HECKE SYMMETRIES AND BRAIDED LIE ALGEBRAS

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Abstract. We consider Hecke symmetries of minimal type, i.e., solutions of the QYBE with two eigenvalues and such that the Poicaré series of the corresponding exterior algebras are polynomials of degree 2. We construct the corresponding quantum cogroups and introduce notion of braided Lie algebra. The examples of Hecke symmetries of minimal type and of braided Lie algebras are given.

Key words: Quantum Yang-Baxter Equation, Hecke symmetry, bi-rank, quantum cogroup, braided Lie algebra

Generalized Lie algebras connected with involutive ($S^2 = 1$) solution of the quantum Yang-Baxter equation (QYBE) have been introduced in our paper [3]. In [5] (see also references therein for our previous papers) we have constructed some explicit examples of generalized Lie algebras (or in other words S-Lie algebras) of *gl* and *sl* types, connected with involutive non-quasiclassical (or non-deformation) solutions of the QYBE. The problem of a proper generalization of this notion to the non-involutive case was open though a lot of papers were devoted to the problem.

This paper is devoted to two questions. On the one hande we continue to study some non-quasiclassical non-involutive solutions S of the QYBE (so called Hecke symmetries). On the other hand we propose the definition of S-Lie algebras (called here braided Lie algebras to stress non-involutivity of the operator S) connected with Hecke symmetries.

The paper consists of three Sections. In Section 1 we recall some usuful facts about Hecke symmetries. We put emphasis on Hecke symmetries of minimal type, i.e. such that the Poincaré series of corresponding exterior algebras are polynomials of degree 2 with leading coefficient 1. Some of such type solutions of the QYBE have been independently constructed in [1].

In Section 2 we introduce quantum cogroups connected with Hecke symmetries of minimal type and compare these objects with Hopf algebras arising from non-degenerated bilinear forms defined in [1]. In Section 3 we introduce a notion of braided groups and give their examples connected with Hecke symmetries of minimal type.

1. HECKE SYMMETRIES: STRUCTURE, EXAMPLES

Let V be a finite-dimensional vector space over a field $\mathbf{k}$ of characteristic 0 and $S : V^{\otimes 2} \to V^{\otimes 2}$ a solution of the QYBE

$$(S \otimes \mathrm{id})(\mathrm{id} \otimes S)(S \otimes \mathrm{id}) = (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(\mathrm{id} \otimes S).$$

Among all the solutions of the QYBE, the most interesting are the so called *closed* ones. Fix a base $\{e_i, 1 \leq i \leq n = \dim V\}$ in the space V and put $S(e_i \otimes e_j) = S_{ij}^{kl} e_k \otimes e_l$. Consider the operator T which in the base $\{e_i\}$ is defined by $S_{ij}^{kl} T_{kn}^{im} = \delta_j^m \delta_n^l$. We call the solution S of the QYBE *closed* if T exists.

It is not difficult to show that a closed solution of the QYBE can be extended up to a braiding operator in a *rigid* quasitensor category $\mathfrak{A}$ containing the space V. According to generally accepted terminology, a quasitensor category is called *rigid* if it satisfies the condition $U \in Ob\,\mathfrak{A} \to U^* \in Ob\,\mathfrak{A}$ and the pairing $U \otimes U^* \to \mathbf{k}$ is $\mathfrak{A}$-morphism. The braiding operator S (or in other words, "commutativity morphism") is a mophism in the category $\mathfrak{A}$ but it is not involutive in general.

In this paper, we deal only with solutions of the QYBE wich have two eigenvalues. We call them Hecke symmeties. More precisely we call a solution S of the QYBE a *Hecke symmetry* if S satisfies the equation

$$(q\,\mathrm{id} - S)(\mathrm{id} + S) = 0.$$

We assume that $q \neq 0$ and $q^n \neq 1, n = 2; 3; ...$

The Hecke symmetries have a great advantage: it is possible to define for them an analogue of the symmetric and exterior algebras. Namely we put

$$\wedge_{\pm}(V) = T(V)/\{I_{\mp}\}$$

where $T(V) = \oplus V^{\otimes k}$ is the tensor algebra of V and $\{I_+\}$ (resp., $\{I_-\}$) is the ideal in $T(V)$ generated by the image I_+ (resp.,I_-) of $S + \mathrm{id}$ (resp., $q\,\mathrm{id} - S$). Denote $\wedge_{\pm}^k(V)$ the homogeneous component of degree k of these algebras and consider the Poincaré series $\mathcal{P}_{\pm}(t)$ of the algebras $\wedge_{\pm}(V)$:

