The generalized Thom conjecture

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THE GENERALIZED THOM CONJECTURE

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ABSTRACT. We prove the generalized Thom conjecture for embedded surfaces with non-negative self-intersection using the Seiberg-Witten monopole invariants.

§1. STATEMENT OF THE RESULT

The purpose of this paper is to show how one can exploit the Kähler geometry of an oriented disk bundle over an oriented 2-manifold of positive Euler class to give new, somewhat simpler proofs of known results regarding the problem of finding lower bounds for the minimal genus of surfaces representing homology classes in four-manifolds.

Let X be a smooth, closed, oriented 4-manifold, $S \in H_2(X;\mathbb{Z})$ be some given two-dimensional homology class. A natural question in 4-manifold topology is to estimate the minimal genus of any smoothly embedded, oriented surface Σ in X representing S. When X is an algebraic surface and Σ is a smooth complex curve C, the canonical class K_X (the first Chern class of the complex cotangent bundle) determines the genus of C through the adjunction formula:

$$2g(C) - 2 = C \cdot C + K_X \cdot C.$$

In particular, if $X = \mathbb{C}P^2$, the genus of a smooth algebraic curve of degree d is then given by the formula $g = (d-1)(d-2)/2$. The Thom conjecture, proven by Kronheimer and Mrowka [KM3] and Morgan, Szabó and Taubes [MST], states that the genus of an algebraic curve in $\mathbb{C}P^2$ gives a lower bound for the genus of any smooth 2-manifold representing the same homology class. Both proofs used the monopole invariants introduced by Seiberg and Witten [W], closely related to Donaldson's polynomial invariants [D].

Given a Riemannian metric on X, a Spin^c structure on X gives rise to an auxiliary Hermitian line bundle L with first Chern class $c_1(L) \equiv w_2(X) \mod 2$. The Seiberg-Witten invariants constitute a map from the set of equivalence classes of Spin^c structures on X (covering the coframe bundle) to the integers. Our main result is:

Theorem 1. *Let X be a 4-manifold with $b^+(X) > 1$. Suppose that the Seiberg-Witten invariant of X is non-zero for the Spin^c structure with auxiliary line bundle L. If Σ is a smoothly embedded, oriented surface representing a homology class S with $c_1(L) \cdot S \neq 0$ and $S \cdot S \geq 0$, then*

$$2g(\Sigma) - 2 - n \geq |c_1(L) \cdot S|,$$

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where $c_1(L) \cdot S$ is the pairing between homology and cohomology classes. □

This theorem also follows from the results in [KM3] for embedded surfaces with self-intersection number zero and the blow-up formula for the Seiberg-Witten invariants [MST]. The purpose of this paper is to prove Theorem 1 without appealing to the blow-up formula.

When X is a minimal algebraic surface of general type, the only Spin^c structures having non-zero Seiberg-Witten invariants are those with auxiliary line bundles the canonical line bundle K_X or its inverse; therefore, we have:

Corollary 2. *Let X be an minimal algebraic surface of general type, then the genus of an algebraic curve with non-negative self-intersection is a lower bound for the genus of any smoothly embedded 2-manifolds representing the same homology class.* □

Although the problem of estimating the minimal genus of embedded surfaces in a smooth 4-manifold was studied by many authors, the first inequality similar to those in Theorem 1 was obtained by Kronheimer and Mrowka in [KM1], which implies the generalized Thom conjecture for the $K3$ surface. Theorem 1 was first proven by Kronheimer and Mrowka [KM2] as part of their structure theorem of Donaldson's polynomial invariants for manifolds of simple type. The immersed sphere version of Theorem 1 was first proven by Fintushel and Stern [FS] using their relation for Donaldson's invariants under rational blow-ups and blow-downs. A proof of Theorem 1 was also given by Morgan, Szabó and Taubes [MST] using the blow-up formula for the Seiberg-Witten invariants. Corollary 2 was first proven by Kronheimer [K] using Donaldson's polynomial invariants for complex surfaces having a smooth canonical divisor.

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§2. THE SEIBERG-WITTEN INVARIANTS

We briefly review the definition of the Seiberg-Witten invariants. For a more detailed exposition, we refer to [M].

Let (X, h) be a m-dimensional smooth manifold X with Riemannian metric h, then h determines the Hodge star operator

$$* = *_h : \Omega^k(X) \to \Omega^{m-k}(X),$$

where $\Omega^k(X) = \Gamma(\bigwedge^k(T^*X))$ is the space of differential k-forms. If X is a complex manifold of dimension $m = 2n$, L is a Hermitian line bundle over X, then the action of the operator $*_h$ can be extended to bundle valued complex differential forms. Following [GH],

$$* : \Omega^{p,q}(X, L) \to \Omega^{n-p,n-q}(X, L^*),$$

where L^* is the dual line bundle of L. Notice that this is different from the convention in [M].

We shall be most concerned with Riemannian 4-manifolds. In this case, $*^2 = 1$ on the space of two-forms $\Omega^2(X)$, so the Hodge star operator $*$ decomposes the space of two-forms into $+1$ and -1 eigenspaces, denoted by $\Omega^2_+(X)$ and $\Omega^2_-(X)$ respectively. When X is closed, the second cohomology of (X, h) has a corresponding decomposition into two eigenspaces $H^2(X, \mathbb{R}) = H^2_+(X, \mathbb{R}) \oplus H^2_-(X, \mathbb{R})$, the dimensions of which are denoted by $b^+(X)$ and $b^-(X)$ respectively.

