Weight function for $U_q(\widehat{\mathfrak{sl}}_3)$

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Abstract

A precise expression of the universal weight function for quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_3)$ is presented. The calculations are done by means of the technique of the projections of products of Drinfeld currents to the intersection of Borel subalgebras.

1 Introduction

The ideology of nested Bethe ansatz [1] prescribes two steps for the procedure of the description of eigenvectors of transfer-matrix in finite-dimensional representations of quantum affine algebra. First, one should construct specific rational functions with values in the representation, and second, solve for them a system of Bethe equations.

These rational vector-valued functions are called off-shell (nested) Bethe vectors. They can serve as a generating system of vectors of finite-dimensional representation of quantum affine algebra. We use an equivalent name 'weight function', that came from its applications in difference Knizhnik-Zamolodchikov equations [9, 11].

A general construction of a weight function for quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ was suggested recently in [4]. It uses the existence of two different types of Borel subalgebras in quantum affine algebra. One type is related to a realization of $U_q(\widehat{\mathfrak{g}})$ as quantized Kac-Moody algebra, another comes from the current realization of $U_q(\widehat{\mathfrak{g}})$ proposed by V.Drinfeld [2]. A weight function is defined as the projection of a product of Drinfeld currents onto the intersection of Borel subalgebras of $U_q(\widehat{\mathfrak{g}})$ of different types, see Section 3.1.

The goal of this paper is to develop a technique for the calculation of the weight function, starting from the definition of [4]. According to this definition, for the calculation of the weight function one should arrange the product of Drinfeld currents in a normal ordered form and then leave only those terms, which belong to the intersection of Borel subalgebras of different types. The normal ordering procedure requires the investigation of current adjoint action and of composed root currents, introduced in [6]. The final result is a precise universal expression for the weight function of $U_q(\widehat{\mathfrak{sl}}_3)$, which then can be specialized to any finite-dimensional representation of $U_q(\widehat{\mathfrak{sl}}_3)$. Note that on a level of a tensor product of evaluation representation, the calculation of the weight function for $U_q(\widehat{\mathfrak{gl}}_N)$ can be found in [10].

The paper is organized as follows. In Section 2 we introduce main objects of the investigation. Section 3 is devoted to the formulation of main results. They contain a precise expression for the weight function of $U_q(\widehat{\mathfrak{sl}}_3)$ (Theorems 1 and 2). As a particular case, we give an expression for the weight function of $U_q(\widehat{\mathfrak{sl}}_2)$ and present this expression in an integral form (Theorem 3). The kernel of the integral is a well known partition function, which coincides with statistical sum of the 6-vertex model on a finite square lattice with the domain walls boundary conditions. We need later a combinatorial identity for this kernel, which we prove, observing a self-adjointness of projection operators (Proposition 3.5).

Sections 4 and 5 are devoted to the proofs of the main statements. They include the study of analytical properties of composed currents and related products (strings) and of current adjoint actions. Note also an important role of symmetrization procedures, based on the properties of analytical continuation of products of the currents and of their projections, see Proposition 5.1. In Appendices we observe necessary properties of opposite projection operator, commutation relations between currents and their projections, and present another proof of the main result.

2 Basic notations

2.1 $U_q(\widehat{\mathfrak{sl}}_3)$ in Chevalley generators

Quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_3)$ is generated by Chevalley[1] generators $e_{\pm\alpha_i}$, $k_{\alpha_i}^{\pm 1}$, where $i=0,1,2$ and $\prod_{i=0}^{2} k_{\alpha_i}=1$, subject to the relations

$$k_{\alpha_i}e_{\pm\alpha_j}k_{\alpha_i}^{-1} = q_i^{\pm a_{ij}}e_{\pm\alpha_j}\,, \quad [e_{\alpha_i},e_{-\alpha_j}] = \delta_{ij}\frac{k_{\alpha_i}-k_{\alpha_i}^{-1}}{q_i-q_i^{-1}}\,, \tag{2.1}$$

$$e^2_{\pm\alpha_i}e_{\pm\alpha_j}+[2]_q e_{\pm\alpha_i}e_{\pm\alpha_j}e_{\pm\alpha_i}+e_{\pm\alpha_j}e^2_{\pm\alpha_i}=0\,,\quad i\neq j\,,\quad (\alpha_i,\alpha_j)=-1\,, \tag{2.2}$$

where $[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$ is Gauss q-number and $a_{ij}=(\alpha_i,\alpha_j)$ is symmetrized Cartan matrix of the affine algebra $\widehat{\mathfrak{sl}}_3$,

$$a_{ij}=(\alpha_i,\alpha_j)=\begin{pmatrix} 2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\,,\quad (\alpha_i,\delta)=0\,. \tag{2.3}$$

One of the possible Hopf structures (which we will call a standard Hopf structure) is given by the formulas:

$$\begin{gathered}\Delta(e_{\alpha_i})=e_{\alpha_i}\otimes 1+k_{\alpha_i}\otimes e_{\alpha_i}\,,\quad \Delta(e_{-\alpha_i})=1\otimes e_{-\alpha_i}+e_{-\alpha_i}\otimes k_{\alpha_i}^{-1}\,,\\ \Delta(k_{\alpha_i}) = k_{\alpha_i}\otimes k_{\alpha_i}\,,\\ \varepsilon(e_{\pm\alpha_i})=0\,,\quad \varepsilon(k_{\alpha_i}^{\pm1})=1\,,\\ a(e_{\alpha_i})=-k_{\alpha_i}^{-1}e_{\alpha_i}\,,\quad a(e_{-\alpha_i})=-e_{-\alpha_i}k_{\alpha_i}\,,\quad a(k_{\alpha_i}^{\pm1})=k_{\alpha_i}^{\mp1}\,,\end{gathered} \tag{2.4}$$

where Δ , ε and a are comultiplication, counit and antipode maps respectively.

2.2 Current realization of the algebra $U_q(\widehat{\mathfrak{sl}}_3)$

As any quantum affine algebra, $U_q(\widehat{\mathfrak{sl}}_3)$ admits a current realization [2]. In this description (we assume again, that the central charge is zero), $U_q(\widehat{\mathfrak{sl}}_3)$ is generated by the elements $e_i[n]$, $f_i[n]$,

[1] In what follows we will not use a gradation operator and set central charge equal to zero. Usually such algebra is denoted as $U'_q(\widetilde{\mathfrak{s}l}_3)$.

where $i=\alpha,\beta,\ n\in\mathbb{Z};\ k_i^{\pm 1},\ h_i[n]$, where $i=\alpha,\beta,\ n\in\mathbb{Z}\backslash\{0\}$. They are gathered into generating functions

$$
\begin{aligned}
e_i(z) \;&=\; \sum_{n\in\mathbb{Z}} e_i[n]z^{-n}\;, \qquad f_i(z) \;=\; \sum_{n\in\mathbb{Z}} f_i[n]z^{-n}\;, \\
\psi_i^{\pm}(z) \;&=\; \sum_{n>0}\psi_i^{\pm}[n]z^{\mp n} = k_i^{\pm 1}\exp\left(\pm(q-q^{-1})\sum_{n>0}h_i[\pm n]z^{\mp n}\right)\;,
\end{aligned}
\tag{2.5}
$$

which satisfy the relations:

$$
\begin{aligned}
(z-q^{(i,j)}w)e_i(z)e_j(w) \;&=\; e_j(w)e_i(z)(q^{(i,j)}z-w)\;, \\
(z-q^{-(i,j)}w)f_i(z)f_j(w) \;&=\; f_j(w)f_i(z)(q^{-(i,j)}z-w)\;,
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
\psi_i^{\pm}(z)e_j(w)\left(\psi_i^{\pm}(z)\right)^{-1} \;&=\; \frac{(q^{(i,j)}z-w)}{(z-q^{(i,j)}w)}e_j(w)\;, \\
\psi_i^{\pm}(z)f_j(w)\left(\psi_i^{\pm}(z)\right)^{-1} \;&=\; \frac{(q^{-(i,j)}z-w)}{(z-q^{-(i,j)}w)}f_j(w)\;,
\end{aligned}
\tag{2.7}
$$

$$
\psi_i^{\mu}(z)\psi_j^{\nu}(w)=\psi_j^{\nu}(w)\psi_i^{\mu}(z)\;, \quad \mu,\nu=\pm\,,
\tag{2.8}
$$

$$
[e_i(z),f_j(w)]=\frac{\delta_{ij}\delta(z/w)}{q-q^{-1}}\left(\psi_i^+(z)-\psi_i^-(w)\right)\;,
\tag{2.9}
$$

where $i,j=\alpha,\beta\,,\ \delta(z)=\sum_{k\in\mathbb{Z}}z^k\,,\ (\alpha,\alpha)=(\beta,\beta)=2\,,\ (\alpha,\beta)=-1$ and

$$
\operatorname*{Sym}_{z_1,z_2}\left(e_i(z_1)e_i(z_2)e_j(w)-(q+q^{-1})e_i(z_1)e_j(w)e_i(z_2)+e_j(w)e_i(z_1)e_i(z_2)\right) \;=\; 0\,,
\tag{2.10}
$$

$$
\operatorname*{Sym}_{z_1,z_2}\left(f_i(z_1)f_i(z_2)f_j(w)-(q+q^{-1})f_i(z_1)f_j(w)f_i(z_2)+f_j(w)f_i(z_1)f_i(z_2)\right) \;=\; 0\,,
\tag{2.11}
$$

where $i,j=\alpha,\beta\,,\ i\neq j\,.$

The assignment

$$
\begin{aligned}
&k_{\alpha_1}\mapsto k_\alpha\,, \qquad k_{\alpha_2}\mapsto k_\beta\,, \qquad\qquad k_{\alpha_0}\mapsto k_\alpha^{-1}k_\beta^{-1}, \\
&e_{\alpha_1}\mapsto e_\alpha[0]\,, \qquad e_{\alpha_2}\mapsto e_\beta[0]\,, \qquad\qquad e_{-\alpha_1}\mapsto f_\alpha[0]\,, \qquad e_{-\alpha_2}\mapsto f_\beta[0]\,, \\
&e_{\alpha_0}\mapsto f_\beta[1]f_\alpha[0]-qf_\alpha[0]f_\beta[1]\,, \qquad e_{-\alpha_0}\mapsto e_\alpha[0]e_\beta[-1]-q^{-1}e_\beta[-1]e_\alpha[0]
\end{aligned}
\tag{2.12}
$$

establishes the isomorphism of two realizations.

The algebra $U_q(\widehat{\mathfrak{sl}}_3)$ admits a natural completion $\overline{U}_q(\widehat{\mathfrak{sl}}_3)=U_q^{(D)}(\widehat{\mathfrak{sl}}_3)$, which can be described as the minimal extension of $U_q(\widehat{\mathfrak{sl}}_3)$, which acts in all highest weight with respect to $U_q(\mathfrak{b}_+)$ representations of $U_q(\widehat{\mathfrak{sl}}_3)$. See [7], Section 2.2, for details.

In a highest weight representations of $U_q(\widehat{\mathfrak{sl}}_3)$ any matrix coefficients of arbitrary product of the currents $a_1(z_1)...,a_n(z_n)$ are formal power series in the space

$$
\mathbf{C}[z_1,z_1^{-1},...,z_m,z_m^{-1}]\left[\left[\frac{z_2}{z_1},\frac{z_3}{z_2},...,\frac{z_m}{z_{m-1}}\right]\right],
$$

and converge in a region $|z_1|\gg|z_2|\gg\cdots\gg|z_m|$ to rational function, see [3, 8]. This observation and commutation relations (2.6), which dictate the rule of analytical continuation from the above region, enable us to consider products of currents as meromorphic functions with values in $\overline{U}_q(\widehat{\mathfrak{sl}}_3)$. We freely use in the following this analytical language and replace within this formalism formal integral by contour integrals. We always specify the contour when work in analytical picture. The integral without a specification of the contour always means a formal integral.

