A small state sum for knots

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Since the discovery of the Jones polynomial and its far reaching generalizations it appeared that many of these new invariants can be obtained by so called state sums, associated to a diagram of the link. These state sums have in common that they are built up by a very high number of summands.

In this paper we introduce a state sum for knots in real line bundles over non-simply connected surfaces in a very simple and effective way. This leads to a new invariant for a certain class of links in the three-sphere. The invariant is a secondary invariant for the linking number and is used to obtain an estimate from below for a generalized unknotting number.

A conjugacy invariant for braids is another application of the new state sum. We use this to show that the exchange move for braids, introduced by Birman & Menasco, indeed changes the conjugacy class of the braid in many cases. This conjugacy invariant can also often be used to show very quickly that a given braid (and for a pure braid even all of its powers) is not conjugate to any positive braid.

Combining our invariant with techniques of Birman & Menasco and Morton we prove that there are infinitely many pairwise non-conjugate presentations of the unknot as (the closure of) a braid with four strings, which are all irreducible, i.e. none of them is conjugate to a stabilization of a braid with three strings. Hence, braid presentations of the unknot are as complicated as they could only be.

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§ 1. The basic construction

Let F^2 be a non-simply connected smooth surface (not necessarily compact or orientable) and let $p : E \to F^2$ be a real line bundle with orientable total space E. We fix an orientation of E. Let $K \hookrightarrow E$ be an oriented knot in general position with respect to p, i.e. $p(K)$ is a connected immersed curve with ordinary double points as the only singularities. The projection p induces, as usual, a diagram of K in F^2. A **writhe** $w(q) = \pm 1$ is well-defined in each double point q of $p(K)$. For this we choose an orientation of the fibre $E_q = p^{-1}(q)$. This determines the undercross and the overcross for the two branches of K intersecting E_q.

Definition 1. *$w(q) = -1$ if the three-frame (undercross, overcross, fibre E_q) agrees with the orientation of E and $w(q) = -1$ otherwise (see Fig. 1).*

Lemma 1. The definition of the writhe is correct.

Proof: If we reverse the orientation of E_q then the undercross and the overcross interchange and, hence, the writhe hasn't changed.

Let $[p(K)]$ denote the homology class in $H_1(F^2; \mathbf{Z})$ represented by $p(K)$. We distinguish two cases.

Case I: $\langle w_1(F^2), [p(K)]\rangle \equiv 1 \bmod 2$, i.e. $p(K)$ is one-sided immersed in F^2. (Here $w_1(F^2)$ denotes the first Stiefel-Whitney class of the tangent bundle of F^2.)

Let $q \in p(K)$ be a crossing. We split the curve $p(K)$in q with respect to the orientation (see Fig. 2) and obtain two oriented curves on F^2. Exactly one of them is again one-sided immersed in F^2. We denote by $\xi(q)$ the class in $H_1(F^2; \mathbf{Z})$represented by this curve. Let H denote the free $\mathbf{Z}$-module generated by $H_1(F^2; \mathbf{Z})$.

Definition 2. *The* **small state sum** $W_K \in H$ *is defined by the sum over all crossings* q

$$W_K = \sum_q w(q)\xi(q) - \left(\sum_q w(q)\right)[p(K)].$$

Case II. $\langle w_1(F^2), [p(K)]\rangle \equiv 0 \bmod 2$.

Let $q \in p(K)$ be a crossing. We again split $p(K)$ at q with respect to the orientation of $p(K)$. There are again two cases: Either both resulting curves are one-sided immersed in F^2 or both are two-sided immersed. We consider only those crossings q for which the second possibility occurs and call them crossings of type II. We orient F^2 along $p(K)$. In crossings of type II this determines a well-defined orientation of F^2. Together with the orientation of E this determines an orientation of E_q. Hence, the overcross and the undercross of K in q are now determined invariantly. The point is, that we can now **distinguish** the two curves which result from the splitting of $p(K)$ at q. Let $\xi^+(q)$ denote the class in $H_1(F^2; \mathbf{Z})$ which is represented by the curve which comes from the undercross and goes to the overcross at q, and let correspondingly, $\xi^-(q)$ denote the class represented by the other curve (see Fig. 3). Let H denote the free $\mathbf{Z}$-module generated by $H_1(F^2; \mathbf{Z})/_{[p(K)]=\{0\}}$ (i.e. we have in $H_1(F^2; \mathbf{Z})$ identified just two elements, namely the class represented by $p(K)$ with the 0-element.)

Definition 3. *The* **small state sum** $W_K \in H$ *is defined as the element which is induced by the sum over all crossings* q *of type II*

$$W_K = \sum_{q \text{ of type II}} w(q)\xi^+(q) - \left(\sum_{q \text{ of type II}} w(q)\right)\{0\}.$$

Theorem 1. W_K **is an isotopy invariant of** $K \hookrightarrow E$ **in each of the both cases.**

Proof: We consider case II. (The proof in case I is similar and is therefore omitted.) We have to check the invariance of W_K under the oriented Reidemeister moves of type I, II and III as in the case of the trivial bundle over $\mathbf{R}^2$ (see Fig. 4). This is in fact sufficient, because the Reidemeister moves correspond to the generical singularities of any one-parameter family of projections of a curve into a surface.

The invariance under moves of type III (i.e. passing a triple point in the projection) is evident, because the writhe $w(q)$ is invariant and the class $\xi^+(q)$ doesn't change under a homotopy of the corresponding curve on F^2.

Under a move of type II (i.e. passing a tac-node in the projection) a pair of crossings q and q' appears or disappears. As easily seen, q and q' are either both of type II or both not, $w(q) = -w(q')$ and $\xi^+(q) = \xi^+(q')$ (see Fig. 3).

Consequently, W_K doesn't change.

A move of type I (i.e. passing a cusp in the projection) adds or eliminates always a crossing q of type II. The crossing q always contributes a summand of the form $w(q)\{0\}$ or $w(q)[p(K)]$. But we have identified $\{0\}$ with $[p(K)]$ and hence the last term in the definition of W_K compensates the change under a move of type I. The theorem is proved.

In the following we will be only interested in the case of orientable surfaces F^2. Hence, the bundle E is trivial and all crossings are of type II.

The most important property of W_K is its very simple "skein relation". Let $q \in p(K)$ be a crossing and let K_+ and K_- denote the associated knots as usual (see, e.g. [10]).

$$W_{K_+} - W_{K_-} = \xi^+(q) + \xi^-(q) - 2\{0\} \tag{1}$$

This follows immediately from the definition, because a crossing change interchanges

$$\xi^+(q) \text{ and } \xi^-(q).$$

Consequently, if we make a crossing change in a crossing q for which both $\xi^+(q)$ and $\xi^-(q)$ are not zero then W_K changes. Hence, it is not an invariant of regular homotopy of K.

Remark: In the definitions and results of this paragraph we could have replaced the homology classes ξ, ξ^+, ξ^- by the free homotopy classes of the corresponding curves on F^2. But we make no use from this in this paper.