Given a Riemannian 4-manifold (X, h), the set of unit cotangent vectors on (X, h) gives a principal SO(4) bundle $P(X) = P(T^*X) \to X$. The structure group SO(4) is isomorphic to

$$(\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm 1\}.$$

By projecting to the first and second factor in the product $\mathrm{SU}(2) \times \mathrm{SU}(2)$, one gets homomorphisms $r_+, r_- : \mathrm{SO}(4) \to \mathrm{SO}(3)$, such that the associated SO(3) vector bundles of r_+ and r_- are the bundles of self-dual 2-forms $\bigwedge^2_+(X)$ and anti-self-dual 2-forms $\bigwedge^2_-(X)$ respectively.

The group $\mathrm{Spin}^c(4)$ is isomorphic to

$$(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(2))/\pm 1,$$

so we have a group homomorphism

$$p : \mathrm{Spin}^c(4) \longrightarrow \mathrm{SO}(4).$$

Corresponding to $r_\pm : \mathrm{SO}(4) \to \mathrm{SO}(3)$, there are homomorphisms

$$\tilde{r}_\pm : \mathrm{Spin}^c(4) \longrightarrow (\mathrm{SU}(2) \times \mathrm{SO}(2))/\pm 1 \cong \mathrm{U}(2),$$

such that the diagram

$$\begin{array}{ccc} \mathrm{Spin}^c(4) & \xrightarrow{\tilde{r}_\pm} & \mathrm{U}(2) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Ad} \\ \mathrm{SO}(4) & \xrightarrow{r_\pm} & \mathrm{SO}(3) \end{array}$$

commutes, where Ad is the adjoint representation.

A Spin^c *structure* W on (X, h) consists of a principal SO(2) bundle

$$p : W \longrightarrow P(X)$$

together with a free $\mathrm{Spin}^c(4)$ action on W, such that the $\mathrm{Spin}^c(4)$ action on W and the SO(4) action on P commutes with the projections p.

Notice that the composition $W \to P(X) \to X$ gives a principal $\mathrm{Spin}^c(4)$ bundle $W \to X$.

Given the Spin^c structure W, the associated complex vector bundles $S^\pm = W \times_{\tilde{r}_\pm} \mathbb{C}^2$ are called the bundles of (positive and negative) spinors, and sections of these bundles are called simply (positive and negative) spinors. There is an isomorphism $End_{\mathbb{C}}(S^+, S^-) \cong T^*(X) \otimes \mathbb{C}$ which induces the Clifford multiplication

$$c : T^*(X) \otimes S^+ \longrightarrow S^-,$$

which is just the restriction of the Clifford multiplication of the Clifford bundle $\mathrm{Cl}(T^*X) : S^+ \oplus S^- \to S^+ \oplus S^-$. The *auxiliary line bundle* L of the Spin^c structure W is $\det(S^+) \cong \det(S^-)$, and the first Chern class of L satisfies $c_1(L) \equiv w_2(X)$ mod 2. Given a Spin^c structure W, the set of all Spin^c structures is identified with $H^2(X;\mathbb{Z})$; given an $\mathrm{SO}(2)$ bundle Q over X with first Chern class $c_1(Q) \in H^2(X;\mathbb{Z})$, the corresponding Spin^c structure has positive spinor bundles $S^+ \otimes Q$, negative spinor bundle $S^- \otimes Q$ and auxiliary line bundle $L \otimes Q^2$.

A Hermitian connection A on L together with the Levi-Čivita connection on (X,h) determine a connection ∇_A on $W \to X$, hence a connection on S^+. The Dirac operator D_A is given by the composition

$$\Gamma(S^+) \xrightarrow{\nabla_A} \Gamma(T^*(X)) \otimes \Gamma(S^+) \xrightarrow{c} \Gamma(S^-).$$

If $\{e_i\}$ is a local orthonormal basis of $T^*(X)$, $\{e^i\}$ the dual basis, then the Dirac operator can be locally written as $D_A = \sum e^i \cdot \nabla_{e_i}$. The action of $\bigwedge^2(X)$ on S^+ is defined by

$$\rho(e^i \wedge e^j) \cdot \xi = \frac{1}{2} e^i e^j \cdot \xi,$$

where ξ is a local section of S^+, $e^i e^j \in \mathrm{Cl}(T^*X)$ acts as Clifford multiplication. Under this action, $\bigwedge^2_+(X)$ maps S^+ to itself. We define $\tau : S^+ \otimes S^+ \to \Omega^2_+(X)$ by requiring $\tau(\xi,\zeta)$, associated to two positive spinors ξ and ζ, to be the unique self-dual two-form with the property that for any other $\omega \in \Omega^2_+(X)$.

$$< \rho(\omega) \cdot \xi, \zeta > = \frac{1}{2} < \omega, \tau(\xi,\zeta) > . \tag{1}$$

