{B}^+(\check{X})$.

2.7 Approximation of weakly Lipschitz measures.

Let A_n, $n \in \mathbb{Z}$ be a sequence of $m \times m$-matrices. Assume that for any $n \in \mathbb{Z}$ we have a decomposition

$$\mathbb{R}^m = E^u_n \oplus E^{cs}_n,$$

where $E^u_n \neq 0$, such that the following is verified:

Assumption 4. *For any n we have $A_n E^u_n = E^u_{n+1}$, $A_n E^{cs}_n \subset E^{cs}_{n+1}$ and the map $A_n : E^u_n \to E^u_{n+1}$ is an isomorphism. There exist constants $C > 0, \alpha > 0$, and, for any $\varepsilon > 0$, a constant C_ε such that for all $n \in \mathbb{Z}, k \in \mathbb{N}$ we have*

1. $||(A_{n+k} \dots A_n)^{-1}|_{E^u_{n+k+1}}||C \leq \exp(-\alpha k)$;
2. $||(A_{n+k} \dots A_n)|_{E^{cs}_n}|| \leq C_\varepsilon \exp(\varepsilon k)$;
3. $||A_n|| \leq C_\varepsilon \exp(\varepsilon |n|)$.

Note that any positive equivariant equivariant sequence $h^{(n)}$ automatically satisfies $h^{(n)} \in E^u_n$.

Proposition 12. *If X is a uniquely ergodic Markov compactum whose sequence of adjacency matrices satisfies Assumption 4, then we have*

$$\mathfrak{B}^+(X) = \{\Phi^+_{\mathbf{v}}, \mathbf{v} = v^{(n)}, v^{(n)} \in E^u_n\}.$$

In particular, the space $\mathfrak{B}^+(X)$ is isomorphic to E^u_0 (since, by Assumption 4, an equivariant sequence $\mathbf{v} = v^{(n)}$, $v^{(n)} \in E^u_n$, is uniquely determined by $v^{(0)}$); we norm the space $\mathfrak{B}^+(X)$ by writing

$$|\Phi^+_v| = |v|, \; v \in E^u_0. \tag{41}$$

Our next aim is to show that, under the assumption of Lipschitz regularity, any finitely-additive measure on $\mathfrak{C}^+_0(X)$ can be approximated by a measure from $\mathfrak{B}^+(X)$.

Let Θ be a finitely-additive complex-valued measure on the semi-ring $\mathfrak{C}_0^+(X)$. Assume that for any $\varepsilon > 0$ there exists a constant $\delta(\Theta, \varepsilon)$ such that for all $x, x' \in X$ and all $n \geq 0$ we have

$$|\Theta(\gamma_n^+(x)) - \Theta(\gamma_n^+(x'))| \leq \delta(\Theta, \varepsilon) \exp(\varepsilon n) \text{ if } F(x_{n+1}) = F(x'_{n+1}). \tag{42}$$

In this case Θ will be called a *weakly Lipschitz measure.*

Lemma 1. *Let Θ be a weakly Lipschitz finitely-additive complex-valued measure on the semi-ring $\mathfrak{C}_0^+$. Then there exists a unique $\Phi^+ \in \mathfrak{B}^+(X)$ and, for any $\varepsilon > 0$, a constant $C = C(\Theta, \varepsilon)$ such that for all $x \in X$ and all $n > 0$ we have*

$$|\Theta(\gamma_n^+(x)) - \Phi^+(\gamma_n^+(x))| \leq C(\Theta, \varepsilon) \exp(\varepsilon n). \tag{43}$$

First we prepare

Lemma 2. *Let A_n be a sequence of matrices satisfying Assumption 4. Let $v_0, v_1, \dots$ be a sequence of vectors such that for any $\varepsilon > 0$ a constant C_ε can be chosen in such a way that for all n we have*

$$|A_n v_n - v_{n+1}| \leq C_\varepsilon \exp(\varepsilon n).$$

Then there exists a unique vector $v \in E_0^+$ such that

$$|A_n \dots A_0 v - v_{n+1}| \leq C'_\varepsilon \exp(\varepsilon n).$$

Proof: Denote $u_{n+1} = v_{n+1} - A_n v_n$ and decompose $u_{n+1} = u_{n+1}^+ + u_{n+1}^-$, where $u_{n+1}^+ \in E_{n+1}^u$, $u_{n+1}^- \in E_{n+1}^{cs}$. Let

$$v_{n+1}^+ = u_{n+1}^+ + A_n u_n^+ + A_n A_{n-1} u_{n-1}^+ + \dots + A_n \dots A_0 u_0^+;$$

$$v_{n+1}^- = u_{n+1}^- + A_n u_n^- + A_n A_{n-1} u_{n-1}^- + \dots + A_n \dots A_0 u_0^-.$$

We have $v_{n+1} \in E_{n+1}^+$, $v_{n+1}^- \in E_{n+1}^-$, $v_{n+1} = v_{n+1}^+ + v_{n+1}^-$. Now introduce a vector

$$v = u_0^+ + A_0^{-1} u_1^+ + \dots + (A_n \dots A_0)^{-1} u_{n+1}^+ + \dots$$

By our assumptions, the series defining v converges exponentially fast and, moreover, we have

$$|A_n \dots A_0 v - v_{n+1}^+| \leq C'_\varepsilon \exp(\varepsilon n)$$

for some constant C'_ε.

Since, by our assumptions we also have $|v_{n+1}^-| \leq C_\varepsilon \exp(\varepsilon n)$, the Lemma is proved completely.

Uniqueness of the vector v follows from the fact that, by our assumptions, for any $\tilde{v} \neq 0$, $\tilde{v} \in E_0^u$ we have

$$|A_n \dots A_0 \tilde{v}| \geq C'' \exp(\alpha n).$$

We proceed to the proof of Lemma 1.

Let $i = 1, \dots, m$ and take arbitrary points $x(i) \in X$ in such a way that

$$F(x(i)_1) = i.$$

Introduce a sequence of vectors $v(n) \in \mathbb{R}^m$ by the formula

$$v(n)_i = \Theta(\gamma_n^+(x(i))).$$

For any $x \in X$ we have

$$|\Theta(\gamma_n^+(x)) - v(n)_{F(x_1)}| \le \delta(\Theta, \varepsilon)\exp(\varepsilon n),$$

whence by additivity it also follows that

$$|A_n v(n) - v(n+1)| \le C_\varepsilon \exp(\varepsilon n).$$

Lemma 1 follows now from Lemma 2.

2.8 Duality.

Given $\Phi^+ \in \mathfrak{V}^+(X)$, $\Phi^- \in \mathfrak{V}^-(X)$, introduce a finitely-additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\mathfrak{C}(X)$ of cylinders in X as follows: for any $C \in \mathfrak{C}$ and $x \in C$, set

$$\Phi^+ \times \Phi^-(C) = \Phi^+(\gamma_\infty^+(x) \cap C) \cdot \Phi^-(\gamma_\infty^-(x) \cap C). \tag{44}$$

Observe that, by holonomy-invariance of Φ^+ and Φ^-, the right-hand side does not depend on the specific choice of $x \in C$ and the left-hand side is thus well-defined.

Introduce a pairing $\langle , \rangle$ between the spaces $\mathfrak{V}^+(X)$ and $\mathfrak{V}^-(X)$ by writing

$$\langle \Phi^+, \Phi^- \rangle = \Phi^+ \times \Phi^-(X). \tag{45}$$

If $\mathbf{v} = v^{(n)}$ is the equivariant sequence corresponding to Φ^+, $\tilde{\mathbf{v}} = \tilde{v}^{(n)}$ the reverse equivariant sequence corresponding to Φ^-, then we clearly have

$$\langle \Phi^+, \Phi^- \rangle = \sum_{i=1}^m v_i^{(0)} \tilde{v}_i^{(0)}. \tag{46}$$

Now consider a uniquely ergodic Markov compactum X whose corresponding sequence of matrices A_n satisfies Assumption 4. Our next aim is to formulate assumptions under which the pairing (45) is nondegenerate on the pair of subspaces $\mathfrak{B}^+(X)$, $\mathfrak{B}^-(X)$.

Consider the annulators

$$\tilde{E}_n^u = Ann(E_n^{cs}); \ \tilde{E}_n^{cs} = Ann(E_n^u).$$

We have

$$\mathbb{R}^m = E_n^u \oplus E_n^{cs}$$

For a matrix A, denote by A^t its transpose. In view of Assumption 4, $A_n^t \tilde{E}_{n+1}^u = \tilde{E}_n^u$, $A_n \tilde{E}_{n+1}^{cs} \subset \tilde{E}_n^{cs}$ and the map

$$A_n^t : \tilde{E}_{n+1}^u \to \tilde{E}_n^u$$

induces an isomorphism.

We also have

$$||A_n^t|| = ||A_n|| \le C_\varepsilon \exp(\varepsilon |n|).$$

We now impose the additional requirements dual to those of Assumption 4.

Assumption 5. *There exist constants $C > 0, \alpha > 0$, and, for any $\varepsilon > 0$, a constant C_ε such that for all $n \in \mathbb{Z}, k \in \mathbb{N}$ we have*

1. $||(A_n^t A_{n+1}^t \dots A_{n+k}^t)^{-1}\big|_{\tilde{E}_n^u}||C \le \exp(-\alpha k)$;
2. $||(A_n^t \dots A_{n+k}^t)\big|_{\tilde{E}_n^{cs}}|| \le C_\varepsilon \exp(\varepsilon k)$.

Again, any positive reverse equivariant equivariant sequence $\lambda^{(n)}$ automatically satisfies $h^{(n)} \in \tilde{E}_n^u$.

Proposition 13. *If X is a uniquely ergodic Markov compactum whose sequence of adjacency matrices satisfies Assumptions 4, 5, then we have*

$$\mathfrak{B}^-(X) = \{\Phi_{\tilde{\mathbf{v}}}^-, \tilde{\mathbf{v}} = \tilde{v}^{(n)}, \tilde{v}^{(n)} \in \tilde{E}_n^u\}.$$

Corollary 3. *If a uniquely ergodic Markov compactum X satisfies Assumptions 4, 5, then the pairing (45) is nondegenerate on the pair of subspaces $\mathfrak{B}^+(X)$, $\mathfrak{B}^-(X)$.*

2.9 Approximation of weakly Lipschitz functions.

Let f be a bounded Borel function such that the measure $fd\Phi_h^+$ on $\mathfrak{C}^+$ is weakly Lipschitz. then the function f is also called weakly Lipschitz. Let $\Phi_f^+ \in \mathfrak{B}^+(X)$ be the measure corresponding to $fd\Phi_h^+$ by Lemma 1. If the Markov compactum X satisfies Assumptions 4, 5, then the measure Φ_f^+ admits the following characterization.

Let $\Phi^- \in \mathfrak{B}^-(X)$ be arbitrary. Then the integral

$$\int_X fd\Phi_h^+ \times \Phi^-$$

can be defined as follows.

Let $\tilde{v} \in \tilde{E}^u$ be the vector assigned to to Φ^-, and let $\tilde{v}^{(n)}$ be the corresponding reverse equivariant sequence. Recall that

$$|\tilde{v}^{(-n)}| \to 0 \text{ exponentially fast as } n \to \infty. \tag{47}$$

Take arbitrary points $x_i^{(n)} \in X$, $n \in \mathbb{N}$, satisfying

$$F((x_i^{(n)})_n) = i, \ i = 1, \dots, m. \tag{48}$$

and consider the expression

$$\sum_{i=1}^{m} \Big(\int_{\gamma_n^+(x_i^{(n)})} f d\Phi_h^+ \Big) \cdot \big(\tilde{v}^{(1-n)}\big)_i. \tag{49}$$

By (47) as $n \to \infty$ the expression (49) tends to a limit which does not depend on the particular choice of $x_i^{(n)}$ satisfying (48). This limit is denoted

$$\int_X f d\Phi^- \times \Phi_h^+.$$

Proposition 14. *The measure Φ_f^+ satisfies, for any $\Phi^- \in \mathfrak{B}^-(X)$, the equality*

$$\langle \Phi_f^+, \Phi^- \rangle = \int_X f d\Phi_h^+ \times \Phi^-.$$

Proof: As above, choose the points $x_i^{(n)} \in X$ satisfying (48) and let $\tilde{v} \in \tilde{E}^u$ be the vector assigned to to Φ^-, and let $\tilde{v}^{(n)}$ be the corresponding reverse equivariant sequence.

Then for any $\varepsilon > 0$ and $n > 0$ sufficiently large, by definition, we have

$$\left| \int_X f d\Phi^- \times \Phi_h^+ - \sum_{i=1}^{m} \left(\int_{\gamma_n^+(x_i^{(n)})} f d\Phi_1^+ \right) \cdot \left(\tilde{v}^{(-n)}\right)_i \right| < \varepsilon_0. \tag{50}$$

By definition of Φ_f^+ and Lemma 1 we have

$$|\sum_{i=1}^{m} \big(\int_{\gamma_n^+(x_i^{(n)})} f d\Phi_h^+ \big) \cdot \big(\tilde{v}^{(-n)}\big)_i - \sum_{i=1}^{m} \big(\Phi_f^+(\gamma_n^+(x_i^{(n)})) \cdot \big(\tilde{v}^{(-n)}\big)_i| < C_\varepsilon \exp(\varepsilon n) \cdot |\tilde{v}_i^{(-n)}|,$$

and, by (47), the right-hand side tends to 0 exponentially fast as $n \to \infty$.

It remains to notice that, by definition,

$$\sum_{i=1}^{m} \left(\Phi_f^+(\gamma_n^+(x_i^{(n)}) \right) \cdot \left(\tilde{v}^{(-n)} \right)_i = \langle \Phi_f^+, \Phi^- \rangle,$$

and the Lemma is proved completely.

2.10 Balanced, Lyapunov Regular and Hyperbolic Markov Compacta.

A Markov compactum will be called *balanced* if the following holds.

Assumption 6. *There exists a positive constant C, a strictly increasing sequence of indices $i_n \in \mathbb{Z}$, $n \in \mathbb{Z}$, such that*

$$\lim_{n \to -\infty} i_n = -\infty, \ \lim_{n \to \infty} i_n = \infty, \tag{51}$$

and, for any $\varepsilon > 0$, a constant C_ε such that

1. *For any $\varepsilon > 0$ and all $l \in \mathbb{Z}$ we have $||A_n|| \le C_\varepsilon \exp(\varepsilon|n|)$.*

2. *For any $n \in \mathbb{Z}$ all entries of the matrix $A_{i_{n+1}} \dots A_{i_n+1}$ are positive, and for all $j, k, l \in \{1, \dots, m\}$ we have*

$$\frac{(A_{i_{n+1}} \dots A_{i_n+1})_{jk}}{(A_{i_{n+1}} \dots A_{i_n+1})_{lk}} \le C.$$

A sufficient condition for the second requirement is that there exist a matrix Q all whose entries are positive and a sequence i_n satisfying (51) such that $A_{i_n} = Q$. All our examples will admit such a matrix Q.

A Markov compactum satisfying Assumption 4 will be called *Lyapunov regular* if the following additional assumption holds.

Assumption 7. *There exist positive numbers $\theta_1 > \theta_2 > \dots > \theta_{l_0} > 0$ and, for any $n \in \mathbb{Z}$, a direct sum decomposition*

$$E_n^u = E_n^1 \oplus E_n^2 \dots \oplus E_n^{l_0}$$

such that $A_n E_n^i = E_{n+1}^i$ and for any nonzero $v \in E_n^i$ we have

$$\lim_{k\to\infty} \frac{\log|A_{n-k} \dots A_n v|}{k} = \theta_i; \tag{52}$$

$$\lim_{k\to\infty} \frac{\log|(A_{n-k} \dots A_n)^{-1} v|}{k} = -\theta_i. \tag{53}$$

The convergence in (52), (53) is uniform on the unit sphere $\{v \in E_n^i, |v| = 1\}$.

Now let X be uniquely ergodic, take $v \in E_0^u$ and consider the corresponding finitely additive measure $\Phi^+ \in \mathfrak{B}^+$. Decompose $v = v^{(1)} + \dots + v^{(l_0)}$, $v^{(i)} \in E_0^i$ and let j be the lowest index such that $v^{(j)} \ne 0$. Then θ_j will be called the *Lyapunov exponent* of the finitely-additive measure Φ^+. For instance, the positive measure has exponent θ_1.

We shall often need the dual of Assumption 7. Take a Markov compactum X satisfying Assumptions 4, 5, 7. For $i = 1, \dots, l_0$, define

$$\tilde{E}_n^i = Ann\left(\oplus_{j\ne i} E_n^j\right).$$

We have then

$$\tilde{E}_n^u = \tilde{E}_n^1 \oplus \tilde{E}_n^2 \dots \oplus \tilde{E}_n^{l_0}$$

and $A_n^t \tilde{E}_{n+1}^i = \tilde{E}_n^i$.

Assumption 8. *For any $n \in \mathbb{Z}$ and any nonzero $v \in \tilde{E}_n^i$ we have*

$$\lim_{k\to\infty} \frac{\log|A_{n-k}^t \dots A_n^t v|}{k} = \theta_i; \tag{54}$$

$$\lim_{k\to\infty} \frac{\log|(A_{n+k}^t \dots A_n^t)^{-1} v|}{k} = -\theta_i. \tag{55}$$

The convergence in (54), (55) is uniform on the unit sphere $\{v \in \tilde{E}_n^i, |v| = 1\}$.

A Markov compactum satisfying Assumptions 7, 8 will be called *Lyapunov bi-regular.*

A case of special interest for us will be when all Lyapunov exponents are simple, i.e., when the following holds.

Assumption 9. *We have* $l_0 = \dim E_n^u = \dim \tilde{E}_n^u$ *and*

$$\dim E_n^i = \dim \tilde{E}_n^i = 1, \ i = 1, \ldots, l_0.$$

In the latter case we shall say that the Markov compactum X has *simple Lyapunov spectrum.*

Finally, a Lyapunov regular Markov compactum X will be called *hyperbolic* if for any equivariant sequence $\mathbf{v}$, $\mathbf{v} = v^{(n)}$, $A_n v^{(n)} = v^{(n+1)}$, satisfying

$$\lim_{n\to\infty} |v^{(n)}| = 0 \tag{56}$$

we have $v^{(0)} \in E_X^u$.

In other words, if a finitely-additive measure $\Phi^+ \in \mathfrak{V}^+(X)$ satisfies the additional requirement

$$\lim_{n\to\infty} \max_{x\in X} \left|\Phi^+\left(\gamma_n^+(x)\right)\right| = 0, \tag{57}$$

then in fact $\Phi^+ \in \mathfrak{B}^+(X)$.

2.11 Extension of measures.

For a Lyapunov regular Markov compactum the finitely-additive measures $\Phi^+ \in \mathfrak{B}^+$ can be extended to a larger family of sets.

Let $\mathcal{R}_n^+$ be the ring generated by the semiring $\mathfrak{C}_n^+$. For an arbitrary subset A of a leaf γ_∞^+ of the foliation $\mathcal{F}^+$, let $\hat{\gamma}_n^+(A)$ be the minimal (by inclusion) element of the ring $\mathcal{R}_n^+$ containing A and similarly let $\check{\gamma}_n^+(A)$ be the maximal (by inclusion) element of the ring $\mathcal{R}_n^+$ containing A. The set difference $\hat{\gamma}_n^+(A)\setminus\check{\gamma}_n^+(A)$ can be represented in a unique way as a finite union of elements of the semi-ring $\mathfrak{C}_n^+$ (that is, of arcs of the type $\gamma_n^+(x)$). The number of these arcs is denoted $\delta_n^+(A)$. We say that a subset A of a leaf γ_∞^+ of the foliation $\mathcal{F}^+$ belongs to the family $\overline{\mathcal{R}}^+$ if for any $\varepsilon > 0$ there exists a constant C_ε such that we have

$$\delta_{-n}^+(A) \le C_\varepsilon \exp(\varepsilon n).$$

Proposition 15. *The family* $\overline{\mathcal{R}}^+$ *is a ring.*

Indeed, this is clear from the inclusion

$$\hat{\gamma}_n^+(A\cup B) \setminus \check{\gamma}_n^+(A\cup B) \subset \left(\hat{\gamma}_n^+(A) \setminus \check{\gamma}_n^+(A)\right) \bigcup \left(\hat{\gamma}_n^+(B) \setminus \check{\gamma}_n^+(B)\right) \tag{58}$$

and similar inclusions for the intersection and difference of sets.

