Extended Moduli Spaces and the Kan Construction

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ABSTRACT. Let Y be a CW-complex with a single 0-cell, let K be its Kan group, a free simplicial group whose realization is a model for the space ΩY of based loops on Y, and let G be a Lie group, not necessarily connected. By means of simplicial techniques involving fundamental results of KAN'S and the standard W- and bar constructions, we obtain a weak G-equivariant homotopy equivalence from the geometric realization $|\mathrm{Hom}(K,G)|$ of the cosimplicial manifold $\mathrm{Hom}(K,G)$ of homomorphisms from K to G to the space $\mathrm{Map}^o(Y,BG)$ of based maps from Y to the classifying space BG of G where G acts on BG by conjugation. Thus when Y is a smooth manifold, the universal bundle on BG being endowed with a universal connection, the space $|\mathrm{Hom}(K,G)|$ may be viewed as a model for the space of based gauge equivalence classes of connections on Y for all topological types of G-bundles on Y thereby yielding a rigorous approach to lattice gauge theory; this is illustrated in low dimensions.

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Introduction

In gauge theory, one usually studies the space of gauge equivalence classes of connections on a principal bundle or suitable subspaces thereof. The geometry of the space of connections is quite simple since it is an affine space. However its analysis is more intricate, and suitable choices of topologies and of completions must be made, depending on the concrete problem under consideration. The miracle is that these analytical problems disappear on the space of gauge equivalence classes of connections. The present paper and its successor [24] provide a step towards an explanation for this. Usual gauge theory could be viewed as non-abelian *singular* cohomology and, in a sense, we offer here a corresponding *cellular* approach: Let Y be a finite CW-complex with a single 0-cell and G a Lie group, not necessarily connected. Let K be the Kan group on Y [30]; this is a simplicial group whose realization is a model for the space ΩY of based loops on Y. By means of simplicial techniques involving fundamental results of KAN'S [30] and the standard W- and bar constructions, we shall obtain a G-equivariant map Φ from the realization $|\mathcal{H}|$ of the cosimplicial G-manifold $\mathcal{H} = \mathrm{Hom}(K, G)$ to the space $\mathrm{Map}^o(Y, BG)$ of based maps from Y to BG where G acts on BG by conjugation, and our main result, Theorem 1.7 below, will say that Φ is a weak homotopy equivalence. One could say the domain of Φ gives a complete set of combinatorial data which determine a bundle with a based gauge equivalence class of connections; the latter is given by the value of the data in $\mathrm{Map}^o(Y, BG)$ under Φ. We do not know whether Φ is in general a genuine homotopy equivalence. For a closed topological surface, in Section 2 below, we briefly indicate a construction of a homotopy inverse of Φ. When Y is a sphere S^q, $q \geq 1$, with the usual CW-decomposition with only two cells, the map Φ boils down to the standard relationship between $\mathrm{Map}^o(S^{q-1}, G)$ and $\mathrm{Map}^o(S^q, BG) \cong \mathrm{Map}^o(S^{q-1}, \Omega BG)$ induced by the standard map from G to the space ΩBG of based loops on BG. In general, every topological type of principal G-bundle on Y gives rise to a group of based gauge transformations; topologically, the space $|\mathcal{H}|$ amounts to the union of the classifying spaces for these groups, one such space for each topological type. Here the universal G-bundle $EG \to BG$ is understood endowed with the universal connection, as exploited by SHULMAN in his thesis [45], see also our follow up paper [24], and hence a based "smooth" map from Y to BG determines a based gauge equivalence class of connections on its induced bundle. For a k-sphere, the space of based maps from a $(k-1)$-sphere to G has already been taken as a model for the space of based gauge equivalence classes of G-connections on the k-sphere at various places in the literature, cf. e. g. [3] (2.3). Our construction offers a generalization thereof, to arbitrary (finite) CW-complexes Y, where it yields a kind of gauge theory on Y. A space similar to $|\mathcal{H}|$ has been studied in [19].

Why do we resort to the space $|\mathcal{H}|$ at all? Apart from the lack of good comparison between $\mathrm{Hom}(\Omega Y, G)$ and $\mathrm{Map}^o(Y, BG)$, for our purposes, there is quite a different reason: The object $\mathcal{H} = \mathrm{Hom}(K, G)$ is a smooth cosimplicial manifold which is *finite dimensional* in each degree; we do not see how this could be manufactured directly from ΩY. This finite dimensionality of $\mathcal{H}$ in each degree will be crucial in [24]: in that paper we shall carry out a purely finite dimensional construction of the generators of the real cohomology of $\mathrm{Map}^o(Y, BG)$ and hence of the generators of the real cohomology of the offspring moduli spaces from which for example

Donaldson polynomials are obtained by evaluation against suitable fundamental classes corresponding to moduli spaces of ASD connections. Another application in [24] will be a purely finite dimensional construction of the Chern-Simons function for an arbitrary 3-manifold. This answers a question raised by ATIYAH in [4] where he comments on a possible combinatorial approach to the path integral quantization of the Chern-Simons function. In fact, our paper [24] may be viewed as a step towards a combinatorial construction of "topological field theories". Perhaps a suitable quantization thereof then yields 3-manifold invariants of the Witten-Reshetikhin-Turaev kind, cf. e. g. [34]. This would provide a rigorous construction of Witten's topological quantum field theory [51]. Our paper [24] generalizes prior constructions in [33] and [48] and, furthermore, the subsequent extensions thereof in [21], [22], [25], [27], [28]; in fact, it yields the "grand unified theory" for a general bundle on an *arbitrary* compact smooth finite dimensional manifold searched for by A. Weinstein [48] and established by L. Jeffrey [28] for the special case of a trivial bundle over a closed surface Y.

Trying to generalize the extended moduli spaces constructed in [21], [22], [25], [27] to arbitrary bundles over arbitrary smooth manifolds, we discovered that these extended moduli spaces may be found as suitable subspaces of the realization of the requisite cosimplicial manifolds; see Section 1 below for details. This suggests that the searched for general extended moduli spaces should be found within the realizations of cosimplicial manifolds of the aformentioned kind. We illustrate this in Sections 2 – 4 below.

In a sense, the extended moduli space constructions carried out in the cited references rely on the fact that a closed topological surface different from the 2-sphere has a combinatorial model which can entirely be described in terms of the fundamental group since such a surface is an Eilenberg-Mac Lane space. Now for a bundle on an *arbitrary* space Y, such a naive approach will fail when Y is *not* an Eilenberg-Mac Lane space. Our *principal innovation* is to take as combinatorial model for Y the *simplicial nerve* (or bar construction) of the Kan group K on Y. This idea is behind the constructions of the present paper, and the structure of the simplicial nerve of K will explicitly be exploited in our follow up paper [24]. In a sense, the cosimplicial manifold of homomorphisms from K to the structure group G generalizes the usual description of based gauge equivalence classes of flat connections in terms of their holonomies to arbitrary connections. This statement can be made much more precise: The geometric realization $|K|$ of K is a topological group and the geometric realization of the cosimplicial manifold $\mathcal{H}$ of homomorphisms from K to G amounts to the space of continuous homomorphisms from $|K|$ to G. In the context of smooth bundles this may look a bit odd at first and it seems difficult to view $|K|$ as a Lie group but a replacement for a missing space of smooth maps from $|K|$ to G is provided by what we call the *smooth* geometric realization $|\mathcal{H}|_{\mathrm{smooth}}$ of the cosimplicial manifold $\mathcal{H}$, cf. Section 1 below; it is (weakly) homotopy equivalent to $|\mathcal{H}|$ and may be viewed as a model for the space of based gauge equivalence classes of connections. The lack of decent smooth structure on $|K|$ is not a problem of infinite dimensions; artificially, $|K|$ can be endowed with a kind of smooth structure by adjointness but for our problem of study there is no need to do so and we do not know what kind of insight such a smooth structure on $|K|$ would provide. Our ultimate hope is that framed moduli spaces for various situations may be found within spaces of the kind $|\mathcal{H}|_{\text{smooth}}$.