Given the Spin^c structure W on (X,h) with auxiliary line bundle L, let $\mathcal{A}_L$ denote the affine space of Hermitian connections on L. Letting the *configuration space* be $\mathcal{C} = \mathcal{A}_L \times \Gamma(S^+)$ and $\mathcal{C}^* = \{(A,\psi) \in \mathcal{A}_L \times \Gamma(S^+) | \psi \neq 0\}$, an element $(A,\psi) \in \mathcal{C}$, is said to satisfy the *Seiberg-Witten monopole equations* if

$$\begin{cases} D_A(\psi) = 0 \\ F_A^+ = i\tau(\psi,\psi); \end{cases}$$

i.e. if it is in the zero set of the map

$$\mathcal{S} : \mathcal{C} \longrightarrow \Omega^2_+(X) \oplus \Gamma(S^-)$$

defined by

$$\mathcal{S}(A,\psi) = \left(F_A^+ - i\tau(\psi,\psi), D_A(\psi)\right).$$

The group $\mathcal{G} = \mathrm{Map}(X,S^1)$, acts in a natural way on the configuration space by letting an element $u \in \mathcal{G}$ act on $\mathcal{A}_L$ by conjugating with u^2, viewed as a gauge transformation of L, and letting u act on $\Gamma(S^+)$ by scalar multiplication. It is easy to see that $\mathcal{G}$ is a symmetry of the solution space $\mathcal{S}^{-1}(0)$ (indeed, $\mathcal{S}$ is a $\mathcal{G}$-equivariant map, for the obvious, linear $\mathcal{G}$ action on the range), so we can consider the moduli space $\mathcal{M}$ of solutions to the Seiberg-Witten equations, the quotient space

$$\mathcal{M} = \mathcal{S}^{-1}(0)/\mathcal{G}.$$

(To set up everything properly, one needs to introduce Sobolev norms on the spaces mentioned above, see [M] for detail.)

The following properties of $\mathcal{M}$ were established in [KM3] for closed Riemannian four-manifolds (X, h):

(1) $\mathcal{M}$ is compact. The virtual dimension of $\mathcal{M}$ is

$$d = \frac{1}{4}\left(c_1(L)^2 - (2\chi(X) + 3\,\mathrm{Sign}(X))\right),$$

where $\chi(X)$ is the Euler characteristic of X, $\mathrm{Sign}(X)$ is the signature of X, $b^+(X) - b^-(X)$.

(2) When $b^+(X) > 0$, for a generic metric h on X, the moduli vspace $\mathcal{M}$ is a smooth manifold of dimension d contained in $\mathcal{C}^*/\mathcal{G}$. In particular, if d is zero, $\mathcal{M}$ is just a finite number of points.

(3) $\mathcal{M}$ is orientable. The orientation of $\mathcal{M}$ is determined by an orientation of $H^0(X) \oplus H^1(X) \oplus H^2_+(X)$.

When the dimension of $\mathcal{M}$ is zero, the Seiberg-Witten invariant is the number of points in $\mathcal{M}$, counted with sign. When $b^+(X) > 1$, this is a smooth invariant of the manifold X. (The invariant can still be defined when the dimension of $\mathcal{M}$ is positive [M].)

When (X, h) is a Kähler surface, the Seiberg-Witten equations can be written more explicitly. Let $K = \det \bigwedge^{1,0}(X) \cong \bigwedge^{2,0}(X)$ be the canonical line bundle, then a Spin^c structure W with auxiliary line bundle L is given by $(K \otimes L)^{1/2}$, with associated spin bundles

$$\begin{aligned} S^+ &= (\bigwedge{}^{0,0} \oplus \bigwedge{}^{0,2})(K \otimes L)^{1/2}, \\ S^- &= \bigwedge{}^{0,1}(K \otimes L)^{1/2}. \end{aligned} \tag{2}$$

A complex spinor $\psi \in \Gamma(S^+)$ can be correspondingly written as a pair

$$(\psi^0, \psi^2) \in (\Omega^{0,0} \oplus \Omega^{0,2})(K \otimes L)^{1/2}.$$

A connection $A \in \mathcal{A}_L$ induces a connection B on the bundle of spinors by coupling it to the Levi-Čivita connection A_0 induced by h on the canonical bundle K. Then, the Dirac operator $D_A : \Gamma(S^+) \to \Gamma(S^-)$ can be written

$$2(\bar\partial_B \oplus \bar\partial_B^*) : (\Omega^{0,0} \oplus \Omega^{0,2})(K \otimes L)^{1/2} \longrightarrow \Omega^{0,1}(K \otimes L)^{1/2}. \tag{3}$$

Letting Φ denote the Kähler form on (X, h), and $\Lambda : \Omega^{1,1}(X) \to \Omega^{0,0}(X)$ denote contraction with (the dual of) Φ, the Seiberg-Witten equations for $(A, \psi) = (A, \psi^0, \psi^2)$ can be written

$$\begin{aligned} &\bar\partial_B\psi^0 + \bar\partial_B^*\psi^2 = 0,\\ &{F_A}^{0,2} = \bar\psi^0\psi^2,\\ &\Lambda F_A = \frac{i}{2}(|\psi^0|^2 - |\psi^2|^2). \end{aligned} \tag{4}$$

(The additional equation

$$F_A^{2,0} = \psi^0\bar\psi^2$$

follows from the fact that A is a Hermitian connection.)