Proposition 16. *Let X be a Lyapunov regular Markov compactum. Then any element Φ^+ extends to a finitely-additive measure on the ring $\overline{\mathcal{R}}^+$.*

Indeed, if $A \in \overline{\mathcal{R}}^+$, then the set $\hat{\gamma}^+_{-n}(A) \setminus \check{\gamma}^+_{-n}(A)$ is a union of elements of the semi-ring $\mathfrak{C}^+_{-n}$, whose number grows subexponentially as $n \to \infty$.

Thus, if the Markov compactum is Lyapunov regular and $\Phi^+ \in \mathfrak{B}^+$, then the quantities

$$\Phi^+_{\mathbf{v}}(\hat{\gamma}^+_{-n-1}(A) \setminus \hat{\gamma}^+_{-n}(A)) \tag{59}$$

$$\Phi^+_{\mathbf{v}}(\hat{\gamma}^+_{-n}(A) \setminus \check{\gamma}^+_{-n}(A)) \tag{60}$$

decay exponentially fast as $n \to \infty$.

We therefore set

$$\Phi^+(A) = \lim_{n\to\infty} \Phi^+(\hat{\gamma}^+_{-n}(A)) = \lim_{n\to\infty} \Phi^+(\check{\gamma}^+_{-n}(A)).$$

The resulting extension is finitely additive: indeed, if $A, B \in \overline{\mathcal{R}}^+$ are disjoint, then

$$\check{\gamma}^+_n(A) \bigsqcup \check{\gamma}^+_n(B) \subset \check{\gamma}^+_n(A \bigsqcup B);$$

$$\hat{\gamma}^+_n(A \bigsqcup B) \subset \hat{\gamma}^+_n(A) \bigcup \hat{\gamma}^+_n(B).$$

Moreover, the set-difference between the right-hand side and the left-hand side in both the above inclusions consists of a finite number of arcs, which, by definition of $\overline{\mathcal{R}}^+$, grows subexponentially with n. Since, by Lyapunov regularity, the value of Φ^+ on each arc decays exponentially, we obtain that the quantity

$$\Phi^+(\hat{\gamma}^+_n(A \bigsqcup B) \setminus \hat{\gamma}^+_n(A) \bigcup \hat{\gamma}^+_n(B)).$$

decays exponentially, and finite additivity is established.

2.12 Vershik's Orderings.

Let $\Gamma \in \mathfrak{G}$. Following S. Ito [15], A.M. Vershik [30, 31], assume that for any $i \in \{1, \ldots, m\}$ a linear ordering is given on the set

$$\{e \in \mathcal{E}(\Gamma) : I(e) = i\}.$$

Such an ordering will be called a *Vershik's ordering* on Γ.

Let X be a uniquely ergodic Markov compactum corresponding to the sequence of graphs Γ_l.

If a Vershik's ordering is given on each Γ_l, $l \in \mathbb{Z}$, then a linear ordering is induced on any leaf of the foliation $\mathcal{F}^+_X$. Indeed, if $x' \in \gamma^+_\infty(x)$, $x' \neq x$, then there exists n such that $x_t = x'_t$ for $t > n$ but $x_n \neq x'_n$. Since $I(x_n) = I(x'_n)$, the edges x_n and x'_n are comparable with respect to our ordering; if $x_n < x'_n$, then we write $x < x'$. This ordering will be called a Vershik's ordering on a Markov compactum X and denoted $\mathfrak{o}$.

An edge will be called *maximal* (with respect to $\mathfrak{o}$) if there does not exist a greater edge; *minimal*, if there does not exist a smaller edge; and an edge e will be called *the successor* of e' if $e > e'$ but there does not exist e'' such that $e > e'' > e'$. We denote by $[x, x']$ the (closed) interval of points x'' satisfying $x \leq x'' \leq x'$; by (x, x') the (open) interval of points x'' satisfying $x < x'' < x'$.

Let $Max(\mathfrak{o})$ be the set of points $x \in X$, $x = (x_n)_{n \in \mathbb{Z}}$, such that each x_n is a maximal edge. Similarly, $Min(\mathfrak{o})$ denotes the set of points $x \in X$, $x = (x_n)_{n \in \mathbb{Z}}$, such that each x_n is a minimal edge. Since edges starting at a given vertex are ordered linearly, the cardinalities of $Max(\mathfrak{o})$ and $Min(\mathfrak{o})$ do not exceed m.

If a leaf γ_∞^+ does not intersect $Max(\mathfrak{o})$, then it does not have a maximal element; similarly, if γ_∞^+ does not intersect $Min(\mathfrak{o})$, then it does not have a minimal element.

Proposition 17. *Let $x \in X$. If $\gamma_\infty^+(x) \cap Max(\mathfrak{o}) = \emptyset$, then for any $t \geq 0$ there exists a point $x' \in \gamma_\infty^+(x)$ such that*

$$\nu_X^+([x, x']) = t. \tag{61}$$

Proof. Let $V(x) = \{t : \exists x' \geq x : \nu_X^+([x, x']) = t\}$. Since $\gamma_\infty^+(x) \cap Max(\mathfrak{o}) = \emptyset$, for any n there exists $x'' \in \gamma_\infty^+(x)$ such that all points in $\gamma_n^+(x'')$ are greater than x. By Assumption 3, the quantity $\nu_X^+(\gamma_n^+(x''))$ goes to ∞ uniformly in x'', as $n \to \infty$. The set $V(x)$ is thus unbounded. Furthermore, by Assumption 3, the quantity $\nu_X^+(\gamma_n^+(x''))$ decays to 0, uniformly in x'', as $n \to -\infty$, whence the set $V(x)$ is dense in $\mathbb{R}_+$. Finally, by compactness of X, the set $V(x)$ is closed, which concludes the proof of the Proposition.

A similar proposition, proved in the same way, holds for negative t.

Proposition 18. *Let $x \in X$. If $\gamma_\infty^+(x) \cap Min(\mathfrak{o}) = \emptyset$, then for any $t \geq 0$ there exists a point $x' \in \gamma_\infty^+(x)$ such that*

$$\nu_X^+([x', x]) = t. \tag{62}$$

Our next aim is to construct a flow h_t^+ such that for all $t \geq 0$ we have $h_t^+ x \in \gamma_\infty^+(x)$ and $\nu_X^+([x, h_t x]) = t$. Note, however, that the above conditions do not determine the point $h_t^+ x$ uniquely. We therefore modify the Markov compactum X by gluing together the points x, x' such that $x < x'$ but $(x, x') = \emptyset$.

Define an equivalence relation $\sim$ on X by writing $x \sim x'$ if $x \in \gamma_\infty^+(x')$ and $(x, x') = (x', x) = \emptyset$. The equivalence classes admit the following explicit description, which is clear from the definitions.

Proposition 19. *Let $x, x' \in X$ be such that $x \in \gamma_\infty^+(x')$, $x < x'$ and $\nu_X^+([x, x']) = 0$. Then there exists $n \in \mathbb{Z}$ such that*

1. *x'_n is a successor of x_n;*
2. *x is the maximal element in $\gamma_n(x)$;*
3. *x' is the minimal element in $\gamma_n(x')$.*

In other words, $\nu_X^+([x, x']) = 0$ if and only if $(x, x') = \emptyset$. In particular, equivalence classes consist at most of two points and, ν-almost surely, of only one point.

Denote $X_{\mathfrak{o}} = X/\sim$, let $\pi_{\mathfrak{o}} : X \to X_{\mathfrak{o}}$ be the projection map and set $\nu_{\mathfrak{o}} = (\pi_{\mathfrak{o}})_*\nu$. The probability spaces $(X_{\mathfrak{o}}, \nu_{\mathfrak{o}})$ and (X, ν) are measurably isomorphic; in what follows, we shall often omit the index $\mathfrak{o}$. The foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ descend to the space $X_{\mathfrak{o}}$; we shall denote their images on $X_{\mathfrak{o}}$ by the same letters and, as before, denote by $\gamma_\infty^+(x)$, $\gamma_\infty^-(x)$ the leaves containing $x \in X_{\mathfrak{o}}$.

Now let $x \in X_{\mathfrak{o}}$ satisfy $\gamma_\infty^+(x) \cap Max(\mathfrak{o}) = \emptyset$. By Proposition 17, for any $t \geq 0$ there exists a unique x' satisfying (61). Denote $h_t^+(x) = x'$. Similarly, if $x \in X_{\mathfrak{o}}$ satisfy $\gamma_\infty^+(x) \cap Min(\mathfrak{o}) = \emptyset$. By Proposition 18, for any $t \geq 0$ there exists a unique x' satisfying (62). Denote $h_{-t}^+(x) = x'$.

We thus obtain a flow h_t^+, which is well-defined on the set

$$X_{\mathfrak{o}} \setminus \Big(\bigcup_{x \in Max(\mathfrak{o}) \cup Min(\mathfrak{o})} \gamma_\infty^+(x) \Big),$$

and, in particular, ν-almost surely on $X_{\mathfrak{o}}$. By definition the flow h_t^+ preserves the measure ν.

A Vershik's ordering on two graphs Γ, Γ' yields an ordering on their concatenation $\Gamma\Gamma'$: one sets $(e, e') < (\tilde{e}, \tilde{e}')$ if $e' < \tilde{e}'$ or if $e' = \tilde{e}'$, $e < \tilde{e}$.

Thus, if a Markov compactum X is endowed with a Vershik's ordering $\mathfrak{o}$, and the Markov compactum $\check{X}$ is obtained from X by concatenation with respect to a strictly increasing sequence (i_n), then $\check{X}$ is automatically also endowed with a Vershik's ordering $\check{\mathfrak{o}}$, and the map $\mathfrak{t}_{Ag(i_n)}$ sends the flow $h_t^{+,\mathfrak{o}}$ on X to the flow $h_t^{+,\check{\mathfrak{o}}}$ on $\check{X}$.

In a similar way, assume that for every graph Γ_n, $n \in \mathbb{Z}$, a linear ordering $\tilde{o}$ is given on all the edges ending at a given vertex. Such an ordering will be called a *reverse Vershik's ordering*. In the same way as above, a reverse Vershik's ordering induces a ν-preserving flow on the leaves of the foliation $\mathcal{F}^-$.

2.13 Hölder cocycles.

As before, we consider a uniquely ergodic Markov compactum X endowed with a Vershik's ordering $\mathfrak{o}$, and we denote by h_t^+ the resulting flow. By an *arc* of the flow h_t^+ we mean a set of the type

$$\gamma(x, t) = \{y \in \gamma^+(x), x \leq y < h_t^+(x)\}, x \in X, t \geq 0. \tag{63}$$

In other words, an arc is the image, under the quotient map by the equivalence relation $\sim_{\mathfrak{o}}$, of an interval $[x, x') = \{x'' : x \leq x'' < x'\}$.

Our first step is to extend all the finitely-additive measures from $\mathfrak{B}^+$ to all arcs of the flow h_t^+, or, in other words, to check the following

Proposition 20. *Any arc of the flow h_t^+ belongs to the ring $\overline{\mathcal{R}}^+$.*

This proposition is based on a variant of the Denjoy-Koksma argument, which in our context becomes the following simple observation.

Proposition 21. *For any $l \in \mathbb{Z}$ there exists an integer N_l satisfying the inequality*

$$N_l \le C_\varepsilon \exp(\varepsilon|l|) \tag{64}$$

and such that the following holds.

Let $l \in \mathbb{Z}$ and let γ be an arc of the flow h_t^+ such that $\check{\gamma}_l(\gamma) = \emptyset$. Then

$$\gamma = \gamma' \bigsqcup \bigsqcup_{k=1}^{N_l} \gamma_{l,k} \bigsqcup \gamma'',$$

where $\gamma_{l,k} \in \mathfrak{C}_{l-1}^+$, and $\check{\gamma}_{l-1}(\gamma') = \check{\gamma}_{l-1}(\gamma'') = \emptyset$ (some of the arcs may be empty).

In other words, if an arc of the flow h_t^+ does not contain arcs from the semi-ring $\mathfrak{C}_l^+$, then it cannot contain more than $C_\varepsilon \exp(\varepsilon|l|)$ arcs of the semi-ring $\mathfrak{C}_{l-1}^+$.

Proof. One may take

$$N_l = 2 \max_k \sum_{i,k=1}^{m} (A_l)_{ik} + 1.$$

It is immediate that if a flow arc contains N_l arcs from $\mathfrak{C}_{l-1}^+$, then it also contains an arc from $\mathfrak{C}_l^+$. The inequality (64) follows from part 3 of Assumption 4.

Proposition 21 immediately implies Proposition 20.

Since every measure $\Phi^+ \in \mathfrak{B}^+$ is defined on every arc of the flow h_t^+, it follows that such a measure defines a cocycle on the orbits of the flow h_t^+ by the formula

$$\Phi^+(x,t) = \Phi^+([x, h_t x]).$$

Slightly abusing notation, we denote the measure and the corresponding cocycle by the same letter; we identify the measure and the cocyle and we speak of the norm of the cocycle, the Lyapunov exponent of the cocycle etc., meaning the norm or the exponent of the corresponding finitely-additive measure.

Our next aim is to verify that cocycles given by measures from $\mathfrak{B}^+$ satisfy the Hölder property.

Lemma 3. *For any $l \in \mathbb{Z}$ there exists an integer M_l satisfying the inequality*

$$M_l \le C_\varepsilon \exp(\varepsilon|l|) \tag{65}$$

such that the following holds.

Let γ be an arc of the flow h_t^+ and let n_0 be the largest integer satisfying $\check{\gamma}_{n_0}(\gamma) \neq \emptyset$.

Then there is a decomposition

$$\gamma = \bigsqcup_{l=n_0}^{-\infty} \bigsqcup_{k=1}^{M_l} \gamma_{l,k} \tag{66}$$

such that $\gamma_{l,k} \in \mathfrak{C}_l^+$.

This is a reformulation of Proposition 21.

Corollary 4. *Let X be a Lyapunov regular balanced Markov compactum with top Lyapunov exponent θ_1. For any $\varepsilon > 0$ there exists a positive constant C_ε depending only on X such that the following is true. Let $\Phi^+ \in \mathfrak{B}^+$ have Lyapunov exponent $\theta > 0$. Then for any $x \in X$ and any $t \in [-1, 1]$ we have*

$$|\Phi^+(x,t)| \le C_\varepsilon \cdot |\Phi^+| \cdot |t|^{\theta/\theta_1 - \varepsilon}. \tag{67}$$

Proof. Indeed, if $\gamma \in \mathfrak{C}_n^+$, then, since our Markov compactum is balanced, we have

$$|\nu^+(\gamma)| \ge C_\varepsilon \exp((\theta_1 - \varepsilon)n),\ |\Phi^+(\gamma)| \le C_\varepsilon \exp((\theta + \varepsilon)n). \tag{68}$$

Combining these two, we obtain

$$|\Phi^+(\gamma)| \le C_\varepsilon |\nu^+(\gamma)|^{\theta/\theta_1 - \varepsilon} \tag{69}$$

for all $\gamma \in \mathfrak{C}_n^+$. The statement of the corollary for an arbitrary flow arc is now immediate from the arc decomposition given by Lemma 3.

A partial converse to this corollary is given by the following Proposition.

Proposition 22. *Let X be a Lyapunov regular balanced Markov compactum with top Lyapunov exponent θ_1, and let $\Phi^+ \in \mathfrak{B}^+$ have Lyapunov exponent $\theta > 0$. Then for any $x \in X$ we have*

$$\limsup_{t \to \infty} \frac{\log |\Phi^+(x,t)|}{\log t} = \theta/\theta_1. \tag{70}$$

Proof. Indeed, by definition of the Lyapunov exponent of the cocycle we have that for any $n \in \mathbb{N}$ there exists i such that

$$|\Phi^+(\gamma_n^+(x))| \ge C_\varepsilon \exp(\theta - \varepsilon)n). \tag{71}$$

Now, given $x \in X$ take the smallest t_1 such that $x' = h_{t_1} x$ satisfies $F(x'_n) = i$. By definition of the top Lyapunov exponent, we must have

$$t_1 \le C_\varepsilon \exp(\theta_1 + \varepsilon)n), \tag{72}$$

and the constants C_ε do not depend on x.

Now write $\gamma_n^+(x') = [x', x'']$, where $x'' = h_{t_2} x'$. Since our Markov compactum is balanced, we have

$$t_2 \ge C_\varepsilon \exp(\theta_1 - \varepsilon)n) \tag{73}$$

Now, we rewrite (71) as

$$|\Phi^+(x', t_2)| \ge C_\varepsilon \exp(\theta - \varepsilon)n).$$

Since

$$\Phi^+(x', t_2) = \Phi^+(x, t_1 + t_2) - \Phi^+(x, t_1),$$

the inequalities (72), (73) yield the proposition.

Proposition 23. *For any $\Phi^+ \in \mathfrak{B}^+$ and any $t_0 \in \mathbb{R}$ we have*

$$\mathbb{E}_\nu(\Phi^+(x, t_0)) = \langle \Phi^+, \nu^- \rangle \cdot t_0.$$

Proof: Since the Proposition is clearly valid for $\Phi^+ = \nu^+$, it suffices to prove it in the case $\langle \Phi^+, \nu^- \rangle = 0$. But indeed, if $\mathbb{E}_\nu(\Phi^+(x, t) \neq 0$, then the Ergodic Theorem implies

$$\limsup_{T \to \infty} \frac{\log |\Phi^+(x, T)|}{\log T} = 1,$$

and then $\langle \Phi^+, \nu^- \rangle \neq 0$.

Proposition 24. *For any $\Phi^+ \in \mathfrak{B}^+$ not proportional to ν^+ and any $t_0 \neq 0$ we have*

$$Var_\nu \Phi^+(x, t_0) \neq 0.$$

Taking $\Phi^+ - \langle \Phi^+, \nu^- \rangle \cdot \nu^+$ instead of Φ^+, we may assume $\mathbb{E}_\nu(\Phi^+(x, t_0)) = 0$. If $Var_\nu \Phi^+(x, t_0) = 0$, then $\Phi^+(x, t_0) = 0$ identically, but then

$$\limsup_{T \to \infty} \frac{\log |\Phi^+(x, T)|}{\log T} = 0,$$

whence $\Phi^+ = 0$, and the Proposition is proved.

Remark. In the context of substitutions, related cocycles have been studied by P. Dumont, T. Kamae and S. Takahashi in [9] as well as by T. Kamae in [16].

2.14 Weakly Lipschitz Functions.

Given a uniquely ergodic Lyapunov biregular balanced Markov compactum X, we introduce a function space $Lip_w^+(X)$ in the following way. A bounded Borel-measurable function $f : X \to \mathbb{R}$ belongs to the space $Lip_w^+(X)$ if there exists a constant $C > 0$ such that for all $n \geq 0$ and any $x, x' \in X$ satisfying $F(x_{n+1}) = F(x'_{n+1})$, we have

$$\left| \int_{\gamma_n^+(x)} f d\Phi_1^+ - \int_{\gamma_n^+(x')} f d\Phi_1^+ \right| \leq C. \tag{74}$$

Let C_f be the infimum of all C satisfying (74) and norm the space $Lip_w^+(X)$ by setting

$$||f||_{Lip_w^+} = \sup_X f + C_f.$$

As before, let $Lip_{w,0}^+(X)$ be the subspace of $Lip_w^+(X)$ of functions whose integral with respect to ν is zero.