For the case of a bundle on a closed surface Σ, the present more general construction involving a model for the full loop space rather than merely a presentation of the fundamental group of the surface [21], [22], [25], [27], [28] already goes beyond the earlier extended moduli space constructions: The realization $|\mathcal{H}|$ of $\mathcal{H} = \mathrm{Hom}(K\Sigma, G)$ contains the spaces of based gauge equivalence classes of *all* central Yang-Mills connections [2], not just those which correspond to the absolute minimum or, equivalently, to projective representations of the fundamental group π of Σ, and hence the space $|\mathcal{H}|$ comes with a kind of Harder-Narasimhan filtration. The latter cannot be obtained from the earlier extended moduli space constructions. See Section 2 below for details. Perhaps information about the multiplicative structure of the cohomology of moduli spaces can be derived from the resulting models in [24].

By means of the simplicial groupoid constructed in [17] for an arbitrary connected simplicial set the present approach can be extended to arbitrary connected simplicial complexes, in particular, to triangulated smooth manifolds. In the non-abelian cohomology spirit, this will amount to a *simplicial* gauge theory. It may be viewed as a rigorous mathematics approach to lattice gauge theory. In fact, the above cosimplicial manifold $\mathcal{H} = \mathrm{Hom}(K, G)$ may be viewed as a space of *parallel transport functions*, cf. e. g. [40] for this notion. More naively, given an ordered simplicial complex, viewed as a simplicial set, contracting a maximal tree yields a simplicial set with a single vertex, and the construction of Kan group can be applied. To keep the present paper to size, we plan to give the details elsewhere. See also the remark at the end of Section 1. Our approach somewhat establishes a link between classical algebraic topology and the more recent gauge theory developments in low dimensional topology: our models for the space of gauge equivalence classes of connections involve classical low dimensional topology notions such as *identity among relations* (Section 3 below) and *universal quadratic group* (Section 4 below). We intend to describe a corresponding rigorous quantum lattice gauge theory elsewhere; see e.g. [12] for renewed interest on this topic in the physics literature.

In a recent paper by Caetano and Picken [13], a certain topological group with a kind of smooth structure has been introduced which serves as a model for the based loop space, and one can then study the space of homomorphisms into the structure group G; this space is weakly homotopy equivalent to the space denoted above by $|\mathcal{H}|$. Our approach in terms of the Kan group has the advantages of being purely finite dimensional and hence being directly related with lattice gauge theory, thereby avoiding any of the technical details of smooth structures in infinite dimensions.

Any unexplained notation is the same as that in our paper [21]. Details about cosimplicial spaces may be found in [8] and [10]. All spaces are assumed to be compactly generated, that is to say, a set that meets every compact set in a closed set is closed.

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1. The finite model

Write Δ for the category of finite ordered sets $[q] = (0, 1, \dots, q)$, $q \geq 0$, and monotone maps. We recall the standard *coface* and *codegeneracy* operators

$$\varepsilon^j \colon [q-1] \to [q], \quad (0,1,\dots,j-1,j,\dots,q-1) \mapsto (0,1,\dots,j-1,j+1,\dots,q),$$
$$\eta^j \colon [q+1] \to [q], \quad (0,1,\dots,j-1,j,\dots,q+1) \mapsto (0,1,\dots,j,j,\dots,q),$$

respectively. As usual, for a simplicial object, the corresponding face and degeneracy operators will be written d_j and s_j. Recall that a *cosimplicial* object in a category $\mathcal{C}$ is a covariant functor from Δ to $\mathcal{C}$. For example, the assignment to $[q]$ of the standard simplex $\nabla[q] = \Delta_q$ yields a cosimplicial space ∇; here we wish to distinguish clearly in notation between the cosimplicial space ∇ and the category Δ. Let K be a free simplicial groupoid, for example a free simplicial group. The simplicial structure of K induces a structure of *cosimplicial* manifold on the groupoid homomorphisms $\mathrm{Hom}(K, G)$ from K to G; here G is viewed as a groupoid with a single object. For $q \geq 0$, we shall henceforth write $\mathrm{H}_q = \mathrm{Hom}(K_q, G)$ so that $\mathcal{H} = \mathrm{Hom}(K, G)$ may be depicted as $\{\mathrm{H}_0, \mathrm{H}_1, \dots, \mathrm{H}_q, \dots\}$ with the requisite smooth maps between the constituents induced by monotone maps between finite sets.

The *geometric realization* $|\mathcal{H}|$ of $\mathcal{H}$, cf. [8], [10], is the space $|\mathcal{H}| = \mathrm{Hom}_\Delta(\nabla, \mathcal{H})$; this is the subspace of the infinite product

$$\mathrm{H}_0 \times \mathrm{Map}(\Delta_1, \mathrm{H}_1) \times \cdots \times \mathrm{Map}(\Delta_q, \mathrm{H}_q) \times \dots \tag{1.1}$$

consisting of all sequences $(\phi_0, \phi_1, \dots, \phi_q, \dots)$ having the property that, for each monotone map $\theta \colon [i] \to [j]$, the diagram

$$\begin{array}{ccc} \Delta_i & \xrightarrow{\theta_*} & \Delta_j \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_j} \\ \mathrm{H}_i & \xrightarrow{\theta_*} & \mathrm{H}_j \end{array} \tag{1.2}$$

commutes.

When K is countable the geometric realization $|K|$ of K is a topological groupoid, cf. e. g. [39] where this is proved for simplicial groups. In general, one has to take compactly generated refinements of the product topologies on the spaces where compositions are defined. Henceforth we suppose K countable. Then the cosimplicial manifold $\mathcal{H} = \mathrm{Hom}(K, G)$ provides a model of the space $\mathrm{Hom}(|K|, G)$ of continuous homomorphisms from $|K|$ to G. In fact, for $q \geq 0$, adjointness yields a canonical map from $\mathrm{Map}(\Delta_q, \mathrm{Hom}(K_q, G))$ to $\mathrm{Map}(K_q \times \Delta_q, G)$, by construction, the space $\mathrm{Hom}(|K|, G)$ canonically embeds into the infinite product of the spaces $\mathrm{Map}(K_q \times \Delta_q, G)$, and we have the following tautology:

Proposition 1.3. *Adjointness induces a homeomorphism between* $\mathrm{Hom}(|K|, G)$ *and* $|\mathrm{Hom}(K, G)|$.

More formally, the geometric realization $|K|$ is the *coend* $K \otimes_\Delta \nabla$, cf. e. g. [36], and we have an adjointness

$$|\mathrm{Hom}(K, G)| = \mathrm{Hom}_\Delta(\nabla, \mathrm{Hom}(K, G)) \to \mathrm{Hom}(K \otimes_\Delta \nabla, G) = \mathrm{Hom}(|K|, G).$$

Henceforth we shall exclusively deal with free simplicial groups. We recall [30] that a graded set $X = \{X_0, X_1, \dots\}$, where $X_q \subseteq K_q$, for $q \geq 0$, is called a set of *(free) generators* for K provided K is freely generated by X as a simplicial group. That is to say:

(1) If $q \geq 1$ and $0 \leq j < q$ then $\partial_j x = e_{q-1}$, the neutral element, for every $x \in X_q$.

(2) For each q, the set X_q together with all the degeneracies $s_u s_v \dots s_w x \in K_q$, for x in some X_r, freely generates K_q (as a free group).

A set X of free generators together with all its degeneracies is then called a CW-*basis* for K, and for every $q \leq 1$ and every $x \in X_q$, the value $\partial_q x \in K_{q-1}$ is called the *attaching element* of x.