$$\mathcal{P}_{\pm}(t) = \sum \dim \wedge_{\pm}^k(V) t^k.$$

We call a Hecke symmetry S (and the corresponding space V) *even* if it is closed and the Poincaré serie $\mathcal{P}_-(t)$ is a polynomial (as it was shown in [5] this condition is equivalent to following one: $\mathcal{P}_-(t)$ is a polynomial with leading coefficient 1). If this polynomial is of degree k we say that V (or S) has bi-rank $k|0$ and denote it bi-rk V. [1]

Now we introduce two important operators $B = B(S) : V \to V$ and $C = C(S) : V \to V$ as follows

$$B(e_i) = B_i^j e_j = T_{ik}^{jk} e_j, \;\; C(e_i) = C_i^j e_j = T_{ki}^{kj} e_j,$$

[1] Note that bi-rank is well-defined for odd objects of Hecke type (it is left to the reader to give a definition of odd spaces). For them we say that bi-rank is equal to $0|l$ and for some objects V composed in some sens from even and odd spaces it is natural to put bi-rank $V = k|l$. We dont want to examine this problem in more detail but stress only that it is not clear yet, whether all *involutive* closed solutions of the QYBE have a bi-rank.

where $\{e_i\}$ is the fixed base in V.

It is easy to see that this definition does not depend on the choice of the base. These operators can be defined for any object in any rigid quasitensor category but we need them only for an inicial space V equipped with a Hecke symmetry S.

The following statements are proved in, or can be easily deduced from, [5].

PROPOSITION 1. *1. For any Hecke symmetry S the relation*

$$\mathcal{P}_+(t)\,\mathcal{P}_-(-t) = 1$$

holds.

2. If S is even then the polynomial $\mathcal{P}_-(t)$ is reciprocal.

3. Moreover if bi-rk$V = k|0$ then the operators B and C satisfy the relation

$$\operatorname{tr} B = \operatorname{tr} C = q^{-k}\, k_q$$

(we denote here and below $k_q = 1 + q + \ldots + q^{k-1}$).

4. If bi-rk$V = 2|0$ then $BC = CB = q^{-3}\mathrm{id}$ and the operators $b = Bq^2$ and $c = Cq^2$ satisfy the following condition

J) if Jordan form of b or c contains a cell with eigenvalue x it contains another cell with eigenvalue qx^{-1} (with the same multiplicity).

5. If an operator $c : V \to V$ satisfies the conditions J) and $\operatorname{tr} c = 1 + q$ then there exists an even closed Hecke symmetry $S : V^{\otimes 2} \to V^{\otimes 2}$ of bi-rank $2|0$ such that $C = C(S) = q^{-2}c$. There exists the one-to-one correspondance between the family of all such Hecke symmetries and matrices v satisfying the condition

$$(c^*)^{-1} q = v^{-1} c v, \;\; v = (v^{ij})$$

(c^* denotes the matrix conjugated to c). If such v is fixed then the corresponding Hecke symmetry is of the form

$$S_{ij}^{kl} = q\delta_i^k\delta_j^l - (1+q)u_{ij}v^{kl},$$

where $u = (u_{ij})$ can be deduced from the equality

$$c = (1+q)vu^* \;\text{ i.e. }\; c_i^j = (1+q)\, v^{jk} u_{ik}.$$

Remark that the quantity $\operatorname{tr} B = \operatorname{tr} C$, which can be defined for any element of a rigid category, is usually called its rank (see for example [6]). So the statement 3 of Proposition 1 establishes the relation for even Hecke symmetries between rank in this sens and bi-rank in our sense. Here and further on, we say that a Hecke symmetry is of *minimal type* if it is even and has bi-rank $2|0$.

Stress also that bi-rank does not change under deformation and therefore, a quasiclassical Hecke symmetry (i.e.,a deformation of the usual tranposition) must have bi-rank $n|0, n = \dim V$.

Let us give two examples of minimal Hecke symmetries.