Another Hopf structure $\Delta^{(D)}$ in $U_q(\widehat{\mathfrak{sl}}_3)$ is naturally related to the current realization. In terms of currents it looks as follows:

$$\Delta^{(D)} e_i(z) = e_i(z)\otimes 1 + \psi_i^-(z)\otimes e_i(z)\,, \tag{2.13a}$$

$$\Delta^{(D)} f_i(z) = 1\otimes f_i(z) + f_i(z)\otimes \psi_i^+(z)\,, \tag{2.13b}$$

$$\Delta^{(D)} \psi_i^{\pm}(z) = \psi_i^{+}(z)\otimes \psi_i^{\pm}(z)\,, \tag{2.13c}$$

$$a\left(e_i(z)\right) = -\left(\psi_i^-(z)\right)^{-1} e_i(z)\,,\quad a\left(f_i(z)\right) = -f_i(z)\,\left(\psi_i^+(z)\right)^{-1}, \tag{2.13d}$$

$$a\left(\psi_i^{\pm}(z)\right) = \left(\psi_i^{\pm}(z)\right)^{-1},\quad \varepsilon(e_i(z)) = \varepsilon(f_i(z)) = 0\,,\quad \varepsilon(\psi_i^{\pm}(z)) = 1\,. \tag{2.13e}$$

The comultiplications Δ of Section 2.1 and $\Delta^{(D)}$ are related by the twist, which can be described explicitly. See [7].

2.3 Borel subalgebras of $U_q(\widehat{\mathfrak{sl}}_3)$

Denote by $U_q(\mathfrak{b}_+)$ the subalgebra of $U_q(\widehat{\mathfrak{sl}}_3)$, generated by the elements e_{α_i} and $k_{\alpha_i}^{\pm1}$, $i=0,1,2$ in Chevalley description of $U_q(\widehat{\mathfrak{sl}}_3)$. Denote also by $U_q(\mathfrak{b}_-)$ the subalgebra of $U_q(\widehat{\mathfrak{sl}}_3)$, generated by the elements $e_{-\alpha_i}$ and $k_{\alpha_i}^{\pm1}$, $i=0,1,2$.

The algebras $U_q(\mathfrak{b}_\pm)$ are Hopf subalgebras of $U_q(\widehat{\mathfrak{sl}}_3)$ with respect to standard comultiplication Δ and serve as q-deformations of the enveloping algebras of opposite Borel subalgebras of Lie algebra $\widehat{\mathfrak{sl}}_3$. We call them standard Borel subalgebras. They contain subalgebras $U_q(\mathfrak{n}_\pm)$, which are generated by the elements $e_{\pm\alpha_i}$, $i=0,1,2$.

The subalgebra $U_q(\mathfrak{n}_+)$ is a left coideal of $U_q(\mathfrak{b}_+)$ with respect to standard comultiplication and the subalgebra $U_q(\mathfrak{n}_-)$ is a right coideal of $U_q(\mathfrak{b}_-)$ with respect to standard comultiplication, that is

$$\Delta(U_q(\mathfrak{n}_+) \subset U_q(\mathfrak{b}_+)\otimes U_q(\mathfrak{n}_+)\,,\qquad \Delta(U_q(\mathfrak{n}_-)) \subset U_q(\mathfrak{n}_-))\otimes U_q(\mathfrak{b}_-)\,.$$

The algebras $U_q(\mathfrak{n}_\pm)$ serve as q-deformed enveloping algebras of standard nilpotent subalgebras of Lie algebra $\widehat{\mathfrak{sl}}_3$.

Borel subalgebras of another type are related to current realization of $U_q(\widehat{\mathfrak{sl}}_3)$.

Denote by U_F the subalgebra of $U_q(\widehat{\mathfrak{sl}}_3)$, generated by the elements $k_i^{\pm1}$, $f_i[n]$, where $i=\alpha,\beta$, $n\in\mathbb{Z}$; $h_i[n]$, $i=\alpha,\beta$, $n>0$. Its completion $\overline{U}_F$ is a Hopf subalgebra of $\overline{U}_q(\widehat{\mathfrak{sl}}_3)$ with respect to comultiplication $\Delta^{(D)}$. We call U_F current Borel subalgebra. It contains the subalgebra U_f, generated by the elements $f_i[n]$, where $i=\alpha,\beta$, $n\in\mathbb{Z}$. The completed algebra $\overline{U}_f$ is a right coideal of $\overline{U}_F$ with respect to comultiplication $\Delta^{(D)}$ and serves as q-deformed enveloping algebra of algebra of currents to $\mathfrak{n}_-$.

The opposite current Borel subalgebra U_E is generated by the elements $k_i^{\pm1}$, $e_i[n]$, where $i=\alpha,\beta$, $n\in\mathbb{Z}$ and by elements $h_i[n]$, $i=\alpha,\beta$, $n<0$.

2.4 Projections $P^{\pm}$ onto intersections of Borel subalgebras

Denote by U_F^+ and U_f^- the following subalgebras of current Borel algebra U_F:

$$U_f^- = U_F \cap U_q(\mathfrak{b}_-) = U_F \cap U_q(\mathfrak{n}_-)\,, \qquad U_F^+ = U_F \cap U_q(\mathfrak{b}_+)\,. \tag{2.14}$$

For any $x \in U_q(\widehat{\mathfrak{sl}}_3)$ denote by $\mathrm{ad}_x : U_q(\widehat{\mathfrak{sl}}_3) \to U_q(\widehat{\mathfrak{sl}}_3)$ the operator of adjoint action of x in $U_q(\widehat{\mathfrak{sl}}_3)$. It is defined by the relation

$$\mathrm{ad}_x(y) = \sum_j a(x_j') \cdot y \cdot x_j'', \qquad \text{if} \qquad \Delta(x) = \sum_j x_j' \otimes x_j''\ .$$

For $i = \alpha, \beta$ denote by S_i the operator $\mathrm{ad}_{f_i[0]}$, such that

$$S_i(y) = y f_i[0] - f_i[0] k_i y k_i^{-1}, \qquad i = \alpha, \beta\,. \tag{2.15}$$

We call S_i *screening operators.*

Proposition 2.1

(i) The algebra U_f^- is generated by the elements $f_i[n]$, where $i = \alpha, \beta$, $n \le 0$; the algebra U_F^+ is generated by the elements $k_i^{\pm 1}$, $f_i[n]$, $h_i[n]$, where $i = \alpha, \beta$, $n > 0$, and by the element

$$f_{\alpha+\beta}[1] = f_\beta[1] f_\alpha[0] - q f_\alpha[0] f_\beta[1] = -\left(f_\alpha[1] f_\beta[0] - q f_\beta[0] f_\alpha[1]\right)\,. \tag{2.16}$$

(ii) Subalgebras U_f^- and U_F^+ are invariant with respect to the action of the screening operators S_i, $i = \alpha, \beta$.

(iii) Subalgebra U_F^+ is a left coideal of U_F with respect to comultiplication $\Delta^{(D)}$; subalgebra U_f^- is a right coideal of U_F with respect to comultiplication $\Delta^{(D)}$.

(iv) The multiplication in U_F establishes an isomorphism of vector spaces U_F and $U_f^- \otimes U_F^+$.

Proof. Statement (ii) can be verified as follows:

$$\begin{aligned} &S_\alpha\left(f_\beta[1] f_\alpha[0] - q f_\alpha[0] f_\beta[1]\right) = \\ &\quad = \left(f_\beta[1] f_\alpha[0] - q f_\alpha[0] f_\beta[1]\right) f_\alpha[0] - q^{-1} f_\alpha[0] \left(f_\beta[1] f_\alpha[0] - q f_\alpha[0] f_\beta[1]\right) = \\ &\quad = f_\beta[1] f_\alpha[0] f_\alpha[0] - (q + q^{-1}) f_\alpha[0] f_\beta[1] f_\alpha[0] + f_\alpha[0] f_\alpha[0] f_\beta[1] = 0\,. \end{aligned}$$

To prove (iii), we use formula (2.13a) written in terms of modes in the form

$$\Delta^{(D)}\left(f_i[n]\right) = 1 \otimes f_i[n] + \sum_{k \ge 0} f_i[n-k] \otimes \psi_i^+[k]\,.$$

We have to show that $\Delta^{(D)}\left(f_\beta[1] f_\alpha[0] - q f_\alpha[0] f_\beta[1]\right) \in U_F \otimes U_F^+$. The formula above shows that it is sufficient to check that $\psi_\beta^+[k] f_\alpha[0] - q f_\alpha[0] \psi_\beta^+[k] \in U_F^+$. But this is true due to the relation

$$f_\alpha[0] \psi_\beta^+(z) - q \psi_\beta^+(z) f_\alpha[0] = (q^2 - 1) \sum_{n=1}^{\infty} (qz)^{-n} \psi_\beta^+(z) f_\alpha[n]\,.$$

$\square$

Define operators $P=P^+:\ U_F\to U_F^+$ and $P^-:\ U_F\to U_f^-$ by the relations

$$P(f_1f_2)\ =\ P^+(f_1f_2)=\varepsilon(f_1)f_2,\qquad P^-(f_1f_2)=f_1\varepsilon(f_2)\ , \tag{2.17}$$

for any $f_1\in U_f^-$ and $f_2\in U_F^+$. Proposition 2.1 implies that the algebras U_f^- and U_F^+ satisfy the conditions *(i)* and *(ii)* of [8] Section 4.1 (see also [5], Section 6) with respect to comultiplication $\left(\Delta^{(D)}\right)^{op}$. So the operators $P^\pm$ are well defined projection operators, $(P^\pm)^2=P^\pm$, which admit extensions to the completed algebra $\overline{U}_F$, such that for any $f\in\overline{U}_F$ the following canonical decomposition is valid:

$$f=\sum_i P^-(f_i'')\cdot P^+(f_i')\,,\quad \text{if}\quad \Delta^{(D)}(f)=\sum_i f_i'\otimes f_i''\,. \tag{2.18}$$

We call expressions of the form $f=\sum_i\tilde f_i\tilde f_i'$, where $\tilde f_i\in\overline{U}_f^-$, $\tilde f_i'\in\overline{U}_F^+$, *normal ordered expansion.* Normal ordered expansion is compatible with the action of the algebra $U_q(\widehat{\mathfrak{sl}}_3)$ in the highest weight representations. Expression (2.18) gives an ordered expansion of the arbitrary element $f\in\overline{U}_F$.

2.5 Composed current and strings

Define the following generating function of the elements in $\overline{U}_q(\widehat{\mathfrak{sl}}_3)$:

$$f_{\alpha+\beta}(z)=\oint f_\alpha(z)f_\beta(w)\,\frac{dw}{w}-\oint\frac{q^{-1}-z/w}{1-q^{-1}z/w}f_\beta(w)f_\alpha(z)\,\frac{dw}{w}\,, \tag{2.19}$$

where the formal integral $\oint g(w)\frac{dw}{w}$ of a Laurent series $g(w)=\sum_{k\in\mathbb{Z}}g_kw^{-k}$ means taking its coefficient g_0.

We can write as well the formal integral in analytical language [6]

$$f_{\alpha+\beta}(z)=-\operatorname*{res}_{w=zq^{-1}}\ f_\alpha(z)f_\beta(w)\,\frac{dw}{w} \tag{2.20}$$

such that the following relation is valid in the algebra $\overline{U}_q(\widehat{\mathfrak{sl}}_3)$:

$$f_\alpha(z)f_\beta(w)=\frac{qz-w}{z-qw}f_\beta(w)f_\alpha(z)+\delta(zq^{-1}/w)f_{\alpha+\beta}(z)\,. \tag{2.21}$$

For any $a,b\in\mathbb{Z}_{\geq0}$ the products

$$f_\alpha(u_1)\cdots f_\alpha(u_a)f_{\alpha+\beta}(u_{a+1})\cdots f_{\alpha+\beta}(u_{a+b}),\quad f_{\alpha+\beta}(u_1)\cdots f_{\alpha+\beta}(u_a)f_\beta(u_{a+1})\cdots f_\beta(u_{a+b}) \tag{2.22}$$

will be called *strings.* The products

$$f_{\alpha+\beta}(u_{a+b})\cdots f_{\alpha+\beta}(u_{a+1})f_\alpha(u_a)\cdots f_\alpha(u_1),\quad f_\beta(u_{a+b})\cdots f_\beta(u_{a+1})f_{\alpha+\beta}(u_a)\cdots f_{\alpha+\beta}(u_1) \tag{2.23}$$

will be called *opposite strings* to the strings (2.22). The strings enjoy nice analytical properties, which are crucial for their use in this paper. These properties are listed in Proposition 4.3, Section 4.1.