REMARK. Here we give preferred treatment to the *last* face operator, as is done in [29], [30]. This turns out to be the appropriate thing to do for principal bundles with structure group acting on the *right* of the total space.

It is proved in [30 (2.2)] that every free simplicial group has a CW-basis. By means of a CW-basis, the geometric realization $|\mathcal{H}|$ of $\mathcal{H}$ may be realized within a space smaller than (1.1) above. In ANDERSON'S terminology [1], the cosimplicial space $\mathcal{H}$ is *primitive* over the projection maps p_q from $\mathrm{H}_q = \mathrm{Hom}(K_q, G)$ to $P_q = G^{X_q}$; this means that, if α runs over the $\binom{q}{k}$ surjections from Δ_q to Δ_k for $k < q$, the product of p_q and the $p_k\mathcal{H}(\alpha)$ provides a homeomorphism

$$\mathrm{H}_q \to P_0 \times P_1^{\binom{q}{1}} \times \dots \times P_{q-1}^{\binom{q}{q-1}} \times P_q.$$

Given $(\phi_0, \phi_1, \dots, \phi_q, \dots)$ in $|\mathcal{H}|$, for $q \geq 0$, write $\psi_q \colon \Delta_q \to P_q$ for the composite of ϕ_q with the projection from H_q onto P_q. For $q \geq 1$, the "last coface map" ε^q from $[q-1]$ to $[q]$ induces the affine map from Δ_{q-1} to Δ_q which identifies Δ_{q-1} with the last face of Δ_q, that is, with the face opposite the last vertex. We now consider the product

$$G^{X_0} \times \mathrm{Map}(\Delta_1, G^{X_1}) \times \dots \times \mathrm{Map}(\Delta_q, G^{X_q}) \times \dots \quad . \tag{1.4}$$

It is finite when Y is compact. Henceforth we write $G^{X_q} = e$ when X_q is empty.

Lemma 1.5. *The assignment to $(\phi_0, \phi_1, \dots, \phi_q, \dots)$ of $(\psi_0, \psi_1, \dots, \psi_q, \dots)$ induces a homeomorphism from $|\mathcal{H}|$ onto the subspace $|\mathcal{H}|'$ of* (1.4) *consisting of all sequences $(\psi_0, \psi_1, \dots, \psi_q, \dots)$ of maps ψ_q whose restriction to all but the last faces of Δ_q has constant value $e \in G^{X_q}$ and which satisfy the recursive requirement that, for each q, the diagram*

$$\begin{array}{ccc} \Delta_{q-1} & \xrightarrow{(\varepsilon^q)_*} & \Delta_q \\ {\scriptstyle\phi_{q-1}}\downarrow & & \downarrow{\scriptstyle\psi_q} \\ \mathrm{H}_q & \xrightarrow{(\varepsilon^q)_*} & G^{X_q} \end{array}$$

commute.

Proof. For $k < q$, each (affine) surjection from Δ_q to Δ_k induces a continuous map from $\mathrm{Map}(\Delta_k, P_k)$ to $\mathrm{Map}(\Delta_q, P_k)$ and these assemble to a continuous map from

$\mathrm{Map}(\Delta_k, P_k)$ to $\mathrm{Map}(\Delta_q, P_k^{\binom{q}{k}})$. These maps, in turn, assemble to a continuous map from $|\mathcal{H}|'$ into (1.1) which yields a continuous inverse of the map from $|\mathcal{H}|$ to $|\mathcal{H}|'$. □

Following [30] we shall say that a CW-complex Y is *reduced* provided it has a single 0-cell and, for every $(q+1)$-cell c, the characteristic map σ_c from Δ_{q+1} to Y has values different from the base point at most on the next to the last face, that is, on the one opposite to the vertex A_q where the vertices of Δ_{q+1} are numbered $A_0, \dots, A_{q+1}$. We note that it is uncommon to have a CW-complex with cells which are images of simplices but the present description is an important ingredient for KAN'S results which we shall subsequently use. A twisting function t from the first Eilenberg subcomplex SY of the total singular complex of Y to a simplicial group K is said to be *regular* provided (i) the elements $t(\sigma_c)$ where c runs through the cells of Y of dimension at least one form the generators of a CW-basis of K and (ii) for every subcomplex Z of Y, the image $t(SZ)$ of its first Eilenberg subcomplex SZ is contained in the simplicial subgroup of K generated by the $t(\sigma_c)$ for c in Z. To any reduced CW-complex Y, Kan's construction [30] assigns a free simplicial group KY together with a regular twisting function t from SY to KY [30] and, furthermore, to any free simplicial group K, the reverse construction of Kan's assigns a reduced CW-complex YK together with a regular twisting function t from SYK to K, so that $YKY \cong Y$ and $KYK \cong K$. For each $(q+1)$-cell c with characteristic map σ_c, since $d_j\sigma_c$ is the base point when $j \neq q$, the twisting function t satisfies

$$\begin{aligned} d_i(t\sigma_c) &= t(d_i\sigma_c) = e, \quad 0 \le i < q, \\ d_q(t\sigma_c) = t(d_q\sigma_c)t(d_{q+1}\sigma_c)^{-1} &= t(d_q\sigma_c) \in K_{q-1}, \\ s_i(t\sigma_c) &= t(s_i\sigma_c), \qquad 0 \le i \le q, \\ e_{q+1} &= t(s_{q+1}\sigma_c). \end{aligned}$$

See [30] for details. The cosimplicial structure of $\mathcal{H}$ may now be described as follows: For each $(q+1)$-cell c, with characteristic map σ_c, write G_c for the factor of $\mathrm{H}_q = \mathrm{Hom}(K_q, G)$ which corresponds to the free generator $t(\sigma_c)$ of K_q. For $0 \le j < q$, the composite of the coface map ε^j from H_{q-1} to H_q with the projection onto G_c is trivial while the composite

$$\mathrm{H}_{q-1} = \mathrm{Hom}(K_{q-1}, G) \to G_c$$

of the coface map ε^q from H_{q-1} to H_q with the projection onto G_c is given by the assignment to $\alpha \in \mathrm{Hom}(K_{q-1}, G)$ of the value $\alpha(t(d_q(\sigma_c)))$. The rest of the structure is now completely determined by the requirement that $\mathcal{H}$ be a cosimplicial space.

The regularity of the twisting function t entails that the total complex of the associated simplicial principal bundle $\pi\colon SY \times_t KY \to SY$ is contractible whence KY is a *loop complex of* SY under t. We now explain what this means for us: The geometric realization of π is a principal $|K|$-bundle with base $|SY|$. Pick a homotopy inverse σ from Y to $|SY|$ of the counit $\varepsilon\colon |SY| \to Y$ of the adjointness between the realization and singular complex functors. When Y is itself the realization of a (reduced) simplicial set there is a canonical such map σ. Whether or not this happens to be the case, σ induces a principal $|K|$-bundle $\kappa: P \to Y$ on Y with contractible total space P. In particular, a standard homotopy theory construction yields a homomorphism from the (Moore) loop space ΩY to the geometric realization $|KY|$ which is a homotopy equivalence. It is in this sense that $|KY|$ is a model for the loop space of Y.

Recall that, for an arbitrary topological group H, the usual *lean* realization $BH = |NH|$ of its *nerve* NH [7], [9], [43] is a classifying space for H, cf. [37], [43], [46]; there is an analoguous construction of contractible total space EH together with a free H-action and projection map onto BH, and this map is locally trivial provided (H, e) is a NDR (neighborhood deformation retract) [47]. Below (H, e) will always be a CW-pair and hence a NDR, cf. e. g. the discussion in the appendix to [44], and we shall exclusively deal with the lean realization $BH = |NH|$.