Now assume that a Vershik's ordering is given on the edges of the graphs defining X, and let h_t^+ be the corresponding flow.

Proposition 21 and the arc decomposition Lemma 3 immediately imply

Proposition 25. *There exists a constant $C > 0$ such that for any $f \in Lip_w^+(X)$, any $T > 0$ and any pair of points x, x' the following is true. If there exists $i \in \{1, \dots, m\}$ and $n \in \mathbb{N}$ such that for all $t : 0 \le t \le T$ we have $F((h_t x)_n) = F((h_t x')_i) = i$, then we have*

$$\left| \int_0^T f \circ h_t^+(x) dt - \int_0^T f \circ h_t^+(x') dt \right| \le C ||f||_{Lip_w^+}. \tag{75}$$

If $f \in Lip_w^+(X)$, then the measure $f d\nu^+$ is a weakly Lipschitz measure on the foliation $\mathcal{F}^+$ in the sense of the definition immediately preceding Lemma 1; moreover, in (42) one may take $\delta(\Theta, \varepsilon) = ||f||_{Lip_w^+}$. Lemma 1 applied to the weakly Lipschitz measure $f d\nu^+$ gives the main result of this subsection:

Theorem 3. *Let X be a uniquely ergodic Lyapunov biregular balanced Markov compactum. There exists a continuous mapping $\Xi^+ : Lip_w^+(X) \to \mathfrak{B}^+(X)$ such that the following holds. For any $\varepsilon > 0$ there exists a constant C_ε such that for any $f \in Lip_w^+(X)$, any $x \in X$ and any $T > 0$ we have*

$$\left| \int_0^T f \circ h_t^+(x) dt - \Xi^+(f; x, T) \right| \le C_\varepsilon ||f||_{Lip_w^+} (1 + T^\varepsilon). \tag{76}$$

Remark. We write $\Xi^+(f; x, T)$ instead of $\Xi^+(f)(x, T)$.

Proof: Indeed, the correspondence $f \to \Xi^+(f)$ is given by Lemma 1; its continuity is clear by construction, and (76) follows from (43).

3 Random Markov Compacta.

3.1 The Space of Markov Compacta.

Recall that $\mathfrak{G}$ is the space of oriented graphs on m vertices, possibly with multiple edges, and such that for any vertex there is an edge coming into it and going out of it.

Now let Ω be the space of sequences of bi-infinite sequences of graphs $\Gamma_n \in \mathfrak{G}$. We write

$$\Omega = \{\omega = \dots \omega_{-n} \dots \omega_n \dots, \omega_i \in \mathfrak{G}, i \in \mathbb{Z}\}.$$

For $\omega \in \Omega$, we denote by $X(\omega)$ the Markov compactum corresponding to ω.

As in the previous sections, for any ω the Markov compactum $X(\omega)$ carries a pair of foliations $\mathcal{F}^+$, $\mathcal{F}^-$, the semirings $\mathfrak{C}^+$ and $\mathfrak{C}^-$ and so forth; to underline the dependence on ω we shall write $\mathcal{F}_\omega^+$, $\mathfrak{C}_\omega^+$ and so forth.

The right shift σ on the space Ω is defined by the formula $(\sigma\omega)_n = \omega_{n+1}$. We have a natural map $\mathfrak{t}_\sigma : X(\omega) \to X(\sigma\omega)$ which to a point $x \in X$ assigns the point $\tilde{x} \in X(\sigma\omega)$ given by $\tilde{x}_n = x_{n+1}$.

The map $\mathfrak{t}_\sigma$ sends the foliations $\mathcal{F}_\omega^+$, $\mathcal{F}_\omega^-$ to $\mathcal{F}_{\sigma\omega}^+$, $\mathcal{F}_{\sigma\omega}^-$; the semirings $\mathfrak{C}_\omega^+$, $\mathfrak{C}_\omega^-$ to $\mathfrak{C}_{\sigma\omega}^+$, $\mathfrak{C}_{\sigma\omega}^-$; and induces an isomorphism $(\mathfrak{t}_\sigma)_* : \mathfrak{V}^+(X_\omega) \to \mathfrak{V}^+(X_{\sigma\omega})$ given by the usual formula

$$(\mathfrak{t}_\sigma)_* \Phi^+(\gamma) = \Phi^+((\mathfrak{t}_\sigma)^{-1}\gamma), \ \gamma \in \mathfrak{C}_{\sigma\omega}^+.$$

Introduce the space

$$\mathfrak{X}\Omega = \{(\omega, x), x \in X(\omega)\}$$

and endow it with a skew-product map $\sigma^{\mathfrak{X}}$ given by the formula

$$\sigma^{\mathfrak{X}}(\omega, x) = (\sigma\omega, \mathfrak{t}_\sigma x).$$

3.2 Cocycles and Measures.

3.2.1 The renormalization cocycle.

We have a natural cocycle $\mathbb{A}$ over the dynamical system (Ω, σ) defined, for $n > 0$, by the formula

$$\mathbb{A}(n, \omega) = A(\omega_n) \dots A(\omega_1).$$

The cocycle $\mathbb{A}$ will be called the *renormalization cocycle.*

Introduce the corresponding skew-product transformation $\sigma^{\mathbb{A}} : \Omega \times \mathbb{R}^m \to \Omega \times \mathbb{R}^m$ by the formula

$$\sigma^{\mathbb{A}}(\omega, v) = (\sigma\omega, \mathbb{A}(1, \omega)v).$$

Let $\Omega_{inv} \subset \Omega$ be the subset of all sequences ω such that all matrices $A(\omega_n)$ are invertible. Assumption 10 implies that $\mu(\Omega \setminus \Omega_{inv}) = 0$.

For $\omega \in \Omega_{inv}$ and $n < 0$ set

$$\mathbb{A}(n, \omega) = A^{-1}(\omega_{-n}) \dots A^{-1}(\omega_0);$$

set $\mathbb{A}(0, \omega)$ to be the identity matrix.

Consider the space

$$\mathfrak{B}^+\Omega = \{(\omega, \Phi^+) : \omega \in \Omega_{inv}, \Phi^+ \in \mathfrak{B}^+(X(\omega))\},$$

and endow $\mathfrak{B}^+\Omega$ with the automorphism $\mathfrak{T}_\sigma$ given by the formula

$$\mathfrak{T}_\sigma(\omega, \Phi^+) = (\sigma\omega, (\mathfrak{t}_\sigma)_*\Phi^+).$$

Let $\omega \in \Omega_{inv}$, $v \in \mathbb{R}^m$. Consider the equivariant sequence $\mathbf{v} = v^{(n)}$ such that $v^{(0)} = v$. Let $\Phi^+_{\mathbf{v}} \in \mathfrak{B}^+(X(\omega))$ be the corresponding finitely-additive measure. Since $v^{(n)}$ is uniquely determined by $v^{(0)}$, we obtain the isomorphism

$$\mathcal{I}^+_\omega : \mathbb{R}^m \to \mathfrak{B}^+(X(\omega))$$

given by $\mathcal{I}^+_\omega v = \Phi^+_{\mathbf{v}}$, $\mathbf{v} = v^{(n)}$, $v^{(0)} = v$.

For any $\omega \in \Omega_{inv}$ the diagram

$$\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{\mathcal{I}^+_\omega} & \mathfrak{B}^+(X(\omega)) \\
\downarrow{\scriptstyle \mathbb{A}(1,\omega)} & & \downarrow{\scriptstyle (\mathfrak{t}_\sigma)_*} \\
\mathbb{R}^m & \xrightarrow{\mathcal{I}^+_{\sigma\omega}} & \mathfrak{B}^+(X(\sigma\omega))
\end{array}$$

is commutative.

Let μ be an ergodic σ-invariant probability measure on Ω satisfying the following

Assumption 10. *1. There exists $\Gamma_0 \in \mathfrak{G}$ such that all entries of the matrix $A(\Gamma_0)$ are positive and that*

$$\mu(\{\omega : \omega_0 = \Gamma_0\}) > 0.$$

2. The matrices $A(\omega_n)$ are almost surely invertible with respect to $\mathbb{P}$.

3. The logarithm of the norm of the renormalization cocycle, as well as that of its inverse, is integrable with respect to μ.

Proposition 26. *If the measure μ satisfies 10, then for almost every ω the Markov compactum $X(\omega)$ is uniquely ergodic and Lyapunov bi-regular.*

Indeed, unique ergodicity is clear by the first condition, while, in view of the second and the third, the Oseledets theorem implies bi-regularity.

Given $\omega \in \Omega$, let E^u_ω be the Lyapunov subspace at the point ω corresponding to positive Lyapunov exponents of $\mathbb{A}$. The first condition in Assumption 10 implies the nontriviality of the subspace E^u_ω: indeed, we have $h^{(0)}_\omega \in E^u_\omega$.

Proposition 27. *For μ-almost every $\omega \in \Omega$ the transformation $\mathcal{I}^+_\omega$ maps the subspace E^u_ω isomorphically onto $\mathfrak{B}^+(X(\omega))$.*

This is immediate from the Oseledets Theorem and Proposition 12.

3.2.2 The transpose cocycle.

The *transpose* cocycle $\mathbb{A}^t$ over the dynamical system $(\Omega, \sigma^{-1}, \mathbb{P})$ is defined, for $n > 0$, by the formula

$$\mathbb{A}^t(n, \omega) = A^t(\omega_{1-n}) \dots A^t(\omega_0).$$

If $\omega \in \Omega_{inv}$, then for $n < 0$ write

$$\mathbb{A}^t(n, \omega) = (A^t)^{-1}(\omega_{-n}) \dots (A^t)^{-1}(\omega_1).$$

and set $\mathbb{A}^t(0, \omega)$ to be the identity matrix.

In the same way as above, for $\omega \in \Omega_{inv}$, $v \in \mathbb{R}^m$, we have the isomorphism

$$\mathcal{I}^-_\omega : \mathbb{R}^m \to \mathfrak{B}^-(X(\omega))$$

given by $\mathcal{I}^-_\omega \tilde{v} = \Phi^+_{\tilde{\mathbf{v}}}$, where $\tilde{\mathbf{v}} = \tilde{v}^{(n)}$ is a reverse equivariant subsequence satisfying $\tilde{v}^{(0)} = \tilde{v}$.

As before, for any $\omega \in \Omega_{inv}$ the diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\mathcal{I}^-_\omega} & \mathfrak{B}^-(X(\omega)) \\ \Big\uparrow {\scriptstyle \mathbb{A}^t(1,\sigma\omega)} & & \Big\downarrow {\scriptstyle (\mathrm{t}_\sigma)_*} \\ \mathbb{R}^m & \xrightarrow{\mathcal{I}^-_{\sigma\omega}} & \mathfrak{B}^-(X(\omega)) \end{array}$$

is commutative.

Continuing, again we let μ satisfy Assumption 10, and, for $\omega \in \Omega$, set $\tilde{E}^u_\omega$ to be the Lyapunov subspace at the point ω corresponding to positive Lyapunov exponents of the cocycle $\mathbb{A}^t$. By the Oseledets Theorem and Proposition 12, we have

Proposition 28. *For μ-almost every $\omega \in \Omega$ the transformation $\mathcal{I}^-_\omega$ maps the subspace $\tilde{E}^u_\omega$ isomorphically onto $\mathfrak{B}^-(X(\omega))$.*

3.2.3 Duality

Take $\omega \in \Omega_{inv}$. For $v, \tilde{v} \in \mathbb{R}^m$ we clearly have

$$\langle \mathcal{I}^+_\omega(v), \mathcal{I}^-_\omega(\tilde{v})\rangle = \sum_{i=1}^m v_i \tilde{v}_i.$$

If μ satisfies Assumption 10, then, by the Oseledets Theorem, for almost every $\omega \in \Omega$ the standard Euclidean inner product yields a nondegenerate pairing of the subspaces E^u_ω and $\tilde{E}^u_\omega$.

Corollary 5. *If a probability σ-invariant ergodic measure μ satisfies Assumption 10, then for μ-almost every $\omega \in \Omega$ the pairing $\langle , \rangle$ is nondegenerate on the pair of subspaces $\mathfrak{B}^+(X(\omega))$, $\mathfrak{B}^-(X(\omega))$.*

3.2.4 Balanced and hyperbolic random Markov compacta.

Proposition 29. *If the measure μ satisfies Assumption 10, then for almost all ω the Markov compactum $X(\omega)$ satisfies Assumption 6.*

Proof: Let $\Gamma_0 \in \mathfrak{G}_+$ have positive probability with respect to μ (the existence of such Γ_0 is given by the first condition of Assumption 10). Let i_n, $n \in \mathbb{Z}$ be consecutive moments of time such that $\omega_{i_n} = \Gamma_0$ (the sequence i_n is almost surely unbounded both in the positive and in the negative direction). The positivity of $A(\Gamma_0)$ immediately implies the second requirement of Assumption 6, and it remains to verify the first requirement. Denote by σ_{Γ_0} the induced map of σ on the set $\{\omega : \omega_0 = \Gamma_0\}$. The renormalization cocycle $\mathbb{A}$ naturally yields the induced cocycle $\mathbb{A}_{\Gamma_0}$ over σ_{Γ_0} (to obtain the matrix of $\mathbb{A}_{\Gamma_0}$ one needs to multiply all the matrices of $\mathbb{A}$ occurring between two consecutive appearances of Γ_0). If the logarithm of of the norm of $\mathbb{A}$ is integrable, then the same is true of $\mathbb{A}_\Gamma$, whence the first requirement of Assumption 6 follows immediately.

Summing up, we see that under Assumption 10 for almost every ω the Markov compactum $X(\omega)$ is uniquely ergodic, Lyapunov biregular and balanced; Theorem 3 is thus applicable and yields

Corollary 6. *Let μ be an ergodic, σ-invariant probability measure on Ω satisfying Assumption 10. For any $\varepsilon > 0$ there exists a constant C_ε depending only on μ such that the following holds for almost every $\omega \in \Omega$. There exists a continuous mapping $\Xi^+_\omega : Lip^+_w(X(\omega)) \to \mathfrak{B}^+(X(\omega))$ such that*

1. *for any* $t_0 \in \mathbb{R}$ *we have* $\Xi^+_\omega(f \circ h^+_{t_0}) = \Xi^+_\omega(f)$*;*

2. *the diagram*

$$\begin{array}{ccc} Lip^+_w(X(\omega)) & \xrightarrow{\Xi^+_\omega} & \mathfrak{B}^+(X(\omega)) \\ \uparrow{\scriptstyle (\mathfrak{t}_\sigma)^*} & & \downarrow{\scriptstyle (\mathfrak{t}_\sigma)_*} \\ Lip^+_w(X(\sigma\omega)) & \xrightarrow{\Xi^+_{\sigma\omega}} & \mathfrak{B}^+(X(\sigma\omega)) \end{array}$$

is commutative

3. *for any* $f \in Lip^+_w(X)$*, any* $x \in X(\omega)$ *and any* $T > 0$ *we have*

$$\left| \int_0^T f \circ h^+_t(x)dt - \Xi^+(f; x, T) \right| \le C_\varepsilon ||f||_{Lip^+_w}(1 + T^\varepsilon). \tag{77}$$

We now give a sufficient condition for hyperbolicity of random Markov compacta. Let $(\mathcal{X}, \mu)$ be a probability space endowed with a μ-preserving transformation T or flow g_s and an integrable linear cocycle A over g_s with values in $GL(m, \mathbb{R})$.

For $p \in \mathcal{X}$ let $E_{0,x}$ be the the *neutral* subspace of A at p, i.e., the Lyapunov subspace of the cocycle A corresponding to the Lyapunov exponent 0. We say that A *acts isometrically on its neutral subspaces* if for almost any p there exists an inner product $\langle\cdot\rangle_p$ on $\mathbb{R}^m$ which depends on p measurably and satisfies

$$\langle A(1, p)v, A(1, p)v\rangle_{g_s p} = \langle v, v\rangle_p,\ v \in E_{0,p}$$

for all $s \in \mathbb{R}$ (again, in the case of a transformation, g_s should be replaced by T in this formula).

The following proposition is clear from the definitions.

Proposition 30. *Let* ν *be a* σ*-invariant ergodic probability measure on* Ω *satisfying Assumption 10 and such that the renormalization cocycle* $\mathbb{A}$ *acts isometrically on its neutral subspaces with respect to* ν*. Then for almost all* ω *the Markov compactum* $X(\omega)$ *is hyperbolic.*

3.3 The renormalization flow on the space of measured Markov compacta.

3.3.1 The space of measured Markov compacta.

Let $\Omega_{ue} \subset \Omega$ be the subset of ω such that the Markov compactum $X(\omega)$ is uniquely ergodic. For $\omega \in \Omega_{ue}$, $r \in \mathbb{R}_+$ define

$$\nu^+_{(\omega,r)} = \frac{\nu^+_\omega}{r};\ \nu^-_{(\omega,r)} = r\nu^-_\omega. \tag{78}$$

It is clear that for any $r > 0$ we have

$$\nu_\omega = \nu^+_{(\omega,r)} \times \nu^-_{(\omega,r)}. \tag{79}$$

Furthermore, from the definitions it is clear that for $l \in \mathbb{Z}$ we have

$$(\mathfrak{t}_\sigma)^l_* \nu^+_{(\omega,r)} = \nu^+_{(\sigma^l\omega, r|\lambda^{(l)}_\omega|)}. \tag{80}$$

We now introduce an equivalence relation $\sim$ on the set of pairs $(\omega, r) \in \Omega_{ue} \times \mathbb{R}_+$ in the following way:

$$(\omega, r) \sim (\omega', r')$$

if there exists $l \in \mathbb{Z}$ such that

$$\omega' = \sigma^l\omega,\ r'/r = |\lambda^{(l)}(\omega)|. \tag{81}$$

Since, by definition,

$$\lambda^{(n)}_{\sigma^l\omega} = \frac{\lambda^{(n+l)}_\omega}{|\lambda^{(l)}_\omega|},$$

we have

$$|\lambda^{(l)}_\omega| = \frac{1}{|\lambda^{(-l)}_{\sigma^l\omega}|},$$

and the fact that $\sim$ is an equivalence relation is clear.

3.3.2 The renormalization flow and the renormalization cocycle.

Let $\overline{\Omega}$ be the set of equivalence classes of $\sim$. Introduce a flow g_s on $\overline{\Omega}$ by the formula

$$g_s(\omega, r) = (\omega, e^s r).$$

The flow g_s will be called the *renormalization* flow.

An explicit fundamental domain for $\sim$ is given by the set

$$\Omega_0 = \{(\omega, r) : \omega \in \Omega_{ue}, 1 \le r < |\lambda^{(1)}(\omega)|^{-1}\}. \tag{82}$$

To every pair $(\omega, r) \in \Omega_0$ we assign

1. the Markov compactum $X(\omega, r) = X(\omega)$;
2. the foliations $\mathcal{F}^+_{(\omega,r)} = \mathcal{F}^+_\omega$, $\mathcal{F}^-_{(\omega,r)} = \mathcal{F}^-_\omega$;
3. the measures $\nu^+_{(\omega,r)} = \frac{\nu^+}{r}$, $\nu^-_{(\omega,r)} = r\nu^+$;
4. the spaces of finitely-additive measures $\mathfrak{B}^+_{(\omega,r)} = \mathfrak{B}^+_\omega$, $\mathfrak{B}^-_{(\omega,r)} = \mathfrak{B}^-_\omega$.