3 Main results

3.1 Universal weight function

Let V be a representation of $U_q(\widehat{\mathfrak{sl}}_3)$ and v be a vector in V. We call v a highest weight vector with respect to current Borel subalgebra U_E, if

$$\begin{aligned} e_i(z)v &= 0\,, \qquad i=\alpha,\beta\,, \\ \psi_i^{\pm}(z)v &= \lambda_i(z)v\,, \qquad i=\alpha,\beta\,, \end{aligned} \tag{3.1}$$

where $\lambda_i(z)$ is a meromorphic function, decomposed in a series over z^{-1} for $\psi_i^+(z)$ and into a series over z for $\psi_i^-(z)$. Representation V is called a representation with highest weight vector $v\in V$ with respect to U_E, if it is generated by v over $U_q(\widehat{\mathfrak{sl}}_3)$.

Let Π denotes the two-element set $\{\alpha,\beta\}$ of positive simple roots of Lie algebra $\mathfrak{s}l_3$. An ordered set $I=a_1,...,a_{|I|}$ together with a map $\iota: I\to\Pi$ is called ordered Π-multiset.

Suppose that for any ordered Π-multiset I, $|I|=n$, it is chosen a formal series $W(t_{i_1},...,t_{i_n})$ $\in$ $U\{t_{i_1},...,t_{i_n}\}$, $i_k\in I$, where

$$U\{t_{i_1},...,t_{i_n}\} \;=\; U_q(\widehat{\mathfrak{sl}}_3)[t_{i_1},t_{i_1}^{-1},...,t_{i_n},t_{i_n}^{-1}]\left[\left[\frac{t_{i_2}}{t_{i_1}},\frac{t_{i_3}}{t_{i_2}},...,\frac{t_{i_n}}{t_{i_{n-1}}},\frac{1}{t_{i_n}}\right]\right], \tag{3.2}$$

that is, $W(t_{i_1},...,t_{i_n})$ is a formal power series over the variables $t_{i_2}/t_{i_1},t_{i_3}/t_{i_2},,...,t_{i_n}/t_{i_{n-1}},1/t_{i_n}$ with coefficients in polynomials $U_q(\widehat{\mathfrak{sl}}_3)[t_{i_1},t_{i_1}^{-1},...,t_{i_n},t_{i_n}^{-1}]$ such that

1) for any highest weight with respect to U_E representation V with highest weight v the function
$$w_V(t_{i_1},...,t_{i_n})=W(t_{i_1},...,t_{i_n})v$$
converges in a region $|t_{i_1}|\gg\cdots\gg|t_{i_n}|$ to a meromorphic V-valued function;

2) if $I=\emptyset$ then $W=1$ and $w_V=v$;

3) let $V=V_1\otimes V_2$ be a tensor product of highest weight representations with highest vectors v_1, v_2 and highest weight series $\{\lambda_i^{(1)}(z)\}$ and $\{\lambda_i^{(2)}(z)\}$, $i=\alpha,\beta$. Then for any ordered Π-multiset I we have

$$\begin{gathered} w_V(\{t_a|_{a\in\ I}\}) = \sum_{I=I_1\coprod I_2} w_{V_1}(\{t_a|_{a\in\ I_1}\})\otimes w_{V_2}(\{t_a|_{a\in\ I_2}\})\times \\ \prod_{a\in I_1}\lambda^{(2)}_{\iota(a)}(t_a)\times \prod_{a<b,\ a\in I_1,\ b\in I_2}\frac{q^{-(\iota(a),\iota(b))}t_a-t_b}{t_a-q^{-(\iota(a),\iota(b))}t_b}\,. \end{gathered} \tag{3.3}$$

A collection $W(t_{i_1},...,t_{i_n})$ is called a *universal weight function.* A collection $w(t_{i_1},...,t_{i_n})$ is called a *weight function.*

The weight function is closely related to off-shell Bethe vectors and is systematically used in investigations of solutions of q-difference Knizhnick-Zamolodchikov equations [4, 9, 11, 10].

Let $I=\{i_1,...,i_n\}$ be an ordered Π-multiset. Put

$$W(t_{i_1},...,t_{i_n}) = P\left(f_{\iota(i_1)}(t_{i_1})\cdots f_{\iota(i_n)}(t_{i_n})\right). \tag{3.4}$$

The main result of the paper [4] can be formulated as follows in the particular case of $U_q(\widehat{\mathfrak{sl}}_3)$.

Theorem. [4] *The collection* $W(t_{i_1},...,t_{i_n})$, *defined in (3.4) is a universal weight function* .

Note once more, that all the expressions for universal weight functions $W(t_{i_1},...,t_{i_n})$ should be understood as formal series over the variables $t_{i_2}/t_{i_1}, t_{i_3}/t_{i_2},,...,t_{i_n}/t_{i_{n-1}}, 1/t_{i_n}$. If we deal with weight function $w(t_{i_1},...,t_{i_n})$, which is a vector valued rational function, there is no difference in a choice of the region in which this function is expanded. See Section 5.1 for more details.

3.2 Reduction to projections of strings

Let S_n be the group of permutations of n elements. For any set $\bar{t} = \{t_1,...,t_n\}$ of variables $t_1,...,t_n$ and any $\sigma \in S_n$ denote by ${}^{\sigma}\bar{t}$ the set $\{t_{\sigma(1)},...,t_{\sigma(n)}\}$. We keep the notation $\hat{\omega}$ for the longest element of the group S_n. In this notation, the set ${}^{\hat{\omega}}\bar{t}$ means the set $\bar{t}$ with reversed order: ${}^{\hat{\omega}}\bar{t} = \{t_n,...,t_1\}$.

The group S_n acts naturally in the space of vector valued meromorphic functions of n variables $\bar{t} = \{t_1,...,t_n\}$ by the rule $F(\bar{t}) \mapsto {}^{\sigma}F(\bar{t})$, where

$$
{}^{\sigma}F(\bar{t}) = {}^{\sigma}F(t_1,...,t_n) = F(t_{\sigma(1)},...,t_{\sigma(n)}) = F({}^{\sigma}\bar{t}) .
$$

Suppose now that $F(\bar{t})$ is a series in a region $|t_1| \gg \cdots \gg |t_n|$ with values in a vector space V, that is $F(\bar{t})$ belongs to a space

$$
V[t_1,t_1^{-1},...,t_n,t_n^{-1}]\left[\left[\frac{t_2}{t_1},\frac{t_3}{t_2},...,\frac{t_n}{t_{n-1}},\frac{1}{t_n}\right]\right] . \tag{3.5}
$$

Suppose that this series converges in a region $|t_1| \gg \cdots \gg |t_n|$ to an analytical function and for any $\sigma \in S_n$ this analytical function admits an analytical continuation to the region $|t_{\sigma(1)}| \gg \cdots \gg |t_{\sigma(n)}|$. Then we put ${}^{\sigma}F(\bar{t})$ to be equal to formal series, representing analytical continuation to the region $|t_1| \gg \cdots \gg |t_n|$ of the function $F(t_{\sigma(1)},...,t_{\sigma(n)})$. So ${}^{\sigma}F(\bar{t})$ is a series in (3.5) again.

With this convention the symmetrization $\mathrm{Sym}_t^n\, F(t_1,...,t_n)$ of the function $F(t_1,...,t_n)$ as well as of a series $F(t_1,...,t_n)$ in a region $|t_1| \gg \cdots \gg |t_n|$ is the sum $\mathrm{Sym}_t^n\, F(t_1,...,t_n) = \sum_{\sigma\in S_n} {}^{\sigma}F(t_1,...,t_n)$. The q-symmetrization of a function $F(\bar{t})$ of n variables or of a series $F(\bar{t})$ in a region $|t_1| \gg \cdots \gg |t_n|$ is defined as

$$
\overline{\mathrm{Sym}}_t^{\,n}\, F(\bar{t}) = \sum_{\sigma\in S_n} \prod_{\substack{\ell<\ell' \\ \sigma(\ell)>\sigma(\ell')}} \frac{q^{-1} - q t_{\sigma(\ell)}/t_{\sigma(\ell')}}{q - q^{-1} t_{\sigma(\ell)}/t_{\sigma(\ell')}}\, {}^{\sigma}F(\bar{t}) . \tag{3.6}
$$

Symmetrization of the series which is convergent in a different asymptotical zone, is defined in analogous manner.

The universal weight function (3.4) allows analytical continuations to different asymptotical zones, since operator P extends to a projection operator in the completed algebra $\overline{U}_F$, where analytical continuation of the products of currents is well defined.

Let $I = \{i_1,...,i_n\}$ be ordered Π-multiset. For any permutation $\sigma \in S_n$ we denote by ${}^{\sigma}I$ an ordered Π-multiset ${}^{\sigma}I = \{i_{\sigma(1)},...,i_{\sigma(n)}\}$, which differs from I by the permutations of the elements, but having the same map $\iota : I \to \Pi$. Let $W(t_{i_{\sigma(1)}} \cdots t_{i_{\sigma(n)}})$ be an universal weight function corresponding to the ordered set ${}^{\sigma}I$ and $\widetilde{W}\,(t_{i_1} \cdots t_{i_n})$ be the analytical continuation of the weight function $W\,(t_{i_1} \cdots t_{i_n})$ in the domain $|t_{i_{\sigma(1)}}| \gg ... \gg |t_{i_{\sigma(n)}}|$.

Proposition 3.1 *The universal weight function (3.4) satisfies relations*

$$W(t_{i_{\sigma(1)}}\cdots t_{i_{\sigma(n)}}) = \prod_{\substack{k<l\\ \sigma^{-1}(k)>\sigma^{-1}(l)}} \frac{q^{(\iota(i_k),\ \iota(i_l))} - \dfrac{t_{i_l}}{t_{i_k}}}{1-q^{(\iota(i_k),\ \iota(i_l))}\dfrac{t_{i_l}}{t_{i_k}}}\widetilde{W}\,(t_1\cdots t_n)\ . \tag{3.7}$$

Proposition 3.1 is a direct consequence of the Proposition 5.1 of the Section 5.1. □

It follows from the Proposition 3.1 that the universal weight function for $U_q(\widehat{\mathfrak{sl}}_3)$ is defined completely by the expression

$$W(t_1,...,t_a,s_1,...,s_b) \;=\; P(f_\alpha(t_1)\cdots f_\alpha(t_a)f_\beta(s_1)\cdots f_\beta(s_b))\,. \tag{3.8}$$

In this paper we suggest an explicit expression for the function (3.8) in terms of current generators of $U_q(\widehat{\mathfrak{sl}}_3)$.