§ 2. A secondary link invariant

Let L be an oriented non-trivial fibred knot and let $\varphi : S^3 \setminus L \to S^1$ be the fibration, i.e. φ is a smooth map without singularities and induces an open book structure near L (see, e.g. [14]). As well-known, φ is unique up to isotopy. Let E be the infinite cyclic covering of $S^3 \setminus L$ corresponding to a meridian of L. φ lifts to a smooth function $\tilde{\varphi} : E \to \mathbf{R}$ and, hence, E has a product structure $F^2 \times \mathbf{R}$, where F^2 is the fiber surface of φ. The action of the group of deck transformations is generated by the monodromy

$$\tau : F^2 \to F^2 \qquad \text{(see, e.g. [14]).}$$

This defines a projection $p : E \to F^2$ which is unique up to isotopy and up to composition with the action of $\tau^m, m \in \mathbf{Z}$, on F^2. We fix such a projection p.

Let now $K \hookrightarrow S^3 \setminus L$ be an oriented knot such that the linking number $lk(K, L) = 0$. The knot K lifts to a closed curve $\tilde{K} \hookrightarrow E$. We apply Definition 3 to $p : \tilde{K} \to F^2$ and obtain a small state sum $W_{\tilde{K}} \in H$ of the form

$$W_{\tilde{K}} = \sum_{i \in I} a_i \eta_i, \text{ where } a_i \in \mathbf{Z} \setminus 0 \text{ and}$$

the η_i are distinct elements in $H_1(F; \mathbf{Z})$ (where we have identified the class $\left[p\left(\tilde{K}\right)\right]$ with 0).

Proposition 1. The unordered set of non-zero integers $\{a_i\}_{i \in I}$ is an isotopy invariant of $K \cup L \hookrightarrow S^3$.

We denote this invariant by $W_{K \cup L}$.

Proof: $W_{\tilde{K}}$ is an isotopy invariant of $\tilde{K} \hookrightarrow E$ for the fixed projection p as follows from Theorem 1. Choosing another projection p' sends $W_{\tilde{K}}$ to $\sum_{i\in I} a_i(\tau_*^m \eta_i)$ for a fixed m. Here we had to identify $\tau_*^m\left[p\left(\tilde{K}\right)\right] = \left[p'\left(\tilde{K}\right)\right]$ with 0. But τ_*^m acts as an isomorphism on $H_1\left(F^2;\mathbf{Z}\right)$ and, hence, the $\tau_*^m\eta_i$ are distinct for distinct i. It follows that the unordered set of coefficients $\{a_i\}_{i\in I}$ remains invariant. ☺

§ 3. A generalized unknotting number

Let $L \cup K \hookrightarrow S^3$ be an oriented link of two components. We assume that L is a non-trivial fibred knot. Let F^2 be a fibre surface for L.

Let $h_t, t \in [0,1]$, be a regular homotopy of K in $S^3 \setminus L$ such that $h_0 = K, h_1$ is embedded in F^2 and $h_t, t \in (0,1)$, is an embedding except for a finite number of values of t where it has an ordinary self-intersection (see, eg.[6]).

Definition 4. *The minimal number of self-intersections among all such homotopies h_t is called the* **unknotting number of** K **with respect to** L *and denoted by $u_L(K)$. If there is no such homotopy at all we set $u_L(K) = \infty$.*

Remark. If we take for L the trivial knot in some ball $B^3 \hookrightarrow S^3$ such that $B^3 \cap K = \phi$ then $u_L(K)$ is the usual unknotting number.

If the linking number $lk(L,K) \neq 0$ then $u_L(K) = \infty$, because $h_t \subset S^3 \setminus L$ and, evidently, $lk\left(h_1, L = \partial F^2\right) = 0$. Therefore we assume in the sequel that $lk(L,K) = 0$.

Let $q \in h_{t_0}$ be a self-intersection point. Let $\gamma^+(q)$ and $\gamma^-(q)$ denote the (unordered) oriented loops obtained from h_{t_0} by splitting h_{t_0} at q with respect to the orientation. We distinguish two cases for the self-intersection q :

Typ I. $lk\left(L,\gamma^+(q)\right) = lk\left(L,\gamma^-(q)\right) = 0$
Typ II. $lk\left(L,\gamma^+(q)\right) \neq 0,\ lk\left(L,\gamma^-(q)\right) \neq 0.$

Definition 5. *The self-intersections of type I are called* **essential**. *In analogy to Definition 4 we denote their minimal number by $u_L^e(K)$.*

Clearly, $u_L(K) \geq u_L^e(K)$.

Let $W_{K\cup L} = \{a_i\}_{i\in I}$ be the isotopy invariant of $K \cup L \hookrightarrow S^3$ defined in the previous paragraph.

Proposition 2. $u_L^e(K) \geq 1/2 \sum_{i\in I} |a_i|$.

Proof: If $h_1 \hookrightarrow F^2$ then the lift $\tilde{h}_1 \hookrightarrow F^2 \times \{\text{const.}\} \hookrightarrow E$. Consequently, $p\left(\tilde{h}_1\right)$ has no double points at all and $W_{h_1\cup L} = \phi$.

Let h_{t_0} have a self-intersection q. We consider how $W_{h_t\cup L}$ changes for t passing through t_0. The lift $\tilde{h}_{t_0}$ has a self-intersection exactly if q is essential. Let $\xi^+(\tilde{q})$ and $\xi^-(\tilde{q})$ be the classes corresponding to the double point $\tilde{q} \in \tilde{h}_{t_0}$ (cf. § 1). It follows form the "skein relation" (1) that $W_{h_t\cup L}$ does not change if $\xi^+(\tilde{q}) = \xi^-(\tilde{q})\left(= \left[p\left(\tilde{h}_t\right)\right]\right)$. If $\xi^+(\tilde{q}) \neq \xi^-(\tilde{q}) \neq 0$ then exactly two numbers $a_i, a_j \in W_{h_t\cup L}$ change by ± 1 and if $\xi^+(\tilde{q}) = \xi^-(\tilde{q}) \neq 0$ then exactly one number a_i changes by ± 2. Consequently, there has to be at least $1/2 \sum_{i\in I} |a_i|$ essential self-intersections in h_t in order to make $W_{h_t\cup L} = \phi$ for $t = 1$. ☺

§ 4. A conjugacy invariant for braids

The conjugacy problem for the braid groups was solved by Garside [8]. However, his algorithm is to too complex to be applicable in practice (see also [1], [2], [7], [9]). So it is very useful to find simple invariants.

Let B_n be the braid group of braids of n strings (see [3]). We represent the closure of a braid $\beta \in B_n$ as an oriented link $\hat{\beta}$ in $\mathbf{R}^3 = \{(x,y,z)\}$ which does not intersect the z-axis and intersects each plane containing the z-axis transversely. As well known , there is a one-to-one correspondence between conjugacy classes in B_n and isotopy classes of closed n-braids in the complement of the z-axis (see [11]). Setting $E = \mathbf{R}^3 \setminus \{z - \text{axis}\}, \mathbf{F}^2 = \mathbf{R}^2 \setminus \{0\}$ and $p(x,y,z) = (x,y)$ we can define the invariant $W_{\hat{\beta}}$. But because we are interested only in conjugacy of braids, no Reidemeister moves of type I occur and we do not need the correction term in the definition of $W_{\hat{\beta}}$. It is also convenient to define the invariant as a Laurent polynomial.