EXAMPLE 1. *Let $\dim V=2$ and $q \neq 1$. Then any pair (c, v) satysfying the conditions above has in some base form*

$$c = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \quad v = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

Then

$$S = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & qm^{-1} & 0 \\ 0 & m & q-1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \text{ where } m = -a/b.$$

Stress that the operator $N = uv$ is scalar iff $m^2 = q$ (the role of this operator will be explained in Proposition 2).

EXAMPLE 2. *Let $\dim V=3$. We put $c = \mathrm{diag}(x, t, q/x)$ where t is one of roots $\pm\sqrt{q}$ and x satisfies the equation $x + t + q/x = 2$. Then assuming v to be as follows we obtain S*

$$v = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}, \ S = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q-x & 0 & -bx/a & 0 & -cx/a & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -at/b & 0 & q-t & 0 & -tc/b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & -qa/cx & 0 & -qb/cx & 0 & q-q/x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}.$$

For this example the operator $N = uv$ is scalar if $a/c = x/t$.

Stress that the last example can be easily generalized to arbitrary dimension $n = \dim V$.

2. HECKE SYMMETRIES ARISING FROM BILINEAR FORM AND QUANTUM COGROUPS

In [1] a method have been introduced to construct a solution of the QYBE by means of a non-degenerated bilinear form. In this Section we show that the family of such solutions coincides with subset of Hecke symmetries of minimal type. We introduce also the quantum groups connected with Hecke symmetries of minimal type and compare them with Hopf algebras defined in [1].

Consider a linear space $L = V \otimes V^*$ with base $\{e_i^j = e_i \otimes e^j\}$ equipped with the operator

$$S_Q : L^{\otimes 2} \to L^{\otimes 2},\ S_Q(e_i^j \otimes e_k^l) = S_{ik}^{ab}\,(S^{-1})_{mn}^{jl}\, e_a^m \otimes e_b^n.$$

Stress here that V^* differs from the dual space (right or left one) in the rigid category mentioned above and moreover the space L does not belong to this category (wich will appear below as the category of comoduls of a quantum cogroup).

It is obvious that this operator S_Q satisfies the QYBE and has eigenvalue 1.

Consider the algebra $A(S) = T(L)/\{I\}$ where $\{I\}$ is the ideal generated by the image I of the operator $\mathrm{id} - S_Q$. Suppose now that the initial operator S is Hecke symmetry of minimal type and introduce the so called *quantum determinant* $det = u_{jl}v^{ik}e_i^j \otimes e_k^l$ (in [5] it was defined for any even Hecke symmetry).

One can see that

$$S_Q(det \otimes e_i^j) = M_{ik}^{jl} e_l^k \otimes det$$

for some operator $M : L \to L$. Introduce the formal inverse element det^{-1} and put

$$S_Q(det^{-1} \otimes e_i^j) = (M^{-1})_{ik}^{jl} e_l^k \otimes det^{-1}$$

(so the element $det\, det^{-1}$ is central) and define the algebra $\mathbf{k}[GL(S)]$ as the quotient of $A(S)$ with the additional generator det^{-1} by the ideal generated by elements

$$det^{-1} \otimes e_i^j - S_Q(det^{-1} \otimes e_i^j).$$

It is natural to do this because

$$S_Q^2(det \otimes e_i^j) = det \otimes e_i^j$$

(see [5]).

If det is a central element of $A(S)$, we introduce also the following algebra $\mathbf{k}[SL(S)] = A(S)/\{I_{det}\}$ where $\{I_{det}\}$ is the ideal in $A(S)$ generated by $det - 1$. The algebras $\mathbf{k}[GL(S)]$ and $\mathbf{k}[SL(S)]$, being equipped with the usual comultiplication ($\Delta e_i^j = e_i^k \otimes e_k^j$) the usual counit ($\gamma e_i^j = \delta_i^j$) and some antipod, are Hopf algebras. We call them *quantum cogroups* because, like in deformation case, it is more natural to use the terme *quantum groups* for dual objects (although we do not have their description similar quasiclassical quatum groups $U_q(\mathfrak{g})$).

These quantum cogroups have been introduced in [4] and [5].

PROPOSITION 2. *(see [4],[5]) If S is Hecke symmetries of minimal type then the element det $\in A(S)$ is central iff the operator $N = uv (N_i^j = u_{ik}v^{kj})$ is scalar.*

Represent now the construction of [1] in a form convenient for our aims.