The twisting function t from SY to K determines a morphism $\bar{t}: SY \to \overline{W}K$ of simplicial sets where $\overline{W}K$ refers to the reduced W-construction [38]. Its realization $|\bar{t}|: |SY| \to |\overline{W}K|$, combined with the chosen map σ from Y to $|SY|$, yields a map ρ from Y to $|\overline{W}K|$. In [6], a canonical homeomorphism between $|\overline{W}K|$ and $B|K|$ has been constructed which is natural in K. By means of it, we identify henceforth $|\overline{W}K|$ and $B|K|$. With these preparations out of the way, the assignment to $\phi \in |\mathrm{Hom}(K, G)| \cong \mathrm{Hom}(|K|, G)$ of the composite $(B\phi)\rho$ yields a G-equivariant map

$$\Phi: |\mathrm{Hom}(K, G)| \to \mathrm{Map}^o(Y, BG) \tag{1.6}$$

where G acts on BG by conjugation. By construction, this map assigns to ϕ a classifying map of the principal G-bundle on Y arising from the principal $|K|$-bundle κ via ϕ. Notice when G is discrete, the space $|\mathrm{Hom}(K, G)|$ boils down to the discrete space $\mathrm{Hom}(\pi_1(Y), G)$ and (1.6) picks the connected components of $\mathrm{Map}^o(Y, BG)$ each of which is contractible.

In general, G-bundles over a classifying space BH of an arbitrary topological group H are *not* classified by representations of H in G. Thus the next result is somewhat surprising and indicates that the realization $|K|$ of the Kan group K has certain special features.

Theorem 1.7. *The map Φ is a weak G-equivariant homotopy equivalence.*

The *only* possible choice the map Φ relies on is that of σ and, as already pointed out, when Y is the realization of a reduced simplicial set, there is a canonical such choice. For example, Y could arise from an ordered simplicial complex by contraction of a maximal tree. We do not pursue these issues here.

We now begin with the preparations for the proof of Theorem 1.7. We shall see the theorem comes down to the canonical map from G to ΩBG. Let $q \geq 1$, and consider the inclusion of the $(q-1)$-skeleton Y^{q-1} into the q-skeleton Y^q. This is a cofibration with cofibre a one point union $\vee S^q$ of as many q-spheres as Y has q-cells.

Lemma 1.8. *The inclusion of the $(q-1)$-skeleton into the q-skeleton induces a Hurewicz fibration*

$$|\mathrm{Hom}(K(\vee S^q), G)| \to |\mathrm{Hom}(KY^q, G)| \to |\mathrm{Hom}(KY^{q-1}, G)| \tag{1.8.1}$$

for the geometric realizations.

For a one-point union $\vee T_j$, the Kan group $K(\vee T_j)$ amounts to the free product $*KT_j$ of the simplicial groups KT_j. When Y^{q-1} is just the base point, the assertion thus amounts to a homeomorphism between $|\mathrm{Hom}(K(\vee_{X_q} S^q), G)| \cong |\mathrm{Hom}(*_{X_q} KS^q, G)|$ and $\times_{X_q} |\mathrm{Hom}(KS^q, G)|$, and there is nothing to prove.

Proof. Let $q \geq 2$, and suppose Y^{q-1} is more than the base point. Consider the cofibration $S^{q-1} \to B^q \to S^q$ of regular CW-complexes, the spheres S^{q-1} and S^q having obvious such CW-decompositions with two cells. Inspection shows that (1.8.1) then boils down to the standard Hurewicz fibration

$$\mathrm{Map}^o(S^{q-1}, G) \to \mathrm{Map}^o(B^{q-1}, G) \to \mathrm{Map}^o(S^{q-2}, G)$$

with contractible total space. In general, the fibration (1.8.1) is induced via the attaching maps for the q-cells of Y from the product of such fibrations involving as many copies as Y has q-cells, as indicated in the commutative diagram

$$\begin{array}{ccc}
|\mathrm{Hom}(K(\vee S^q), G)| & \xrightarrow{\cong} & \times_{X_{q-1}} \mathrm{Map}^o(S^{q-1}, G) \\
\downarrow & & \downarrow \\
|\mathrm{Hom}(KY^q, G)| & \longrightarrow & \times_{X_{q-1}} \mathrm{Map}^o(B^{q-1}, G) \\
\downarrow & & \downarrow \\
|\mathrm{Hom}(KY^{q-1}, G)| & \longrightarrow & \times_{X_{q-1}} \mathrm{Map}^o(S^{q-2}, G)
\end{array} \tag{1.8.2}$$

whose bottom map is induced by the attaching maps of the q-cells of Y. □

Proof of (1.7). The map Φ is compatible with the CW-structures and hence induces, for $n \geq 1$, a commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}^o(\vee S^{n+1}, BG) & \longleftarrow & |\mathrm{Hom}(K(\vee S^{n+1}), G)| \\
\downarrow & & \downarrow \\
\mathrm{Map}^o(Y^{n+1}, BG) & \xleftarrow{\Phi^{n+1}} & |\mathrm{Hom}(KY^{n+1}, G)| \\
\downarrow & & \downarrow \\
\mathrm{Map}^o(Y^n, BG) & \xleftarrow{\Phi^n} & |\mathrm{Hom}(KY^n, G)|
\end{array}$$

of fibrations. Since for a one-point union the Kan group equals the free product of the Kan groups for the factors, in degree one, the map Φ^1 amounts to a product of copies of maps of the kind $G \to \Omega BG$, the number of factors being given by the number of 1-cells of Y. Likewise, on the fibres, the map comes down to a product of copies of maps of the kind

$$|\mathrm{Hom}(KS^{n+1}, G)| \to \mathrm{Map}^o(S^{n+1}, BG),$$

the number of factors being given by the number of $(n+1)$-cells of Y. However, $\mathrm{Map}^o(S^{n+1}, BG)$ equals $\mathrm{Map}^o(SS^n, BG)$ where 'S' refers to the based suspension operator, adjointness identifies $\mathrm{Map}^o(SS^n, BG)$ with $\mathrm{Map}^o(S^n, \Omega BG)$, and again we are left with a standard homotopy equivalence

$$|\mathrm{Hom}(KS^{n+1}, G)| = \mathrm{Map}^o(S^n, G) \to \mathrm{Map}^o(S^n, \Omega BG).$$

By induction we can therefore conclude that Φ is a weak homotopy equivalence. This proves the assertion. □

As usual, we shall say that a map from Δ_n to a smooth manifold M is *smooth* when it is defined and smooth on a neighborhood of Δ_n in the ambient space. Henceforth we write $\mathrm{Smooth}(\cdot,\cdot)$ for spaces of *smooth maps*. We define the promised *smooth realization* by

$$|\mathcal{H}|_{\mathrm{smooth}} = |\mathcal{H}| \cap G^{X_0} \times \mathrm{Smooth}(\Delta_1, G^{X_1}) \times \cdots \times \mathrm{Smooth}(\Delta_q, G^{X_q}) \times \ldots .$$

It is weakly homotopy equivalent to $|\mathcal{H}|$ and may be viewed as a model for the space of based gauge equivalence classes of connections on Y when the latter is a smooth manifold.

Finally we explain briefly the notion of attaching element: We recall [14] that the homotopy groups of K may be described as the homology groups of the MOORE complex

$$MK\colon M_0 \xleftarrow{d_1} M_1 \xleftarrow{d_2} M_2 \xleftarrow{d_3} \ldots$$

of K. Here, for $k \geq 1$, the group M_k is the intersection $\cap_{j=0}^{k-1}\ker(d_j)$ and the operator d_k in the Moore complex is the restriction of the last face operator (denoted by the same symbol). It is also customary in the literature to take the intersection of the kernels of the last face operators and to take the first face operator as boundary in the Moore complex. Let $t(\sigma_c) \in X_q$ be a free generator corresponding to a $(q+1)$-cell c of Y, attached via the map σ_c, restricted to the boundary of Δ_{q+1}; the latter represents an element of $\pi_q(Y^q)$ which, under the standard isomorphisms between $\pi_q(Y^q)$, $\pi_{q-1}(\Omega Y^q)$, and $\pi_{q-1}(KY^q)$, passes to the class in $\pi_{q-1}(KY^q)$ represented by the value $\partial_q x \in K_{q-1}$ of the *attaching element* $t(\sigma_c)$.