A naturally oriented meridian m of the z-axis represents a generator of $H_1(\mathbf{R}^2 \setminus \{0\}) \cong \mathbf{Z}$. Here the orientation is chosen in such a way that $\left[\hat{\beta}\right] = n[m]$. We assume that $\hat{\beta}$ is a knot. Let $\xi^+(q) = n^+(q)[m]$ and $\xi^-(q) = n^-(q)[m]$ for any crossing q of β, where $\xi^+(q)$ and $\xi^-(q)$ are defined as in section 1. Here $n^+(q)$ and $n^-(q)$ are positive integers and, clearly, $n^+(q) + n^-(q) = n$. Hence, they are in fact an ordered splitting of the string number n associated to the crossing q.

Definition 6. *The invariant $\hat{W}_{\hat{\beta}}(x) \in \mathbf{Z}[x, x^{-1}]$, where x is a variable, is defined as the sum over all crossings q (or letters in a word representing β)*

$$\hat{W}_{\hat{\beta}}(x) = \sum_q w(q) x^{n^+(q) - n^-(q)}.$$

Proposition 3. $\hat{W}_{\hat{\beta}}(x)$ is a conjugacy invariant of $\beta \in B_n$ and has the following properties:

i) **$\hat{W}_{\hat{\beta}}(x)$ is a symmetric Laurent polynomial,** *i.e.*

$$\hat{W}_{\hat{\beta}}(x^{-1}) = \hat{W}_{\hat{\beta}}(x).$$

ii) **the maximal degree of monomials**

$$\max \deg \hat{W}_{\hat{\beta}}(x) \le n - 2$$

iii) **if β is conjugate to a positive braid (i.e. one which can be represented by a word using only the standard generators σ_1 and not their inverses) then $\max \deg \hat{W}_{\hat{\beta}}(x) = n - 2$ and all coefficients are non-negative.**

iv) **$\hat{W}_{\hat{\beta}}(1)$ is equal to the exponent sum $e(\beta)$.**

Remark. As well known, every homomorphism of B_n into an abelian group factors through the homomorphism $e : B_n \to \mathbf{Z}$ given by $e(\beta)$. The map into the abelian group $\hat{W}_{\hat{\beta}} : B_n \to \mathbf{Z}[x, x^{-1}]$ is not a homomorphism but it is well defined on conjugacy classes in B_n. Together with property iv) this shows that $\hat{W}_{\hat{\beta}}$ is a refinement of e.

Proof: Invariance follows directly from Theorem 1 and the remark in front of Definition 5. ii) and iv) follow immediately from the definition. To prove i) we notice that according to (1) a crossing change changes $\hat{W}_{\hat{\beta}}(x)$ by a symmetric polynomial. With crossing changes and conjugations every braid $\beta \in B_n$ (such that $\hat{\beta}$ is a knot) can be transformed into the braid $0_n = \sigma_1\sigma_2\ldots\sigma_{n-1}$. A direct calculation shows $\hat{W}_{\hat{0}_n}(x) = x^{n-2} + x^{n-4} + \ldots + x^{4-n} + x^{2-n}$ and, hence, $\hat{W}_{\hat{\beta}}(x)$ is always symmetric. ☺

Let β be a positive braid. Each crossing contributes to $\hat{W}_{\hat{\beta}}(x)$ a monomial with coefficient $+1$ and, hence, no coefficient of $\hat{W}_{\hat{\beta}}(x)$ is negative. It is an elementary geometrical fact (which we will not prove) that there are always two crossings q, q' such that $|n^+(q) - n^-(q)| = |n^+(q') - n^-(q')| = n - 2$. The contributions of these crossings can not cancel because all other crossings contribute monomials with positive coefficients too. The proposition is proved.

Example.

$$\beta = \sigma_1\sigma_2\sigma_3^{-1}\sigma_2\sigma_4\sigma_5^{-1}\sigma_4\sigma_3\sigma_4^{-1}\sigma_3^6 \in B_6.$$

$\hat{W}_{\hat{\beta}}(x) = 4x^2 + 1 + 4x^{-2}$ and, consequently, β is not conjugate to any positive braid.

$\hat{W}_{\hat{\beta}}(x)$ can be calculated by hand in a few minutes!

Remarks: 1. It seems to be difficult to extend $\hat{W}_{\hat{\beta}}(x)$ to a knot invariant because it behaves unpredictable under stabilization (i.e. the second Markov move [3]).

2. In a forthcoming joint paper with C.-F. Bödigheimer we extend $\hat{W}_{\hat{\beta}}(x)$ to a conjugacy invariant for hyperelliptic mapping class groups.

§ 5. Characteristic classes for the group of pure braids

Let $S : B_n \to \Sigma_n$ be the projection of the braid group onto the symmetric group, induced by the additional relations $\sigma_i^2 = 1$. Let $\{\alpha_i\}_{i=1,\ldots,(n-1)!}$ be the set of all elements of maximal cycle length $(n-1)$ in Σ_n. To each α_i corresponds a unique positive braid of exponent sum $(n-1)$ in $S^{-1}(\alpha_i)$. We denote this braid by α_i too. (The closure $\hat{\alpha}_i$ of each α_i represents the unknot).

Let $P_n \subset B_n$ be the subgroup of pure braids, i.e. braids which induce the trivial permutation in the symmetric group (see [3]).

Definition 7. *The class* $W_i \in H^1(P_n; \mathbb{Z}[x^{\pm 1}])$, $i = 1, \ldots, (n-1)!$, *is defined by* $W_i(\beta) = \hat{W}_{\widehat{\alpha_i\beta}}(x) - \hat{W}_{\hat{\alpha}_i}(x)$ *for* $\beta \in P_n$.

Lemma 2. *The definition of* W_i *is correct.*

Proof: $\widehat{\alpha_i\beta}$ is a knot for $\beta \in P_n$ and, hence, $\hat{W}_{\widehat{\alpha_i\beta}}(x)$ is defined. For any braid $\gamma \in B_n$ (such that $S(\gamma) = 1$) the contribution of a crossing to $\hat{W}_{\hat{\gamma}}(x)$ is the same as the contribution of the same crossing to $\hat{W}_{\widehat{\gamma\beta}}(x)$ for any $\beta \in P_n$. For W_i only the crossings in β give contributions and, consequently, W_i is a homomorphism into the additive group of Laurent polynomials. ☺

The following lemma is proved with the same arguments.

Lemma 3. *Let* $\beta \in P_n$. *The unordered set* $\{W_i(\beta)\}_{i=1,\ldots,(n-1)!}$ *is a conjugacy invariant of* β *in* B_n.

Remarks: 1. It would be interesting to find out how the classes W_i are related to each other. Are they really different?

2. $H_1(P_n;\mathbf{Z}) \cong \mathbf{Z}$ and the map $P_n \to H_1(P_n;\mathbf{Z})$ is given by $\beta \mapsto e(\beta)$, where $e(\beta)$ is the exponent sum of β. $W_i(\beta)$ evaluated at $x = 1$ is just $e(\beta)$. Consequently, the set $\{W_i(\beta)\}_{i=1,\dots,(n-1)!}$ can be considered as a refinement of the abelian invariant $e(\beta)$.