PROPOSITION 3. *Let $B = (B_{ij})$ be a non-degenerated bilinear form. Then the operator S_{DL}*

$$(S_{DL})_{ij}^{kl} = \delta_i^k \delta_j^l + aB_{ij}(B^{-1})^{kl}$$

where $B_{ik}(B^{-1})^{kl} = \delta_i^l$ is a solution of the QYBE iff $a+a^{-1}+B_{ij}(B^{-1})^{ij} = 0$.

To establish the relation between the construction from [1] and ours, consider the operator

$$S = qS_{DL} = q\,\mathrm{id} + qaB \otimes B^{-1} \qquad (S_{ij}^{kl} = q\delta_i^k \delta_j^l + qaB_{ij}(B^{-1})^{kl})$$

and put $u_{ij} = B_{ij}, v^{kl} = -qa(1+q)^{-1}(B^{-1})^{kl}$. It is easy to see that the operator S satisfies the conditions of Proposition 1 iff for B, a and q the relation above and relation $qa^2 = 1$ hold.

Hence S is a Hecke symmetry of minimal type with eigenvalues -1 and a^{-2} and the operator S_{DL} has eigenvalues $-a^2$ and 1. The operator $N = uv$ is scalar in case under consideration. Therefore the map

$$\{S_{DL}\} \rightarrow \{\text{Hecke symmetries of minimal type with central } det\}$$

is constructed. It is invertible because assuming *det* to be central we have $v = bu^{-1}$ with some $b \in \mathbf{k}$.

PROPOSITION 4. *In the algebra $\mathbf{k}[SL(S)]$ the relations*

$$u_{kl}e_i^k \otimes e_j^l = u_{ij}, \quad v^{ij}e_i^k \otimes e_j^l = v^{kl}$$

hold.

The first relation arrises from the follow chain of equalities

$$u_{kl}e_i^k \otimes e_j^l = (1+q)^{-1}(q\mathrm{id} + S_Q)u_{kl}e_i^k \otimes e_j^l =$$

$$(1+q)^{-1}(qu_{kl}e_i^k \otimes e_j^l + u_{kl}S_{ij}^{ab}e_a^m \otimes e_b^n (S^{-1})_{mn}^{kl}) =$$

$$(1+q)^{-1}(qu_{kl}e_i^k \otimes e_j^l - u_{mn}S_{ij}^{ab}e_a^m \otimes e_b^n) = u_{ij}det = u_{ij}.$$

The second relation can be proved in the similar way. Here we use the following lemma.

LEMMA 1. *The relations*

$$S(u_{kl}e^k\otimes e^l)=u_{kl}S^{kl}_{ab}e^a\otimes e^b=-u_{kl}e^k\otimes e^l,$$

$$S(v^{ij}e_i\otimes e_j)=v^{ij}S^{ab}_{ij}e_a\otimes e_b=-v^{ij}e_i\otimes e_j$$

hold

Vice versa any of the relations from Proposition 4 yields the equality $det=1$.

In [1] some Hopf algebras have been introduced as quotients of $T(L)$ by the relations from Proposition 4. Due to this Proposition we can conclude that these algebras coincid with quantum cogroups $\mathbf{k}[SL(S)]$ defined above.

3. BRAIDED LIE ALGEBRAS

Let us recall first the definition of S-Lie algebras in the case when the operator S is an involutive solution of the QYBE. We say that the space V is equipped with a structure of S-Lie algebra if there exists an operator (S-Lie bracket) $[,]:V^{\otimes 2}\to V$ satisfying the axioms

1. $[,]S=-[,]$;
2. $[,][,]^{12}(\mathrm{id}+S^{12}S^{23}+S^{23}S^{12})=0$;
3. $S[,]^{12}=[,]^{23}S^{12}S^{23}$ with usual notation $S^{12}=S\otimes\mathrm{id}$ and so on.

To introduce a braded counterpart of this notion we consider first a notion of quadratic algebras. Let the space V be fixed. Consider a subspace $I\subset V^{\otimes 2}$ and so called *quadratic algebra* corresponding to $I:\wedge_+(V)=T(V)/\{I\}$ where $\{I\}$ is the ideal in $T(V)$ generated by I.