REMARK. In [17], the relationship between reduced simplicial sets and simplicial groups has been extended to one between connected simplicial sets and simplicial groupoids. By means of it, we intend to generalize elsewhere the above constructions to arbitrary simplicial complexes and in particular to triangulated smooth manifolds. This will enable us to remove the seemingly fuzzy notion of reduced CW-complex which is somewhat unnatural for smooth manifolds. More naively, cf. [31], given an ordered simplicial complex Y, viewed as a simplicial set, contracting a maximal tree T yields a simplicial set Y/T, and the Kan construction applied to it yields a free simplicial group K and a twisting function $\widetilde{t}$ from Y/T to K so that K is a loop complex for Y/T; composing with the projection from Y to Y/T we obtain a twisting function t from Y to K so that K is a loop complex for Y, that is, the resulting simplicial principal bundle $Y \times_t K \to Y$ has contractible total space. A similar theory can be made for K with the modification that the simplices of Y will not constitute a CW-basis of K. It remains to be seen which approach is the most suitable one for what kind of problem.

2. Closed surfaces

Let Σ be a closed topological surface of genus $\ell \geq 0$, endowed with the usual CW-decomposition with a single 0-cell o, with 1-cells $u_1, v_1, \ldots, u_\ell, v_\ell$, and with a single 2-cell c. We suppose the decomposition regular in the above sense. For $1 \leq j \leq \ell$, write x_j and y_j for the based homotopy class of u_j and v_j respectively, and denote by r the based homotopy class of the attaching map for c. Then

$$\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle$$

is a presentation for the fundamental group π of Σ. We suppose things have been arranged in such a way that $r = \Pi[x_j, y_j]$ in the free group F on the generators. When the genus is zero, $\mathcal{P}$ is to be interpreted as a non-trivial presentation of the trivial group, with F the trivial group. The Kan group $K = K\Sigma$ for Σ is the free simplicial group with $K_0 = F$, with K_1 the free group on $2\ell + 1$ generators $r, s_0(x_1), s_0(y_1), \ldots, s_0(x_\ell), s_0(y_\ell)$ where only r is non-degenerate and, for $q \geq 2$, K_q is the free group on the $(2\ell + q)$ degenerate generators

$$s_q s_{q-1} \ldots s_0(x_j), \quad s_q s_{q-1} \ldots s_0(y_j), \quad s_{j_q} s_{j_{q-1}} \ldots s_{j_1} r, \quad q \geq j_q > j_{q-1} > \cdots > j_1 \geq 0.$$

Moreover, the *only* face operators which are *not* determined by the simplicial identities are

$$d_0(r) = e, \quad d_1(r) = \Pi[x_j, y_j],$$

and the degeneracy operators are completely determined by the construction itself. In particular, for genus $\ell \geq 1$, the *Moore* complex of K has zero'th homology group $\pi_1(\Sigma)$ and is exact in higher dimensions.

The relator r induces a smooth map from $G^{2\ell}$ to G in the usual way, where $G^{2\ell}$ is interpreted to be a single point when ℓ is zero; abusing notation, we denote this map by r as well. The geometric realization $|\mathcal{H}|$ of the resulting cosimplicial space $\mathcal{H} = \mathrm{Hom}(K, G)$ is the *fibre* of r. In fact, the diagram (1.8.2), with $q = 2$, now boils down to

$$\begin{array}{ccc} |\mathrm{Hom}(K(S^2), G)| & \xrightarrow{\cong} & \Omega G \\ \downarrow & & \downarrow \\ |\mathrm{Hom}(K\Sigma, G)| & \longrightarrow & \mathrm{Map}^o(B^1, G) \\ \downarrow & & \downarrow \\ G^{2\ell} & \xrightarrow{r} & G \end{array} \tag{2.1}$$

where ΩG refers to the space $\mathrm{Map}^o(S^1, G)$ of based loops as usual and B^1 to the closed interval. In particular, when the genus ℓ is zero, $|\mathrm{Hom}(K\Sigma, G)|$ amounts to ΩG. This illustrates once more the well known relationship between moduli spaces over a complex curve and the loop group, cf. [41].

The topological type of the corresponding bundles, that is, the connected components of the realization $|\mathcal{H}|$, may be described as follows: We take the description of $|\mathcal{H}|$ as the fibre of r, that is to say, $|\mathcal{H}|$ is now the space of pairs (w, ϕ) where $w \in G^{2\ell}$ and $\phi: I \to G$ is a path in G from e to $r(w)$. Given a point $(w,\phi) = (w_1, w_2, \dots, w_{2\ell-1}, w_{2\ell}, \phi)$ of $|\mathcal{H}|$, pick paths u_j in G from e to w_j and let $\psi: I \to G$ be the path in G from e to $r(w)$ given by $\psi(t) = [u_1(t), u_2(t)] \dots [u_{2\ell-1}(t), u_{2\ell}(t)]$. Then the composite $\psi^{-1} + \phi$ is a closed path in G from e to e; its class in $\pi_1(G)$ represent the topological type or connected component of $|\mathcal{H}|$ in which (w, ϕ) lies.

We now show how the based gauge equivalence classes of the critical sets of the Yang-Mills functional for the gauge theory over Σ with reference to the group G [2] can be found in the space $|\mathcal{H}|$: Suppose at first that Σ is not a 2-sphere. Write $\pi = \pi_1(\Sigma)$ and view π as a simplicial group $\{\pi_q\}$ with $\pi_q = \pi$ for each q and all face and degeneracy operators the identity map. The canonical projection of simplicial groups from $K = K\Sigma$ to π has kernel the Kan group KY where Y arises from the universal covering $\widetilde{\Sigma}$ of Σ by contraction of a maximal tree to a point, and there results an extension

$$1 \to KY \to K \to \pi \to 1$$

of simplicial groups. Their realizations yield the extension

$$1 \to |KY| \to |K| \to \pi \to 1$$

of topological groups and the group $\pi_0|KY|$ of connected components of $|KY|$ may be identified with the kernel N of the projection from $F = K_0$ to π; notice $\pi_0|KY|$ amounts to the fundamental group of the 1-skeleton of $\widetilde{\Sigma}$. Dividing out $[N, F]$ we obtain the groups $N/[N,F] \cong \mathbf{Z}$ and $\Gamma = N/[N,F]$ which yield the universal central extension

$$1 \to \mathbf{Z} \to \Gamma \to \pi \to 1$$

of π, the central copy $\mathbf{Z}$ being generated by $r[N,F]$. The injection of $\mathbf{Z}$ into the reals $\mathbf{R}$ then induces the central extension

$$1 \to \mathbf{R} \to \Gamma_{\mathbf{R}} \to \pi \to 1$$

of π. The projection of $|KY|$ onto its group $\pi_0|KY|$ of connected components, combined with the projection onto $\mathbf{Z} \cong N/[F,N]$, extends to a continuous homomorphism ϑ from $|KY|$ to $\mathbf{R}$. In fact, adjointness yields a bijection

$$\mathrm{Hom}(KY, S\mathbf{R}) \to \mathrm{Hom}(|KY|, \mathbf{R})$$

where $S\mathbf{R}$ is the singular complex of $\mathbf{R}$, viewed as a simplicial group, and it is straightforward to extend the assignment of 1 to r to a morphism of simplicial groups from KY to $S\mathbf{R}$. The homomorphism ϑ, in turn, induces a continuous surjective homomorphism Θ from $|K|$ to $\Gamma_{\mathbf{R}}$ as indicated in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & |KY| & \longrightarrow & |K| & \longrightarrow & \pi & \longrightarrow & 1 \\ & & \vartheta\big\downarrow & & \Theta\big\downarrow & & \mathrm{Id}\big\downarrow & & \\ 1 & \longrightarrow & \mathbf{R} & \longrightarrow & \Gamma_{\mathbf{R}} & \longrightarrow & \pi & \longrightarrow & 1 \end{array}$$

of extensions of topological groups. The homomorphism Θ induces an injection

$$\mathrm{Hom}(\Gamma_{\mathbf{R}}, G) \to \mathrm{Hom}(|K|, G) = |\mathcal{H}|.$$