Proposition 4. *Let $\beta \in P_n$. If at least one of the polynomials $W_i(\beta)$ has a negative coefficient then none of the braids $\beta^m, m-$ any positive integer, is conjugate in P_n to a positive braid. If all of the polynomials $W_i(\beta)$ have a negative coefficient then none of the braids $\beta^m, m-$ any positive integer, is conjugate in B_n to a positive braid.*

Proof: $W_i(\beta^m) = mW_i(\beta)$ and, hence, $W_i(\beta^m)$ has a negative coefficient. The proposition follows then from Lemma 3 and Proposition 3 iii). ☺

§ 6. Exchange moves and conjugacy classes

Birman and Menasco introduced an important new move for closed braids in order to avoid stabilization in the study of link types as closed braids [4]. Following them we call this move exchange move. It is illustrated in Fig. 5.

One strand is weighted with a positive integer n, so the whole braid β belongs to B_{n+2}. The X and Y are braids in B_{n+1}. We denote the n new negative crossings (i.e. after the move) which are nearest before the box X by $p_1, \dots, p_n$, and we denote the n new crossings just behind the box by $q_1, \dots, q_n$.

Definition 8. *Let $\hat{\beta}$ be a knot. The defect $\Delta(x) \in \mathbf{Z}[x^{\pm 1}]$ of the exchange move is defined by*

$$\begin{aligned}\Delta(x) \; &= \sum_{i=1}^{n} \left(x^{n^+(q_i)-n^-(q_i)} + x^{n^-(q_i)-n^+(q_i)}\right) \\ &- \sum_{i=1}^{n} \left(x^{n^+(p_i)-n^-(p_i)} + x^{n^-(p_i)-n^+(p_i)}\right).\end{aligned}$$

Repeated applications of the exchange move create infinitely many presentations of the same link type as a $(n+2)-$braid.

Proposition 5. *If $\Delta(x) \not\equiv 0$ then all braids obtained from β by repeated applications of the exchange move are pairwise non conjugate.*

Proof: Changing all crossings $p_1, \dots, p_n, q_1, \dots, q_n$ we obtain a braid conjugate to β. Consequently, with respect to (1) the exchange move adds $\Delta(x)$ to $\hat{W}_{\hat{\beta}}(x)$ and k times repeated applications add $k \cdot \Delta(x)$. The proposition follows then from Proposition 3. ☺

Example 1. As well known, the number of pairwise non conjugate presentations of a link as a 3-braid is always finite (see [13]). So the simplest examples should be 4-braids.

Setting in Fig. 5 $n = 2, X = \sigma_1\sigma_2, Y = \sigma_2$ we obtain presentations of the unknot. Let $\beta(m)$ denote the braid which is the result of applying m times the exchange move to β. An easy calculation shows

$$\hat{W}_{\hat{\beta}(m)} = 2x^2 - 1 + 2x^{-2} + m\Delta(x),$$

where

$$\Delta(x) = 2x^2 - 4 + 2x^{-2}.$$

Consequently, all $\beta(m)$ are pairwise non-conjugate.

Example 2. The first examples of infinitely many pairwise non-conjugate presentations of the unknot as a braid with four strings where obtained by Morton [11]. For his examples

$$\beta_i = \sigma_1\sigma_2^{2i+1}\sigma_3\sigma_2^{-2i} \in B_4,\ i \geq 0,$$

one obtains

$$\hat{W}_{\hat{\beta}_i} = x^2 + 1 + x^{-2} + i\left(-2x^2 + 4 - 2x^{-2}\right)$$

and, hence, all the braids β_i are non-conjugate to all the braids $\beta(m)$ from Example 1.

§ 7. Irreducible braid presentations of the unknot

If a braid $\beta \in B_n$ is conjugate to $\gamma\sigma_{n-1}^{\pm 1}$, for some $\gamma \in B_{n-1}$ then β is said to be **reducible.** So, one can pass from β to γ without using Markov moves which increase the string index (cf. [1], [5], [12], [15]). The examples of the previous paragraph are all reducible. Morton [12] gave the first example of an irreducible presentation of the unknot as a braid with four strings. In this paragraph we use Mortons approach (which uses an idea of Rudolph [15] and Casson) to show that there are infinitely many such presentation.

Theorem 2. The braids

$$\beta_n = \left(\sigma_3^{-1}\sigma_2^{-2}\sigma_3^{-1}\right)^n\left(\sigma_3^{-1}\sigma_2^{-2}\sigma_3^{-3}\sigma_2^{3}\sigma_3\right)\left(\sigma_3\sigma_2^{2}\sigma_3\right)^n\left(\sigma_2\sigma_1^{3}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\right) \in B_4,\ n \geq 0,$$

have unknotted closure and are pairwise non-conjugate. The braids β_n for $n \equiv 4 \bmod 5$ are all irreducible.

Proof: The starting point is Mortons example

$$\beta = \sigma_3^{-2}\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^{3}\sigma_2^{-1}\sigma_1\sigma_2^{-1} \in B_4.$$

β is irreducible and has unknotted closure. We take the arc α and push it through the hatched region in its previous position (see Fig. 6). This is an isotopy of the knot. The resulting braid has the same exponent sum as β and will be our braid β_0. It allows some kind of exchange move, namely rotating the arc α around the first three strings. An easy calculation shows that the defect of this move $\Delta(x) \equiv 0$. Therefore we make a "partial exchange move", namely rotating the arc α only around the second and third strings (see Fig. 7). Iterating this move leads to the braids β_n. The defect of this move $\Delta(x) = 4 - 2x^2 - 2x^{-2}$, and, hence, all the braids β_n are pairwise non-conjugate. ☺

Following Morton [12] we consider the representation

$$\begin{aligned}\phi\quad &: B_4 \to SL(2,\mathbb{Z}),\ \text{defined by}\\ \phi(\sigma_1)\ &= \phi(\sigma_3) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\\ \phi(\sigma_2)\ &= \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.\end{aligned}$$

A direct calculation shows

$$tr(\phi(\beta_n)) = 140n^2 + 106n + 22.$$

If β_n is reducible then it follows from the classification up to conjugacy of the presentations of the unknot in B_3 that either

$$tr(\phi(\beta_n)) = 3 + c^2 + cd - d^2$$

or

$$tr(\phi(\beta_n)) = 1 + a^2 - ac + c^2$$

for same $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ (cf. [12]). Consequently, either

$$4n + 1 \equiv (2c + d)^2 \bmod 5$$

or

$$2c^2 + 4n + 4 \equiv (2a - c)^2 \bmod 5.$$

An easy analysis shows that exactly for $n \equiv 4 \bmod 5$ none of the both cases is possible. This completes the proof.

Remark. The trace tr is clearly a conjugacy invariant and, hence, it shows again that the braids β_n are pairwise non-conjugate. But tr doesn't behave additively under iteration of the move in difference to the defect $\Delta(x)$, and, hence, the calculation of tr is more tedious.

§ 8. An equivalence relation for braided surfaces

A positive band in the braid group B_n is a conjugate of one of the standard generators, and a negative band is the inverse of a positive band. Each representation of a braid as a product of bands yields a handle decomposition of a certain ribbon surface in the 4–ball bounded by the corresponding closed braid. These surfaces are called braided surfaces. They were introduced and studied by Rudolph in the beautiful paper [15]. Following Rudolph, let $b = (b(1), \ldots, b(k))$, where each $b(i)$ is a positive or negative band in B_n, denote a band representation of the braid $\beta = b(1) \ldots b(k) \in B_n$. There are four natural operations that relate different band representations of the same braid β.