Recall now that a quadratic algebra $\wedge_+(V)$ is called Koszul algebra if the complex

$$\ldots\xrightarrow{d}\wedge^k_+(V)\otimes\wedge^l_-(V)\xrightarrow{d}\wedge^{k+1}_+(V)\otimes\wedge^{l-1}_-(V)\xrightarrow{d}\ldots$$

is exact. [2] Here $\wedge^k_+(V)$ is as usually the k-homogenous component of $\wedge_+(V)$; $\wedge^l_-(V)$ are defined as follows $\wedge^1_-(V)=V,\wedge^2_-(V)=I,\wedge^3_-(V)=I\otimes V\cap V\otimes I$ and so on, and d is a natural differential (see [7] for details).

Let a map $[,]:I\to V$ be given. Define *a quadratic-linear* algebra (an analogue of envelopping algebra) in the natural way $U(\mathfrak{g})=T(V)/\{J\}$ where $\{J\}\subset T(V)$ is ideal generated by elements $I-[,]I$. Since in this algebra there exists a natural filtration, it is possible to consider the graded algebra $\mathrm{Gr}U(\mathfrak{g})$.

PROPOSITION 5. *Let us assume that the algebra $\wedge_+(V)$ is Koszul algebra and that the following conditions*

$$([,]^{12}-[,]^{23})(I\otimes V\cap V\otimes I)\subset I$$

[2] In some papers another complex connected with quadratic algebra is considered and the algebra is called Koszul algebra if the last complex is exact (see [5] where the both complexes are considered).

and

$$[,]([,]^{12} - [,]^{23})(I \otimes V \cap V \otimes I) = 0$$

hold. Then $GrU(\mathfrak{g})$ *is isomorphic to* $\wedge_+(V)$.

This Proposition is proved in [7] where the first condition is called *correctness* and the second one is called *Jacoby identity.*

Suppose now that we have an algebra $A = \mathbf{k}[GL(S)]$ or $A = \mathbf{k}[SL(S)]$ as above. Consider the category $\mathfrak{A}$ of left comodules of A, i.e., for any $V \in \mathfrak{A}$ there exists a coaction $\Delta : V \to A \otimes V$ with usual properties.

Let $V \in \mathfrak{A}$. Suppose that there exists a map $[,] : V^{\otimes 2} \to V$.

DEFINITION 1. *The agregate* $(V, I \oplus I^* = V^{\otimes 2}, [,])$ *will be called a braided Lie algebra if the following axioms hold*

0. the algebra $\wedge_+(V) = T(V)/\{I\}$ *is Koszul algebra;*

1. $[,]I^* = 0$;

2. the relations from Proposition 5 are satisfied;

3. I, $I^* \in Ob\,\mathfrak{A}$ *and the map* $[,]$ *is a morphism in* $\mathfrak{A}$.

Let us explain that the last condition means that

$$\Delta[,] = (\mu \otimes [,])(\Delta \otimes \Delta)$$

where $\Delta \otimes \Delta : V^{\otimes 2} \to A^{\otimes 2} \otimes V^{\otimes 2}$ and $\mu : A^{\otimes 2} \to A$ is the multiplication in the algebra A.

Stress that a S-Lie algebra for involutive S is a particular case of a braided Lie algebra. If we put $I = I_-$ and $I^* = I_+$ where $I_\pm \in V^{\otimes 2}$ is as in Section 1 (assuming $q = 1$), all axioms of braided Lie algebras are satisfied for any S-Lie algebra. The verification of this fact is left to the reader. We note only that "koszulity" of the algebras $\wedge_+(V)$ have been proved (in more general context) in [5].

Note also that it is natural to introduce the axiom 0 if we want to obtain a "good" envelopping algebra (see Proposition 5). In the forthcoming publications we hope to elucidate the important role of this axiom in the quantization procedure.

Consider now an example of a braided Lie algebra constructed by means of a Hecke symmetry of minimal type.

Let $S : V^{\otimes 2} \to V^{\otimes 2}$ be a Hecke symmetry of minimal type such that *det* is central and put $A = \mathbf{k}[SL(S)]$. Fix the base $\{e_i,\ 1 \le i \le n = \dim V\}$. Consider one-dimensional A-comodule $V_0 = \mathbf{k}\, e_0$ ($\Delta e_0 = 1 \otimes e_0$) and denote $V' = V \oplus V_0$. We put $I = I_- \oplus I_0$ (resp., $I^* = I_+ \oplus I_0^*$) where $I_\pm \subset V^{\otimes 2}$ are the same spaces as in Section 1 and $I_0, I_0^* \subset V_0^{\otimes 2} \oplus V_0 \otimes V \oplus V \otimes V_0$ are generated by elements $\{e_0 \otimes e_i - e_i \otimes e_0\}$ (resp., $\{e_0 \otimes e_0,\ e_0 \otimes e_i + e_i \otimes e_0\}$).