The space $\mathrm{Hom}(\Gamma_{\mathbf{R}}, G)$ is well known to be that of based gauge equivalence classes of the critical sets of the Yang-Mills functional (for all topological types of bundles) [2]. Formally, the subspace $\mathrm{Hom}(\Gamma_{\mathbf{R}}, G)$ of $|\mathcal{H}|$ decomposes the latter into G-equivariant "Morse strata"; in fact, it yields a kind of Harder-Narasimhan filtration of $|\mathcal{H}|$, and the resulting decomposition of the latter is a kind of generalized Birkhoff decomposition, cf. [41] and what is said below. There is even an obvious candidate for a Morse function arising from the energy of the paths in $\mathrm{Smooth}^o(B^1, G) \subseteq \mathrm{Map}^o(B^1, G)$, cf. (2.1); note that B^1 is just the unit interval, and we run into a certain variational problem with additional boundary constraints coming from the word map r. Details have not been worked out yet.

We now explain briefly how under the present circumstances a homotopy inverse of the map Φ, cf. (1.6) above, may be obtained. Given a smooth principal bundle ξ on Σ, the holonomy yields a smooth map from the space $\mathcal{A}(\xi)$ of connections to $\mathrm{Hom}(|K|, G) = |\mathcal{H}|$ which, after a suitable choice of ϑ has been made, restricts to a map from the space of Yang-Mills connections to $\mathrm{Hom}(\Gamma_{\mathbf{R}}, G)$. More precisely, with the present conventions, the surface Σ is obtained from a 2-simplex Δ_2 with vertices A_0, A_1, A_2 in such a way that its characteristic map σ sends the faces (A_0, A_1) and (A_1, A_2) to the 0-cell and the face (A_0, A_2) to the boundary path $\Pi[u_j, v_j]$. For each point p of the first face (A_1, A_2), let w_p be the linear path in Δ_2 joining the vertex A_0 with p. The assignment to a connection A on ξ of the holonomies of the closed paths $\sigma \circ u_j$ and $\sigma \circ v_j$ yields a smooth map from the space $\mathcal{A}(\xi)$ of connections to $G^{2\ell}$, and the assignment to A of the holonomies of the closed paths $\sigma \circ w_p$ yields a lift of this map to the space $|\mathcal{H}|_{\mathrm{smooth}}$ which is smooth in a suitable sense. Assembling these maps over all topological types of bundles we obtain in fact a homotopy inverse of the above map Φ. The existence of this map is due to the fact that we are working over a topological surface where the combinatorics of the situation is simple. Finer combinatorial tools will perhaps yield a homotopy inverse of Φ in general.

When Σ is the 2-sphere, in view of the identification of $\mathrm{Hom}(|K|, G)$ with $\mathrm{Map}^o(S^1, G) = \Omega G$, with $G = S^1$, we see there is a surjective homomorphism from $|K|$ to S^1 which classifies the universal cover of $|K|$. For general G, this surjection induces an embedding of $\mathrm{Hom}(S^1, G)$ into $\mathrm{Hom}(|K|, G) = \Omega G$. The space $\mathrm{Hom}(S^1, G)$ is well known to be that of based gauge equivalence classes of the critical sets of the Yang-Mills functional over the 2-sphere [2], cf. also [20], yielding the Birkhoff decomposition, cf. [41].

We conclude this Section with a topological remark. The fibrations (1.8.1) are known to be rationally trivial, at least for G simply connected. Some hints for the general case may be found in [15], and, for the present special case, where (1.8.1) boils down to the left-hand vertical fibration in (2.1), the rational triviality may be found in [2]. It also follows from Theorem 7.1 in [24]. However, over the integers, the fibration under discussion is in general certainly not trivial. We briefly explain this for Y a closed surface Σ: It is well known that the word map r from $G^{2\ell}$ to G map is not null homotopic unless G is abelian, cf. [26]. For example, for $G = \mathrm{SU}(2)$, the commutator map from $G \times G$ to G factors through $S^6 = S^3 \wedge S^3$ and generates $\pi_6(S^3)$ which is finite cyclic of order 12. In fact, this generator is the SAMELSON product $[a, a]$ where a refers to the generator of $\pi_3(S^3)$. A general word map r produces higher degree generators in the homotopy of G whence r will certainly not be null homotopic. In particular, with coefficients in a finite field, the spectral sequence of the fibration will in general be non-trivial.

3. 3-complexes and 3-manifolds

Let Y be a 3-complex with a single 3-cell, for example, a closed compact 3-manifold, endowed with a regular CW-decomposition with a single 0-cell o, with 1-cells $u_1, \ldots, u_\ell$, 2-cells $c_1, \ldots, c_\ell$, and a single 3-cell c. For $1 \leq j \leq \ell$, write x_j and r_j for the based homotopy classes of u_j and c_j, respectively, and denote by σ the based homotopy class of the attaching map for c. Then

$$\mathcal{S} = \langle x_1, \ldots, x_\ell; r_1, \ldots, r_\ell; \sigma \rangle$$

is a *spine* for Y; in particular, (i) the data $\mathcal{P} = \langle x_1, \ldots, x_\ell; r_1, \ldots, r_\ell \rangle$ constitute a presentation of the fundamental group π of Y so that the attaching maps of the 2-cells assign a word w_j in the free group F on the generators to each relator r_j, and (ii) the attaching map σ of the single 3-cell assigns an *identity among relations*

$$i = z_1 r_{j_1}^{\varepsilon_1} z_1^{-1} \ldots z_m r_{j_m}^{\varepsilon_m} z_m^{-1} \tag{3.1}$$

to c representing the element of the second homotopy group $\pi_2(Y^2)$ of the 2-skeleton Y^2 of Y which is killed by the 3-cell c; here each z_k is an element of F, and the meaning of "identity among relations" will be made clear below in terms of the structure of the Kan group. See [11] for more details on the notion of identity among relations.

To spell out the Kan group $K = KY$ for Y, we do not distinguish in notation between the values of the characteristic maps of the cells under the twisting function t from SY to K and the based homotopy classes in the spine $\mathcal{S}$ they correspond to. With these preparations out of the way, the group K is the free simplicial group with $K_0 = F$, with K_1 the free group on 2ℓ generators $r_1, \ldots, r_\ell, s_0(x_1), \ldots, s_0(x_\ell)$, the $r_1, \ldots, r_\ell$ being non-degenerate, with K_2 the free group on 3ℓ degenerate generators

$$s_0(r_1), \ldots, s_0(r_\ell), s_1(r_1), \ldots, s_1(r_\ell), s_1 s_0(x_1), \ldots, s_1 s_0(x_\ell)$$

together with a single non-degenerate generator σ, and, for $q \geq 3$, K_q is free on a certain number of degenerate generators. Moreover, the *only* face operators which are *not* determined by the simplicial identities are

$$\begin{aligned} &d_0(r_j) = e, \quad d_1(r_j) = w_j \in K_0, \quad d_0(\sigma) = e, \quad d_1(\sigma) = e, \\ &d_2(\sigma) = (s_0 z_1) r_{j_1}^{\varepsilon_1} (s_0 z_1)^{-1} \ldots (s_0 z_m) r_{j_m}^{\varepsilon_m} (s_0 z_m)^{-1} \in K_1, \quad \varepsilon_j = \pm 1, \end{aligned}$$