I. If for some j $b(j)b(j+1) = 1 \in B_n$ then $b \mapsto (b(1), \ldots, b(j-1), b(j+1), \ldots, b(k))$ is called an elementary contraction.

II. The opposite operation to I, called elementary expansion.

III. $b \mapsto \Big(b(1), \ldots, b(j-1), b(j)b(j+1)b(j)^{-1}, b(j), b(j+2), \ldots\Big)$ a forward slide.

IV. $b \mapsto \Big(b(1), \ldots, b(j-1), b(j+1), b(j+1)^{-1}b(j)b(j+1), b(j+2), \ldots\Big)$ a backward slide (this move is opposite to III).

A theorem of Rudolph says, that two band representations of β in B_n may always be joined by a finite sequence in which adjacent band representations differ by one of the four moves above [15].

Let $S : B_n \to \Sigma_n$ be the projection onto the symmetric group.

Definition 9. *A handle slide is called* **permutation preserving** *if it doesn't change the image of the handle in the symmetric group, i.e. for forward slides* $S\Big(b(j)b(j+1)b(j)^{-1}\Big) =$

$S(b(j+1))$ and for backward slides $S\Big(b(j+1)^{-1}b(j)b(j+1)\Big) = S(b(j))$. Two band representations (or braided surfaces) are called **permutation preserving equivalent** *if they can be joined by a finite sequence in which adjacent band representations differ by move of type I or II or a permutation preserving slide of type III or IV or a conjugation by bands in B_n.*

Every band $b(j)$ is of the form $a\sigma_i^{\pm 1}a^{-1}, a \in B_n, i \in \{1, \ldots, n-1\}$. We call the $\sigma_i^{\pm 1}$ the **centre** of the band.

Definition 10. *Let $\hat{\beta}$ be a knot. The invariant $V_b(x) \in \mathbb{Z}[x]$ is defined as the sum over the centres p of all bands $b(j)$ of the band representation b of β*

$$V_b(x) = \sum_p w(p) x^{|n^+(p) - n^-(p)|}.$$

Proposition 6. *$V_b(x)$ is invariant under permutation preserving equivalence.*

Proof: The images in the plane of the braid obtained by splitting the centre of a band $b(j)$ and of the braid obtained by splitting the centre of the adjacent band $b(j)^{-1}$ are the same. Consequently, the moves I and II don't change $V_b(x)$.

Let p be the centre of a band $b(j)$ and let p' be the centre of the resulting band $b(j+1)^{-1}b(j)b(j+1)$ after a handle slide. The braid obtained by splitting p is identical to the braid obtained by splitting p'.

Let q be the centre of the band $b(j+1)$ before the handle slide and let q' be the centre of the resulting band $b(j+1)$ after the (now assumed) permutation preserving handle slide. Clearly, $|n^+(q) - n^-(q)|$ is determined by $s(b(1) \ldots b(j-1)b(j)b(j+2) \ldots b(k))$ up to conjugation in Σ_n. Analogous, $|n^+(q') - n^-(q')|$ is determined by

$$S\Big(b(1) \ldots b(j-1)\Big(b(j+1)^{-1}b(j)b(j+1)\Big)\, b(j+2) \ldots b(k)\Big).$$

But $S(b(j)) = S\Big(b(j+1)^{-1}b(j)b(j+1)\Big)$ and, consequently, $|n^+(q) - n^-(q)| = |n^+(q') - n^-(q')|$. An example easily shows that $n^+(q) - n^-(q) = n^-(q') - n^+(q')$ and, hence, taking the absolute value of $n^+(q) - n^-(q)$ in the definition of $V_b(x)$ is really necessary. The rest of the proof is the same as the proof of Theorem 1. ☺

Let $V_b(x) = \sum_{i=0}^{n-2} a_i x^i, a_i \in \mathbb{Z}$. For any band representation b let M_b^2 denote the associated braided surface. The Euler characteristic $\chi(M_b^2) = n - k$, where k is the number of bands (compare [15]).

Proposition 7. *Let b' be any band representation which is permutation preserving equivalent to the band representation b of $\beta \in B_n$ ($\hat{\beta}$ is a knot). Then $\chi(M_{b'}^2) \leq n - \sum_{i=0}^{n-2} |a_i|$.*

Proof: $V_{b'}(x) \equiv V_b(x)$ and the proposition follows from the evident inequality $k' \geq \sum_{i=0}^{n-2} |a_i|$ (because each band contributes only a monomial $\pm x^i$ into $V_{b'}(x)$). ☺

Remark. It would be very interesting to compare $V_b(x)$ with $V_{b'}(x)$ for band representations b and b' of β which are not permutation preserving equivalent. This problem seems to be of the same sort as Remark 1 in paragraph 4.

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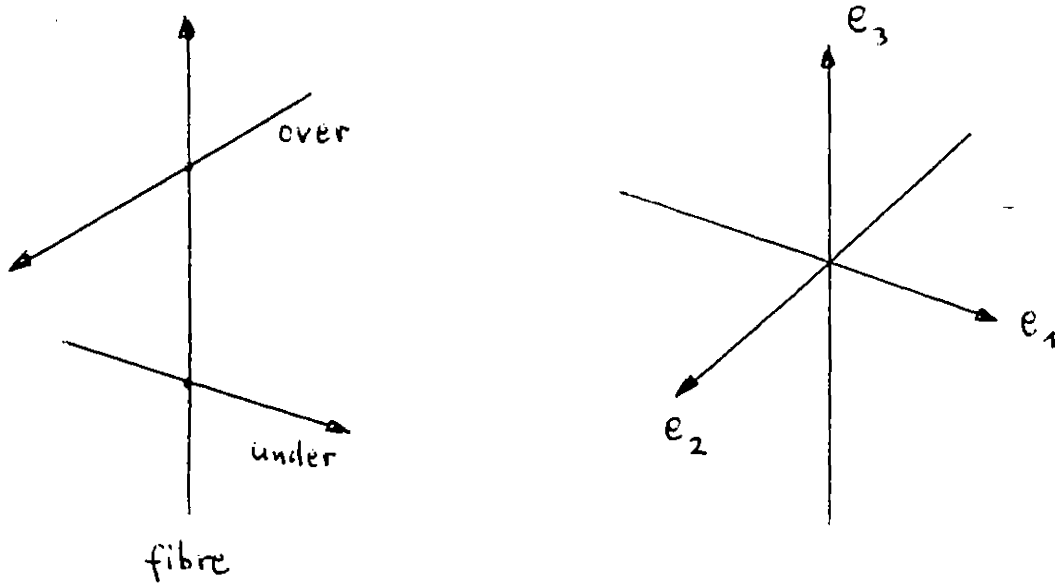