We identify $\overline{\Omega}$ with Ω_0 and for $\overline{\omega} \in \overline{\Omega}$ we speak of the corresponding Markov compactum $X(\overline{\omega})$, the foliations, the measures etc., meaning those objects for the corresponding point $(\omega, r) \in \Omega_0$.

The identification of $\overline{\Omega}$ with Ω_0 yields a representation of g_s as a suspension flow over the shift σ with roof function

$$\tau^1(\omega) = -\log|\lambda^{(1)}(\omega)|. \tag{83}$$

It is important to note that the roof function $\tau^1(\omega)$ given by (83) only depends on the *future* of the sequence ω.

Given $(\omega, r) \in \Omega_0$ and $s \in \mathbb{R}$, define an integer $\tilde{n}(\omega, r, s)$ by the formula

$$(\omega, e^s r) \sim (\sigma^{\tilde{n}(\omega,r,s)}\omega, r'),\ (\sigma^{\tilde{n}(\omega,r,s)}\omega, r') \in \Omega_0 \tag{84}$$

For every $s \in \mathbb{R}$, we have a natural map

$$\mathfrak{t}_s : X(\overline{\omega}) \to X(g_s\overline{\omega})$$

given, for $\overline{\omega} = (\omega, r)$, $(\omega, r) \in \Omega_0$, by the formula

$$\mathfrak{t}_s(\overline{\omega}) = \mathfrak{t}_\sigma^{\tilde{n}(\omega,r,s)}.$$

Introduce the space

$$\mathfrak{X}\overline{\Omega} = \{(\overline{\omega}, x), x \in X(\overline{\omega})\}$$

and endow it with the skew-product flow $g_s^{\mathfrak{X}}$ given by the formula

$$g_s^{\mathfrak{X}}(\overline{\omega}, x) = (g_s^{\mathfrak{X}}\overline{\omega}, \mathfrak{t}_s x).$$

It is immediate from the definitions that if $(\omega, r) \sim (\omega', r')$ and $\omega' = \sigma^l\omega$, then

$$\mathfrak{t}_\sigma^l \nu^+_{(\omega,r)} = \nu^+_{(\omega',r')};\ \mathfrak{t}_\sigma^l \nu^-_{(\omega,r)} = \nu^-_{(\omega',r')}, \tag{85}$$

and we thus have

$$(\mathfrak{t}_s)_*\nu^+_{\overline{\omega}} = \nu^+_{g_s\overline{\omega}};\ (\mathfrak{t}_s)_*\nu^-_{\overline{\omega}} = \nu^-_{g_s\overline{\omega}}. \tag{86}$$

For $\overline{\omega} = (\omega, r)$, $(\omega, r) \in \Omega_0$, write

$$\overline{\mathbb{A}}(s, \overline{\omega}) = \mathbb{A}(\tilde{n}(\omega, r, s), \omega). \tag{87}$$

We thus obtain a matrix cocycle $\overline{\mathbb{A}}$ over the flow g_s.

Let $g_s^{\overline{\mathbb{A}}} : \overline{\Omega} \times \mathbb{R}^m \to \overline{\Omega} \times \mathbb{R}^m$ be the skew-product transformation corresponding to the cocycle $\overline{\mathbb{A}}$ by the formula

$$g_s^{\overline{\mathbb{A}}}(\overline{\omega}, v) = (g_s\overline{\omega}, \overline{\mathbb{A}}v).$$

As before, for any $\overline{\omega} = (\omega, r)$, we have the isomorphism

$$\mathcal{I}^+_{\overline{\omega}} : \mathbb{R}^m \to \mathfrak{Y}^+(X(\overline{\omega}))$$

given by $\mathcal{I}^+_{\overline{\omega}} v = \Phi^+_{\mathbf{v}}$, $\mathbf{v} = v^{(n)}$, $v^{(0)} = v$ (recall that, by definition, $\omega \in \Omega_{inv}$).

For any $s \in \mathbb{R}$ the diagram

$$\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{\mathcal{I}^+_{\overline{\omega}}} & \mathfrak{Y}^+(X(\overline{\omega})) \\
\Big\downarrow{\scriptstyle \mathbb{A}(s,\overline{\omega})} & & \Big\downarrow{\scriptstyle (\mathfrak{t}_s)_*} \\
\mathbb{R}^m & \xrightarrow{\mathcal{I}^+_{g_s\overline{\omega}}} & \mathfrak{Y}^+(X(g_s\overline{\omega}))
\end{array}$$

is commutative.

3.3.3 Characterization of finitely-additive measures.

Introduce the space $\mathfrak{V}^+\overline{\Omega}$ by the formula

$$\mathfrak{V}^+\overline{\Omega} = \{(\overline{\omega}, \Phi^+), \Phi^+ \in \mathfrak{V}^+_{\overline{\omega}}\}$$

and a flow $\mathfrak{T}_s$ on $\mathfrak{V}^+\overline{\Omega}$ by the formula

$$\mathfrak{T}_s(\overline{\omega}, \Phi^+) = (g_s\overline{\omega}, (\mathfrak{t}_s)_* \Phi^+).$$

The trivialization map

$$\mathrm{Triv} : \mathfrak{V}^+\overline{\Omega} \to \overline{\Omega} \times \mathbb{R}^m$$

is given by the formula

$$\mathrm{Triv}(\overline{\omega}, \Phi^+) = (\overline{\omega}, (\mathcal{I}^+_{\overline{\omega}})^{-1}\Phi^+).$$

The diagram

$$\begin{array}{ccc} \mathfrak{V}^+\overline{\Omega} & \xrightarrow{\mathrm{Triv}} & \overline{\Omega} \times \mathbb{R}^m \\ \Big\downarrow{\scriptstyle \mathfrak{T}_s} & & \Big\downarrow{\scriptstyle g_s^{\overline{\mathbb{A}}}} \\ \mathfrak{V}^+\overline{\Omega} & \xrightarrow{\mathrm{Triv}} & \overline{\Omega} \times \mathbb{R}^m \end{array}$$

is commutative.

Recall that $W(\mathfrak{G})$ stands for the set of all finite words over the alphabet $\mathfrak{G}$, and that for a word $w \in W(\mathfrak{G})$, $w = w_0 \dots w_n$, $w_i \in \mathfrak{G}$, we write $\Gamma(w) = \Gamma(w_0)\dots\Gamma(w_n)$.

Let $\mathfrak{G}_+$ be the set of all $\Gamma \in \mathfrak{G}$ such that all entries of the matrix $A(\Gamma)$ are positive. Take $\Gamma \in \mathfrak{G}_+$ and a word $w \in W(\mathfrak{G})$, $w = w_0 \dots w_n$ satisfying $\Gamma(w) = \Gamma$. Let $\mathscr{M}(w, \infty)$ be the family of Borel ergodic σ-invariant measures μ on Ω (finite or infinite) such that

1. $\mu(\{\omega : \omega_0 = w_0, \dots, \omega_n = w_n\}) > 0$;
2. $\int_\Omega \tau^1(\omega) d\mu(\omega) = 1$.

The first condition together with ergodicity of μ implies that $\mu(\Omega \setminus \Omega_{ue}) = 0$. The second condition yields that the measure

$$\mathbb{P}_\mu = \mu \times \frac{dr}{r}$$

is a well-defined probability g_s-invariant measure on $\overline{\Omega}$.

Denote

$$\mathscr{M}(\Gamma, \infty) = \bigcup_{w \in W(\mathfrak{G}), \Gamma(w)=\Gamma} \mathscr{M}(w, \infty),$$

and let $\mathscr{P}^+$ be the space of ergodic probability g_s-invariant measures $\mathbb{P}$ on $\overline{\Omega}$ having the form $\mathbb{P} = \mathbb{P}_\mu$, $\mu \in \mathscr{M}(\Gamma, \infty)$, $\Gamma \in \mathfrak{G}_+$.

By definition, for any $\mathbb{P} \in \mathscr{P}^+$ the cocycle $\overline{\mathbb{A}}$ is integrable with respect to $\mathbb{P}$, and the Oseledets Theorem is applicable to $\overline{\mathbb{A}}$. For $\overline{\omega} \in \overline{\Omega}$, let $E^u_{\overline{\omega}}$ be the strictly unstable space of the cocycle $\overline{\mathbb{A}}$ at the point $\overline{\omega}$.

Proposition 31. *For almost every $\overline{\omega}$ the map $\mathcal{I}^+_{\overline{\omega}}$ induces an isomorphism from the strictly unstable space $E^u_{\overline{\omega}}$ of the cocycle $\overline{\mathbb{A}}$ at the point $\overline{\omega}$ to the space $\mathfrak{B}^+(X(\overline{\omega}))$.*

We start with the case when the measure μ is finite. In this case, for $\overline{\omega} \in \overline{\Omega}$, $\overline{\omega} = (\omega, r)$, the unstable space $E^u_{\overline{\omega}}$ of the cocycle $\overline{\mathbb{A}}$ at $\overline{\omega}$ coincides with the unstable space E^u_{ω} of the cocycle $\mathbb{A}$ at ω. Concatenating the graphs between consecutive occurrences of the word w and considering the induced map of the shift σ on the cylinder $\{\omega : \omega_0 = w_0, \dots, \omega_n = w_n\}$, we obtain Proposition 31 as an immediate corollary of Proposition 27.

In this case the Lyapunov exponents of $\mathbb{A}$ and $\overline{\mathbb{A}}$ are related by the following

Proposition 32. *If the positive Lyapunov exponents of $\mathbb{A}$ are*

$$\theta_1 > \theta_2 \cdots > \theta_{l_0}, \tag{88}$$

then the positive Lyapunov exponents $\overline{\theta}_i$ of $\overline{\mathbb{A}}$ are

$$\overline{\theta}_i = \theta_i / \theta_1. \tag{89}$$

Remark. In particular, we always have $\overline{\theta}_1 = 1$.

We proceed to the case when $\mu \in \mathscr{M}(w, \infty)$ is infinite. We will again consider the induced map of the shift σ on the cylinder $\{\omega : \omega_0 = w_0, \dots, \omega_n = w_n\}$, and check that the induced measure is finite.

More precisely, let $\mathfrak{G}_\Gamma \subset \mathfrak{G}$ be defined by the formula

$$\mathfrak{G}_\Gamma = \{\Gamma' \in \mathfrak{G} : \Gamma' = \Gamma\Gamma'' \text{ for some } \Gamma'' \in \mathfrak{G}\}.$$

Let $\Omega_\Gamma \subset \Omega$ be the subspace of sequences such that all their symbols lie in $\mathfrak{G}_\Gamma$, and let $\overline{\Omega}_\Gamma \subset \overline{\Omega}$ be the set of all pairs $(\omega, r) \in \overline{\Omega}$ satisfying $\omega \in \Omega_\Gamma$.

Let $\mathscr{M}_\Gamma$ be the space of Borel ergodic σ-invariant measures μ on Ω_Γ such that

$$\int_{\Omega_\Gamma} \tau^1(\omega) d\mu(\omega) = 1.$$

If $\Gamma \in \mathfrak{G}_+$, then any measure in $\mathscr{M}_\Gamma$ is necessarily finite since the function $\tau(\omega)$ is bounded away from 0 on Ω_Γ.

Let $\Omega'(w, \infty) \subset \Omega$ be the subset of bi-infinite sequences containing infinitely many occurrences of w, both in the past and in the future, and let $\Omega(w, \infty) \subset \Omega'(w, \infty)$ be the subset of $\omega \in \Omega'(w, \infty)$ satisfying the additional condition $\omega_0 = w_0, \dots, \omega_n = w_n$. Let $\sigma(w, \infty)$ be the induced map of the shift σ to $\Omega(w, \infty)$.

Write

$$\overline{\Omega}(w, \infty) = \{(\omega, r) \in \overline{\Omega}, \omega \in \Omega'(w, \infty)\}.$$

For a measure $\mu \in \mathscr{M}(w, \infty)$, let $\mu(w, \infty)$ be its restriction to $\Omega(w, \infty)$.

Concatenating the graphs between consecutive occurrences of w, we obtain a natural aggregating surjection

$$Ag^w : \Omega(w, \infty) \to \Omega_\Gamma$$

such that the diagram

$$\begin{tikzcd} \Omega(w,\infty) \arrow[r, "\sigma(w,\infty)"] \arrow[d, "Ag^w"] & \Omega(w,\infty) \arrow[d, "Ag^w"] \\ \Omega_\Gamma \arrow[r, "\sigma"] & \Omega_\Gamma \end{tikzcd}$$

is commutative.

The map Ag^w lifts to a map $\overline{Ag^w} : \overline{\Omega}(w,\infty) \to \overline{\Omega}_\Gamma$ such that the diagram

$$\begin{tikzcd} \overline{\Omega}(w,\infty) \arrow[r, "g_s"] \arrow[d, "\overline{Ag^w}"] & \overline{\Omega}(w,\infty) \arrow[d, "\overline{Ag^w}"] \\ \overline{\Omega}_\Gamma \arrow[r, "g_s"] & \overline{\Omega}_\Gamma \end{tikzcd}$$

is commutative.

The map $\overline{Ag^w}$ preserves the cocycle $\overline{\mathbb{A}}$ in the following sense: if the map $\overline{Ag^w} \times \mathrm{Id} : \overline{\Omega}(w,\infty) \times \mathbb{R}^m \to \overline{\Omega}_\Gamma \times \mathbb{R}^m$ is given by

$$\overline{Ag^w} \times \mathrm{Id}(\overline{\omega}, v) = (\overline{Ag^w}\overline{\omega}, v),$$

then the diagram

$$\begin{tikzcd} \overline{\Omega}(w,\infty) \times \mathbb{R}^m \arrow[r, "g_s^{\overline{\mathbb{A}}}"] \arrow[d, "\overline{Ag^w}\times\mathrm{Id}"] & \overline{\Omega}(w,\infty) \times \mathbb{R}^m \arrow[d, "\overline{Ag^w}\times\mathrm{Id}"] \\ \overline{\Omega}_\Gamma \times \mathbb{R}^m \arrow[r, "g_s^{\overline{\mathbb{A}}}"] & \overline{\Omega}_\Gamma \times \mathbb{R}^m \end{tikzcd}$$

is commutative.

For $\mu \in \mathscr{M}(w,\infty)$, set

$$\mu^w = (Ag^w)_* \mu.$$

The correspondence

$$\mu \to \mu^w \tag{90}$$

induces an affine map from $\mathscr{M}(w,\infty)$ to $\mathscr{M}_\Gamma$. From the definitions we clearly have

Proposition 33. *For any $\mu \in \mathscr{M}(w,\infty)$ the dynamical systems $(\overline{\Omega}, \mathbb{P}_\mu, g_s)$ and $(\Omega, \mathbb{P}_{\mu^w}, g_s)$ are measurably isomorphic.*

We have thus reduced the case of infinite measures to the case of finite measures, in which Proposition 27 is applicable. Proposition 31 is proved completely.

3.3.4 The space of translation flows.

Now assume that for almost all ω there is a Vershik's ordering $\mathfrak{o}(\omega)$ on the edges of each graph ω_n, $n \in \mathbb{Z}$. Furthermore, assume that the ordering is shift-invariant in the following sense: the ordering $\mathfrak{o}(\omega)$ on the edges of the graph ω_{n+1} is the same as the ordering $\mathfrak{o}(\sigma\omega)$ on the edges of the graph $(\sigma\omega)_n = \omega_{n+1}$.

In this case for almost every $\omega \in \Omega$ we obtain a flow $h_t^{+,\omega}$ on $X(\omega)$. Given $(\omega, r) \in \Omega_0$, set $h_t^{+,(\omega,r)} = h_{t/r}^{+,\omega}$.

Identifying, as before, the spaces Ω_0 and $\overline{\Omega}$ we obtain, for almost every $\overline{\omega}$, a flow $h_t^{+,\overline{\omega}}$ on $X(\overline{\omega})$ in such a way that the following diagram is commutative:

$$\begin{array}{ccc} X(\overline{\omega}) & \xrightarrow{h^+_{\exp(s)t,\overline{\omega}}} & X(\overline{\omega}) \\ \Big\downarrow \mathfrak{t}_s & & \Big\downarrow \mathfrak{t}_s \\ X(g_s\overline{\omega}) & \xrightarrow{h^+_{t,\overline{\omega}}} & X(g_s\overline{\omega}) \end{array}$$

Recall that $\mathscr{P}^+$ is the space of ergodic probability g_s-invariant measures $\mathbb{P}$ on $\overline{\Omega}$ having the form $\mathbb{P} = \mathbb{P}_\mu$, $\mu \in \mathscr{M}(\Gamma, \infty)$, $\Gamma \in \mathfrak{G}_+$. Corollary 6 implies

Theorem 4. *Let $\mathbb{P} \in \mathscr{P}^+$. For any $\varepsilon > 0$ there exists a constant C_ε depending only on $\mathbb{P}$ such that the following holds for almost every $\overline{\omega} \in \overline{\Omega}$. There exists a continuous mapping $\Xi^+_{\overline{\omega}} : Lip^+_w(X(\overline{\omega})) \to \mathfrak{B}^+(X(\overline{\omega}))$ such that*

1. *for any $t_0 \in \mathbb{R}$ we have $\Xi^+_{\overline{\omega}}(f \circ h^+_{t_0}) = \Xi^+_{\overline{\omega}}(f)$;*
2. *the diagram*

$$\begin{array}{ccc} Lip^+_w(X(\overline{\omega})) & \xrightarrow{\Xi^+_{\overline{\omega}}} & \mathfrak{B}^+(X(\overline{\omega})) \\ \Big\uparrow (\mathfrak{t}_s)^* & & \Big\downarrow (\mathfrak{t}_s)_* \\ Lip^+_w(X(g_s\overline{\omega})) & \xrightarrow{\Xi^+_{g_s\overline{\omega}}} & \mathfrak{B}^+(X(g_s\overline{\omega})) \end{array}$$

   is commutative

3. *for any $f \in Lip^+_w(X)$, any $x \in X(\overline{\omega})$ and any $T > 0$ we have*

$$\left| \int_0^T f \circ h_t^+(x)dt - \Xi^+(f; x, T) \right| \le C_\varepsilon ||f||_{Lip^+_w}(1 + T^\varepsilon). \qquad (91)$$

3.4 Limit Theorems.

3.4.1 The leading term in the asymptotic for the ergodic integral.

We assume that the first and the second Lyapunov exponents of the cocycle $\overline{\mathbb{A}}$ are simple, and we consider the corresponding subspaces $E^u_{1,\overline{\omega}} = \mathbb{R}h_{\overline{\omega}}$ and $E^u_{2,\overline{\omega}}$. Furthermore, let $E^u_{\geq 3,\overline{\omega}}$ be the subspace corresponding to the remaining Lyapunov exponents.

We have then the decomposition

$$E^u_{\overline{\omega}} = E^u_{1,\overline{\omega}} \oplus E^u_{2,\overline{\omega}} \oplus E^u_{\geq 3,\overline{\omega}}.$$

A similar decomposition holds for $\tilde{E}^u$:

$$\tilde{E}^u_{\overline{\omega}} = \tilde{E}^u_{1,\overline{\omega}} \oplus \tilde{E}^u_{2,\overline{\omega}} \oplus \tilde{E}^u_{\geq 3,\overline{\omega}}.$$

Choose $\Phi_2^+ \in \mathcal{I}_{\overline{\omega}}(E^u_{2,\overline{\omega}})$, $\Phi_2^- \in \mathcal{I}_{\overline{\omega}}(\tilde{E}^u_{2,\overline{\omega}})$ in such a way that

$$\langle \Phi_2^+, \Phi_2^- \rangle = 1.$$

Take $f \in Lip^+_w(X)$, $x \in X$, $T \in \mathbb{R}$ and observe that the expression

$$m_{\Phi_2^-}(f)\Phi_2^+(x,T) \tag{92}$$

does not depend on the precise choice of $\Phi_2^{\pm}$ (we have the freedom of multiplying Φ_2^+ by an arbitrary scalar, but then Φ_2^- is divided by the same scalar).