For the sets of variables $\bar{t}=\{t_1,...,t_k\}$ and $\bar{s}=\{s_1,...,s_k\}$ we define the following series:

$$\begin{aligned} Y(\bar{t};\bar{s}) &= \prod_{i=1}^{k}\frac{1}{1-s_i/t_i}\prod_{j=1}^{i-1}\frac{q-q^{-1}s_j/t_i}{1-s_j/t_i} \;=\; \prod_{i=1}^{k}\frac{1}{1-s_i/t_i}\prod_{j=i+1}^{k}\frac{q-q^{-1}s_i/t_j}{1-s_i/t_j} \\ Z(\bar{t};\bar{s}) &= Y(\bar{t};\bar{s})\prod_{i=1}^{k}\frac{s_i}{t_i} \end{aligned} \tag{3.9}$$

Theorem 1 *The universal weight function (3.8) can be written in the following form:*

$$\begin{aligned} W(t_1,...,t_a,s_1,...,s_b) &= P\left(f_\alpha(t_1)\cdots f_\alpha(t_a)f_\beta(s_1)\cdots f_\beta(s_b)\right) = \\ = \sum_{k=0}^{\min\{a,b\}}\frac{1}{k!(a-k)!(b-k)!}&\;\overline{\mathrm{Sym}}_t^{\,a}\,\overline{\mathrm{Sym}}_s^{\,b}\left(P\left(f_\alpha(t_1)\cdots f_\alpha(t_{a-k})f_{\alpha+\beta}(t_{a-k+1})\cdots f_{\alpha+\beta}(t_a)\right)\right. \\ &\times\left. P\left(f_\beta(s_{k+1})\cdots f_\beta(s_b)\right)Y(q^{-1}t_{a-k+1},\dots,q^{-1}t_a;s_1,\dots,s_k)\right)\,. \end{aligned} \tag{3.10}$$

Theorem 1 reduces the calculation of the weight function to the calculation of the projections of strings.

3.3 Projections of strings

We describe first projections of single currents. For any current $a(z)=\sum_{n\in\mathbb{Z}}a_nz^{-n}$ denote by $a^{\pm}(z)$ the currents ($a(z)=a^+(z)-a^-(z)$)

$$\begin{aligned} a^+(z) &= \oint\frac{a(w)}{1-\frac{w}{z}}\frac{dw}{z} \;=\; \sum_{n>0}a_nz^{-n}, \\ a^-(z) &= -\oint\frac{a(w)}{1-\frac{z}{w}}\frac{dw}{w} \;=\; -\sum_{n\le 0}a_nz^{-n}. \end{aligned} \tag{3.11}$$

Proposition 3.2 *Projections of the currents* $f_\alpha(z)$, $f_\beta(z)$, $f_{\alpha+\beta}(z)$ *can be written as follows*

$$\begin{aligned} &P\left(f_\alpha(t)\right)=f_\alpha^+(t)\,, \qquad P\left(f_\beta(t)\right)=f_\beta^+(t)\,, \\ &P\left(f_{\alpha+\beta}(t)\right)=S_\beta(f_\alpha^+(t)) \;=\; f_\alpha^+(t)f_\beta[0]-qf_\beta[0]f_\alpha^+(t)\,. \end{aligned} \tag{3.12}$$

There are also analogous formulas for an opposite projection:

$$\begin{gathered} P^-\left(f_\alpha(t)\right) = -f_\alpha^-(t)\,, \qquad P^-\left(f_\beta(t)\right) = -f_\beta^-(t)\,, \\ P^-\left(f_{\alpha+\beta}(t)\right) = q^{-1}S_\alpha(f_\beta^-(q^{-1}t)) = q^{-1}f_\beta^-(q^{-1}t)f_\alpha[0] - f_\alpha[0]f_\beta^-(q^{-1}t)\,. \end{gathered} \tag{3.13}$$

Define a set of rational functions of the variable s, depending on parameters $s_1,\ldots,s_b$:

$$\varphi_{s_j}(s;s_1,\ldots,s_b) = \prod_{i=1,\ i\neq j}^{b} \frac{s-s_i}{s_j-s_i} \prod_{i=1}^{b} \frac{q^{-1}s_j-qs_i}{q^{-1}s-qs_i}\,. \tag{3.14}$$

As functions of the variable s, they have simple poles at the points $s=q^2s_i$, $i=1,\ldots,b$, tend to zero when $s\to\infty$ and have properties: $\varphi_{s_j}(s_i;s_1,\ldots,s_b)=\delta_{ij}$. The set (3.14) is uniquely defined by these properties.

Let us define the following combination of currents:

$$f_\gamma(t;t_1,\ldots,t_b) = f_\gamma(t) - \sum_{m=1}^{b} \varphi_{t_m}(t;t_1,\ldots,t_b) f_\gamma(t_m) \tag{3.15}$$

where γ coincides either with the simple root α or β or with the composed root $\alpha+\beta$.

Theorem 2 *Projection of the string (2.22) has the following factorized expression:*

$$\begin{aligned} &P\left(f_\alpha(t_1)\cdots f_\alpha(t_{a-k})f_{\alpha+\beta}(t_{a-k+1})\cdots f_{\alpha+\beta}(t_a)\right) = \\ &= \prod_{1\le i\le a-k<j\le a} \frac{qt_i-q^{-1}t_j}{t_i-t_j} \prod_{1\le i<j\le a} \frac{q^{-1}t_i-qt_j}{qt_i-q^{-1}t_j} \\ &\quad\times P\left(f_{\alpha+\beta}(t_a)\right) P\left(f_{\alpha+\beta}(t_{a-1};t_a)\right)\cdots P\left(f_{\alpha+\beta}(t_{a-k+1};t_{a-k+2},\ldots,t_a)\right) \\ &\quad\times P\left(f_\alpha(t_{a-k};t_{a-k+1},\ldots,t_a)\right)\cdots P\left(f_\alpha(t_1;t_2,\ldots,t_a)\right). \end{aligned} \tag{3.16}$$

3.4 Examples

Let us give several explicit examples, illustrating Theorems 1 and 2. The second, the third and the forth examples are given taking into account the Corollary 3.3 to the Theorem 3.

$$\begin{aligned} P\left(f_\alpha(t_1)f_{\alpha+\beta}(t_2)\right) &= \frac{q^{-1}t_1-qt_2}{t_1-t_2}\, P\left(f_{\alpha+\beta}(t_2)\right)\left(f_\alpha^+(t_1) - \frac{(q-q^{-1})t_2}{qt_2-q^{-1}t_1} f_\alpha^+(t_2)\right), \\ P\left(f_\alpha(t_1)f_\alpha(t_2)\right) &= f_\alpha^+(t_1)\left(f_\alpha^+(t_2) - \frac{(q-q^{-1})t_1}{qt_1-q^{-1}t_2} f_\alpha^+(t_1)\right), \\ P\left(f_{\alpha+\beta}(t_1)f_{\alpha+\beta}(t_2)\right) &= P\left(f_{\alpha+\beta}(t_1)\right)\left(P\left(f_{\alpha+\beta}(t_2)\right) - \frac{(q-q^{-1})t_1}{qt_1-q^{-1}t_2} P\left(f_{\alpha+\beta}(t_1)\right)\right), \\ P\left(f_\alpha(t_1)f_\alpha(t_2)f_\alpha(t_3)\right) &= f_\alpha^+(t_1)\left(f_\alpha^+(t_2) - \frac{(q-q^{-1})t_1}{qt_1-q^{-1}t_2} f_\alpha^+(t_1)\right)\times \\ &\times \left(f_\alpha^+(t_3) - \frac{t_1-t_3}{t_2-t_3}\frac{(qt_1-q^{-1}t_3)(q-q^{-1})t_3}{(qt_1-q^{-1}t_3)(qt_2-q^{-1}t_3)} f_\alpha^+(t_2) - \right. \\ &\left. - \frac{t_2-t_3}{t_2-t_1}\frac{(qt_3-q^{-1}t_1)(q-q^{-1})t_1}{(qt_1-q^{-1}t_3)(qt_2-q^{-1}t_3)} f_\alpha^+(t_1)\right) \end{aligned}$$

and

$$
\begin{aligned}
&P\left(f_\alpha(t_1)f_\alpha(t_2)f_\beta(s_1)f_\beta(s_2)\right) = P\left(f_\alpha(t_1)f_\alpha(t_2)\right)P\left(f_\beta(s_1)f_\beta(s_2)\right)+\\
&\quad+\overline{\mathrm{Sym}}_{s_1,s_2}\left(\overline{\mathrm{Sym}}_{t_1,t_2}\left(P\left(f_\alpha(t_1)f_{\alpha+\beta}(t_2)\right)\frac{t_2}{t_2-qs_1}\right)f_\beta^+(s_2)\right)+\\
&\quad+\frac{1}{2}\,\overline{\mathrm{Sym}}_{t_1,t_2}\left(P\left(f_{\alpha+\beta}(t_1)f_{\alpha+\beta}(t_2)\right)\overline{\mathrm{Sym}}_{s_1,s_2}\left(\frac{t_1}{t_1-qs_1}\frac{t_2}{t_2-qs_2}\frac{qt_2-s_1}{t_2-qs_1}\right)\right).
\end{aligned}
\tag{3.17}
$$

Note that in the Theorem 2 and in the examples considered above, the normal ordering of the roots is changed from $\alpha,\alpha+\beta,\beta$ to $\alpha+\beta,\alpha,\beta$. The correct normal ordering $\alpha,\alpha+\beta,\beta$ can be restored by use of the commutation relations given in the Proposition 5.8. The result of this calculations is that the second line in the formula (3.17) can be replaced by the the expression

$$
\overline{\mathrm{Sym}}_{s_1,s_2}\left(\overline{\mathrm{Sym}}_{t_1,t_2}\left(f_\alpha^+(t_1)P(f_{\alpha+\beta}(t_2;t_1))\,\frac{q^{-1}t_1-qt_2}{t_1-t_2}\,\frac{qt_1-s_1}{t_1-qs_1}\,\frac{t_2}{t_2-qs_1}\right)f_\beta^+(s_2)\right).
$$

3.5 Universal weight function for $U_q(\widehat{\mathfrak{sl}}_2)$

The currents $e_\alpha(z)$, $f_\alpha(z)$, $\psi_\alpha^\pm(z)$, as well as Chevalley generators $e_{\pm\alpha_i}$, $k_{\alpha_i}^\pm$, $i=0,1$ generate Hopf subalgebra $U_q(\widehat{\mathfrak{sl}}_2)$ in $U_q(\widehat{\mathfrak{sl}}_3)$. For this algebra the weight function and projection operators can be defined independently. One can observe, that the corresponding projection of the product $f_\alpha(t_1)\cdots f_\alpha(t_n)$ coincides with its projection inside the algebra $U_q(\widehat{\mathfrak{sl}}_3)$. As a corollary of Theorem 2, we get the description of the weight function for $U_q(\widehat{\mathfrak{sl}}_2)$. It admits also a simple integral presentation (3.19).

Theorem 3

(i) The universal weight function (3.8) can be written in the following form

$$
\begin{aligned}
W(t_1,\dots,t_a) &= P\left(f_\alpha(t_1)\cdots f_\alpha(t_a)\right) = \\
&= \prod_{1\le i<j\le a}\frac{q^{-1}t_i-qt_j}{qt_i-q^{-1}t_j}\; f_\alpha^+(t_a)f_\alpha^+(t_{a-1};t_a)\cdots f_\alpha^+(t_1;t_2,\dots,t_a)\,.
\end{aligned}
\tag{3.18}
$$

(ii) The weight function (3.18) admits the integral presentation

$$
P\left(f_\alpha(t_1)\cdots f_\alpha(t_a)\right)=\prod_{i<j}\frac{t_i-t_j}{qt_i-q^{-1}t_j}\oint\cdots\oint Z(\overline{t};\overline{w})\,f_\alpha(w_a)\frac{dw_a}{w_a}\cdots f_\alpha(w_1)\frac{dw_1}{w_1}\,. \tag{3.19}
$$

The currents $f_\alpha^+(t_\ell;t_{\ell+1},\dots,t_a)$ are defined by the formula (3.15) and the kernel $Z(\overline{t};\overline{w})$ of the integral transform are defined by the definition (3.9).