Fig. 1

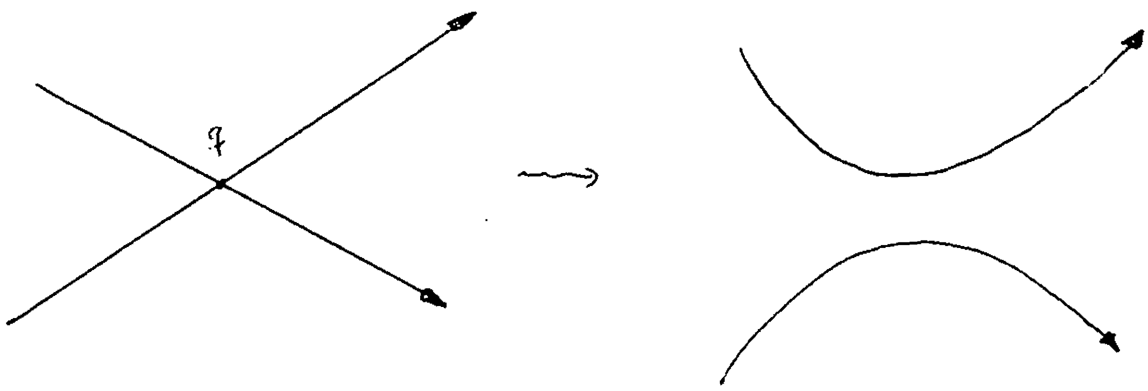

Fig. 2

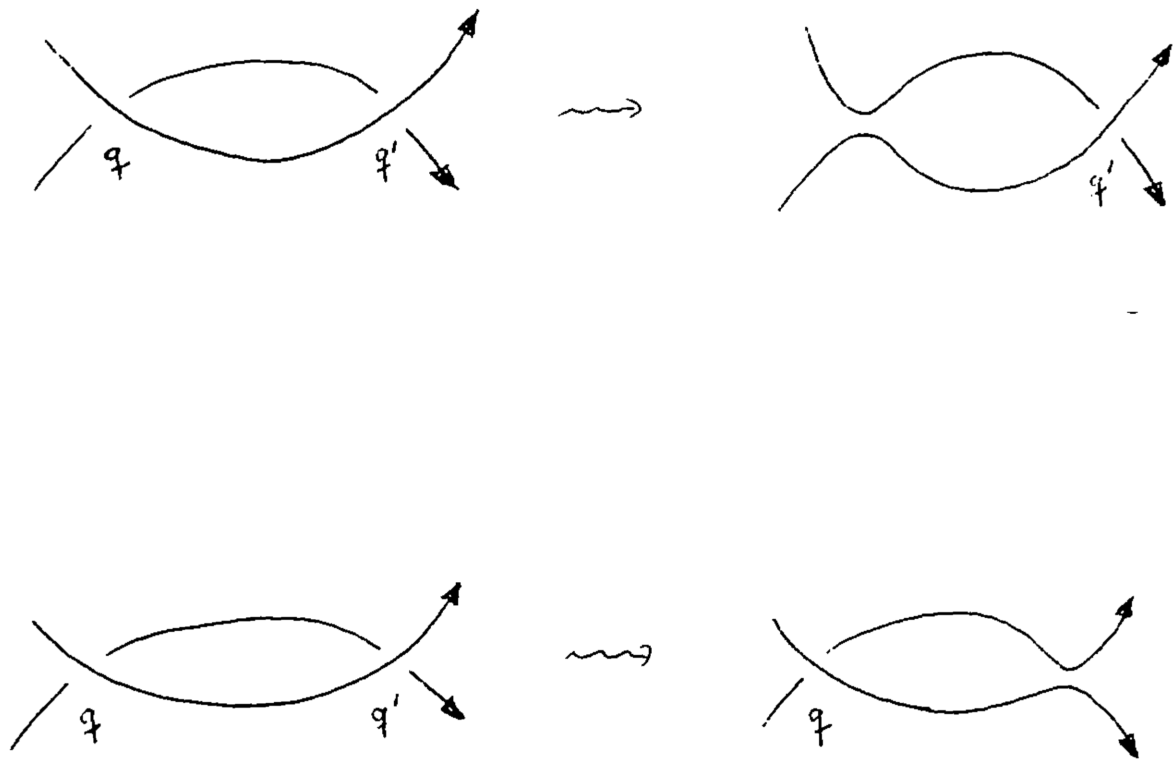

Fig. 3