Now for $f \in Lip^+_w(X)$ write

$$\Phi_f^+(x,T) = (\int_X f d\nu) \cdot T + m_{\Phi_2^-}(f)\Phi_2^+(x,T) + \Phi^+_{3,f}(x,T),$$

where $\Phi^+_{3,f} \in \mathcal{I}_{\overline{\omega}}(\tilde{E}^u_{\geq 3,\overline{\omega}})$.

In particular, there exist two positive constants C and α depending only on $\mathbb{P}$ such that for any $f \in Lip^+_w(X)$, $\int_X f d\nu = 0$, we have

$$\left| \int_0^T f \circ h_t^+(x)dt - m_{\Phi_2^-}(f)\Phi_2^+(x,T) \right| \leq C||f||_{Lip} T^{\overline{\theta}_2 - \alpha}. \tag{93}$$

3.4.2 The growth of the variance.

In order to estimate the variance of the random variable $\int_0^T f \circ h_t^+(x)dt$, we start by studying the growth of the variance of the random variable $\Phi^+_{2,\overline{\omega}}(x,T)$ as $T \to \infty$.

Recall that $\mathbb{E}_{\nu(\overline{\omega})}\Phi^+_{2,\overline{\omega}}(x,T) = 0$ for all T, while $Var_{\nu(\overline{\omega})}\Phi^+_{2,\overline{\omega}}(x,T) \neq 0$ for $T \neq 0$. Recall that for a cocycle $\Phi^+ \in \mathfrak{B}^+_{\overline{\omega}}$, $\Phi^+ = \mathcal{I}^+_{\overline{\omega}}(v)$, we have defined its norm $|\Phi^+|$ by the formula $|\Phi^+| = |v|$. Introduce a multiplicative cocycle $H_2(\overline{\omega}, s)$ over the flow g_s by the formula

$$H_2(\overline{\omega}, s) = \frac{|\overline{\mathbb{A}}(s,\overline{\omega})v|}{|v|}, \ v \in E^u_{2,\overline{\omega}}, \ v \neq 0. \tag{94}$$

Observe that the right-hand side does not depend on the specific choice of $v \neq 0$.

By definition, we now have

$$\lim_{s\to\infty} \frac{\log H_2(s,\overline{\omega})}{s} = \overline{\theta}_2. \tag{95}$$

Proposition 34. *There exists a positive measurable function $V : \overline{\Omega} \to \mathbb{R}_+$ such that the following holds for $\mathbb{P}$-almost all $\overline{\omega} \in \overline{\Omega}$.*

$$Var_\nu \Phi_2^+(x,T) = V(g_s\overline{\omega})|\Phi_2^+|^2(H_2(\overline{\omega},s))^2. \tag{96}$$

Indeed, the function $V(\overline{\omega})$ is given by

$$V(\overline{\omega}) = \frac{Var_\nu \Phi_2^+(x,1)}{|\Phi_2^+|^2}.$$

Observe that the right-hand side does not depend on a particular choice of $\Phi_2^+ \in \mathfrak{B}^+_{2,\overline{\omega}}$, $\Phi^+ \neq 0$.

Using (93), we now proceed to estimating the growth of the variance of $\int_0^T f \circ h_t^+(x)dt$.

Proposition 35. *There exists $\alpha > 0$ depending only on $\mathbb{P}$ and a positive measurable function $C : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}_+$ such that the following holds for $\mathbb{P}$-almost all $\overline{\omega} \in \overline{\Omega}$. Let $\Phi^+_{2,\overline{\omega}} \in \mathfrak{B}^+$, $\Phi^-_{2,\overline{\omega}} \in \mathfrak{B}^-$ be such that $\langle \Phi^+_{2,\overline{\omega}}, \Phi^-_{2,\overline{\omega}} \rangle = 1$. Let $f \in Lip^+_{w,\overline{\omega}}$ be such that*

$$\int_{X(\overline{\omega})} f d\nu(\overline{\omega}) = 0,\ m_{\Phi^-_{2,\overline{\omega}}}(f) \neq 0.$$

Then

$$\left| \frac{Var_{\nu(\overline{\omega})} \mathfrak{S}(f,x,e^s)}{V(g_s\overline{\omega})(m_{\Phi_2^-}(f)|\Phi_2^+|H_2(\overline{\omega},s))^2} - 1 \right| \leq C(\overline{\omega}, g_s\overline{\omega}) \exp(-\alpha s). \tag{97}$$

Remark. Observe that the quantity $(m_{\Phi_2^-}(f)|\Phi_2^+|)^2$ does not depend on the specific choice of $\Phi_2^+ \in \mathfrak{B}_2^+$, $\Phi_2^- \in \mathfrak{B}_2^-$ such that $\langle \Phi^+, \Phi^- \rangle = 1$.

Indeed, the proposition is immediate from the inequality

$$|\mathbb{E}(\xi_1^2) - \mathbb{E}(\xi_2^2)| \leq \sup|\xi_1 + \xi_2| \cdot \mathbb{E}|\xi_1 - \xi_2|,$$

which holds for any two bounded random variables ξ_1, ξ_2 on any probability space, and the following clear Proposition.

Proposition 36. *There exists a and a positive measurable function $C : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}_+$ and a positive measurable function $V' : \overline{\Omega} \to \mathbb{R}_+$ such that*

$$\max |\Phi_2^+(x,e^s)| = V'(g_s\overline{\omega})H_2(s,\overline{\omega}); \tag{98}$$

$$\left| \frac{\max \mathfrak{S}(f,x,e^s)}{V'(g_s\overline{\omega})(m_{\Phi_2^-}(f)|\Phi^+|H_2(\overline{\omega},s))^2} - 1 \right| \leq C(\overline{\omega}, g_s\overline{\omega}) \exp(-\alpha s). \tag{99}$$

3.4.3 Formulation and proof of the limit theorem.

We now turn to the asymptotic behaviour of the distribution of the random variable $\mathfrak{S}(f, x, e^s)$ as $s \to \infty$.

As in the Introduction, we take the space $C[0,1]$ of continuous functions on the unit interval endowed with the Tchebyshev topology, we let $\mathfrak{M}$ be the space of Borel probability measures on the space $C[0,1]$ endowed with the Lévy-Prohorov metric.

Consider the space $\mathcal{H}'$ given by the formula

$$\overline{\Omega}' = \{\overline{\omega}' = (\overline{\omega}, v), v \in E^+_{2,\overline{\omega}}, |v| = 1\}.$$

The flow g_s is lifted to $\overline{\Omega}'$ by the formula

$$g'_s(X, v) = \left(g_s X, \frac{\overline{\mathbb{A}}(s, \overline{\omega})v}{|(\overline{\mathbb{A}}(s, \overline{\omega})v|} \right).$$

Given $\overline{\omega}' \in \overline{\Omega}'$, $\overline{\omega}' = (\overline{\omega}, v)$, write

$$\Phi^+_{2,\overline{\omega}'} = \mathcal{I}_{\overline{\omega}}(v).$$

As before, write

$$V(\overline{\omega}') = Var_{\nu(\overline{\omega})} \Phi^+_{2,\overline{\omega}'}(x, 1).$$

Now introduce the map

$$\mathcal{D}^+_2 : \overline{\Omega}' \to \mathfrak{M}$$

by setting $\mathcal{D}^+_2(\overline{\omega}')$ to be the distribution of the $C[0,1]$-valued normalized random variable

$$\frac{\Phi_{2,\overline{\omega}'}(x, \tau)}{\sqrt{V(\overline{\omega}')}}, \ \tau \in [0,1].$$

Note here that for any $\tau_0 \neq 0$ we have $Var_{\nu(\overline{\omega})} \Phi_{2,\overline{\omega}}(x, \tau_0) \neq 0$, so, by definition, we have $\mathcal{D}^+_2(\overline{\omega}') \in \mathfrak{M}_1$.

Now, as before, we take a function $f \in Lip^+_{w,\overline{\omega}}$ such that

$$\int_{X(\overline{\omega})} f d\nu(\overline{\omega}) = 0, \ m_{\Phi^-_{2,\overline{\omega}}}(f) \neq 0$$

and denote by $\mathfrak{m}[f, s]$ the distribution of the normalized random variable

$$\frac{\int_0^{\tau \exp(s)} f \circ h^+_t(x) dt}{\sqrt{Var_{\nu(\overline{\omega})} \int_0^{\exp(s)} f \circ h^+_t(x) dt}}. \tag{100}$$

As before, d_{LP} stands for the Lévy-Prohorov metric on $\mathfrak{M}$.

Theorem 5. *Let $\mathbb{P} \in \mathscr{P}^+$ be such that both the first and the second Lyapunov exponents of the renormalization cocycle $\overline{\mathbb{A}}$ are positive and simple with respect to $\mathbb{P}$. There exists a positive measurable function $C : \overline{\Omega}' \times \overline{\Omega}' \to \mathbb{R}_+$ and a positive constant α depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $X' \in \overline{\Omega}'$ and any $f \in Lip^+_{w,0}(X)$ satisfying $m^-_{2,X'}(f) > 0$ we have*

$$d_{LP}(\mathfrak{m}[f,s], \mathcal{D}_2^+(g'_s\overline{\omega}')) \leq C(\overline{\omega}', g'_s\overline{\omega}') \exp(-\alpha s). \tag{101}$$

Proof: For any pair of random variables ξ_1, ξ_2 and on any probability space and any positive real numbers M_1, M_2 we have

$$\sup\left|\frac{\xi_1}{M_1} - \frac{\xi_2}{M_2}\right| \leq \sup|\xi_1| \cdot \left|\frac{M_1 - M_2}{M_1 M_2}\right| + \frac{\sup|\xi_1 - \xi_2|}{M_2}. \tag{102}$$

Let $\varepsilon > 0$ and let ξ_1, ξ_2 be two random variables on an arbitrary probability space $(\Omega, \mathbb{P})$ taking values in a complete metric space and such that the distance between their values does not exceed ε. Then, by definition of the Lévy-Prohorov metric, the Lévy-Prohorov distance between their distributions $(\xi_1)_*\mathbb{P}$, $(\xi_2)_*\mathbb{P}$ also does not exceed ε.

Theorem 5 is now immediate from Equation (93) and Proposition 35.

3.4.4 Atoms of limit distributions.

Proposition 37. *Let $\omega \in \Omega$ satisfy $\lambda_1^{(0,\omega)} > 1/2$. Then there exists a set $\Pi \subset X(\omega)$ such that*

1. *$\nu_\omega(\Pi) \geq (2\lambda_1^{(0,\omega)} - 1)h_1^{(0,\omega)}$;*
2. *for any $\Phi^+ \in \mathfrak{B}^+(X(\omega))$, the function $\Phi^+(x, h_1^{(0,\omega)})$ is constant on Π.*

Proof. We consider ω fixed and omit it from notation. As before, consider the flow transversal

$$I = \{x \in X : x_n = \min\{e : I(e) = F(x_{n+1}) \text{ for all } n \leq 0\}.$$

and decompose it into "subintervals" $I_k = \{x \in I : I(x_0) = k\}$, $k = 1, \dots, m$.

The transversal I carries a natural conditional measure ν_I invariant under the first-return map of the flow h_t^+ on I. The measure ν_I is given by the formula

$$\nu_I(\{x \in I : x_1 = e_1, \dots, x_n = e_n\}) = \lambda_{I(e_n)}^{(n+1)},$$

provided $I(e_k) = F(e_{k+1})$, $k = 1, \dots, n$. In particular, $\nu_I(I_k) = \lambda_k^{(0)}$. For brevity, denote $t_1 = h_1^{(0)}$. By definition, $h^+_{t_1} I_1 \subset I$ and we have

$$\nu_I(I_1 \bigcap h^+_{t_1} I_1) \geq 2\lambda_1^{(0)} - 1 > 0.$$

Introduce the set

$$\Pi = \{h^+_\tau x, 0 < \tau < t_1, x \in I_1, h_{t_1} x \in I_1\}.$$

The first statement of the Proposition is clear, and we proceed to the proof of the second. Note first that for any $\Phi^+ \in \mathfrak{B}^+(X(\omega))$ and any $\tau, 0 \le \tau \le t_1$ the quantity $\Phi^+(x, \tau)$ is constant as long as x varies in I_1.

Fix $\Phi^+ \in \mathfrak{B}^+(X(\omega))$ and take an arbitrary $\tilde{x} \in \Pi$. Write $\tilde{x} = h^+_{\tau_1} x_1$, where $x_1 \in I_1$, $0 < \tau_1 < t_1$. We have $h^+_{t_1-\tau_1}\tilde{x} \in I_1$, whence

$$\Phi^+(h^+_{t_1-\tau_1}\tilde{x}, \tau_1) = \Phi^+(x_1, \tau_1)$$

and

$$\Phi^+(\tilde{x}, t_1) = \Phi^+(\tilde{x}, t_1-\tau_1)+\Phi^+(h^+_{t_1-\tau_1}\tilde{x}, \tau_1) = \Phi^+(h^+_{\tau_1}x_1, t_1-\tau_1)+\Phi^+(x_1, \tau_1) = \Phi^+(x_1, t_1),$$

and the Proposition is proved.

For $\overline{\omega}' \in \overline{\Omega}'$, let $\mathfrak{m}_2(\overline{\omega}), \tau)$ be the distribution of the $\mathbb{R}$-valued random variable

$$\frac{\Phi^+_{2,\overline{\omega}'}(x, \tau)}{\sqrt{Var_\nu \Phi^+_{2,\overline{\omega}'}(x, \tau)}}.$$

For brevity, write $\mathfrak{m}_2(\overline{\omega}', 1) = \mathfrak{m}_2(\overline{\omega}')$.

Corollary 7. *Let $\mathbb{P} \in \mathscr{P}^+$ be such that both the first and the second Lyapunov exponents of the renormalization cocycle $\overline{\mathbb{A}}$ are positive and simple with respect to $\mathbb{P}$. Assume that for any $\varepsilon > 0$ we have*

$$\mathbb{P}(\{\overline{\omega} : \lambda_1^{(\overline{\omega})} > 1-\varepsilon, h_1^{(\overline{\omega})} > 1-\varepsilon\}) > 0.$$

Then the measure δ_0 is an accumulation point, in the weak topology, for the set

$$\{\mathfrak{m}_2(\overline{\omega}'), \overline{\omega}' \in \overline{\Omega}'\}.$$

Here, as before, δ_0 stands for the delta-measure at zero.

Proposition 38. *Let $\tilde{x}, \hat{x} \in X$ be distinct and satisfy $\tilde{x} \in \gamma^+_\infty(\hat{x})$, $\tilde{x} \in \gamma^-_\infty(\hat{x})$. Let t_0 be such that $\hat{x} = h^+_{t_0}\tilde{x}$. Then there exists a set Π of positive measure such that for any $x \in \Pi$ and any $\Phi^+ \in \mathfrak{B}^+(X)$ we have*

$$\Phi^+(x, t_0) = \Phi^+(\tilde{x}).$$

Proof. There exist $n_0, n_1 \in \mathbb{Z}$ such that

$$\tilde{x}_t = \hat{x}_t, \ t \in (-\infty, n_0] \cup [n_1, \infty).$$

Let Π be the set of all x' satisfying the equality $x'_t = \tilde{x}_t$ for $t \in (n_0, n_1)$, and, additionally, the equalities

$$F(x'_{n_1-1}) = F(\tilde{x}_{n_1-1}), \ I(x'_{n_0}) = I(\tilde{x}_{n_0}),$$

By holonomy-invariance, for any $x' \in \Pi$ and any $\Phi^+ \in \mathfrak{B}^+(X)$ we have

$$\Phi^+(x', t_0) = \Phi^+(\tilde{x}).$$

To estimate from below the measure of the set Π, set $I(\tilde{x}_{n_0}) = i$, $F(\tilde{x}_{n_1-1}) = j$ and note that by definition we have

$$\nu(\Pi) \geq \lambda_i^{(n_0)} h_j^{(n_1-1)}.$$

For a fixed $\overline{\omega}$, the set of "homoclinic times" t_0 for which there exist $\tilde{x}, \hat{x} \in X$ satisfying $\tilde{x} \in \gamma_\infty^+(\hat{x})$, $\tilde{x} \in \gamma_\infty^-(\hat{x})$, $\hat{x} = h_{t_0}^+ \tilde{x}$, is countable and dense in $\mathbb{R}$.

Corollary 8. *For almost every $\overline{\omega} \in \overline{\Omega}$, there exists a dense set of times $t_0 \in \mathbb{R}$ such that for any $\Phi^+ \in \mathfrak{B}^+$ the distribution of the random variable $\Phi^+(x, t_0)$ has an atom.*

3.5 Ergodic averages of Vershik's automorphisms.

3.5.1 The space of one-sided Markov compacta.

Assume we are given a sequence $\{\Gamma_n\}$, $n \in \mathbb{N}$, of graphs belonging to $\mathfrak{G}$. To this sequence we assign the one-sided Markov compactum of infinite one-sided paths in our sequence of graphs:

$$Y = \{y = y_1 \dots y_n \dots,\ y_n \in \mathcal{E}(\omega_n), F(y_{n+1}) = I(y_n)\}. \tag{103}$$

As before, we write $A_n(Y) = A(\Gamma_n)$.

The Markov compactum Y is endowed with a natural tail equivalence relation: $y \sim y'$ if there exists n_0 such that $y_n = y'_n$ for all $n > n_0$.

Cylinders in Y are subsets of the form $\{y : y_{n+1} = e_1, \dots, y_{n+k} = e_k\}$, where $n \in \mathbb{N}$, $k \in \mathbb{N}$, $e_1 \in \mathcal{E}(\Gamma_{n+1}), \dots, e_k \in \mathcal{E}(\Gamma_{n+k})$ and $F(e_i) = I(e_{i+1})$. The family of all cylinders forms a semi-ring which we denote by $\mathfrak{C}(Y)$.

Let $\mathfrak{V}(Y)$ be the vector space of all real-valued finitely-additive measures Φ defined on the semi-ring $\mathfrak{C}(Y)$ and invariant under the tail equivalence relation in the following precise sense: if $e_1 \in \mathcal{E}(\Gamma_1), \dots, e_k \in \mathcal{E}(\Gamma_k)$ and $F(e_i) = I(e_{i+1})$, then the measure

$$\Phi(\{y : y_1 = e_1, \dots, y_k = e_k\})$$

only depends on $I(e_k)$.

As before, a sequence of vectors $\mathbf{v} = v^{(l)}$, $v^{(l)} \in \mathbb{R}^m$, $l \in \mathbb{N}$ satisfying

$$v^{(l)} = A_l^t v^{(l+1)},$$

will be called *reverse equivariant*. By definition, the vector space of all reverse equivariant sequences is isomorphic to $\mathfrak{V}(Y)$.

Now, in analogy to the bi-infinite case, let Ω_+ be the space of sequences of one-sided infinite sequences of graphs $\Gamma_n \in \mathfrak{G}$. As before, we write

$$\Omega_+ = \{\omega = \omega_1 \dots \omega_n \dots, \omega_i \in \mathfrak{G}, i \in \mathbb{N}\},$$

and, for $\omega \in \Omega$, we denote by $Y(\omega)$ the Markov compactum corresponding to ω. The right shift σ on the space Ω is defined by the formula $(\sigma\omega)_n = \omega_{n+1}$.