Proof. The statement (i) is an particular case of the Theorem 2. The assertion (ii) is obtained from (i) by the substitution of expressions (3.11) and an elementary identity

$$
\frac{1}{t-w}-\sum_{m=1}^{b}\varphi_{t_m}(t;t_1,\dots,t_b)\,\frac{1}{t_m-w}=\frac{1}{t-w}\prod_{i=1}^{b}\frac{t-t_i}{w-t_i}\,\frac{q^{-1}w-qt_i}{q^{-1}t-qt_i}\,.
$$

Note, that since a factor before integral in (3.19) has the same properties of the q-symmetry as the product of currents $f_\alpha(t_1)f_\alpha(t_2)\cdots f_\alpha(t_a)$, the integral itself in (3.19) is symmetric with respect to permutations of the parameters $t_1,\ldots,t_a$. It means that we can use in (3.19) instead of the kernel $Z(t_1,\ldots,t_a;w_1,\ldots,w_a)$ the kernel $Z(t_{\sigma(1)},\ldots,t_{\sigma(a)};w_1,\ldots,w_a)$ for any permutation $\sigma\in S_a$.

Corollary 3.3 *The projection of the product of currents can be written in the "direct" order*

$$\begin{aligned}P\left(f_\alpha(t_1)\cdots f_\alpha(t_a)\right) &= \prod_{1\le i<j\le a}\frac{t_i-t_j}{qt_i-q^{-1}t_j}\oint\cdots\oint Z(^{\hat\omega}\bar t;\,^{\hat\omega}\overline{w})\, f_\alpha(w_1)\frac{dw_1}{w_1}\cdots f_\alpha(w_a)\frac{dw_a}{w_a}\\ &= f^+_\alpha(t_1)f^+_\alpha(t_2;t_1)\cdots f^+_\alpha(t_{a-1};t_1,\ldots,t_{a-2})f^+_\alpha(t_a;t_1,\ldots,t_{a-1}).\end{aligned} \tag{3.20}$$

To prove this corollary, it is sufficient to rename parameters in the integral (3.19) $t_i\to t_{a+1-i}$, $i=1,\ldots,a$ and calculate the integral, or rename variables in (3.18) and continue analytically the result to the original domain.

3.6 A combinatorial identity for the kernels $Y(\bar t;\bar s)$ and $Z(\bar t;\bar s)$

The opposite current Borel subalgebra U_E (see Section 2.3) also admits a decomposition into a product of its intersections with Borel subalgebras

$$U^+_e=U_E\cap U_q(\mathfrak{b}_+)=U_E\cap U_q(\mathfrak{n}_+)\,,\qquad U^-_E=U_E\cap U_q(\mathfrak{b}_-)\ ,$$

such that the relations

$$\tilde P^+(e_1e_2)=e_1\varepsilon(e_2)\,,\qquad \tilde P(e_1e_2)=\ \tilde P^-(e_1e_2)=\varepsilon(e_1)e_2\,, \tag{3.21}$$

where $e_1\in U^+_e$, $e_2\in U^-_E$ define projection operators $\tilde P^\pm$, which map U_E to their subalgebras U^-_E and U^+_e and satisfy properties, analogous to the properties of projectors $P^\pm$.

In particular, the projection $\tilde P\left(e_\alpha(s_1)\cdots e_\alpha(s_b)\right)$ admits an integral presentation with factorized kernel $Z(\overline{w};\,\bar t)$, see (3.9):

$$\tilde P\left(e_\alpha(s_1)\cdots e_\alpha(s_a)\right)=\prod_{i<j}\frac{s_i-s_j}{q^{-1}s_i-qs_j}\oint\cdots\oint Z(\overline{w};\bar s)e_\alpha(w_1)\frac{dw_1}{w_1}\cdots e_\alpha(w_a)\frac{dw_a}{w_a}\,. \tag{3.22}$$

Define by the symbol $\widetilde{\mathrm{Sym}}{}^n_s g(s_1,\ldots,s_n)$ the q^{-1}-symmetrization of the function $g(s_1,\ldots,s_n)$:

$$\begin{aligned}\widetilde{\mathrm{Sym}}{}^n_s\ g(s_1,\ldots,s_n) &= \sum_{\nu\in S_n}\prod_{\substack{\ell<\ell'\\ \nu^{-1}(\ell)>\nu^{-1}(\ell')}}\frac{qs_\ell-q^{-1}s_{\ell'}}{q^{-1}s_\ell-qs_{\ell'}}\ g(s_{\nu(1)},\ldots,s_{\nu(n)})=\\ &=\prod_{i<j}\frac{qs_i-q^{-1}s_j}{q^{-1}s_i-qs_j}\ \overline{\mathrm{Sym}}{}^n_s\ g(s_n,\ldots,s_1)\,.\end{aligned} \tag{3.23}$$

Current Borel subalgebras U_F and U_E are Hopf dual with respect to the Hopf structure $\Delta^{(D)}$. Let $\langle\ ,\ \rangle:U_E\otimes U_F\to\mathbb{C}$ be corresponding Hopf pairing. It satisfies the properties: $\langle ab,x\rangle=\langle b\otimes a,\Delta^{(D)}(x)\rangle$, $\langle a,xy\rangle=\langle\Delta^{(D)}(a),x\otimes y\rangle$ and for the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ looks as follows

$$\langle e_\alpha(s_1)\cdots e_\alpha(s_n),f_\alpha(t_1)\cdots f_\alpha(t_n)\rangle=(q^{-1}-q)^{-n}\ \overline{\mathrm{Sym}}{}^n_t\left(\prod_{k=1}^n\delta\left(\frac{s_k}{t_k}\right)\right). \tag{3.24}$$

Note that by obvious reasons the r.h.s. of the equality (3.24) can be rewritten as q^{-1}-symmetrization over the variables $s_1,\ldots,s_n$ in the domain $|s_1|\ll|s_2|\ll\ldots\ll|s_n|$.

Proposition 3.4 *Operators $P^{\pm}$ and $\tilde{P}^{\mp}$ are adjoint with respect to the Hopf pairing $\langle\ ,\ \rangle$: for any $f \in U_F$ and $e \in U_E$ we have*

$$\langle e, P^+(f)\rangle = \langle \tilde{P}^-(e), f\rangle\,, \qquad \langle e, P^-(f)\rangle = \langle \tilde{P}^+(e), f\rangle\,.$$

Proof. Denote by $\tilde{\mathcal{R}} \in U_E \otimes U_F$ the tensor of the Hopf pairing (3.24). In the notations of the paper [8], $\tilde{\mathcal{R}}$ coincides with $(\mathcal{R}^{21})^{-1}$. It was established in the Section 4.2 of the paper [8] that two pairs (U_f^-, U_F^+), (U_e^+, U_E^-) of subalgebras of current Borel algebras form a biorthogonal decomposition of quantum affine algebra (see [8], section 4.1 for the definition). It implies that the tensor $\tilde{\mathcal{R}} \in U_E \otimes U_F$ of the Hopf pairing admits a decomposition $\tilde{\mathcal{R}} = \mathcal{R}_1\mathcal{R}_2$, where

$$\mathcal{R}_1 = (1\otimes P^-)\tilde{\mathcal{R}} = (\tilde{P}^+ \otimes 1)\tilde{\mathcal{R}}\,, \quad \mathcal{R}_2 = (1\otimes P^+)\tilde{\mathcal{R}} = (\tilde{P}^- \otimes 1)\tilde{\mathcal{R}}\,,$$

such that $e \in U_E$, $f \in U_F$ the following equalities are valid:

$$\langle e, P^+(f)\rangle = \langle \tilde{P}^-(e), f\rangle = \langle \mathcal{R}_2, f\otimes e\rangle\,, \quad \langle e, P^-(f)\rangle = \langle \tilde{P}^+(e), f\rangle = \langle \mathcal{R}_1, f\otimes e\rangle\,. \tag{3.25}$$

□

Proposition 3.5

(i) For any sets of variables $\overline{t} = \{t_1, ..., t_n\}$ and $\overline{s} = \{s_1, ..., s_n\}$ we have equalities

$$\langle e_\alpha(s_1)\cdots e_\alpha(s_n), P(f_\alpha(t_1)\cdots f_\alpha(t_n))\rangle = (q^{-1}-q)^{-n}\prod_{i<j}\frac{t_i - t_j}{qt_i - q^{-1}t_j}\widetilde{\mathrm{Sym}}_s^{\,n} Z(\overline{t};\, {}^{\hat{\omega}}\overline{s})\,, \tag{3.26}$$

$$\langle \tilde{P}(e_\alpha(s_1)\cdots e_\alpha(s_n)), f_\alpha(t_1)\cdots f_\alpha(t_n)\rangle = (q^{-1}-q)^{-n}\prod_{i<j}\frac{s_i - s_j}{q^{-1}s_i - qs_j}\overline{\mathrm{Sym}}_t^{\,n} Z(\overline{t};\overline{s})\,. \tag{3.27}$$

(ii) For any two permutations $\sigma, \sigma' \in S_n$ we have the following identity in the ring of the functions, symmetric over both groups of variables $\overline{t}$ and $\overline{s}$:

$$\prod_{i<j}\frac{qt_i - q^{-1}t_j}{t_i - t_j}\ \overline{\mathrm{Sym}}_t^{\,n}\ Z(\overline{t};\, {}^{\sigma}\overline{s}) = \prod_{i<j}\frac{qs_i - q^{-1}s_j}{s_i - s_j}\ \overline{\mathrm{Sym}}_s^{\,n}\ Z({}^{\sigma'}\overline{t};\overline{s})\,, \tag{3.28}$$

$$\prod_{i<j}\frac{qt_i - q^{-1}t_j}{t_i - t_j}\ \overline{\mathrm{Sym}}_t^{\,n}\ Y(\overline{t};\, {}^{\sigma}\overline{s}) = \prod_{i<j}\frac{qs_i - q^{-1}s_j}{s_i - s_j}\ \overline{\mathrm{Sym}}_s^{\,n}\ Y({}^{\sigma'}\overline{t};\overline{s})\,. \tag{3.29}$$

Proof. The statement (i) can be obtained by the substitution of integral presentations (3.19) and (3.22) into corresponding Hopf pairings. From (i) after replacement of q^{-1}- symmetrization by q-symmetrization according (3.23) we get (ii) for $\sigma = \sigma' = 1$ (remind, that the functions $Z(\overline{t};\overline{s})$ $Y(\overline{t};\overline{s})$ differ by a simple factor, symmetric over both groups of variables, see (3.9)). Next, both parts of equality (3.26) are q- symmetric over variables t. Since the product $\prod_{i<j}\frac{t_i-t_j}{qt_i-q^{-1}t_j}$ is also q- symmetric, the remaining factor $\widetilde{\mathrm{Sym}}_s^{\,n} Z(\overline{t},{}^{\hat{\omega}}\,\overline{s})$ is symmetric over variables t, such that for any $\sigma \in S_n$ we have

$$\overline{\mathrm{Sym}}_s^{\,n} Z(\overline{t};\overline{s}) = \overline{\mathrm{Sym}}_s^{\,n} Z({}^{\sigma}\overline{t};\overline{s}) \quad\text{and}\quad \overline{\mathrm{Sym}}_t^{\,n} Z(\overline{t};\overline{s}) = \overline{\mathrm{Sym}}_t^{\,n} Z(\overline{t};\,{}^{\sigma}\overline{s})\ , \tag{3.30}$$

which imply the statement (ii). □

The identity (3.31) has several proofs by means of direct calculations. Our proof is based on the interpretation of both sides of this identity as of specific matrix elements of $U_q(\widehat{\mathfrak{sl}}_2)$ weight function.