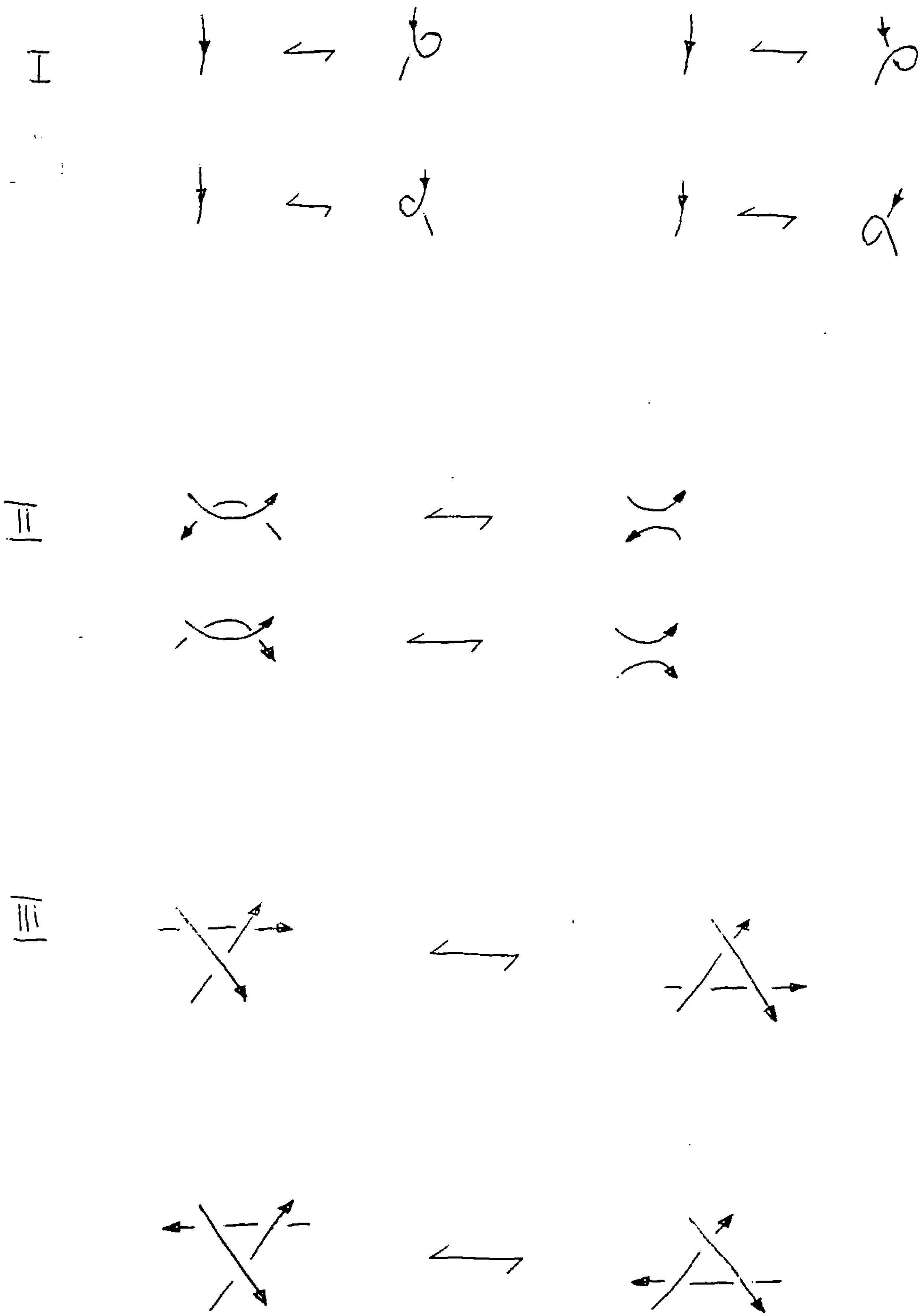

Fig. 4

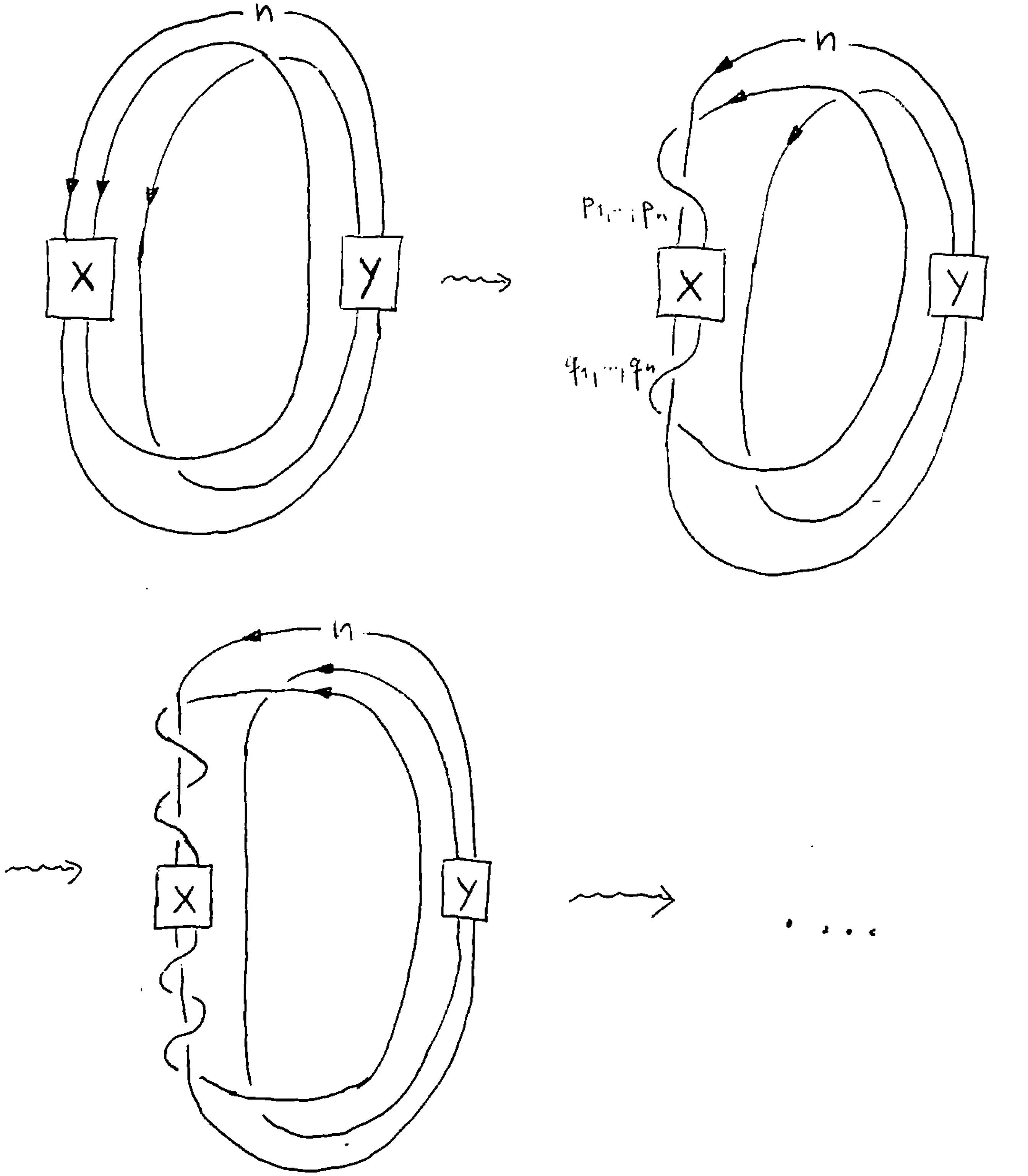

Fig. 5

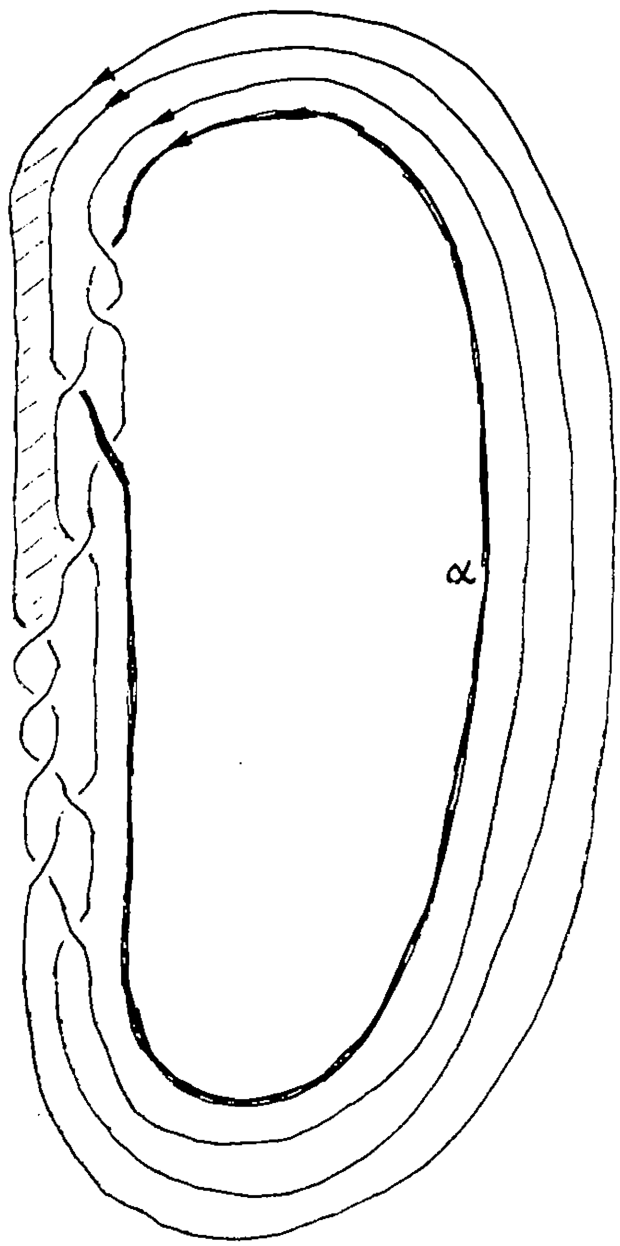