In [5] we have proved that $\wedge_+(V)$ is Koszul algebra. Using this result it is not difficult to show that the algebra $\wedge_+(V') = T(V')/\{I\}$ is Koszul algebra as well. We introduce in V' an A-modul structure putting $\Delta e_i = e_i^p \otimes e_p$, $\Delta e_0 = 1 \otimes e_0$ and extend this structure on $T(V')$ in a natural way. It is obvious that I, I^* are A-comodules and $I \oplus I^* = V'^{\otimes 2}$. Introduce a bracket:

$$[e_i, e_j] = g u_{ij} e_0,\ 1 \le i, j \le n, [e_i, e_0] = -[e_0, e_i] = c_i e_i$$

where $u = (u_{ij})$ is as in Proposition 1 and $g, c_i \in \mathbf{k}$.

Verify now that this bracket is a morphism in the category $\mathfrak{A}$ of A-comodules. First we will check compatibility of the bracket $[e_i, e_j]$ with coaction of A. Indeed by virtue of Proposition 4

$$(\mu \otimes [,])(\Delta e_i, \Delta e_j) = (\mu \otimes [,])(e_i^p \otimes e_p, e_j^q \otimes e_q) = \mu(e_i^p \otimes e_j^q) \otimes [e_p, e_q] =$$

$$g\mu(e_i^p \otimes e_j^q) u_{pq} \otimes e_0 = g u_{ij} v^{mn} e_m^p \otimes e_n^q u_{pq} \otimes e_0 = 1 \otimes g u_{ij} e_0 = \Delta[e_i, e_j].$$

It is obvious that the bracket $[e_i, e_0]$ is compatible with coaction of A iff $c_i = c$ for any i. The verification of the axiom 1 is left to the reader. Verify now the axiom 2. Since S is a Hecke symmetry of minimal type one has $I_- \otimes V \cap V \otimes I_- = \{0\}$. Hence the space $I \otimes V' \cap V' \otimes I$ is generated be the elements

$$\{v^{ij}(e_i \otimes e_j \otimes e_0 - e_i \otimes e_0 \otimes e_j + e_0 \otimes e_i \otimes e_j)\}.$$

Applying the operator $[,]^{12} - [,]^{23}$ to an element from this family we have

$$([,]^{12} - [,]^{23}) v^{ij}(e_i \otimes e_j \otimes e_0 - e_i \otimes e_0 \otimes e_j + e_0 \otimes e_i \otimes e_j) =$$

$$v^{ij}(g u_{ij} e_0 \otimes e_0 - c e_i \otimes e_j - c e_i \otimes e_j - c e_i \otimes e_j - c e_i \otimes e_j - g u_{ij} e_0 \otimes e_0) =$$

$$-4 v^{ij} c e_i \otimes e_j \in I.$$

Axiom 2 is satisfied if $cg = 0$. Therefore under this condition all axioms of braided Lie algebra are satisfied.

Consider the particular case $n = 2$. In terms of the "envelopping algebra" the relations between the generators e_0, e_1, e_2 are of the form

$$a e_1 \otimes e_2 + b e_2 \otimes e_1 = g(1+q) e_0,\ e_1 \otimes e_0 - e_0 \otimes e_1 = 2 c e_1,$$

$$e_2 \otimes e_0 - e_0 \otimes e_2 = 2 c e_2$$

where we assume that a, b from Example 1 satisfy the condition $m^2 = (a/b)^2 = q$ and eigher $c = 0$ or $g = 0$. As result we obtaine a braided deformation of usual Lie algebras, namely of Heisenberg algebra when $c = 0$ and of the algebra $[e_1, e_2] = 0$, $[e_1, e_0] = 2ce_1$, $[e_2, e_0] = 2ce_2$ when $g = 0$ (in fact only the first relation is deformed).

Stress that these relations differ from ones arising from representation of quantum group $U_q(sl_2)$ of spine 1 (see [2]). The last example will be considered elsewhere from the point of view of our definition of braided Lie algebras.

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