Let μ be an ergodic σ-invariant probability measure on Ω_+ whose natural extension to the space Ω satisfies Assumption 10. Again we have two natural cocycles over the system (Ω, σ, μ):

1. the renormalization cocycle $\mathbb{A}$ defined, for $n > 0$, by the formula

$$\mathbb{A}(n, \omega) = A(\omega_n) \dots A(\omega_1).$$

2. the reverse transpose cocycle $\mathbb{A}^{-t}$ defined, for $n > 0$, by the formula

$$\mathbb{A}^{-t}(n, \omega) = (A^t)^{-1}(\omega_n) \dots (A^t)^{-1}(\omega_1).$$

Our assumptions imply that both these cocycles satisfy the Oseledets Theorem.

For $\omega \in \Omega$, let $\check{E}_\omega$ be the strictly stable Lyapunov subspace of the cocycle $\mathbb{A}^{-t}$. Each $v \in \check{E}_\omega$ gives rise to a reverse-equivariant sequence of vectors and thus to a measure $\Phi_v \in \mathfrak{V}(Y)$; denote

$$\mathfrak{B}(\omega) = \mathfrak{B}(Y(\omega)) = \{\Phi_v,\ v \in \check{E}_\omega\}.$$

Let $-\theta_{l_0} > -\theta_{l_0-1} > \dots > -\theta_1$ be the distinct negative Lyapunov exponents of the inverse transpose cocycle $\mathbb{A}^{-t}$. By the Oseledets Theorem, for almost every ω we have the corresponding flag of subspaces

$$\check{E}_{\theta_1} \subset \check{E}_{\theta_2} \subset \dots \subset \check{E}_{\theta_{l_0}},$$

where for any $v \in \check{E}_{\theta_i} \setminus \check{E}_{\theta_{i-1}}$ we have

$$\lim_{n\to\infty} \frac{\log |\mathbb{A}^{-t}(n, \omega)v|}{n} = -\theta_i.$$

Set

$$\mathfrak{B}_{\theta_i}(\omega) = \mathfrak{B}_{\theta_i}(Y(\omega)) = \{\Phi_v,\ v \in \check{E}_{\theta_i,\omega}\}.$$

For instance, our assumptions imply that almost every ω admits a unique (up to scaling) positive reverse equivariant sequence; the space $\mathfrak{V}(Y(\omega))$ then contains a unique positive countably additive probability measure ν_ω; we have then $\nu_\omega \in \mathfrak{B}_{\theta_1}(Y(\omega))$.

As before, a bounded measurable function $f : Y \to \mathbb{R}$ will be called *weakly Lipschitz* if there exists a constant $C > 0$ such that for any cylinder $\mathcal{C} \in \mathfrak{C}(Y)$ and any points $y(1), y(2) \in \mathcal{C}$ we have

$$|f(y(1)) - f(y(2))| \le C\nu_\omega(\mathcal{C}). \tag{104}$$

If C_f is the infimum of all constants in the right-hand side of (104), then the norm of f is given by

$$||f||_{Lip_w} = C_f + \sup_Y f.$$

For almost all ω, all $f \in Lip_w(Y(\omega))$ and all $\Phi \in \mathfrak{B}(Y(\omega))$, the Riemann-Stieltjes integral $\int_Y f d\Phi$ is well-defined. Given $f \in Lip_w(Y(\omega))$, set

$$\theta(f) = \max\{\theta_i : \text{ there exists } \Phi \in \mathfrak{B}_{\theta_i}(Y(\omega)) \text{ such that } \int_Y f d\Phi \ne 0\}$$

(if $\int f d\Phi = 0$ for all $\Phi \in \mathfrak{B}(Y(\omega))$, then we set $\theta(f) = 0$).

3.5.2 Vershik's automorphisms.

As before, assume that for almost every $\omega \in \Omega_+$ there is a Vershik's ordering $\mathfrak{o}(\omega)$ on the edges of each graph ω_n, $n \in \mathbb{N}$. Furthermore, assume that the ordering is shift-invariant in the following sense: the ordering $\mathfrak{o}(\omega)$ on the edges of the graph ω_{n+1} is the same as the ordering $\mathfrak{o}(\sigma\omega)$ on the edges of the graph $(\sigma\omega)_n = \omega_{n+1}$. Almost every Markov compactum $Y(\omega)$ is now endowed with a Vershik's automorphism T_Y with respect to the ordering $\mathfrak{o}(\omega)$, and Theorem 4 implies

Corollary 9. *Let μ be an ergodic σ-invariant probability measure on Ω_+ whose natural extension to the space Ω satisfies Assumption 10. Then for μ-almost any $\omega \in \Omega_+$, any $f \in Lip_w(Y(\omega))$ and any $y \in Y(\omega)$ we have*

$$\limsup_{N\to\infty} \frac{\log \left| \sum\limits_{k=0}^{N-1} f(T_Y^k y) \right|}{\log N} = \theta(f). \tag{105}$$

Note that for any $\omega \in \Omega$ the automorphism $T_{Y(\sigma\omega)}$ on $Y(\sigma\omega)$ can be realized as an induced automorphism of $T_{Y(\omega)}$ on $Y(\omega)$ in the following way. Take $\Gamma(\omega_1)$ and let $\mathcal{E}_{min}(\Gamma(\omega_1))$ be the set of minimal edges with respect to the ordering $\mathfrak{o}$. Consider the subset $Y'(\omega) \subset Y(\omega)$ given by

$$Y'(\omega) = \{x \in Y(\omega) : x_1 \in \mathcal{E}_{min}(\Gamma(\omega_1))\}.$$

The shift σ maps $Y'(\omega)$ bijectively onto $Y(\sigma\omega)$; the induced map of $T_{Y(\omega)}$ on $Y'(\omega)$ is isomorphic to $T_{Y(\sigma\omega)}$. For $\Phi \in \mathfrak{V}(Y(\omega))$, consider its restriction $\Phi|_{Y'(\omega)}$; we have $\sigma_* \left(\Phi|_{Y'(\omega)}\right) \in \mathfrak{V}(Y(\sigma\omega))$, and if $\Phi \in \mathfrak{B}(Y(\omega))$, then we have $\sigma_* \left(\Phi|_{Y'(\omega)}\right) \in \mathfrak{B}(Y(\sigma\omega))$.

4 Abelian Differentials and Markov Compacta.

4.1 The Mapping into Cohomology.

First we show that for an arbitrary abelian differential $\mathbf{X} = (M, \omega)$ the map

$$\check{\mathcal{I}}_{\mathbf{X}} : \mathfrak{B}_c^+(M, \omega) \to H^1(M, \mathbb{R})$$

given by (5) is well-defined. Then, in the following subsections we show that for almost all (M, ω) the image of $\mathfrak{B}^+(\mathbf{X})$ under this mapping is the unstable space of the Kontsevich-Zorich cocycle.

Proposition 39. *Let γ_i, $i = 1, \ldots, k$, be rectangular closed curves such that the cycle $\sum_{i=1}^k \gamma_i$ is homologous to 0. Then for any $\Phi^+ \in \mathfrak{B}^+$ we have*

$$\sum_{i=1}^k \Phi^+(\gamma_i) = 0.$$

Take a fundamental polygon Π for M such that all its sides are simple closed rectangular curves on M. Let $\partial\Pi$ be the boundary of Π, oriented counterclockwise. By definition,

$$\Phi^+(\partial\Pi) = 0, \tag{106}$$

since each curve of the boundary enters $\partial\Pi$ twice and with opposite signs.

Proposition 40. *Let $\gamma \subset \Pi$ be a simple rectangular closed curve. Then*

$$\Phi^+(\gamma) = 0.$$

Proof of Proposition 40.

We may assume that γ is oriented counterclockwise and does not contain zeros of the form ω. By Jordan's theorem, γ is the boundary of a domain $N \subset \Pi$. Let $p_1, \ldots, p_r$ be zeros of ω lying inside N; let κ_i be the order of p_i. Choose an arbitrary $\varepsilon > 0$, take $\delta > 0$ such that $|\Phi^+(\gamma)| \leq \varepsilon$ as soon as the length of γ does not exceed δ and consider a partition of N given by

$$N = \Pi_1^{(\varepsilon)} \bigsqcup \cdots \bigsqcup \Pi_n^{(\varepsilon)} \bigsqcup \tilde{\Pi}_1^{(\varepsilon)} \bigsqcup \cdots \bigsqcup \tilde{\Pi}_r^{(\varepsilon)}, \tag{107}$$

where all $\Pi_i^{(\varepsilon)}$ are admissible rectangles and $\tilde{\Pi}_i^{(\varepsilon)}$ is a $4(\kappa_i + 1)$-gon containing p_i and no other zeros and satisfying the additional assumption that all its sides are no longer than δ. Let $\partial\Pi_i^{(\varepsilon)}$, $\partial\tilde{\Pi}_i^{(\varepsilon)}$ stand for the boundaries of our polygons oriented counterclockwise.

We have

$$\Phi^+(\gamma) = \sum \Phi^+(\partial\Pi_i^{(\varepsilon)}) + \sum \Phi^+(\partial\tilde{\Pi}_i^{(\varepsilon)}).$$

In the first sum, each term is equal to 0 by definition of Φ^+, whereas the second sum does not exceed, in absolute value, the quantity

$$C(\kappa_1, \ldots, \kappa_r)\varepsilon,$$

where $C(\kappa_1, \ldots, \kappa_r)$ is a positive constant depending only on $\kappa_1, \ldots, \kappa_r$. Since ε may be chosen arbitrarily small, we have

$$\Phi^+(\gamma) = 0,$$

which is what we had to prove.

For $A, B \in \partial\Pi$, let $\partial\Pi_A^B$ be the part of $\partial\Pi$ going counterclockwise from A to B.

Proposition 41. *Let $A, B \in \partial\Pi$ and let $\gamma \subset \Pi$ be an arbitrary rectangular curve going from A to B. Then*

$$\Phi^+(\partial\Pi_A^B) = \Phi^+(\gamma).$$

We may assume that γ is simple in Π, since, by Proposition 40, self-intersections of γ (whose number is finite) do not change the value of $\Phi^+(\gamma)$. If γ is simple, then γ and $\Phi^+(\partial\Pi_B^A)$ together form a simple closed curve, and the proposition follows from Proposition 40.

Corollary 10. *If $\gamma \subset \Pi$ is a rectangular curve which yields a closed curve in M homotopic to zero in M, then*

$$\Phi^+(\gamma) = 0.$$

Indeed, by the previous proposition we may think that $\gamma \subset \partial\Pi$, in which case the statement follows from (106).

4.2 Veech's Space of Zippered Rectangles

4.2.1 Rauzy-Veech Induction

Let π be a permutation of m symbols, which will always be assumed irreducible in the sense that $\pi\{1, \ldots, k\} = \{1, \ldots, k\}$ implies $k = m$. The Rauzy operations a and b are defined by the formulas

$$a\pi(j) = \begin{cases} \pi j, & \text{if } j \leq \pi^{-1}m, \\ \pi m, & \text{if } j = \pi^{-1}m + 1, \\ \pi(j-1), & \text{if } \pi^{-1}m + 1 < j \leq m; \end{cases}$$

$$b\pi(j) = \begin{cases} \pi j, & \text{if } \pi j \leq \pi m, \\ \pi j + 1, & \text{if } \pi m < \pi j < m, \\ \pi m + 1, & \text{if } \pi j = m. \end{cases}$$

These operations preserve irreducibility. The *Rauzy class* $\mathcal{R}(\pi)$ is defined as the set of all permutations that can be obtained from π by application of the transformation group generated by a and b. From now on we fix a Rauzy class $\mathcal{R}$ and assume that it consists of irreducible permutations.

For $i, j = 1, \ldots, m$, denote by E^{ij} the $m \times m$ matrix whose $(i,j)th$ entry is 1, while all others are zeros. Let E be the identity $m \times m$-matrix. Following Veech [24], introduce the unimodular matrices

$$\mathcal{A}(a, \pi) = \sum_{i=1}^{\pi^{-1}m} E^{ii} + E^{m,\pi^{-1}m+1} + \sum_{i=\pi^{-1}m}^{m-1} E^{i,i+1}, \tag{108}$$

$$\mathcal{A}(b, \pi) = E + E^{m,\pi^{-1}m}. \tag{109}$$

For a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$, we write

$$|\lambda| = \sum_{i=1}^{m} \lambda_i.$$

Let

$$\Delta_{m-1} = \{\lambda \in \mathbb{R}^m : |\lambda| = 1, \ \lambda_i > 0 \text{ for } i = 1, \ldots, m\}.$$

One can identify each pair (λ, π), $\lambda \in \Delta_{m-1}$, with the *interval exchange map* of the interval $I := [0, 1)$ as follows. Divide I into the sub-intervals $I_k :=$

$[\beta_{k-1}, \beta_k)$, where $\beta_0 = 0$, $\beta_k = \sum_{i=1}^k \lambda_i$, $1 \le k \le m$, and then place the intervals I_k in I in the following order (from left to write): $I_{\pi^{-1}1}, \dots, I_{\pi^{-1}m}$. We obtain a piecewise linear transformation of I that preserves the Lebesgue measure.

The space $\Delta(\mathcal{R})$ of interval exchange maps corresponding to $\mathcal{R}$ is defined by

$$\Delta(\mathcal{R}) = \Delta_{m-1} \times \mathcal{R}.$$

Denote

$$\Delta_\pi^+ = \{\lambda \in \Delta_{m-1} | \ \lambda_{\pi^{-1}m} > \lambda_m\}, \quad \Delta_\pi^- = \{\lambda \in \Delta_{m-1} | \ \lambda_m > \lambda_{\pi^{-1}m}\},$$

$$\Delta^+(\mathcal{R}) = \cup_{\pi \in \mathcal{R}} \{(\pi, \lambda) | \ \lambda \in \Delta_\pi^+\},$$

$$\Delta^-(\mathcal{R}) = \cup_{\pi \in \mathcal{R}} \{(\pi, \lambda) | \ \lambda \in \Delta_\pi^-\},$$

$$\Delta^\pm(\mathcal{R}) = \Delta^+(\mathcal{R}) \cup \Delta^-(\mathcal{R}).$$

The *Rauzy-Veech induction map* $\mathscr{T} : \Delta^\pm(\mathcal{R}) \to \Delta(\mathcal{R})$ is defined as follows:

$$\mathscr{T}(\lambda, \pi) = \begin{cases} (\frac{\mathcal{A}(a,\pi)^{-1}\lambda}{|\mathcal{A}(a,\pi)^{-1}\lambda|}, a\pi), & \text{if } \lambda \in \Delta_\pi^+, \\ (\frac{\mathcal{A}(b,\pi)^{-1}\lambda}{|\mathcal{A}(b,\pi)^{-1}\lambda|}, b\pi), & \text{if } \lambda \in \Delta_\pi^-. \end{cases} \tag{110}$$

One can check that $\mathscr{T}(\lambda, \pi)$ is the interval exchange map induced by (λ, π) on the interval $J = [0, 1-\gamma]$, where $\gamma = \min(\lambda_m, \lambda_{\pi^{-1}m})$; the interval J is then stretched to unit length.

Denote

$$\Delta^\infty(\mathcal{R}) = \bigcap_{n \ge 0} \mathscr{T}^{-n} \Delta^\pm(\mathcal{R}). \tag{111}$$

Every $\mathscr{T}$-invariant probability measure is concentrated on $\Delta^\infty(\mathcal{R})$. On the other hand, a natural Lebesgue measure defined on $\Delta(\mathcal{R})$, which is finite, but non-invariant, is also concentrated on $\Delta^\infty(\mathcal{R})$. Veech [24] showed that $\mathscr{T}$ has an absolutely continuous ergodic invariant measure on $\Delta(\mathcal{R})$, which is, however, infinite.

We have two matrix cocycles $\mathcal{A}^t$, $\mathcal{A}^{-1}$over $\mathscr{T}$ defined by

$$\mathcal{A}^t(n, (\lambda, \pi)) = \mathcal{A}^t(\mathscr{T}^n(\lambda, \pi)) \cdot \ldots \cdot \mathcal{A}^t(\lambda, \pi),$$

$$\mathcal{A}^{-1}(n, (\lambda, \pi)) = \mathcal{A}^{-1}(\mathscr{T}^n(\lambda, \pi)) \cdot \ldots \cdot \mathcal{A}^{-1}(\lambda, \pi).$$

We introduce the corresponding skew-product transformations $\mathscr{T}^{\mathcal{A}^t} : \Delta(\mathcal{R}) \times \mathbb{R}^m \to \Delta(\mathcal{R}) \times \mathbb{R}^m$, $\mathscr{T}^{\mathcal{A}^{-1}} : \Delta(\mathcal{R}) \times \mathbb{R}^m \to \Delta(\mathcal{R}) \times \mathbb{R}^m$,

$$\mathscr{T}^{\mathcal{A}^t}((\lambda, \pi), v) = (\mathscr{T}(\lambda, \pi), \mathcal{A}^t(\lambda, \pi)v);$$

$$\mathscr{T}^{\mathcal{A}^{-1}}((\lambda, \pi), v) = (\mathscr{T}(\lambda, \pi), \mathcal{A}^{-1}(\lambda, \pi)v).$$

4.2.2 The construction of zippered rectangles

Here we briefly recall the construction of the Veech space of zippered rectangles. We use the notation of [8].

Zippered rectangles associated to the Rauzy class $\mathcal{R}$ are triples (λ, π, δ), where $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$, $\lambda_i > 0$, $\pi \in \mathcal{R}$, $\delta = (\delta_1, \dots, \delta_m) \in \mathbb{R}^m$, and the vector δ satisfies the following inequalities:

$$\delta_1 + \cdots + \delta_i \leq 0, \quad i = 1, \dots, m-1, \tag{112}$$

$$\delta_{\pi^{-1}1} + \cdots + \delta_{\pi^{-1}i} \geq 0, \quad i = 1, \dots, m-1. \tag{113}$$

The set of all vectors δ satisfying (112), (113) is a cone in $\mathbb{R}^m$; we denote it by $K(\pi)$.

For any $i = 1, \dots, m$, set

$$a_j = a_j(\delta) = -\delta_1 - \cdots - \delta_j, \; h_j = h_j(\pi, \delta) = -\sum_{i=1}^{j-1} \delta_i + \sum_{l=1}^{\pi(j)-1} \delta_{\pi^{-1}l}. \tag{114}$$

4.2.3 Zippered rectangles and abelian differentials.

Given a zippered rectangle (λ, π, δ), Veech [24] takes m rectangles $\Pi_i = \Pi_i(\lambda, \pi, \delta)$ of girth λ_i and height h_i, $i = 1, \dots, m$, and glues them together according to a rule determined by the permutation π. This procedure yields a Riemann surface M endowed with a holomorphic 1-form ω which, in restriction to each Π_i, is simply the form $dz = dx + idy$. The union of the bases of the rectangles is an interval $I^{(0)}(\lambda, \pi, \delta)$ of length $|\lambda|$ on M; the first return map of the vertical flow of the form ω is precisely the interval exchange $\mathbf{T}_{(\lambda,\pi)}$.

The *area* of a zippered rectangle (λ, π, δ) is given by the expression

$$\mathit{Area}\,(\lambda, \pi, \delta) := \sum_{r=1}^{m} \lambda_r h_r = \sum_{r=1}^{m} \lambda_r \Big(-\sum_{i=1}^{r-1} \delta_i + \sum_{i=1}^{\pi r - 1} \delta_{\pi^{-1}i}\Big). \tag{115}$$

(Our convention is $\sum_{i=u}^{v} \dots = 0$ when $u > v$.)