The canonical bundle K is holomorphic, so $\bar\partial_B\bar\partial_B = \frac{1}{2}F_A^{0,2}$. If (A, ψ^0, ψ^2) solves the Seiberg-Witten equations, then

$$0 = \bar\partial_B(\bar\partial_B\psi^0 + \bar\partial_B^*\psi^2) = \frac{1}{2}F_A^{0,2}\psi^0 + \bar\partial_B\bar\partial_B^*\psi^2 = \frac{1}{2}|\psi^0|^2\psi^2 + \bar\partial_B\bar\partial_B^*\psi^2 = 0.$$

When (X, h) is closed, taking inner product of the last identity with ψ^2 and integrating over (X, h), one gets

$$\frac{1}{2}\int_X |\psi^0|^2|\psi^2|^2 * 1 + \int_X |\bar\partial_B^*\psi^2|^2 * 1 = 0,$$

which, by the non-negativity of each term on the right hand side, implies that

$$|\psi^0||\psi^2| = 0, \quad \bar\partial_B\psi^0 = 0, \quad \bar\partial_B^*\psi^2 = 0.$$

Unique continuation theorem for elliptic operators implies that either ψ^0 or ψ^2 must be identically zero, so $F_A^{0,2} = 0$; hence, by the Newlander-Nirenberg theorem, A induces a holomorphic structure on L. If $\psi^2 = 0$, then ψ^0 is a holomorphic section of $(K \otimes L)^{1/2}$ satisfying the Kähler vortex equation

$$\Lambda F_A = \frac{i}{2}|\psi^0|^2.$$

The roles of ψ^0 and ψ^2 are symmetric in the following sense: following the convention in [GH], $\bar\partial^* = - * \bar\partial *$, where

$$* : \Omega^{0,2}(K \otimes L)^{1/2} \to \Omega^{2,0}(K^* \otimes L^*)^{1/2} \cong \Omega^{0,0}(K \otimes L^*)^{1/2}$$

is an isomorphism. Hence if $\psi^0 = 0$, $*\psi^2$ is a holomorphic section of $(K \otimes L^*)^{1/2}$. The curvature of L^* is $-F_A$, so $*\psi^2$ satisfies a Kähler vortex equation.

Using the above observations, Kronheimer and Mrowka, Tian and Yau, Morgan and Friedman, and D. Morrison proved that for a minimal algebraic surface of general type X, the only Spin^c structures having non-zero Seiberg-Witten invariants are those with auxiliary line bundles K_X or its inverse, see [M].

§3. Stretching the neck

Let X be a smooth, closed oriented 4-manifold, Σ be a smoothly embedded, oriented surface of genus $g(\Sigma)$ in X representing a nontrivial homology class $S = [\Sigma]$. Suppose that the self-intersection number of Σ is

$$n = S \cdot S \geq 0,$$

and let N denote a tubular neighborhood of Σ; then $p : Y = \partial N \to \Sigma$ is an S^1-bundle with first Chern class $c_1(Y) = n$.

We briefly outline the arguments which prove Theorem 1. First, we will show that, by stretching out the metric in a neighborhood of Σ, there must be a solution to the Seiberg-Witten equations in a cylindrical-end model for the neighborhood of Σ. Moreover, this solution will be bounded in a certain sense (Corollary 5). Then, by passing to a Kähler model for the neighborhood of Σ, we can reexpress the Seiberg-Witten equations (Lemma 6) in a more explicit form. The technical heart of this paper (Proposition 7) then is to exploit this version of the equations and the boundedness results to prove a vanishing result for part of the spinor (as in the discussion from the previous section), allowing us to identify bounded solutions with certain vortices over the Kähler model. This identification will allow us, in the next section, to prove Theorem 1.

We begin with some notation. Let h_Σ be a Riemannian metric on the surface Σ, with volume V and Hodge star operator $*_\Sigma$. Let η be a S^1-invariant one form dual to the S^1 action on Y, such that

$$d\eta = -\frac{2n\pi}{V} p^*(*1);$$

then the metric on Y can be chosen to be

$$h = \eta^2 + p^* h_\Sigma.$$

The manifold X is diffeomorphic to $X_R = N \cup ([0, R) \times Y) \cup (X \setminus N)$, where the metric on the neck $[0, R) \times Y$ is the product metric $dt^2 + h$. As in the study of Donaldson's polynomial invariants and [KM3], we will investigate the behavior of the solutions to the Seiberg-Witten equations on X_R when the length of the neck R goes to infinity.

When R goes to infinity, the open manifolds $W_R = N \cup ([0, R) \times Y)$ have geometric limit

$$W^o = N \cup ([0, \infty) \times Y),$$

which has a conformally Kähler, cylindrical-end metric g by [KM2].

Since the Seiberg-Witten invariant is independent of the metric on X, we can assume that the restriction of the metric on the manifold X_R to the subset $N \cup ([0, R) \times Y)$ agrees with the restriction of g. Thus we can assume that the scalar curvature of the manifolds X_R has a uniform bound.

This is important in light of the pointwise estimate proved in [KM3] coming from the Weitzenböck formula, which bounds the norm of spinor in a solution to the Seiberg-Witten equations by the scalar curvature. This estimate is especially powerful when combined with the following weak compactness result (Lemma 4 of [KM3]):

Lemma 3. *(Kronheimer-Mrowka) If Z is a compact, oriented Riemannian 4-manifold with boundary equipped with a* Spin^c *structure, and if (A_i, Φ_i) is a sequence of solutions on Z with $|\Phi_i|$ unformly bounded, then there is a subsequence $\{i'\} \subset \{i\}$ and gauge transformations $g_{i'}$ such that the sequence $\{g_{i'}(A_{i'}, \Phi_{i'})\}$ converges in C^∞.*

We collect here two consequences.