We will use the identity (3.29) in the following form:

$$\prod_{i<j}\frac{qt_i-q^{-1}t_j}{t_i-t_j}\ \overline{\mathrm{Sym}}_t^{\,n}\ Y(t_1,...,t_n;s_n,...,s_1)=\prod_{i<j}\frac{qs_i-q^{-1}s_j}{s_i-s_j}\ \overline{\mathrm{Sym}}_s^{\,n}\ Y(t_1,...,t_n;s_1,...,s_n)\,. \tag{3.31}$$

Let us denote the left or the right hand side of the identity (3.31), divided by the product $\prod_{i=1}^n t_i$ as function $\mathcal{Z}(\bar t,\bar s)$. As it follows from above considerations, this function is symmetric over both set of variables $\bar t$ and $\bar s$ and has a “physical” meaning. It coincides with statistical sum of the complete inhomogeneous 6-vertex model with domain walls boundary conditions. As it was pointed out to us by N. Slavnov, this function has a determinant representation

$$\mathcal{Z}(\bar t,\bar s)=\frac{\prod\limits_{i,j=1}^{n}(qt_i-q^{-1}s_j)}{\prod\limits_{i<j}(t_i-t_j)(s_j-s_i)}\det\left|\frac{1}{(t_i-s_j)(qt_i-q^{-1}s_j)}\right|_{i,j=1,\dots,n}. \tag{3.32}$$

4 Analytical properties of strings

4.1 Properties of the current $f_{\alpha+\beta}(z)$

The current $f_{\alpha+\beta}(z)$ was defined in Section 2.5 by the relation (2.20). Note first of all that due to relations (2.6) it admits, besides (2.20), the following analytical presentation:

$$f_{\alpha+\beta}(z)\ =\operatorname*{res}_{w=z} f_\alpha(w)f_\beta(q^{-1}z)\,\frac{dw}{w}\ =\ (q-q^{-1})f_\beta(q^{-1}z)f_\alpha(z)\,. \tag{4.1}$$

We can also obtain the current $f_{\alpha+\beta}(z)$ as a result of adjoint action, related to the comultiplication $\Delta^{(D)}$.

Define left and right adjoint actions with respect to coalgebra structure $\Delta^{(D)}$:

$$\mathrm{ad}_x^{(D)}(y)=\sum_j a(x_j')\cdot y\cdot x_j''\,,\qquad \tilde{\mathrm{ad}}_x^{(D)}(y)=\sum_j x_j''\cdot y\cdot a^{-1}(x_j')\,, \tag{4.2}$$

if $\Delta^{(D)}(x)=\sum_j x_j'\otimes x_j''$. We call them *current adjoint actions*. We have

$$\begin{aligned}\mathrm{ad}_{f_i(z)}^{(D)}(y)&=yf_i(z)-f_i(z)\left(\psi_i^+(z)\right)^{-1}y\,\psi_i^+(z)\,,\\ \tilde{\mathrm{ad}}_{f_i(z)}^{(D)}(y)&=f_i(z)y-\psi_i^+(z)\,y\,\psi_i^+(z)^{-1}f_i(z)\,.\end{aligned} \tag{4.3}$$

Proposition 4.1 *We have equalities*

$$\mathrm{ad}_{f_\beta(w)}^{(D)}\left(f_\alpha(z)\right)=\delta(zq^{-1}/w)f_{\alpha+\beta}(z)\,,\qquad \tilde{\mathrm{ad}}_{f_\alpha(w)}^{(D)}\left(f_\beta(z)\right)=\delta(qz/w)f_{\alpha+\beta}(qz) \tag{4.4}$$

such that

$$f_{\alpha+\beta}(z)=\oint\frac{dw}{w}\ \mathrm{ad}_{f_\beta(w)}\left(f_\alpha(z)\right)\ =\ \oint\frac{dw}{w}\ \tilde{\mathrm{ad}}_{f_\alpha(w)}\left(f_\beta(q^{-1}z)\right)\,. \tag{4.5a}$$

Proposition 4.2 *[6] The following relations are valid in* $\overline{U}_q(\widehat{\mathfrak{sl}}_3)$ *:*

$$\begin{aligned} f_\alpha(z)f_{\alpha+\beta}(w) &= \frac{q^{-1}z-qw}{z-w}f_{\alpha+\beta}(w)f_\alpha(z)\,,\\ f_{\alpha+\beta}(qz)f_\beta(w) &= \frac{q^{-1}z-qw}{z-w}f_\beta(w)f_{\alpha+\beta}(qz)\,,\\ \frac{qz-q^{-1}w}{z-w}f_{\alpha+\beta}(z)f_{\alpha+\beta}(w) &= \frac{zq^{-1}-qw}{z-w}f_{\alpha+\beta}(w)f_{\alpha+\beta}(z)\,. \end{aligned} \tag{4.6}$$

Note that in analytical language both sides of all the relations (4.6) are analytical functions in $(\mathbb{C}^*)^2$. It means, for instance, that the product $f_\alpha(z)f_{\alpha+\beta}(w)$ has no zeroes and poles, while the product $f_{\alpha+\beta}(w)f_\alpha(z)$ has simple zero at $z=w$ and simple pole at $z=q^2w$.

Proof. The proof combines the relations (2.6) and Serre relations in analytical form [3]. Namely, in the algebra $\overline{U}_q(\widehat{\mathfrak{sl}}_3)$

(i) the products $f_\alpha(z)f_\alpha(w)$ and $f_\beta(z)f_\beta(w)$ have a simple zero at $z=w$;

(ii) the products $(z_1-qz_2)(z_2-qz_3)(z_1-q^{-2}z_3)f_\alpha(z_1)f_\beta(z_2)f_\alpha(z_3)$ and $(z_1-qz_2)(z_2-qz_3)(z_1-q^{-2}z_3)f_\beta(z_1)f_\alpha(z_2)f_\beta(z_3)$ vanish on the lines $z_2=qz_1=q^{-1}z_3$ and $z_2=q^{-1}z_1=qz_3$.

The properties of products of currents, given in Proposition 4.2, admit straightforward generalization to strings.

Proposition 4.3

(i) The strings (2.22) have simple poles at hyperplanes $u_i=q^{-2}u_j$ *and simple zeros at hyperplanes* $u_i=u_j$*, where* $i<j$ *and either* $1\le i,j\le a$ *or* $a+1\le i,j\le a+b$ *and no other poles or zeros.*

(ii) The opposite strings (2.23) have simple poles at hyperplanes $u_i=q^2u_j$ *and simple zeros at hyperplanes* $u_i=u_j$ *for all pairs* $i<j$ *and no other poles or zeros.*

(iii) The strings and opposite strings are related. In particular,

$$\begin{aligned} &f_\alpha(t_{k+1})\cdots f_\alpha(t_a)f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)=\\ &=\prod_{i,j:\ 1\le i\le k<j\le a}\frac{q^{-1}t_i-qt_j}{t_i-t_j}f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)f_\alpha(t_{k+1})\cdots f_\alpha(t_a)\,. \end{aligned} \tag{4.7}$$

4.2 Screening operators and projections of $f_{\alpha+\beta}(z)$

Let $\tilde{S}_i$ denote the screening operators $\tilde{S}_i=\tilde{\mathrm{ad}}_{f_i[0]}$ with respect to the adjoint action $\tilde{\mathrm{ad}}$ in $\overline{U}_q(\widehat{\mathfrak{sl}}_3)$: $\tilde{\mathrm{ad}}_x(y)=\sum_i x_i''\cdot y\cdot a(x_i')$, where $\Delta(x)=\sum_i x_i'\otimes x_i''$, such that

$$\tilde{S}_i\,(y)=f_i[0]\,y-k_i^{-1}\,y\,k_i\,f_i[0]\,. \tag{4.8}$$

We have the following relations between screening and projection operators.

Proposition 4.4

(i) For any $y\in U_F$ *and* $i=\alpha,\beta$ *we have equalities*

$$P^+\left(\oint\frac{dw}{w}\,\mathrm{ad}^{(D)}_{f_i(w)}(y)\right)=S_i\left(P^+(y)\right)\,,\quad P^-\left(\oint\frac{dw}{w}\,\tilde{\mathrm{ad}}^{(D)}_{f_i(w)}(y)\right)=\tilde{S}_i\left(P^-(y)\right)\,. \tag{4.9}$$

(ii) The screenings operators S_i and $\tilde{S}_i$ are related as follows:

$$\tilde{S}_i\,(y) = -q^{-2}k_i^{-1}\,S_i\,(y)\;k_i\,. \tag{4.10}$$

(iii) Screening operators S_i and $\tilde{S}_i$ commute with projections $P^{\pm}$: for any $y\in U_F$

$$P^{\pm}S_i\,(y) = S_iP^{\pm}\,(y)\;, \quad P^{\pm}\tilde{S}_i\,(y) = \tilde{S}_iP^{\pm}\,(y)\;. \tag{4.11}$$

Proof. Statements (i) and (ii) are obvious. Let us prove the equality $PS_{f_i[0]}\,(y) = S_{f_i[0]}P\,(y)$, where $y = y_1y_2$, and $y_1\in U_f^-$ and $y_2\in U_F^+$. The adjoint action satisfy the property $\mathrm{ad}_x(y_1\cdot y_2) = \sum_i \mathrm{ad}_{x_i'}(y_1)\cdot\mathrm{ad}_{x_i''}(y_2)$, if $\Delta(x) = \sum_i x_i'\otimes x_i''$. This implies the relation

$$S_i\,(y) = S_i\,(y_1)\,k_iy_2k_i^{-1} + y_1S_i\,(y_2) \tag{4.12}$$

and

$$PS_i\,(y) \;\;=\;\; \varepsilon\,(S_i\,(y_1))\,k_iy_2k_i^{-1} + \varepsilon(y_1)S_i\,(y_2) = S_iP\,(y)\;,$$

since the screening operator S_i preserves the subalgebras U_F^+ and U_f^- and $\varepsilon(S_i(y)) = 0$ for any $y\in\overline{U}_f$ except $y=1$. □

The properties of the screenings and of adjoint actions allow to calculate the projections of the current $f_{\alpha+\beta}(z)$ and establish for it the corresponding normal ordered decomposition (2.18).

Proposition 4.5

(i) The projections of the current $f_{\alpha+\beta}(z)$ are

$$P^+\left(f_{\alpha+\beta}(z)\right) = S_\beta\left(f_\alpha^+(z)\right)\;, \quad P^-\left(f_{\alpha+\beta}(z)\right) = -\tilde{S}_\alpha\left(f_\beta^-(q^{-1}z)\right)\;. \tag{4.13}$$

(ii) We have the following normal ordered expansion:

$$f_{\alpha+\beta}(z) - P\left(f_{\alpha+\beta}(z)\right) = (q^{-1}-q)\left(f_\beta^-(q^{-1}z)f_\alpha(z)\right)^+ - f_{\alpha+\beta}^-(z)\,. \tag{4.14}$$

Proof. The statement (i) follows from proposition 4.4 and relation (4.5a). Next, formal integral (2.19) can be written as

$$\begin{aligned} f_{\alpha+\beta}(w) &= f_\alpha(w)f_\beta[0] - q^{-1}f_\beta[0]f_\alpha(w) - (q^{-1}-q)\sum_{k>0} f_\beta[-k]f_\alpha(w)\left(q^{-1}w\right)^k = \\ &= f_\alpha(w)f_\beta[0] - qf_\beta[0]f_\alpha(w) - (q^{-1}-q)\sum_{k\geq 0} f_\beta[-k]f_\alpha(w)\left(q^{-1}w\right)^k = \\ &= S_\beta\left(f_\alpha(w)\right) + (q^{-1}-q)f_\beta^-(q^{-1}w)f_\alpha(w)\,. \end{aligned} \tag{4.15}$$

The application of the operation $\oint\frac{dw}{z(1-w/z)}$, see (3.11), to both sides of (4.15), gives the relation (4.14). □

4.3 Proof of Theorem 2

We calculate first the projection of the opposite string $P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)f_\alpha(t_{k+1})\cdots f_\alpha(t_a)\right)$.

Proposition 4.6
(i) The projection of the string $f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)f_\alpha(t_{k+1})\cdots f_\alpha(t_a)$ *with* $a>k$ *can be written as*

$$\begin{aligned} &P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)f_\alpha(t_{k+1})\cdots f_\alpha(t_a)\right)= \\ &\quad = P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)f_\alpha(t_{k+1})\cdots f_\alpha(t_{a-1})\right) f^+_\alpha(t_a)+\sum_{j=1}^{a-1}\frac{X_j(t_1,\ldots,t_{a-1})}{t_a-q^2t_j}\,. \end{aligned} \tag{4.16}$$

(ii) The projection of the string $f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)$ *can be presented in a form*

$$P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)\right)=P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_{k-1})\right)P(f_{\alpha+\beta}(t_k))+\sum_{j=1}^{k-1}\frac{X'_j(t_1,...,t_{k-1})}{t_a-q^2t_j}\,. \tag{4.17}$$

Proof of the relation (4.16) is based on the inductive use of the following Lemma, which shows, that during the move of the current $f^-_\alpha(z_a)$ to the left of the string only the simple the poles $z_a=q^2z_j$ appear, such that the corresponding coefficient at $(z_a-q^2z_j)^{-1}$ does not depend on z_a.