Fig. 6 a

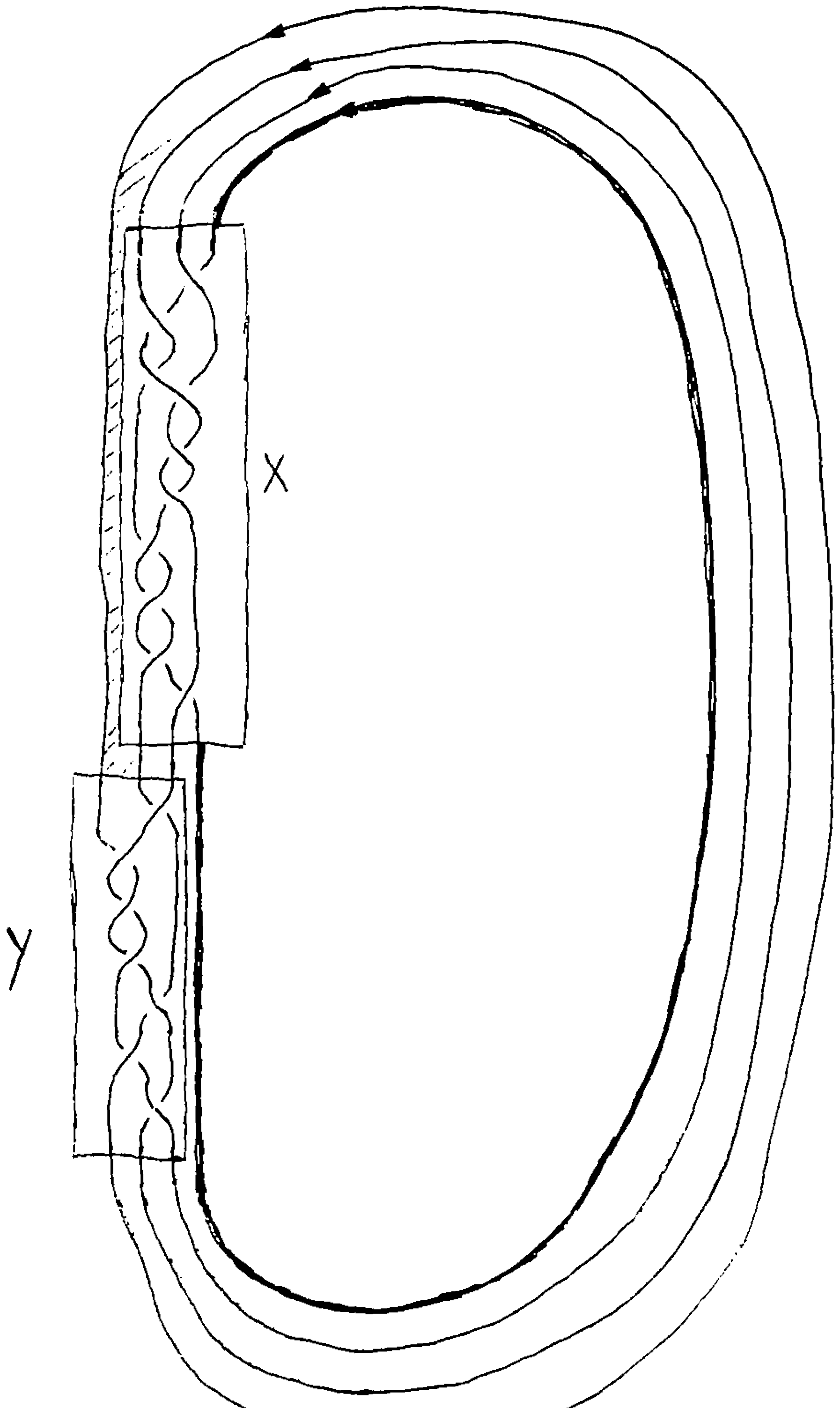

Fig. 6b

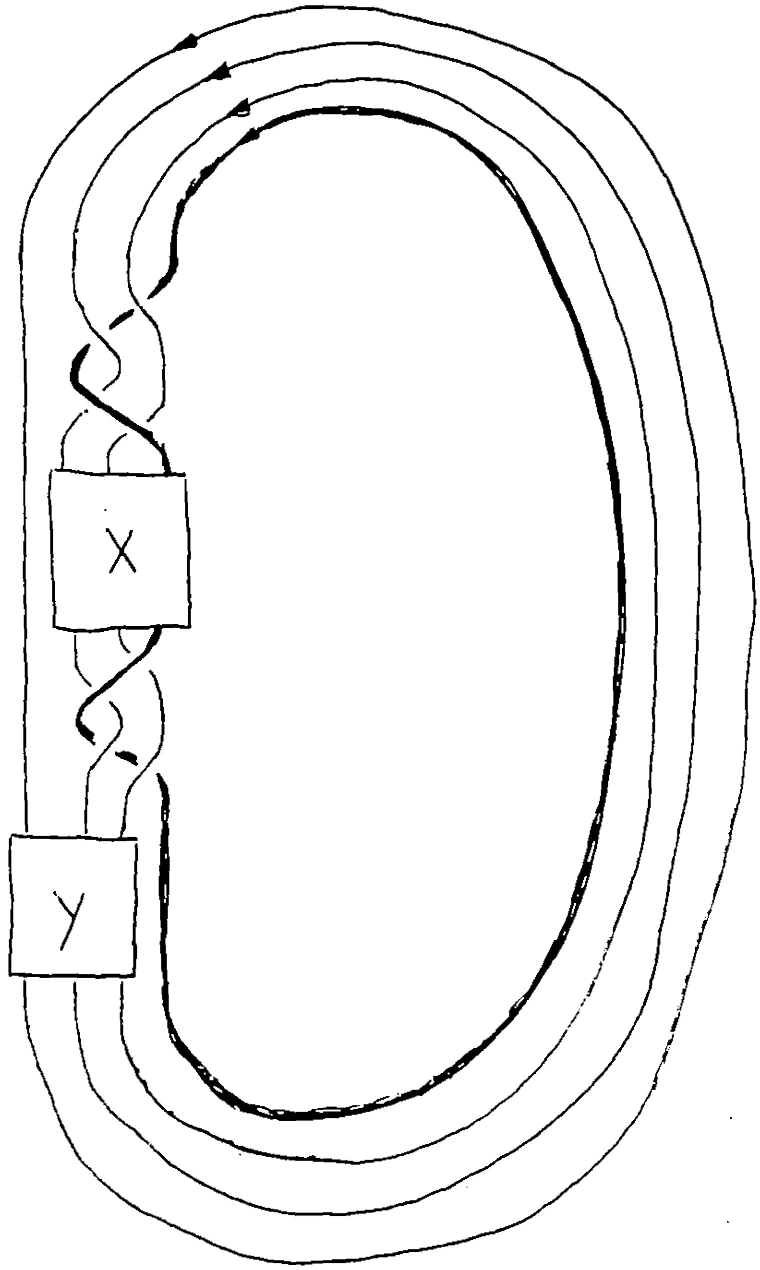

Fig. 7