Corollary 4. *Suppose the Seiberg-Witten invariant of X is non-trivial for the* Spin^c *structure with auxiliary line bundle L, let $\{R(i)\}$ be a sequence of real numbers with $R(i)$ goes to infinity, (A_i, ψ_i) be a solution to the Seiberg-Witten equation on the manifold $X_{R(i)}$. Then there is a subsequence $\{i'\} \subset \{i\}$ and gauge transformations $g_{i'}$ defined over $W_{R(i')}$, such that the sequence $\{g_{i'}(A_{i'}, \psi_{i'})|_{W_{R(i')}}\}$ converges in C^∞ on compact sets to a solution (A, ψ) on the cylindrical-end manifold (W^o, g).*

Proof. This follows from the weak compactness result stated above and a diagonal argument, applied to the nested increasing family of compact sets $W_{R(i)} \subset (W^o, \mathrm{g})$. □

Note that the solution (A, ψ) constructed above has a C^0-bounded spinor ψ. This bound, along with weak compactness, allows us to prove a near-periodicity result for the spinor.

Corollary 5. *Consider a solution (A, ψ) over (W^o, g) with bounded $|\psi|$. There is a sequence of real numbers $\{T_i\}$ with the property that the restriction of the solution $(A, \psi)|_{[T_i-1, T_i+1]\times Y}$ is uniformly bounded in C^∞.*

Proof. We can view the sequence $\{(A, \psi)|_{[T-1, T+1]\times Y}\}_{T\in\mathbb{N}}$ as a sequence of solutions over $[0, 2]\times Y$ with uniformly C^0 bounded spinor. Then, extract a subsequence according to Lemma 3.

Putting A into the temporal gauge, we can think of (A, ψ) on the cylindrical region $[0, \infty) \times Y$ as a path of connections and spinors $(A_0(t), \psi(t))$ in the configuration space for the three-manifold Y. The previous result can be interpreted as saying that there is some point in that configuration space which is an accumulation point for that path.

As in [KM3], this path is the downward gradient flow for a Chern-Simons type functional on the configuration space of the three-manifold

$$C(A, \psi) = \int_Y (B - A)F_B - \frac{1}{2}\int_Y (A - B)d(A - B) + \frac{1}{2}\int_Y < \psi, \mathrm{D}_A\, \psi >,$$

where B is some reference connection on $L|_Y$. In the proof of Proposition 8 in [KM3], it is shown that this functional changes by a bounded amount (independent of R) over each of the tubes $[0, R) \times Y$ in the manifolds W_R. It follows then that for the limiting solution, too, the difference $C(A(T), \psi(T)) - C(A(0), \psi(0))$ is bounded independently of T. It is also worth pointing out that in the non-zero self-intersection case, the Chern-Simons function is actually real-valued, i.e. independent of the gauge of A, once a base connection B is chosen, since in general

$$C(u(A, \psi)) - C(A, \psi) = 4\pi^2 < c_1(L) \cup [u], [Y] >,$$

where $[u]$ denotes the cohomology class induced by pulling back the volume form of S^1. But any line bundle over Y which extends over Σ is necessarily a torsion class, so the above difference must vanish.

As mentioned before, the manifold (W^o, g) has is conformal to a Kähler manifold $(W, \widehat{\mathrm{g}})$. The latter metric is given by $\widehat{\mathrm{g}} = \sigma^2 \mathrm{g}$, where the conformal factor is of the form

$$\sigma^2 = e^{-2n\pi\tau/V},$$

for a real function τ which agrees with the first coordinate on the region $[10, \infty) \times Y$ of W^o (of course, the choice of constant 10 here is arbitrary). Such a metric can be written down explicitly by describing the Kähler form Φ for g. We set

$$\Phi = -f'(t)dt \wedge \eta + \left(\frac{2n\pi}{V}\right) f(t) p^* \Phi_\Sigma,$$

where t is the standard coordinate on the interval $(-\pi, \infty)$, Φ_Σ is the Kähler form of the metric h_Σ on Σ. Take f to be a smooth, monotone decreasing function on $(-\pi, \infty)$ satisfying

$$f(t) = \begin{cases} 1 - \cos t, & \text{when } -\pi < t < -\pi/2 \\ e^{-2n\pi t/V}, & \text{when } t > 10, \end{cases}$$

then the form Φ is closed, the corresponding metric is positive and can be completed at $t = -\pi$ by attaching a copy of Σ, and that $\sigma^{-2} g$ is cylindrical in the region $t > 10$. Notice that because the conformal factor is decaying exponentially, g has finite volume.

Recall (Equation (4)) that the Seiberg-Witten equations have a particularly nice form on Kähler manifold. We now consider the equations on a metric which is conformal to a Kähler metric g. Let K denote the canonical line bundle of $(W^o, \widehat{\mathrm{g}})$, A_0 denote the Levi-Čivita connection on K with respect to the Kähler metric $\widehat{\mathrm{g}}$. Given a Hermitian connection A on L, let denote B denote the connection on $(K \otimes L)^{1/2}$ induced by A_0 and A.

Rescaling the orthonormal coframe gives an identification between

$$P(W^o, \widehat{\mathrm{g}}) \cong P(W^o, \mathrm{g}).$$

Composing with this identification, we get a correspondence between Spin^c structures for $(W^o, \widehat{\mathrm{g}})$ and those for (W^o, g). In particular, we get an identification between the bundle of spinors for $(W^o, \widehat{\mathrm{g}})$ and the bundle of spinors for (W^o, g). In particular, there is an identification between the Hermitian bundles

$$S^+ \oplus S^-(W^o, \mathrm{g}) \cong \Omega^{0,*}((K \otimes L)^{1/2}), \tag{5}$$

where the Hermitian metric on the bundle on the right hand side comes from the Kähler metric $(W^o, \widehat{\mathrm{g}})$.