Lemma 4.7 *The following relations are valid:*

$$f_\alpha(z)f^-_\alpha(w) \;=\; \frac{q^{-1}z-qw}{qz-q^{-1}w}f^-_\alpha(w)f_\alpha(z)+\frac{z(q-q^{-1})}{qz-q^{-1}w}(1+q^2)f^+_\alpha(q^2z)f_\alpha(z)\,, \tag{4.18}$$

$$f_{\alpha+\beta}(z)f^-_\alpha(w) \;=\; \frac{z-w}{qz-q^{-1}w}f^-_\alpha(w)f_{\alpha+\beta}(z)+\frac{(q-q^{-1})z}{qz-q^{-1}w}f_{\alpha+\beta}(z)f^-_\alpha(z)\,. \tag{4.19}$$

Proof. The relations (4.18) and (4.19) follow from the application of the integral transform $-\oint\frac{du}{u}\frac{1}{1-w/u}$ to the relations

$$f_\alpha(z)f_\alpha(u)=\frac{q^{-1}z-qu}{qz-q^{-1}u}f_\alpha(u)f_\alpha(z)+(q^{-2}-q^2)\delta(q^2z/u)f_\alpha(q^2z)f_\alpha(z) \tag{4.20}$$

and

$$f_\alpha(u)f_{\alpha+\beta}(z)=\frac{q^{-1}u-qz}{u-z}f_{\alpha+\beta}(z)f_\alpha(u)\,.$$

□

For the proof of statement (ii) of Proposition 4.6 we use the relation (4.14) and substitute into r.h.s. of the relation (4.17) normal ordered expansion

$$f_{\alpha+\beta}(z_k)=P\left(f_{\alpha+\beta}(z_k)\right)+(q^{-1}-q)\left(f^-_\beta(q^{-1}z_k)f_\alpha(z_k)\right)^+-f^-_{\alpha+\beta}(z_k)\,.$$

Then we move inductively currents $f^-_\beta(q^{-1}z_k)$ and $f^-_{\alpha+\beta}(z_k)$ to the left, using relations (B.4) and observing that $P(f^-_\beta(z_k)\cdot F)=P(f^-_{\alpha+\beta}(z_k)\cdot F)=0$ for any element $F\in U_F$. □

We use now the statement (ii) of Proposition 4.3. It says that the string $f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)$ $f_\alpha(t_{k+1})\cdots f_\alpha(t_a)$ has simple zeroes at hyperplanes $t_i=t_j$. Substitution of these conditions to

(4.16) and (4.17) gives the systems of $a-1$ linear equations over the field of rational functions $\mathbb{C}(t_1,...,t_{a-1})$ for the operators $X_j(t_1,\ldots,t_{a-1})$:

$$\sum_{j=1}^{a-1}\frac{X_j(t_1,\ldots,t_{a-1})}{t_i-q^2t_j}=X\cdot f^+_\alpha(t_i)\,,\quad i=1,...,a-1\,, \tag{4.21}$$

where $X=P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)f_\alpha(t_{k+1})\cdots f_\alpha(t_{a-1})\right)$. The matrix $B_{i,j}=(t_i-q^2t_j)^{-1}$ of this system has nonzero in $\mathbb{C}(t_1,...,t_{a-1})$ determinant,

$$\det(B)=(-q^2)^{\frac{a(a-1)}{2}}\frac{\prod_{i\neq j}(t_i-t_j)^2}{\prod_{i,j}(t_i-q^2t_j)}\,,$$

so the system has unique solution over $\mathbb{C}(t_1,...,t_{a-1})$. This implies that operators X_j are linear combinations over $\mathbb{C}(t_1,...,t_{a-1})$ of operators $X\cdot f^+_\alpha(t_j)$, $j=1,...,a-1$, so the projection of the string can be presented as

$$P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)f_\alpha(t_{k+1})\cdots f_\alpha(t_a)\right)=X\cdot f^+_\alpha(t_a)-\sum_{j=1}^{a-1}\varphi_{t_j}(t_a;t_1,\ldots,t_{a-1})X\cdot f^+_\alpha(t_j)\,, \tag{4.22}$$

where $\varphi_{t_j}(t_a;t_1,\ldots,t_{a-1})=A_j(t_a;t_1,\ldots,t_{a-1})/\prod_{m=1}^{a-1}(t_a-q^2t_m)$ are rational functions which nominators $A_j(t_a;t_1,\ldots,t_{a-1})$ are polynomials over t_a of degree less then $a-1$. The system (4.21) is satisfied if rational functions $\varphi_{t_j}(t_a;t_1,\ldots,t_{a-1})$ enjoy the property

$$\varphi_{t_j}(t_i;t_1,\ldots,t_{a-1})=\delta_{i,j}\,,\qquad i,j=1,...,a-1\,.$$

This interpolation problem has unique solution given by the formula (3.14).

The relation (4.22) looks now as a recurrence relation between projections of strings of different length. The corresponding relation for strings $f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_k)$ looks the same:

$$P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_a)\right)=X'\cdot P(f_{\alpha+\beta})(t_a)+\sum_{j=1}^{a-1}\varphi_{t_j}(t_a;t_1,\ldots,t_{a-1})X'\cdot P(f_{\alpha+\beta})(t_j)\,,$$

where $X'=P\left(f_{\alpha+\beta}(t_1)\cdots f_{\alpha+\beta}(t_{a-1})\right)$ and rational functions $\varphi_{t_j}(t_a;t_1,\ldots,t_{a-1})$ are given by the relation (3.14). Successive applications of recurrence relations and the relation (4.7) give the statement of Theorem 2. $\square$

5 Current adjoint action and symmetrization

5.1 Projections and analytical continuation

Let us remind [8], that in the completed algebra $\overline{U}_q(\widehat{\mathfrak{sl}}_3)$ any product of currents $f_{\iota(1)}(t_1)\cdots f_{\iota(n)}(t_n)$ can be considered as an analytical function in a region $|t_1|\gg|t_2|\gg\cdots\gg|t_n|$, which admits an analytical continuation to a meromorphic function in $(\mathbb{C}^*)^n$. Due to commutation relations (2.6) for any $\sigma\in S_n$ the analytical continuation of this product to a region $|t_{\sigma(1)}|\gg|t_{\sigma(2)}|\gg\cdots\gg|t_{\sigma(n)}|$ is given by a series

$$\prod_{\substack{k>l\\ \sigma(k)<\sigma(l)}}\frac{q^{(\iota(\sigma(k)),\ \iota(\sigma(l)))}-\dfrac{t_{\sigma(l)}}{t_{\sigma(k)}}}{1-q^{(\iota(\sigma(k)),\ \iota(\sigma(l)))}\dfrac{t_{\sigma(l)}}{t_{\sigma(k)}}}f_{\iota(\sigma(1))}(t_{\sigma(1)})\cdots f_{\iota(\sigma(n))}(t_{\sigma(n)})\,. \tag{5.1}$$

Let ${}^{\sigma}P\left(f_{\iota(1)}(t_1)\cdots f_{\iota(n)}(t_n)\right)$ denotes the analytical continuation of the projection of the product of currents $P\left(f_{\iota(\sigma(1))}(t_{\sigma(1)})\cdots f_{\iota(\sigma(n))}(t_{\sigma(n)})\right)$ from the region $|t_{\sigma(1)}|\gg|t_{\sigma(2)}|\gg\cdots\gg|t_{\sigma(n)}|$ to the region $|t_1|\gg|t_2|\gg\cdots\gg|t_n|$.

Proposition 5.1

(i) Projections $P^{\pm}$ commute with the analytical continuation.

(ii) We have the following identity of formal series in $U\{t_1,...,t_n\}$, see (3.2):

$$ {}^{\sigma}P^{\pm}\left(f_{\iota(1)}(t_1)\cdots f_{\iota(n)}(t_n)\right)=\prod_{\substack{k<l\\ \sigma^{-1}(k)>\sigma^{-1}(l)}}\frac{q^{(\iota(k),\ \iota(l))}-t_l/t_k}{1-q^{(\iota(k),\ \iota(l))}t_l/t_k}P^{\pm}\left(f_{\iota(1)}(t_1)\cdots f_{\iota(n)}(t_n)\right). \tag{5.2}$$

Proof. Assertion (i) is based on the fact that the projection operator conserve normal ordering in the algebra U_F (with respect to the action in the category of highest weight representations), in other words, it is continuous in the topology, which defines completion $\overline{U}_F$. Assertion (ii) follows from (i) and (5.1). □

Proposition 5.1 provides a powerful tool for the computation of the weight functions. The crucial point in its application is that, contrary to product of currents, projection of product of currents admits a simple analytical continuation, equivalent to the analytical continuation of rational functions.

Example. Consider $P\left(f_\alpha(t_1)f_\alpha(t_2)\right)$. We have (see Section 3.4)

$$P\left(f_\alpha(t_1)f_\alpha(t_2)\right) = f_\alpha^+(t_1)f_\alpha^+(t_2)-\frac{(q-q^{-1})t_1}{qt_1-q^{-1}t_2}\left(f_\alpha^+(t_1)\right)^2, \tag{5.3}$$

$$P\left(f_\alpha(t_2)f_\alpha(t_1)\right) = f_\alpha^+(t_2)f_\alpha^+(t_1)-\frac{(q-q^{-1})t_2}{qt_2-q^{-1}t_1}\left(f_\alpha^+(t_2)\right)^2. \tag{5.4}$$

The equality (5.3) is an equality of formal series in a region $|t_1|\gg|t_2|$, which means that the rational function $t_1/(qt_1-q^{-1}t_2)$ is expanded into a power series over t_2/t_1, the equality (5.4) is an equality of formal series in a region $|t_2|\gg|t_1|$, which means that the rational function $t_2/(qt_2-q^{-1}t_1)$ is expanded into a power series over t_1/t_2. The analytical continuation to the region $|t_1|\gg|t_2|$ of the right hand side of (5.4) consists of analytical continuation of the rational function $t_2/(qt_2-q^{-1}t_1)$, which should be expanded now into a power series over t_2/t_1. So the equality

$$ {}^{(12)}P\left(f_\alpha(t_1)f_\alpha(t_2)\right)=\frac{q^2-t_2/t_1}{1-q^2t_2/t_1}P\left(f_\alpha(t_1)f_\alpha(t_2)\right)$$

means the relation between formal series in a region $|t_2|\gg|t_1|$

$$f_\alpha^+(t_2)f_\alpha^+(t_1)-\frac{(q-q^{-1})t_2}{qt_2-q^{-1}t_1}\left(f_\alpha^+(t_2)\right)^2=\frac{qt_1-q^{-1}t_2}{q^{-1}t_1-qt_2}f_\alpha^+(t_1)f_\alpha^+(t_2)-\frac{(q-q^{-1})t_1}{q^{-1}t_1-qt_2}\left(f_\alpha^+(t_1)\right)^2. \tag{5.5}$$

This is a basic relation in Borel subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$. It is valid as well in a region $|t_1|\gg|t_2|$ and can be generalized to a multiple product. See Corollary 3.3 to the Theorem 3.

According to our definition of q-symmetrization, see Section 3.2, statements of Proposition 5.1 and the rule of analytical continuation of product of currents (5.1), projections of products of currents with the same index of simple root, as well as products of currents themselves are q-symmetric:

$$
\begin{aligned}
f_\alpha(t_1)\cdots f_\alpha(t_n) &= \frac{1}{n!}\,\overline{\mathrm{Sym}}{}_t^n\, f_\alpha(t_1)\cdots f_\alpha(t_n)\,,\\
P^\pm\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right) &= \frac{1}{n!}\,\overline{\mathrm{Sym}}{}_t^n\, P^\pm\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right)\,.
\end{aligned}
\tag{5.6}
$$

We apply now these arguments in order to write a symmetrized version of the canonical decomposition (2.18) of a product of currents.