Furthermore, to each rectangle Π_i Veech [25] assigns a cycle $\gamma_i(\lambda, \pi, \delta)$ in the homology group $H_1(M, \mathbb{Z})$: namely, if P_i is the left bottom corner of Π_i and Q_i the left top corner, then the cycle is the union of the vertical interval P_iQ_i and the horizontal subinterval of $I^{(0)}(\lambda, \pi, \delta)$ joining Q_i to P_i. It is clear that the cycles $\gamma_i(\lambda, \pi, \delta)$ span $H_1(M, \mathbb{Z})$.

4.2.4 The space of zippered rectangles.

Denote by $\mathcal{V}(\mathcal{R})$ the space of all zippered rectangles corresponding to the Rauzy class $\mathcal{R}$, i.e.,

$$\mathcal{V}(\mathcal{R}) = \{(\lambda, \pi, \delta) : \lambda \in \mathbb{R}_+^m, \; \pi \in \mathcal{R}, \; \delta \in K(\pi)\}.$$

Let also

$$\mathcal{V}^+(\mathcal{R}) = \{(\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : \lambda_{\pi^{-1}m} > \lambda_m\},$$
$$\mathcal{V}^-(\mathcal{R}) = \{(\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : \lambda_{\pi^{-1}m} < \lambda_m\},$$
$$\mathcal{V}^\pm(\mathcal{R}) = \mathcal{V}^+(\mathcal{R}) \cup \mathcal{V}^-(\mathcal{R}).$$

Veech [24] introduced the flow $\{P^t\}$ acting on $\mathcal{V}(\mathcal{R})$ by the formula

$$P^t(\lambda, \pi, \delta) = (e^t\lambda, \pi, e^{-t}\delta),$$

and the map $\mathcal{U} : \mathcal{V}^\pm(\mathcal{R}) \to \mathcal{V}(\mathcal{R})$, where

$$\mathcal{U}(\lambda, \pi, \delta) = \begin{cases} (\mathcal{A}(\pi, a)^{-1}\lambda, a\pi, \mathcal{A}(\pi, a)^{-1}\delta), & \text{if } \lambda_{\pi^{-1}m} > \lambda_m, \\ (\mathcal{A}(\pi, b)^{-1}\lambda, b\pi, \mathcal{A}(\pi, b)^{-1}\delta), & \text{if } \lambda_{\pi^{-1}m} < \lambda_m. \end{cases}$$

(The inclusion $\mathcal{U}\mathcal{V}^\pm(\mathcal{R}) \subset \mathcal{V}(\mathcal{R})$ is proved in [24].) The map $\mathcal{U}$ and the flow $\{P^t\}$ commute on $\mathcal{V}^\pm(\mathcal{R})$ and both preserve the measure determined on $\mathcal{V}(\mathcal{R})$ by the volume form $Vol = d\lambda_1 \ldots d\lambda_m d\delta_1 \ldots d\delta_m$. They also preserve the area of a zippered rectangle (see (115)) and hence can be restricted to the set

$$\mathcal{V}^{1,\pm}(\mathcal{R}) := \{(\lambda, \pi, \delta) \in \mathcal{V}^\pm(\mathcal{R}) : Area(\lambda, \pi, \delta) = 1\}.$$

The restriction of the volume form Vol to $\mathcal{V}^{1,\pm}(\mathcal{R})$ induces on this set a measure $\mu_{\mathcal{R}}$ which is invariant under $\mathcal{U}$ and $\{P^t\}$.

For $(\lambda, \pi) \in \Delta(\mathcal{R})$, denote

$$\tau^0(\lambda, \pi) =: -\log(|\lambda| - \min(\lambda_m, \lambda_{\pi^{-1}m})). \tag{116}$$

From (108), (109) it follows that if $\lambda \in \Delta_\pi^+ \cup \Delta_\pi^-$, then

$$\tau^0(\lambda, \pi) = -\log|\mathcal{A}^{-1}(c, \pi)\lambda|, \tag{117}$$

where $c = a$ when $\lambda \in \Delta_\pi^+$, and $c = b$ when $\lambda \in \Delta_\pi^-$.

Next denote

$$\mathscr{Y}_1(\mathcal{R}) := \{x = (\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : |\lambda| = 1,\ Area(\lambda, \pi, \delta) = 1\},$$
$$\tau(x) := \tau^0(\lambda, \pi) \text{ for } x = (\lambda, \pi, \delta) \in \mathscr{Y}_1(\mathcal{R}),$$
$$\mathcal{V}_{1,\tau}(\mathcal{R}) := \bigcup_{x \in \mathscr{Y}_1(\mathcal{R}),\ 0 \le t \le \tau(x)} P^t x. \tag{118}$$

Let

$$\mathcal{V}_{\neq}^{1,\pm}(\mathcal{R}) := \{(\lambda, \pi, \delta) \in \mathcal{V}^{1,\pm}(\mathcal{R}) : a_m(\delta) \neq 0\},$$
$$\mathcal{V}_\infty(\mathcal{R}) := \bigcap_{n \in \mathbb{Z}} \mathcal{U}^n \mathcal{V}_{\neq}^{1,\pm}(\mathcal{R}).$$

Clearly $\mathcal{U}^n$ is well-defined on $\mathcal{V}_\infty(\mathcal{R})$ for all $n \in \mathbb{Z}$.

We now set

$$\mathcal{Y}'(\mathcal{R}) := \mathcal{Y}_1(\mathcal{R}) \cap \mathcal{V}_\infty(\mathcal{R}), \quad \tilde{\mathcal{V}}(\mathcal{R}) := \mathcal{V}_{1,\tau}(\mathcal{R}) \cap \mathcal{V}_\infty(\mathcal{R}).$$

The above identification enables us to define on $\tilde{\mathcal{V}}(\mathcal{R})$ a natural flow, for which we retain the notation $\{P^t\}$. (Although the bounded positive function τ is not separated from zero, the flow $\{P^t\}$ is well defined.)

Note that for any $s \in \mathbb{R}$ we have a natural "tautological" map

$$\mathfrak{t}_s : M(\mathscr{X}) \to M(P^s\mathscr{X})$$

which on each rectangle Π_i is simply expansion by e^s in the horizontal direction and contraction by e^s in the vertical direction. By definition, the map $\mathfrak{t}_s$ sends the vertical and the horizontal foliations of $\mathscr{X}$ to those of $P^s\mathscr{X}$.

Introduce the space

$$\mathfrak{X}\tilde{\mathcal{V}}(\mathcal{R}) = \{(\mathscr{X}, x) : \mathscr{X} \in \tilde{\mathcal{V}}(\mathcal{R}), x \in M(\mathscr{X})\}$$

and endow the space $\mathfrak{X}\tilde{\mathcal{V}}(\mathcal{R})$ with the flow $P^{s,\mathfrak{X}}$ given by the formula

$$P^{s,\mathfrak{X}}(\mathscr{X}, x) = (P^s\mathscr{X}, \mathfrak{t}_s x).$$

The flow P^s induces on the transversal $\mathcal{Y}(\mathcal{R})$ the first-return map $\overline{\mathscr{T}}$ given by the formula

$$\overline{\mathscr{T}}(\lambda, \pi, \delta) = \mathcal{U}P^{\tau^0(\lambda,\pi)}(\lambda, \pi, \delta). \tag{119}$$

Observe that, by definition, if $\overline{\mathscr{T}}(\lambda, \pi, \delta) = (\lambda', \pi', \delta')$, then $(\lambda', \pi') = \mathscr{T}(\lambda, \pi)$.

For $(\lambda, \pi, \delta) \in \tilde{\mathcal{V}}(\mathcal{R})$, $s \in \mathbb{R}$, let $\tilde{n}(\lambda, \pi, \delta, s)$ be defined by the formula

$$\mathcal{U}^{\tilde{n}(\lambda,\pi,\delta,s)}(e^s\lambda, \pi, e^{-s}\delta) \in \mathcal{V}_{1,\tau}(\mathcal{R}).$$

Endow the space $\tilde{\mathcal{V}}(\mathcal{R})$ with a matrix cocycle $\overline{\mathcal{A}}^t$ over the flow P^s given by the formula

$$\overline{\mathcal{A}}^t(s, (\lambda, \pi, \delta)) = \mathcal{A}^t(\tilde{n}(\lambda, \pi, \delta, s), (\lambda, \pi)).$$

4.2.5 The correspondence between cocycles.

To a connected component $\mathcal{H}$ of the space $\mathcal{M}_\kappa$ there corresponds a unique Rauzy class $\mathcal{R}$ in such a way that the following is true [24, 18].

Theorem 6 (Veech)**.** *There exists a finite-to-one measurable map $\pi_{\mathcal{R}} : \tilde{\mathcal{V}}(\mathcal{R}) \to \mathcal{H}$ such that $\pi_{\mathcal{R}} \circ P^t = g_t \circ \pi_{\mathcal{R}}$. The image of $\pi_{\mathcal{R}}$ contains all abelian differentials whose vertical and horizontal foliations are both minimal.*

Following Veech [25], we now describe the correspondence between the cocycle $\mathcal{A}^t$ and the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$.

As before, let $\mathbb{H}^1(\mathcal{H})$ be the fibre bundle over $\mathcal{H}$ whose fibre at a point (M, ω) is the cohomology group $H^1(M, \mathbb{R})$. The Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$ induces a skew-product flow $g_s^{\mathbf{A}_{KZ}}$ on $\mathbb{H}^1(\mathcal{H})$ given by the formula

$$g_s^{\mathbf{A}_{KZ}}(\mathbf{X}, v) = (\mathbf{g}_s\mathbf{X}, \mathbf{A}_{KZ}v), \ \mathbf{X} \in \mathcal{H}, v \in H^1(M, \mathbb{R}).$$

Following Veech [24], we now explain the connection between the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$ and the cocycle $\overline{\mathcal{A}}^t$.

For any irreducible permutation π Veech [25] defines an alternating matrix L^π by setting $L^\pi_{ij} = 0$ if $i = j$ or if $i < j, \pi i < \pi j$, $L^\pi_{ij} = 1$ if $i < j, \pi i > \pi j$, $L^\pi_{ij} = -1$ if $i > j, \pi i < \pi j$ and denotes by $N(\pi)$ the kernel of L^π and by $H(\pi) = L^\pi(\mathbb{R}^m)$ the image of L^π. The dimensions of $N(\pi)$ and $H(\pi)$ do not change as π varies in $\mathcal{R}$, and, furthermore, Veech [25] establishes the following properties of the spaces $N(\pi)$, $H(\pi)$.

Proposition 42. *Let $c = a$ or b. Then*

1. $H(c\pi) = \mathcal{A}^t(c,\pi)H(\pi)$, $N(c\pi) = \mathcal{A}^{-1}(c,\pi)N(\pi)$;
2. *the diagram*

$$\begin{array}{ccc} \mathbb{R}^m/N(\pi) & \xrightarrow{L^\pi} & H(\pi) \\ \Big\downarrow {\scriptstyle \mathcal{A}^{-1}(\pi,c)} & & \Big\downarrow {\scriptstyle \mathcal{A}^t(\pi,c)} \\ \mathbb{R}^m/N(c\pi) & \xrightarrow{L^{c\pi}} & H(c\pi) \end{array}$$

   is commutative and each arrow is an isomorphism.

3. *For each π there exists a basis $\mathbf{v}_\pi$ in $N(\pi)$ such that the map $\mathcal{A}^{-1}(\pi,c)$ sends every element of $\mathbf{v}_\pi$ to an element of $\mathbf{v}_{c\pi}$.*

Each space H^π is thus endowed with a natural anti-symmetric bilinear form $\mathcal{L}_\pi$ defined, for $v_1, v_2 \in H(\pi)$, by the formula

$$\mathcal{L}_\pi(v_1, v_2) = \langle v_1, (L^\pi)^{-1} v_2\rangle. \tag{120}$$

(The vector $(L^\pi)^{-1}v_2$ lies in $\mathbb{R}^m/N(\pi)$; since for all $v_1 \in H(\pi)$, $v_2 \in N(\pi)$ by definition we have $\langle v_1, v_2\rangle = 0$, the right-hand side is well-defined.)

Consider the $\mathscr{T}^{\mathcal{A}^t}$-invariant subbundle $\mathscr{H}(\Delta(\mathcal{R})) \subset \Delta(\mathcal{R}) \times \mathbb{R}^m$ given by the formula

$$\mathscr{H}(\Delta(\mathcal{R})) = \{((\lambda,\pi), v), (\lambda,\pi) \in \Delta(\mathcal{R}), v \in H(\pi)\}.$$

as well as a quotient bundle

$$\mathscr{N}(\Delta(\mathcal{R})) = \{((\lambda,\pi), v), (\lambda,\pi) \in \Delta(\mathcal{R}), v \in \mathbb{R}^m/N(\pi)\}.$$

The bundle map $\mathscr{L}_{\mathcal{R}} : \mathscr{H}(\Delta(\mathcal{R})) \to \mathscr{N}(\Delta(\mathcal{R}))$ given by $\mathscr{L}_{\mathcal{R}}((\lambda,\pi), v) = ((\lambda,\pi), L^\pi v)$ induces a bundle isomorphism between $\mathscr{H}(\Delta(\mathcal{R}))$ and $\mathscr{N}(\Delta(\mathcal{R}))$.

Both bundles can be naturally lifted to bundles $\mathscr{H}(\tilde{\mathcal{V}}(\mathcal{R}))$, $\mathscr{N}(\tilde{\mathcal{V}}(\mathcal{R}))$ over the space $\tilde{\mathcal{V}}(\mathcal{R})$ of zippered rectangles; they are naturally invariant under the corresponding skew-product flows $P^{s,\overline{\mathcal{A}}^t}$, $P^{s,\overline{\mathcal{A}}^{-1}}$, and the map $\mathscr{L}_{\mathcal{R}}$ lifts to a bundle isomorphism between $\mathscr{H}(\tilde{\mathcal{V}}(\mathcal{R}))$ and $\mathscr{N}(\tilde{\mathcal{V}}(\mathcal{R}))$.

Take $\mathscr{X} \in \tilde{\mathcal{V}}(\mathcal{R})$ and write $\pi_{\mathcal{R}}(\mathscr{X}) = (M(\mathscr{X}), \omega(\mathscr{X}))$. Veech [26] has shown that the map $\pi_{\mathcal{R}}$ lifts to a bundle epimorphism $\tilde{\pi}_{\mathcal{R}}$ from $\mathscr{H}(\tilde{\mathcal{V}}(\mathcal{R}))$ onto $\mathbb{H}^1(\mathcal{H})$ that intertwines the cocyle $\overline{\mathcal{A}}^t$ and the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$:

Proposition 43 (Veech)**.** *For almost every $\mathscr{X} \in \tilde{\mathcal{V}}(\mathcal{R})$, $\mathscr{X} = (\lambda, \pi, \delta)$, there exists an isomorphism $\mathcal{I}_{\mathscr{X}} : H(\pi) \to H^1(M(\mathscr{X}), \mathbb{R})$ such that*

1. *the map $\tilde{\pi}_{\mathcal{R}} : \mathscr{H}(\Delta(\mathcal{R})) \to \mathbb{H}^1(\mathcal{H})$ given by*
$$\tilde{\pi}_{\mathcal{R}}(\mathscr{X}, v) = (\pi_{\mathcal{R}}(\mathscr{X}), \mathcal{I}_X v)$$
induces a measurable bundle epimorphism from $\mathscr{H}(\Delta(\mathcal{R}))$ onto $\mathbb{H}^1(\mathcal{H})$;

2. *the diagram*
$$\begin{array}{ccc} \mathscr{H}(\tilde{\mathcal{V}}(\mathcal{R})) & \xrightarrow{\tilde{\pi}_{\mathcal{R}}} & \mathbb{H}^1(\mathcal{H}) \\ \Big\downarrow{\scriptstyle P^{s,\overline{\mathcal{A}}^t}} & & \Big\downarrow{\scriptstyle g_s^{\mathbf{A}_{KZ}}} \\ \mathscr{H}(\tilde{\mathcal{V}}(\mathcal{R})) & \xrightarrow{\tilde{\pi}_{\mathcal{R}}} & \mathbb{H}^1(\mathcal{H}) \end{array}$$
is commutative;

3. *for $\mathscr{X} = (\lambda, \pi, \delta)$, the isomorphism $\mathcal{I}_X$ takes the bilinear form $\mathcal{L}_\pi$ on $H(\pi)$, defined by (120), to the cup-product on $H^1(M(\mathscr{X}), \mathbb{R})$.*

Proof: Recall that to each rectangle Π_i Veech [25] assigns a cycle $\gamma_i(\lambda, \pi, \delta)$ in the homology group $H_1(M, \mathbb{Z})$: if P_i is the left bottom corner of Π_i and Q_i the left top corner, then the cycle is the union of the vertical interval P_iQ_i and the horizontal subinterval of $I^{(0)}(\lambda, \pi, \delta)$ joining Q_i to P_i. It is clear that the cycles $\gamma_i(\lambda, \pi, \delta)$ span $H_1(M, \mathbb{Z})$; furthermore, Veech shows that the cycle $t_1\gamma_1 + \cdots + t_m\gamma_m$ is homologous to 0 if and only if $(t_1, \ldots, t_m) \in N(\pi)$. We thus obtain an identification of $\mathbb{R}^m/N(\pi)$ and $H_1(M, \mathbb{R})$. Similarly, the subspace of $\mathbb{R}^m$ spanned by the vectors $(f(\gamma_1), \ldots, f(\gamma_m))$, $f \in H^1(M, \mathbb{R})$, is precisely $H(\pi)$. The identification of the bilinear form $\mathcal{L}_\pi$ with the cup-product is established in Proposition 4.19 in [29].

The third statement of Proposition 42 has the following important

Corollary 11. *Let $\mathbb{P}_{\mathcal{V}}$ be an ergodic P^s-invariant probability measure for the flow P^s on $\mathcal{V}(\mathcal{R})$ and let $\mathbb{P}_{\mathcal{H}} = (\pi_{\mathcal{R}})_* \mathbb{P}_{\mathcal{V}}$ be the corresponding $\mathbf{g}_s$-invariant measure on $\mathcal{H}$. If the Kontsevich-Zorich cocycle acts isometrically on its neutral subspace with respect to $\mathbb{P}_{\mathcal{H}}$, then the cocycle $\overline{\mathcal{A}}^t$ also acts isometrically on its neutral subspace with respect to $\mathbb{P}_{\mathcal{V}}$.*

4.3 Zippered Rectangles and Markov Compacta.

4.3.1 The main lemma

Given a finite set $\mathfrak{G}_0 \subset \mathfrak{G}$, denote

$$\Omega_{\mathfrak{G}_0} = \{\omega \in \Omega : \omega_n \in \mathfrak{G}_0, n \in \mathbb{Z}\};$$

$$\overline{\Omega}_{\mathfrak{G}_0} = \{\overline{\omega} = (\omega, r) : \omega \in \Omega_{\mathfrak{G}_0}\}.$$

Lemma 4. *Let $\mathcal{R}$ be a Rauzy class of irreducible permutations. There exists a finite set $\mathfrak{G}_{\mathcal{R}} \subset \mathfrak{G}$ and a Vershik's ordering $\mathfrak{o}$ as well as a reverse Vershik's ordering $\tilde{\mathfrak{o}}$ on each $\Gamma \in \mathfrak{G}_{\mathcal{R}}$ such that the following is true. There exists a map*

$$\overline{\mathfrak{Z}}_{\mathcal{R}} : \tilde{\mathcal{V}}_{ue}(\mathcal{R}) \to \overline{\Omega}_{\mathfrak{G}_{\mathcal{R}}},$$

and, for any $\mathscr{X} \in \tilde{\mathcal{V}}_{ue}(\mathcal{R})$, an $\mathbf{m}_{\mathscr{X}}$-almost surely defined map

$$\overline{\mathscr{I}}_{\mathscr{X}} : M(\mathscr{X}) \to X(\overline{\omega}_{\mathscr{X}})$$

such that the following is true.