and the degeneracy operators are completely determined by the simplicial identities as well. We note that i to be an identity among relations means precisely that $d_1 d_2(\sigma) = e$ or, equivalently, that the word i in the generators of F arising from substituting each w_j for r_j reduces to the trivial element of F. In particular, the part $M_0 \xleftarrow{d_1} M_1 \xleftarrow{d_2} M_2$ of the *Moore* complex of K determines an exact sequence

$$1 \leftarrow \pi_1(Y) \leftarrow M_0 \leftarrow M_1/(d_2 M_2) \leftarrow \pi_2(Y) \leftarrow 1$$

which is just the part

$$1 \leftarrow \pi_1(Y) \leftarrow \pi_1(Y^1) \leftarrow \pi_2(Y, Y^1) \leftarrow \pi_2(Y) \leftarrow 1$$

of the long exact homotopy sequence of the pair (Y, Y^1). The resulting cosimplicial manifold $\mathcal{H} = \mathrm{Hom}(K, G)$ has $\mathrm{H}_0 = G^\ell$, $\mathrm{H}_1 = G^{2\ell}$, $\mathrm{H}_2 = G^{3\ell+1}$, etc., the coface and codegeneracy maps being determined by the simplicial structure of K spelled out above.

The ℓ-tuple $(r_1, \ldots, r_\ell)$ of relators induces a smooth map from G^ℓ to G^ℓ in the usual way which we denote by r with an abuse of notation where G^ℓ is interpreted to be a single point when ℓ is zero, and the geometric realization of the cosimplicial space $\mathrm{Hom}(KY^2, G)$ is the *homotopy fibre* of r, as inspection of the diagram (1.8.2) with $q = 2$ shows. Consequently, the geometric realization of $\mathcal{H} = \mathrm{Hom}(K, G)$ admits the following description: The attaching map of the single 3-cell of Y induces a homomorphism of free simplicial groups from KS^2 to KY^2 which is given by the assignment to a free generator of the free cyclic group $K_1(S^2)$ of

$$(s_0 z_1) r_{j_1}^{\varepsilon_1} (s_0 z_1)^{-1} \ldots (s_0 z_m) r_{j_m}^{\varepsilon_m} (s_0 z_m)^{-1} \in K_1 = K_1(Y^2) = K_1(Y).$$

This homomorphism induces a map σ^* from $|\mathrm{Hom}(KY^2, G)|$ to $|\mathrm{Hom}(KS^2, G)| = \Omega G$, and the realization $|\mathrm{Hom}(KY, G)|$ is the homotopy fibre of the map σ^*.

4. Simply connected polyhedra and 4-manifolds

A simply connected 4-manifold Y may be written as the cofibre of a map f from the 3-sphere S^3 to a bunch $\vee_\ell S_j^2$ of ℓ copies of the 2-sphere. In general this construction yields a simply connected 4-complex with a single 4-cell c, with characteristic map σ from Δ_4 to Y; it is of the homotopy type of a 4-manifold if and only if the attaching map f induces a non-degenerate quadratic form on $\mathrm{H}_2 Y \cong \mathbf{Z}^\ell$. However, for non-degenerate intersection form, it may *not* yield all *smooth* simply connected 4-manifolds and finer decompositions might be necessary to recover these. Moreover, working with more general CW-complexes, we could model arbitrary non-simply connected 4-manifolds but we concentrate here on the present situation and momentarily work with a general 4-complex as above Y arising from an arbitrary attaching map f of the mentioned kind. The corresponding Kan group $K = KY$ has K_0 trivial, K_1 the free group on ℓ generators $t(\sigma_1), \ldots, t(\sigma_\ell)$, where $\sigma_1, \ldots, \sigma_\ell$ are the characteristic maps of the 2-cells of Y, K_2 the free group on the 2ℓ degenerate generators

$$s_0 t(\sigma_1), \ldots, s_0 t(\sigma_\ell), s_1 t(\sigma_1), \ldots, s_1 t(\sigma_\ell) \tag{4.1}$$

and K_3 the free group on the 3ℓ degenerate generators

$$s_1 s_0 t(\sigma_1), \ldots, s_1 s_0 t(\sigma_\ell), s_2 s_0 t(\sigma_1), \ldots, s_2 s_0 t(\sigma_\ell), s_2 s_1 t(\sigma_1), \ldots, s_2 s_1 t(\sigma_\ell)$$

together with a single non-degenerate generator $t(\sigma)$. The only face operators which are not determined by the simplicial identities are

$$d_0(t\sigma) = d_1(t\sigma) = d_2(t\sigma) = e, \quad d_3(t\sigma) = t(d_3(\sigma)).$$

Notice $d_3(\sigma)$ is a singular 3-simplex of Y and $t(d_3(\sigma)) \in K_2$ is a word in the free generators (4.1); in analogy with what was said in previous Sections, we write $r = t(d_3(\sigma)) \in K_2$.

We now explain how this element r may be made explicit. To this end we recall that, by a result of J. H. C. Whitehead, $\pi_3(Y^2) = \pi_3(\vee_\ell S^2_j)$ equals the universal quadratic group $\Gamma(\pi_2(Y))$ [18] on $\pi_2(Y)$ and hence is free abelian of rank $\binom{\ell+1}{2}$, cf. [5, 49, 50]; more explicitly, after a choice $a_j \in \pi_2(S^2_j) \cong \mathbf{Z}$ and $b_j \in \pi_3(S^2_j) \cong \mathbf{Z}$ of generators has been made, where $1 \leq j \leq n$, a basis of $\pi_3(Y^2)$ is given by $b_1, \ldots, b_\ell$ and the Whitehead products $[a_i, a_j]$ for $i < j$. We now translate this to the second homotopy group $\pi_2(KY^2)$ ($\cong \pi_3(Y^2)$) of the Kan group on the 2-skeleton Y^2: For a single 2-sphere S^2, the Kan group KS^2 has K_1 free cyclic with generator $x = t(\sigma_1)$ and the Moore complex MS^2 has $M_0 = e$, $M_1 = \mathbf{Z}$, and M_2 the commutator subgroup of $K_2 = s_0K_1 * s_1K_1$. The first homology group of MS^2 equals K_1; this is a copy of the integers as it should be since it is just $\pi_2 S^2$ and, likewise, the second homology group of MS^2 is a copy of the integers, generated by the commutator $[s_0(x), s_1(x)] \in [K_2, K_2]$; this element corresponds to the Hopf map from S^3 to S^2 which generates $\pi_3(S^2)$. See [31] (p. 310) for details.

We now return to our 2-complex $Y^2 = \vee_\ell S^2_j$. For simplicity, for $1 \leq j \leq \ell$, write $x_j = t(\sigma_j) \in K_1$ for the free generators corresponding to the 2-spheres in Y. The Kan group KY^2 has $K_2 = s_0K_1 * s_1K_1$ and $K_3 = s_1s_0K_1 * s_2s_0K_1 * s_2s_1K_1$ etc., and the Moore complex of Y^2 has first homology group the group K_1 made abelian and second homology group generated by the classes of the commutators

$$v_j = [s_0(x_j), s_1(x_j)] \in [K_2, K_2], \quad 1 \leq j \leq \ell,$$

and of the elements

$$w_{i,j} = s_0(x_i)v_j(s_0(x_i))^{-1} \in [K_2, K_2], \quad 1 \leq i < j \leq \ell.$$