It is important to notice that the actual bundle identification between the S^+ for $\widehat{\mathrm{g}}$ and g does not quite preserve the Clifford module structure. Rather, if $(\theta \cdot)$

and $(\theta\widehat{\cdot})$ denote Clifford multiplication by $\theta \in T^*(W^o)$ with respect to the Clifford module structures for g and $\widehat{\mathrm{g}}$ respectively, then we must have

$$(\theta\,\cdot) = \sigma(\theta\widehat{\cdot}),$$

so that if $\omega \in \Omega^+(W^o)$, then

$$\rho(\omega) = \sigma^2\,\widehat{\rho}(\omega). \tag{6}$$

(In keeping with this notational trend, when comparing metric-dependent objects, such as the Dirac operator, Clifford multiplication, the map τ, the map ρ, etc. for the metric g and $\widehat{\mathrm{g}}$, we will let D, $\cdot$, τ, ρ denote these objects for the metric g, and $\widehat{\mathrm{D}}$, $\widehat{\cdot}$, $\widehat{\tau}$, $\widehat{\rho}$ denote the corresponding objects for the metric $\widehat{\mathrm{g}}$.)

With these observations in place, we turn to the proof of the following.

Lemma 6. *Solutions (A, ψ) to the Seiberg-Witten equations for (W, g) correspond, under the above correspondence, to data*

$$(A, \phi^0, \phi^2) \in \mathcal{A}_L \times \Omega^{0,0}((K \otimes L)^{1/2}) \times \Omega^{0,2}((K \otimes L)^{1/2})$$

over $(W, \widehat{\mathrm{g}})$, which satisfy the equations

$$\begin{aligned} 0 &= \bar{\partial}_B(\sigma^{-3/2}\phi^0) + \bar{\partial}_B^*(\sigma^{-3/2}\phi^2) \\ F_A^{0,2} &= \sigma^{-2}\bar{\phi}^0\phi^2 \\ \Lambda F_A &= \frac{i}{2}\sigma^{-2}(|\phi^0|^2 - |\phi^2|^2) \end{aligned} \tag{7}$$

Proof. Under the above correspondence, the positive spinor ψ corresponds to the pair $\phi = (\phi^0, \phi^2)$, thought of as a positive spinor on $(W^o, \widehat{\mathrm{g}})$. The first Seiberg-Witten equation, which says that

$$\mathrm{D}_A(\psi) = 0,$$

is equivalent to the condition that

$$\widehat{\mathrm{D}}_A(\sigma^{-3/2}\phi) = 0.$$

This is a straightforward exercise in the definitions together with the computation of how the Levi-Čivita connection changes under conformal changes of metric. The computations are done in both [H] and [LM]. Combining this with the complex interpretation of the Dirac operator $\widehat{\mathrm{D}}_A$ for a Kähler manifold, as in Equation (3), we get the first equation.

The other equations arise from an analysis of how the formula defining the map τ, Equation (1), changes with a conformal change of metric. The claim is, of course, that

$$\tau = \sigma^{-2}\,\widehat{\tau}.$$

This is true because of Equation (6), together with the fact that the norm $\widehat{\mathrm{g}}$ induces on two-forms, which we write by a slight abuse of notation simply as $<,>_{\widehat{\mathrm{g}}}$, differs from the norm g induces on the same space $<,>_{\mathrm{g}}$ by a factor of σ^{-4}. More explicitly, for any choice of $\omega \in \Omega^+(W^o)$, $\xi, \sigma \in \Gamma(S^+)$, we have that

$$\begin{aligned} < \omega, \sigma^{-2}\,\widehat{\tau}(\chi,\zeta) >_{\mathrm{g}} &= \sigma^4 < \omega, \sigma^{-2}\,\widehat{\tau}(\chi,\zeta) >_{\widehat{\mathrm{g}}} \\ &= < \sigma^2\,\widehat{\rho}(\omega)\chi, \zeta > \\ &= < \rho(\omega)\chi, \zeta > . \end{aligned}$$

□

In the above discussion, we have repeatedly used the identification of the spinors for g with those for $\widehat{\mathrm{g}}$, hence with differential forms over $(W^o, \widehat{\mathrm{g}})$ with values in $(K \otimes L)^{1/2}$. But the natural almost-complex structure for (W^o, g) is the same as the complex structure on (W^o, g), so this this latter bundle is naturally identified with the bundle of forms over (W^o, g) with values in $(K \otimes L)^{1/2}$. This natural identification is, of course, *not* an isometry. Writing a spinor $\psi \in S^+(W^o, \mathrm{g})$ as a form (ϕ^0, ϕ^2), we have that the norm of the spinor

$$\|\psi\|_{S+} = \|(\phi^0, \phi^2)\|_{\Lambda^{0,\bullet}_{\widehat{\mathrm{g}}}((L\otimes K)^{1/2})} = \|(\sigma^{-1}\phi^0, \sigma^{-3}\phi^2)\|_{\Lambda^{0,\bullet}_{\mathrm{g}}((L\otimes K)^{1/2})}. \tag{8}$$

Exploiting this different norm on the space of spinors, along with an argument along the lines of the vanishing result for the Kähler case outlined in the previous section, we can prove the following vanshing result.