1. *the diagram*

$$\begin{CD} \tilde{\mathcal{V}}_{ue}(\mathcal{R}) @>{\overline{\mathfrak{Z}}_{\mathcal{R}}}>> \overline{\Omega} \\ @V{P^s}VV @VV{g_s}V \\ \tilde{\mathcal{V}}_{ue}(\mathcal{R}) @>{\overline{\mathfrak{Z}}_{\mathcal{R}}}>> \overline{\Omega} \end{CD}$$

   is commutative and if $\mathbb{P}_{\mathcal{V}}$ is an ergodic probability P^s-invariant measure, then we have

$$(\overline{\mathfrak{Z}}_{\mathcal{R}})_* \mathbb{P}_{\mathcal{V}} \in \mathscr{P}^+. \tag{121}$$

2. *if a map*

$$\overline{\mathfrak{Z}}_{\mathcal{R}}^{\mathfrak{X}} : \mathfrak{X}\tilde{\mathcal{V}}_{ue}(\mathcal{R}) \to \mathfrak{X}\overline{\Omega}$$

   is given by the formula

$$\overline{\mathfrak{Z}}_{\mathcal{R}}^{\mathfrak{X}}(\mathscr{X}, x) = (\overline{\mathfrak{Z}}_{\mathcal{R}}\mathscr{X}, \overline{\mathscr{I}}_{\mathscr{X}} x),$$

   then the diagram

$$\begin{CD} \mathfrak{X}\tilde{\mathcal{V}}_{ue}(\mathcal{R}) @>{\overline{\mathfrak{Z}}_{\mathcal{R}}^{\mathfrak{X}}}>> \mathfrak{X}\overline{\Omega} \\ @V{P^{s,\mathfrak{X}}}VV @VV{g_s^{\mathfrak{X}}}V \\ \mathfrak{X}\tilde{\mathcal{V}}_{ue}(\mathcal{R}) @>{\overline{\mathfrak{Z}}_{\mathcal{R}}^{\mathfrak{X}}}>> \mathfrak{X}\overline{\Omega} \end{CD}$$

   is commutative;

3. *if the map*

$$\overline{\mathfrak{Z}}'_{\mathcal{R}} : \tilde{\mathcal{V}}_{ue}(\mathcal{R}) \times \mathbb{R}^m \to \overline{\Omega} \times \mathbb{R}^m$$

   is given by the formula

$$\overline{\mathfrak{Z}}'_{\mathcal{R}}(\mathscr{X}, v) = (\overline{\mathfrak{Z}}_{\mathcal{R}}\mathscr{X}, v),$$

   then the diagram

$$\begin{CD} \tilde{\mathcal{V}}_{ue}(\mathcal{R}) \times \mathbb{R}^m @>{\overline{\mathfrak{Z}}'_{\mathcal{R}}}>> \overline{\Omega} \times \mathbb{R}^m \\ @V{P^{s,\mathcal{A}^t}}VV @VV{g_s^{\mathbb{A}}}V \\ \tilde{\mathcal{V}}_{ue}(\mathcal{R}) \times \mathbb{R}^m @>{\overline{\mathfrak{Z}}'_{\mathcal{R}}}>> \overline{\Omega} \times \mathbb{R}^m \end{CD}$$

   is commutative.

4. *the map $\overline{\mathscr{J}}_{\mathscr{X}}$ sends the vertical flow h_t^+ on $\mathscr{X}$ to the flow $h_t^{+,\mathfrak{o}}$ on $X(\overline{\mathfrak{Z}}_{\mathcal{R}}(\mathscr{X}))$; the horizontal flow h_t^- on $\mathscr{X}$ to the flow $h_t^{-,\tilde{\mathfrak{o}}}$ on $X(\overline{\mathfrak{Z}}_{\mathcal{R}}(\mathscr{X}))$ and induces isomorphisms between the space $\mathfrak{B}^+_{\mathscr{X}}$ and the space $\mathfrak{B}^+_{(\overline{\mathfrak{Z}}_{\mathcal{R}}(\mathscr{X}))}$; the space $\mathfrak{B}^-_{\mathscr{X}}$ and the space $\mathfrak{B}^-_{(\overline{\mathfrak{Z}}_{\mathcal{R}}(\mathscr{X}))}$.*

Informally, the map $\overline{\mathfrak{Z}}_{\mathcal{R}}$ is constructed as follows. First, to a zippered rectangle one assigns its *Rauzy-Veech expansion*, the bi-infinite sequence of pairs (π, c), $\pi \in \mathcal{R}$, $c = a$ or b. To each pair (π, c) we have assigned a unimodular matrix $\mathcal{A}(\pi, c)$. Now to each such matrix we assign a graph Γ in $\mathfrak{G}$. The resulting sequence of graphs yields the desired Markov compactum.

Remark. In an unpublished note Veech has shown that for a minimal interval exchange (λ, π) the sequence of its Rauzy-Veech renormalization matrices $\mathcal{A}(\mathscr{T}^n(\lambda, \pi))$ uniquely determines the permutation π. In particular, if (λ, π) is uniquely ergodic, then the sequence of Rauzy-Veech renormalization matrices uniquely determines the interval exchange transformation. The result of Veech implies that the map $\overline{\mathfrak{Z}}_{\mathcal{R}}$ is in fact *injective*.

4.3.2 Rauzy-Veech expansions of zippered rectangles.

Given a Rauzy class $\mathcal{R}$ of irreducible permutations, introduce an alphabet

$$\mathfrak{A}_{\mathcal{R}} = \{(\pi, c), c = a \text{ or } b, \pi \in \mathcal{R}\}.$$

To each letter $\mathbf{p}_1 \in \mathfrak{A}_{\mathcal{R}}$ assign a set $\Delta_{\mathbf{p}_1} \subset \Delta(\mathcal{R})$ given by

$$\Delta_{\mathbf{p}_1} = \Delta_\pi^+ \text{ if } \mathbf{p}_1 = (\pi, a);\ \Delta_{\mathbf{p}_1} = \Delta_\pi^- \text{ if } \mathbf{p}_1 = (\pi, b).$$

Take a zippered rectangle $\mathscr{X} \in \mathcal{Y}'(\mathcal{R})$, $\mathscr{X} = (\lambda, \pi, \delta)$, $|\lambda| = 1$. For $n \in \mathbb{Z}$ write $\overline{\mathscr{T}}\mathscr{X} = (\lambda^{(n)}, \pi^{(n)}, \delta^{(n)})$ and assign to $\mathscr{X}$ a sequence $\mathbf{p}_n(\mathscr{X})_{n\in\mathbb{Z}}$ given by

$$(\lambda^{(n)}, \pi^{(n)}) \in \Delta_{\mathbf{p}_n}. \tag{122}$$

The sequence $\mathbf{p}_n(\mathscr{X})_{n\in\mathbb{Z}}$ is the *Rauzy-Veech expansion* of the zippered rectangle $\mathscr{X}$.

4.3.3 A Markov compactum corresponding to a zippered rectangle

To each letter $\mathbf{p}_1 \in \mathcal{A}_{\mathcal{R}}$, we assign an oriented graph $\Gamma(\mathbf{p}_1)$ on m vertices in the following way.

Case 1. Assume $\mathbf{p}_1 = (\pi, a)$. Then the graph $\Gamma(\mathbf{p}_1)$ has $m + 1$ edges

$$e_{ii},\ 1 \le i \le \pi^{-1}m;\ e_{\pi^{-1}m+1,m};\ e_{i,i-1},\ \pi^{-1}m + 1 \le i \le m.$$

We set $I(e_{ij}) = i, F(e_{ij}) = j$.

Case 2. $\mathbf{p} = (\pi, \mathbf{b})$. The graph $\Gamma(\pi, b)$ has $m+1$ edges

$$e_{ii}, i = 1, \ldots, m;\ e_{\pi^{-1}m,m}.$$

Again we set $I(e_{ij}) = i, F(e_{ij}) = j$.

The incidence matrix of a graph $\Gamma(\mathbf{p}_1)$, $\mathbf{p}_1 = (\pi, c)$, is the transpose of the Rauzy matrix assigned to (π, c).

A canonical Vershik's ordering on the graphs $\Gamma(\mathbf{p}_1)$ is given by the rule: $e_{ij} < e_{ik}$ if and only if $j < k$.

A word $\mathbf{p}$ in the alphabet $\mathfrak{A}_{\mathcal{R}}$,

$$\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_l),\ \mathbf{p}_i = (\pi_i, c_i),$$

will be called *admissible* if $\pi_{i+1} = c_i\pi_i$. The set of all admissible words will be denoted $\mathscr{W}_{\mathcal{R}}$. Similarly, an infinite sequence will be called admissible if its every finite subsequence is admissible. The set of all admissible bi-infinite sequences will be denoted $\Sigma_{\mathcal{R}}$. We have a natural map $Gr_{\mathcal{R}} : \Sigma_{\mathcal{R}} \to \Omega$ given by

$$Gr_{\mathcal{R}} : (\mathbf{p}_n)_{n\in\mathbb{Z}} \to \Gamma((\mathbf{p}_n))_{n\in\mathbb{Z}}.$$

There is a natural map $Code^+_{\mathcal{R}} : \Delta(\mathcal{R}) \to \Sigma^+_{\mathcal{R}}$ which sends (λ, π) to a sequence $\mathbf{p}_n$, $n \in \mathbb{N}$ given by

$$\mathscr{T}^n(\lambda, \pi) \in \Delta_{\mathbf{p}_n}.$$

This map is extended to a map $Code_{\mathcal{R}} : \mathcal{Y}'(\mathcal{R}) \to \Sigma_{\mathcal{R}}$ which sends (λ, π, δ) to a sequence $\mathbf{p}_n$, $n \in \mathbb{Z}$ given by

$$\overline{\mathscr{T}}^n(\lambda, \pi, \delta) = (\lambda^{(n)}, \pi^{(n)}, \delta^{(n)}), (\lambda^{(n)}, \pi^{(n)}) \in \Delta_{\mathbf{p}_n}.$$

We thus obtain the desired composition map:

$$\mathfrak{Z}_{\mathcal{R}} = Gr_{\mathcal{R}} \circ Code_{\mathcal{R}} : \mathcal{Y}'(\mathcal{R}) \to \Omega.$$

4.3.4 Properties of the symbolic coding.

We have thus constructed a measurable coding mapping $\mathfrak{Z}_{\mathcal{R}} : \mathcal{Y}'(\mathcal{R}) \to \Omega$. The diagram

$$\begin{array}{ccc} \mathcal{Y}'(\mathcal{R}) & \xrightarrow{\mathfrak{Z}_{\mathcal{R}}} & \Omega \\ \downarrow{\scriptstyle \overline{\mathscr{T}}} & & \downarrow{\scriptstyle \sigma} \\ \mathcal{Y}'(\mathcal{R}) & \xrightarrow{\mathfrak{Z}_{\mathcal{R}}} & \Omega \end{array}$$

is commutative by construction.

For a zippered rectangle $\mathscr{X} \in \mathcal{Y}'(\mathcal{R})$, consider the corresponding abelian differential $\mathbf{X} = \pi_{\mathcal{R}}(\mathscr{X})$ with underlying surface $M(\mathscr{X}$ and let $\mathbf{m}_{\mathscr{X}}$ be the Lebesgue measure on $M(\mathscr{X})$. Write $\omega_{\mathscr{X}} = \mathfrak{Z}_{\mathcal{R}}(\mathscr{X})$. We then have a "tautological" coding mapping from the Markov compactum $X(\omega_{\mathscr{X}})$ to $M(\mathscr{X})$. The foliaitons $\mathcal{F}^+_{X(\omega_{\mathscr{X}})}$ and $\mathcal{F}^-_{X(\omega_{\mathscr{X}})}$ are taken, respectively, to the vertical and the horizontal foliations on $M(\mathscr{X})$; unique ergodicity of the Markov compactum $X(\omega_{\mathscr{X}})$ is equivalent to the unique ergodicity of both the horizontal and the vertical flows on $M(\mathscr{X})$.

Now assume that the Markov compactum $X(\omega_{\mathscr{X}})$ is indeed uniquely ergodic. Then the coding mapping is $\nu_{\omega_{\mathscr{X}}}$-almost surely invertible, and we obtain a $\mathbf{m}_{\mathscr{X}}$-almost surely defined map

$$\mathscr{J}_{\mathscr{X}} : M(\mathscr{X}) \to X(\omega_{\mathscr{X}}).$$

Recall that $\mathbf{X} = \pi_{\mathcal{R}}(\mathscr{X})$. By definition, the mapping $\mathscr{J}_{\mathscr{X}}$ induces a linear isomorphism between the space $\mathfrak{B}^+_{\mathbf{X}}$ and the space $\mathfrak{B}^+_{X(\omega_{\mathscr{X}})}$; and similarly between the space $\mathfrak{B}^-_{\mathbf{X}}$ and the space $\mathfrak{B}^-_{X(\omega_{\mathscr{X}})}$. We have $(\mathscr{J}_{\mathscr{X}})_* \mathbf{m}_{\mathscr{X}} = \nu_{\omega_{\mathscr{X}}}$. The mapping $\mathscr{J}_{\mathscr{X}}$ takes the space of weakly Lipschitz functions on $M(\mathscr{X})$ to the space of weakly Lipschitz functions on $X(\omega_{\mathscr{X}})$.

The map $\mathfrak{Z}_{\mathcal{R}}$ lifts to a natural map

$$\overline{\mathfrak{Z}}_{\mathcal{R}} : \tilde{\mathcal{V}}_{ue}(\mathcal{R}) \to \overline{\Omega},$$

and, again, the diagram

$$\begin{array}{ccc} \tilde{\mathcal{V}}_{ue}(\mathcal{R}) & \xrightarrow{\overline{\mathfrak{Z}}_{\mathcal{R}}} & \overline{\Omega} \\ \downarrow{\scriptstyle P^s} & & \downarrow{\scriptstyle g_s} \\ \tilde{\mathcal{V}}_{ue}(\mathcal{R}) & \xrightarrow{\overline{\mathfrak{Z}}_{\mathcal{R}}} & \overline{\Omega} \end{array}$$

is commutative.

If $\mathbb{P}_{\mathcal{V}}$ is an ergodic probability P^s-invariant measure, then we have

$$(\overline{\mathfrak{Z}}_{\mathcal{R}})_* \mathbb{P}_{\mathcal{V}} \in \mathscr{P}^+. \tag{123}$$

Indeed, (121) is a reformulation of a Lemma due to Veech [24] which states that every finite P^s-invariant measure assigns positive probability to a Rauzy matrix with positive entries.

For any $\mathscr{X} \in \tilde{\mathcal{V}}_{ue}(\mathcal{R})$ we again obtain $\mathbf{m}_{\mathscr{X}}$-almost surely defined map

$$\overline{\mathscr{J}}_{\mathscr{X}} : M(\mathscr{X}) \to X(\overline{\omega}_{\mathscr{X}}).$$

Introduce a map

$$\overline{\mathfrak{Z}}^{\mathfrak{X}}_{\mathcal{R}} : \mathfrak{X}\tilde{\mathcal{V}}_{ue}(\mathcal{R}) \to \mathfrak{X}\overline{\Omega}$$

by the formula

$$\overline{\mathfrak{Z}}^{\mathfrak{X}}_{\mathcal{R}}(\mathscr{X}, x) = (\overline{\mathfrak{Z}}_{\mathcal{R}}\mathscr{X}, \overline{\mathscr{J}}_{\mathscr{X}} x).$$

The diagram

$$\begin{array}{ccc} \mathfrak{X}\tilde{\mathcal{V}}_{ue}(\mathcal{R}) & \xrightarrow{\overline{\mathfrak{Z}}^{\mathfrak{X}}_{\mathcal{R}}} & \mathfrak{X}\overline{\Omega} \\ \downarrow{\scriptstyle P^{s,\mathfrak{X}}} & & \downarrow{\scriptstyle g^{\mathfrak{X}}_s} \\ \mathfrak{X}\tilde{\mathcal{V}}_{ue}(\mathcal{R}) & \xrightarrow{\overline{\mathfrak{Z}}^{\mathfrak{X}}_{\mathcal{R}}} & \mathfrak{X}\overline{\Omega} \end{array}$$

is commutative.

The map $\overline{\mathfrak{Z}}_{\mathcal{R}}$ intertwines the cocycles $\mathcal{A}^t$ and $\mathbb{A}$ in the following sense. Take $\mathscr{X} \in \tilde{\mathcal{V}}_{ue}(\mathcal{R})$, $v \in \mathbb{R}^m$ and write

$$\overline{\mathfrak{Z}}'_{\mathcal{R}}(\mathscr{X}, v) = (\overline{\mathfrak{Z}}_{\mathcal{R}}\mathscr{X}, v).$$

The resulting map

$$\overline{\mathfrak{Z}}'_{\mathcal{R}} : \tilde{\mathcal{V}}_{ue}(\mathcal{R}) \times \mathbb{R}^m \to \overline{\Omega} \times \mathbb{R}^m$$

intertwines the cocycles $\mathcal{A}^t$ and $\mathbb{A}$: indeed, by definition, the diagram

$$\begin{array}{ccc} \tilde{\mathcal{V}}_{ue}(\mathcal{R}) \times \mathbb{R}^m & \xrightarrow{\overline{\mathfrak{Z}}'_{\mathcal{R}}} & \overline{\Omega} \times \mathbb{R}^m \\ \Big\downarrow P^{s,\mathcal{A}^t} & & \Big\downarrow g_s^{\mathbb{A}} \\ \tilde{\mathcal{V}}_{ue}(\mathcal{R}) \times \mathbb{R}^m & \xrightarrow{\overline{\mathfrak{Z}}'_{\mathcal{R}}} & \overline{\Omega} \times \mathbb{R}^m \end{array}$$

is commutative.

Denote $\mathfrak{G}_{\mathcal{R}} = \{\Gamma(\mathbf{p}_1), \mathbf{p}_1 \in \mathfrak{A}_{\mathcal{R}}\}$ and set

$$\Omega_{\mathfrak{G}_{\mathcal{R}}} = \{\omega \in \Omega : \omega_n \in \mathfrak{G}_{\mathcal{R}}, n \in \mathbb{Z}\};$$

$$\overline{\Omega}_{\mathfrak{G}_{\mathcal{R}}} = \{\overline{\omega} = (\omega, r) : \omega \in \Omega_{\mathfrak{G}_{\mathcal{R}}}\}.$$

By construction, $\mathfrak{Z}_{\mathcal{R}}(\mathcal{Y}'_{\mathcal{R}}) \subset \Omega_{\mathfrak{G}_{\mathcal{R}}}$. Every graph $\Gamma \in \mathfrak{G}_{\mathcal{R}}$ is endowed with a Vershik's ordering constructed in the previous subsection, and we obtain a σ-equivariant Vershik's ordering $\mathfrak{o}_{\mathcal{R}}$ on $\mathfrak{Z}_{\mathcal{R}}(\mathcal{Y}'_{\mathcal{R}})$.

By definition, the mapping $\overline{\mathscr{J}}_{\mathscr{X}}$ sends the vertical flow h_t^+ on $M(\mathscr{X})$ to the flow $h_t^{+,\omega_{\mathscr{X}}}$ and the horizontal flow h_t^- on $M(\mathscr{X})$ to the flow $h_t^{-,\omega_{\mathscr{X}}}$.

Lemma 4 is proved. Theorems 1, 2 follow now from their symbolic counterparts, Theorems 4, 5.

Theorems 1, 2 are proved completely.

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