The elements v_j and $w_{i,j}$ correspond to the generators written b_j and $[a_i, a_j]$ above, respectively. The former assertion is obvious and the latter one may be seen by inspection of the long exact homotopy sequence of the extension

$$1 \to [K,K]^{\mathrm{Ab}} \to K/[[K,K],[K,K]] \to K^{\mathrm{Ab}} \to 1$$

of simplicial groups: In fact, the canonical maps from $[K, K]$ to K and $[K, K]$ to $[K, K]^{\mathrm{Ab}}$ induce an isomorphism from $\pi_2(K)$ onto $\pi_2([K, K]^{\mathrm{Ab}})$, and the action of $\pi_1(K^{\mathrm{Ab}})$ on $\pi_2([K, K]^{\mathrm{Ab}})$ corresponds to the operation of Whitehead product in $\pi_2(Y^2)$. The attaching element r is now a word in the v_j and the $w_{i,j}$. Moreover the quadratic form on the second integral cohomology may be described in the following way: The relevant part of WHITEHEAD'S exact sequence [50] looks like

$$\mathrm{H}_4(Y) \xrightarrow{b} \Gamma(\pi_2(Y)) \to \pi_3(Y)$$

and the quadratic map from $\pi_2(Y) = \mathrm{H}_2(Y)$ to $\mathrm{H}_2(Y) \otimes \mathrm{H}_2(Y)$ given by the assignment to a of $a \otimes a$ factors through a homomorphism of abelian groups from $\Gamma(\pi_2(Y))$ to $\mathrm{H}_2(Y) \otimes \mathrm{H}_2(Y)$. The composite thereof with the boundary b yields a homomorphism from $H_4(Y)$ to $H_2(Y) \otimes H_2(Y)$ the dual of which is the intersection pairing on Y. In particular Y models the homotopy type of a simply connected 4-manifold if and only if this pairing is non-degenerate. The non-degeneracy of the intersection pairing now translates in an obvious way to a condition on the word map r. Notice the similarity of the situation with that over a surface, cf. Section 2 above.

The cosimplicial manifold $\mathcal{H} = \mathrm{Hom}(K, G)$ has $H_0 = e$, $H_1 = G^\ell$, $H_2 = G^{2\ell}$, $H_3 = G^{3\ell+1}$, and the only part of the cosimplicial structure which is not determined by the structure itself is the composite of ε^3 from H_2 to H_3 with the projection onto the primitive part G of H_3 which corresponds to the single 4-cell of Y. We write this as a word map r from $G^\ell \times G^\ell$ to G. This makes perfect sense since it is given by the assignment to $(c_1, \ldots, c_\ell, d_1, \ldots, d_\ell) \in G^{2\ell}$ of the element of G which is obtained by substituting c_j and d_j for each occurrence of $s_0(x_j)$ and of $s_1(x_j)$, respectively, in $r = t(d_3(\sigma)) \in K_2$. The smooth geometric realization of $\mathcal{H}$ may now be described as the space of pairs (ϕ_1, ϕ_3) of smooth maps $\phi_1 \colon \Delta_1 \to H_1 = G^\ell$ and $\phi_3 \colon \Delta_3 \to G$ subject to the conditions

(1) $\phi_1(0) = \phi_1(1) = e$,
(2) ϕ_3 has constant value e on the first three faces of Δ_3, and
(3) the diagram

$$\begin{array}{ccc} \Delta_2 & \xrightarrow{\varepsilon^3} & \Delta_3 \\ {\scriptstyle (\phi_1 \circ \eta^0, \phi_1 \circ \eta^1)} \downarrow & & \downarrow {\scriptstyle \phi_3} \\ G^\ell \times G^\ell & \xrightarrow[r]{} & G \end{array} \tag{4.2}$$

is commutative. In some more detail, realize the standard simplex Δ_q in $\mathbf{R}^{q+1}$ as usual as the subset of points $(t_0, \ldots, t_q)$ defined by $t_j \geq 0$ and $\sum t_j = 1$ so that the maps η^0 and η^1 from Δ_2 to Δ_1 are given by

$$\eta^0(t_0, t_1, t_2) = (t_0 + t_1, t_2), \quad \eta^1(t_0, t_1, t_2) = (t_0, t_1 + t_2).$$

Notice the resulting map (η^0, η^1) from Δ_2 to $\Delta_1 \times \Delta_1$ identifies Δ_2 with one of the two simplices in the triangulation of $\Delta_1 \times \Delta_1$ coming into play in the *shuffle map*, cf. p. 243 of [35]. When we take $(t_1, \ldots, t_q)$ as independent variables on Δ_q, the realization of $\mathcal{H}$ appears as the space of pairs of G-valued smooth maps (ϕ_1, ϕ_3), where ϕ_1 is a smooth function of a single variable $t \in I$ while ϕ_3 is a smooth function of three variables t_1, t_2, t_3 defined for $t_1 + t_2 + t_3 \leq 1$ and $t_j \geq 0$, subject to the conditions

(1) $\phi_1(0) = \phi_1(1) = e$,
(2) $\phi_3(t_1, t_2, t_3) = e$ if $t_1 = 0$, $t_2 = 0$, or if $t_1 + t_2 + t_3 = 1$, and
(3) $\phi_3(t_1, t_2, 0) = r(\phi_1(t_1), \phi_1(t_1 + t_2))$.

When Y underlies a smooth 4-manifold, this space of maps is a model for the space of based gauge equivalence classes of connections on all topological types of bundles on Y. Perhaps moduli spaces of based gauge equivalence classes of ASD-connections can be found within this space.

We conclude this Section with a remark on the topology of the space of based gauge equivalence classes of all connections: Under the present circumstances, $|\mathrm{Hom}(KY^2,G)| = \times_\ell \Omega G$, and the diagram (1.8.2) boils down to

$$\begin{array}{ccc} |\mathrm{Hom}(K(S^4),G)| & \xrightarrow{\cong} & \mathrm{Map}^o(S^3,G) \\ \downarrow & & \downarrow \\ |\mathrm{Hom}(KY,G)| & \longrightarrow & \mathrm{Map}^o(B^3,G) \\ \downarrow & & \downarrow \\ \times_\ell \Omega G & \xrightarrow{\tau} & \mathrm{Map}^o(S^2,G) \end{array}$$

where the map τ admits the following description: An element of $\times_\ell \Omega G$ is a map ϕ from Δ_1 to G^ℓ which sends the end points to e; now τ is given by the assignment to ϕ of the composite

$$\Delta_2 \xrightarrow{(\phi\circ\eta^0,\phi\circ\eta^1)} G^\ell \times G^\ell \xrightarrow{r} G. \tag{4.3}$$

Inspection shows that this composite indeed vanishes on the boundary of Δ_2 and hence passes to a based map from the 2-sphere to G. As already pointed out, the left-hand vertical fibration in (4.3) is rationally trivial, cf. [15], see also [24], whence τ is rationally homotopically trivial. However, over the integers, τ will in general *not* be trivial. For example, let Y be complex projective 2-space with the obvious cell decomposition, so that $\ell = 1$ and the attaching map is the Hopf map from S^3 to S^2. Homotopically, the map τ then amounts to the map from $\mathrm{Map}^o(S^2,BG)$ to $\mathrm{Map}^o(S^3,BG)$ induced by the Hopf map. For simplicity, let $G = \mathrm{SU}(2)$. Now $\pi_4(\mathrm{Map}^o(S^2,BG)) \cong \pi_6(BG) \cong \pi_5(S^3)$ which is cyclic of order 2, and we can represent the non-trivial element by a non-trivial principal SU(2)-bundle on $S^4 \times S^2$. In particular, its long exact homotopy sequence will have non-trivial boundary operators. Under the map from $S^4 \times S^3$ to $S^4 \times S^2$ induced by the Hopf map, the bundle passes to a principal SU(2)-bundle on $S^4 \times S^3$ having essentially the same long exact homotopy sequence as the bundle on $S^4 \times S^2$; in particular, the bundle on $S^4 \times S^3$ is non-trivial either. A little thought reveals that this implies that the map from $\mathrm{Map}^o(S^2,BG)$ to $\mathrm{Map}^o(S^3,BG)$ induced by the Hopf map is not null homotopic.

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