Proposition 7. *A solution (A, ψ) to the Seiberg-Witten equations for (W^o, g) with bounded C^1 (spinor) norm corresponds to a triple (A, ϕ^0, ϕ^2) as above, with one of ϕ^0 or ϕ^2 identically zero. Hence, A determines a holomorphic structure on L.*

Proof. Applying $\bar\partial_B$ to the first equation in 6, and then using the third equation, we get

$$\begin{aligned} 0 &= \bar\partial_B\,\bar\partial_B(\sigma^{-3/2}\phi^0) + \bar\partial_B\,\bar\partial_B^*(\sigma^{-3/2}\phi^2) \\ &= \frac{1}{2}\sigma^{-2}\bar\phi^0\phi^2(\sigma^{-3/2}\phi^0) + \bar\partial_B\,\bar\partial_B^*(\sigma^{-3/2}\phi^2) \\ &= \frac{1}{2}\sigma^{-7/2}|\phi^0|^2\phi^2 + \bar\partial_B\,\bar\partial_B^*(\sigma^{-3/2}\phi^2), \end{aligned} \tag{9}$$

since $\bar\partial_B\,\bar\partial_B = \frac{1}{2}F_A^{0,2}$, because the canonical line bundle K is holomorphic on the Kähler manifold $(W^o, \widehat{\mathrm{g}})$.

We would like to rewrite this equation purely in terms of data for forms on (W^o, g). In particular, we must reexpress the operator $\bar\partial^*$ appearing above, as the adjoint here is taken with respect to the inner product on forms coming from the Kähler metric (though this was not reflected in our notation).

To do this, recall that the Hodge star operators $*$ and $\widehat{*}$ on the space of p-forms are related by

$$\widehat{*} = \sigma^{4-2p}\,*\,.$$

Following [GH], we have $\bar\partial^* = -\widehat{*}\,\bar\partial\,\widehat{*}$ on any Kähler manifold, so that, on the space of two forms, we have that

$$\bar\partial_B^* = -\sigma^{-2} * \bar\partial_B\, * .$$

Let

$$(z^0, z^2) \in (\Omega^{0,0} \oplus \Omega^{0,2})((K \otimes L)^{1/2})$$

be given by

$$z^0 = \sigma^{-1}\phi^0, \qquad z^2 = \sigma^{-3}\phi^2.$$

By Equation (8), we see that the hypothesis of the Proposition implies that the pair (z^0, z^2) is C^0 bounded with respect to the norm on forms induced by g.

Moreover, by Corollary 5, we see that the C^∞ (in particular, the C^1) norm of the restriction of (z^0, z^2) to the sequence generalized annuli $\{[T_i - 1, T_i + 1] \times Y\}_{i \in \mathbb{N}}$ is bounded uniformly. The point here is that the Levi-Čivita connection for $\widehat{\mathrm{g}}$, which is used in the the definition of the spinor C^k norm, differs from the Levi-Čivita connection for g by a zeroth-order operator whose pointwise norm grows like $|d\log\sigma|_{\Lambda^1(W^o, \mathrm{g})}$, which is evidently uniformly bounded.

Given the above relations, we can rewrite Equation (9) as

$$\frac{1}{2}\sigma^{-1/2}|z^0|^2 z^2 - \bar{\partial}_B\left(\sigma^{-2} * \bar{\partial}_B *(\sigma^{3/2}z^2)\right) = 0.$$

Taking inner product with $\sigma^{3/2}z^2$ with respect to the cylindrical metric on two-forms, and integrating over the compact subset $t \le T$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{t\le T} \sigma|z^0|^2 z^2 \wedge * \, z^2 - \int_{t \le T} \bar{\partial}_B\left(\sigma^{-2} * \bar{\partial}_B *(\sigma^{3/2}z^2)\right) \wedge *(\sigma^{3/2}z^2) \\
=&\frac{1}{2}\int_{t\le T} \sigma|z^0|^2|z^2|^2 * 1 + \int_{t\le T} \sigma^{-2}\left|* \bar{\partial}_B *(\sigma^{3/2}z^2)\right|^2 * 1 \\
&- \int_{t=T} \sigma^{-1/2} * \bar{\partial}_B *(\sigma^{3/2}z^2) \wedge * \, z^2 \\
=& I_1(T) + I_2(T) = 0,
\end{aligned}$$

where

$$\begin{aligned}
I_2(T) &= -\int_{t=T} \sigma^{-1/2} * \bar{\partial}_B *(\sigma^{3/2}z^2) \wedge * \, z^2 \\
&= -\int_{t=T} \sigma\left(\sigma^{-3/2} * \bar{\partial}_B *(\sigma^{3/2}z^2)\right) \wedge * \, z^2.
\end{aligned}$$

It is now apparent that

$$\lim_{i \mapsto \infty} I_2(T_i) = 0,$$

as $\lim_{t\mapsto\infty}\sigma(t) = 0$, and the two forms $(\sigma^{-3/2} * \bar{\partial}_B *(\sigma^{3/2}z^2))$ and z^2 are uniformly bounded on the $t = T_i$ slices.

This forces $I_1(T)$, which is *a priori* non-negative, to vanish identically; i.e.

$$|z^0||z^2| = 0, \qquad * \bar{\partial}_B *(\sigma^{3/2}z^2) = 0.$$

Rewriting the above identities on the Kähler manifold $(W^o, \widehat{\mathrm{g}})$, one gets

$$|\phi^0||\phi^2| = 0, \quad \bar{\partial}_B^*(\sigma^{-3/2}\phi^2) = 0, \quad \bar{\partial}_B(\sigma^{-3/2}\phi^0) = 0.$$

Unique continuation theorem [DK] implies that one of ϕ^0 and ϕ^2 must be identically zero. Then $F_A^{0,2} = 0$, so L is a holomorphic line bundle.

The integration-by-parts argument given above works when $n > 0$. The case of $n = 0$ is similar but simpler; we leave it to the reader. □

§4. Proof of the main theorem

Now we are ready to prove Theorem 1 and Corollary 2.

Proof of Theorem 1. Let X be a 4-manifold with $b^+(X) > 1$ and suppose that the Seiberg-Witten invariant of X is non-zero for the Spin^c structure with auxiliary line bundle L. Let Σ be a smoothly embedded, oriented surface representing a homology class S with $c_1(L) \cdot S \neq 0$ and self-intersection number $S \cdot S = n \geq 0$. As in Section 3, we study the limiting behavior of solutions to the Seiberg-Witten equations on the Riemannian manifolds $X_{R(i)}$ when the length of the neck $R(i)$ goes to infinity.

Let (A_i, ψ_i) be a solution to the Seiberg-Witten equations on $X_{R(i)}$, by proposition 3, we can suppose that (A_i, ψ_i) converges in C^∞ on compact supports to a solution (A, ψ) on the cylindrical-end manifold (W^o, g).

We exclude the case of $\psi = 0$ as follows. Recall that for our solution, there is a constant independent of R which bounds the difference

$$C(A(R), 0) - C(A(0), 0) = -\int_{Y \times [0,R]} F_A \wedge F_A = -\int_{Y \times [0,R]} |F|^2 (*1).$$

In other words, the closed differential form F_A is in $L^2(W^o)$. By [APS], $iF_A/2\pi$ represents a class in the image of $H_c^2(W^o)$ in $H^2(W^o)$, so $[iF_A/2\pi]$ represents a multiple of $[\Sigma]$. This then forces $(iF_A/2\pi) \wedge (iF_A/2\pi)$ to be non-negative, hence identically zero, because it is also anti-self-dual. In particular, $c_1(L) \cdot S = 0$, violating our assumption.

By Proposition 7, written on the Kähler manifold $(W^o, \widehat{\mathrm{g}})$, (A, ψ) is given by a triple (A, ϕ^0, ϕ^2) with one of ϕ^0 or ϕ^2 identically zero, and L is a holomorphic line bundle. We first assume that $\phi^0 \neq 0$, then (A, ϕ^0) satisfies a modified version of the Kähler vortex equations:

$$\Lambda F_A = \frac{i}{2}\sigma^{-2}|\phi^0|_g^2 \tag{10}$$

$$F_A^{0,2} = 0$$

$$0 = \bar{\partial}_B(\sigma^{-3/2}\phi^0) \tag{11}$$

where $\phi^0 \in \Gamma((K \otimes L)^{1/2})$ is a C^0-bounded section.

Equation (10) forces the line bundle to have negative degree on S. Indeed,

$$\int_{W^o} i\Lambda F_A \,\widehat{*}\, 1 = -\int_{W^o} \frac{1}{2}\sigma^{-2}|\phi^0|_g^2 \,\widehat{*}\, 1 = -\int_{W^o} \frac{1}{2}\sigma^2 |z^0|_{cyl}^2 * 1 < 0.$$

On the other hand (as in Proposition 5.11 in [KM2]), Φ is finite-energy, self-dual closed two-form on W^o with positive integral over $\Sigma \subset W^o$, so it must represent a positive multiple of the Poincaré dual of Σ. So, it follows that

$$\int_{W^o} i\Lambda F_A \,\widehat{*}\, 1 = \int_{W^o} iF_A \wedge \Phi = 2\pi c_1(L) \cdot S < 0.$$

In particular, the restriction of the form F_A, hence the restriction of ϕ^0, to Σ cannot vanish identically.

Equation (11) guarantees then that the line bundle $(K \otimes L)^{1/2}|_\Sigma$ has a non-zero holomorphic section. Hence, it must have positive degree. Since topologically K is isomorphic to $K_\Sigma \otimes N^*$, where K_Σ is the canonical line bundle for the Riemannian surface Σ, N is the normal bundle, we see that

$$\begin{aligned} c_1(K \otimes L) \cdot S &= c_1(K_\Sigma \otimes N^* \otimes L) \cdot S \\ &= 2g(\Sigma) - 2 - S \cdot S + c_1(L) \cdot S \geq 0. \end{aligned}$$

Thus, when the solution is given by $(A, \phi^0, 0)$,

$$2 - 2g(\Sigma) + S \cdot S \leq c_1(L) \cdot S < 0.$$

When the solution is given by $(A, 0, \phi^2)$, the same argument gives

$$0 < c_1(L) \cdot S \leq 2g(\Sigma) - 2 - S \cdot S.$$

These two inequalities are eqivalent to the inequality stated in Theorem 1. □

Proof of Corollary 2. For surfaces of general type, we always have that $c_1(K) \cdot S$ is non-negative if the homology class S can be represented by an algebraic curve. When $c_1(K) \cdot S = 0$, the result follws from [KM1]. □

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