Proposition 5.2 *There is an equality of formal series in a region* $|t_1|\gg|t_2|\gg ...\gg|s_b|$ *:*

$$
\begin{aligned}
&f_\alpha(t_1)\cdots f_\alpha(t_a)f_\beta(s_1)\cdots f_\beta(s_b) =\\
&= \sum_{\substack{0\le m\le a\\ 0\le k\le b}} \frac{1}{m!(a-m)!k!(b-k)!}\,\overline{\mathrm{Sym}}{}_t^a\,\overline{\mathrm{Sym}}{}_s^b\,\Bigg(\prod_{\substack{m+1\le \ell\le a\\ 1\le \ell'\le k}}\frac{qt_\ell - s_{\ell'}}{t_\ell - qs_{\ell'}}\times\\
&P^-\left(f_\alpha(t_1)\cdots f_\alpha(t_m)f_\beta(s_1)\cdots f_\beta(s_k)\right)\cdot P^+\left(f_\alpha(t_{m+1})\cdots f_\alpha(t_a)f_\beta(s_{k+1})\cdots f_\beta(s_b)\right)\Bigg).
\end{aligned}
\tag{5.7}
$$

In particular, for currents of one type we have:

$$
f_\beta(s_1)\cdots f_\beta(s_b) = \sum_{0\le k\le b}\frac{1}{k!(b-k)!}\,\overline{\mathrm{Sym}}{}_s^b\left(P^-\left(f_\beta(s_1)\cdots f_\beta(s_k)\right)\cdot P^+\left(f_\beta(s_{k+1})\cdots f_\beta(t_b)\right)\right).
\tag{5.8}
$$

Proof. Direct application of the expansion (2.18) for the product of currents gives the following relation for formal series in a region $|t_1|\gg\cdots\gg|t_n|$ gives the relation:

$$
\begin{aligned}
f_{\iota(1)}(t_1)\cdots f_{\iota(n)}(t_n) &= \sum_{J\subset I}\prod_{\substack{\ell<\ell'\\ \ell\in J,\ \ell'\not\in J}}\frac{t_\ell - q^{(\iota(\ell),\iota(\ell'))}t_{\ell'}}{q^{(\iota(\ell),\iota(\ell'))}t_\ell - t_{\ell'}}\times\\
&\times P^-\left(f_{\iota(j_1')}(t_{j_1'})\cdots f_{\iota(j_k')}(t_{j_k'})\right)\cdot P^+\left(f_{\iota(j_1'')}(t_{j_1'})\cdots f_{\iota(j_k')}(t_{j_l''})\right)\,.
\end{aligned}
\tag{5.9}
$$

Here the set $I=\{1,...,n\}$, its subset $J=\{j_1',...,j_k'\}$, where $j_1'<\cdots<j_k'$, and $I\setminus J=\{j_1'',...,j_l''\}$, where $j_1''<\cdots<j_l''$. The application of symmetrization procedure, based on Proposition 5.1, gives the statement of Proposition 5.2. $\square$

5.2 Current adjoint action

Serre relations (2.10), (2.11) admit different presentations. In Section 4.1 we presented them as properties of composed currents and strings. Here we reformulate Serre relations via current adjoint action, see (4.2).

Lemma 5.3 *Serre relations (2.11) can be written in the following form:*

$$
\mathrm{ad}^{(D)}_{f_\beta(s_1)f_\beta(s_2)}\left(f_\alpha(t)\right) = 0\,.
\tag{5.10}
$$

Proof. The statement follows from the chain of equalities

$$\begin{aligned}
\mathrm{ad}^{(D)}_{f_\beta(s_1)f_\beta(s_2)}\left(f_\alpha(t)\right) &= \mathrm{ad}^{(D)}_{f_\beta(s_2)}\left(\mathrm{ad}^{(D)}_{f_\beta(s_1)}\left(f_\alpha(t)\right)\right) = \mathrm{ad}^{(D)}_{f_\beta(s_2)}\left(f_{\alpha+\beta}(t)\right)\delta(t/qs_1) = \\
&= \left(f_{\alpha+\beta}(t)f_\beta(s_2) - f_\beta(s_2)\psi^+_\beta(s_2)^{-1}f_{\alpha+\beta}(t)\psi^+_\beta(s_2)\right)\delta(t/qs_1) \\
&= \left(f_{\alpha+\beta}(t)f_\beta(s_2) - f_\beta(s_2)f_{\alpha+\beta}(t)\frac{t-q^3s_2}{qt-q^2s_2}\right)\delta(t/qs_1) = 0\,.
\end{aligned}$$

□

Lemma 5.3 implies the following

Proposition 5.4 *For $a\geq k$ an identity of formal series in a domain $|t_1|\gg\cdots\gg|t_a|$*

$$\begin{aligned}
P\left(\mathrm{ad}^{(D)}_{f_\beta(u_1)\cdots f_\beta(u_k)}\left(f_\alpha(t_1)\cdots f_\alpha(t_a)\right)\right) &= \frac{1}{k!(a-k)!} \\
\times\ \overline{\mathrm{Sym}}^{\,a}_{t}\ &\Bigg(P\left(f_\alpha(t_1)\cdots f_\alpha(t_{a-k})f_{\alpha+\beta}(t_{a-k+1})\cdots f_{\alpha+\beta}(t_a)\right) \\
\times\prod_{i<j}^{k}\frac{q^{-1}\tilde t_i - q\tilde t_j}{\tilde t_i-\tilde t_j}\ &\widetilde{\mathrm{Sym}}^{\,k}_{\tilde t}\left(\prod_{i=1}^k\delta\left(\frac{\tilde t_i}{qu_i}\right)\right)\Bigg)\,,
\end{aligned}\tag{5.11}$$

where $\tilde t_i = t_{a-k+i}$, $i=1,\ldots,k$.

Proof is a combinatorial exercise with definition of current adjoint action (4.2), its properties, including (5.10) and the relation

$$\mathrm{ad}^{(D)}_{f_\beta(s)}(F_1\cdot F_2) = F_1\cdot\mathrm{ad}^{(D)}_{f_\beta(s)}(F_2) + \mathrm{ad}^{(D)}_{f_\beta(s)}(F_1)\cdot\psi^+_\beta(s)^{-1}\,F_2\,\psi^+_\beta(s)\,.$$

Note also the active use of Proposition 5.1, which allows to use under the projections commutation relation between total currents without paying attention to δ-function terms. Taking this into account and after necessary combinatorics we obtain

$$\begin{aligned}
P\left(\mathrm{ad}^{(D)}_{f_\beta(u_1)\cdots f_\beta(u_k)}\left(f_\alpha(t_1)\cdots f_\alpha(t_a)\right)\right) &= \frac{1}{k!(a-k)!}\ \overline{\mathrm{Sym}}^{\,a}_{t}\ \overline{\mathrm{Sym}}^{\,k}_{u}\ \Bigg(\prod_{i=1}^{k}\prod_{\ell=1}^{i-1}\frac{qt_{a-k+i}-u_\ell}{t_{a-k+i}-qu_\ell}\times \\
\times P\Big(f_\alpha(t_1)\cdots f_\alpha(t_{a-k})\mathrm{ad}^{(D)}_{f_\beta(u_1)}&\left(f_\alpha(t_{a-k+1})\right)\cdots\mathrm{ad}^{(D)}_{f_\beta(u_k)}\left(f_\alpha(t_a)\right)\Big)\Bigg)\,.
\end{aligned}$$

Using now the fact that the adjoint action in the last formula produces a product of δ-functions $\prod_i\delta(t_{a-k+i}/qu_i)$, $i=1,\ldots,k$ (see (4.4)), one can move out the rational function, which is under q-symmetrization over variables u_i , from this symmetrization and replace the latter by the q^{-1}-symmetrization over the variables $t_{a-k+i}=\tilde t_i$. Proposition 5.4 is proved. □

5.3 Proof of Theorem 1

Our goal is to reduce an expression

$$P\left(f_\alpha(t_1)\cdots f_\alpha(t_a)f_\beta(s_1)\cdots f_\beta(s_b)\right)$$

to projections of strings. Let us substitute into it the decomposition (5.8) for the product of currents $f_\beta(s_1)\cdots f_\beta(s_b)$:

$$
\begin{aligned}
P\left(f_\alpha(t_1)\cdots f_\alpha(t_\alpha)f_\beta(s_1)\cdots f_\beta(s_b)\right) = \sum_{k=0}^{b}\ \frac{1}{k!(b-k)!}\times \\
\times\overline{\mathrm{Sym}}_s^{\,b}\left(P^+\left(f_\alpha(t_1)\cdots f_\alpha(t_a)P^-\left(f_\beta(s_1)\cdots f_\beta(s_k)\right)\right)\cdot P^+\left(f_\beta(s_{k+1})\cdots f_\beta(s_b)\right)\right).
\end{aligned}
\tag{5.12}
$$

We use now a strengthened coideal property of the subalgebra U_f^-.

Proposition 5.5

(i) For any element $F\in U_f^-$ *we have*

$$
\Delta^{(D)}\,F = 1\otimes F + F'\otimes F'',\quad \textit{such that}\quad F'\in U_f^-\quad \textit{and}\quad \varepsilon(F')=0\,. \tag{5.13}
$$

(ii) For any product $f_{\iota(1)}(t_1)\cdots f_{\iota(n)}(t_n)$ *we have the equality of series in* $U\{t_1,...,t_n\}$ *(see (3.2)):*

$$
\begin{aligned}
&P\left(f_{\iota(1)}(t_1)\cdots f_{\iota(m)}(t_m)P^-\left(f_{\iota(m+1)}(t_{m+1})\cdots f_{\iota(n)}(t_n)\right)\right) = \\
&\quad = P\left(\mathrm{ad}^{(D)}_{P^-\left(f_{\iota(m+1)}(t_{m+1})\cdots f_{\iota(n)}(t_n)\right)} f_{\iota(1)}(t_1)\cdots f_{\iota(m)}(t_m)\right).
\end{aligned}
\tag{5.14}
$$

Proof. It is sufficient to check the statement (i) for generators of the algebra U_f^-, where it is a direct observation. The statement (ii) is a direct consequence of (i). $\square$

We apply the particular case of (5.14):

$$
P\left(f_\alpha(t_1)\cdots f_\alpha(t_a)P^-\left(f_\beta(s_1)\cdots f_\beta(s_k)\right)\right) = P\left(\mathrm{ad}^{(D)}_{P^-\left(f_\beta(s_1)\cdots f_\beta(s_k)\right)} f_\alpha(t_1)\cdots f_\alpha(t_a)\right) \tag{5.15}
$$

and substitute into it an integral presentation of the projection $P^-\left(f_\beta(s_1)\cdots f_\beta(s_k)\right)$:

$$
P^-\left(f_\beta(s_1)\cdots f_\beta(s_k)\right) = \prod_{i<j}\frac{s_i-s_j}{qs_i-q^{-1}s_j}\oint\cdots\oint Y({}^{\hat\omega}\overline{u};\,{}^{\hat\omega}\overline{s})f_\beta(u_1)\frac{du_1}{u_1}\cdots f_\beta(u_k)\frac{du_k}{u_k}\,. \tag{5.16}
$$

Then the r.h.s. of the equality (5.15) takes the form

$$
\begin{aligned}
&\prod_{i<j}^{k}\frac{s_i-s_j}{qs_i-q^{-1}s_j}\oint\cdots\oint\prod_{i=1}^{k}\frac{du_i}{u_i}\ Y({}^{\hat\omega}\overline{u},\,{}^{\hat\omega}\overline{s})\ P\left(\mathrm{ad}^{(D)}_{f_\beta(u_1)\cdots f_\beta(u_k)}\left(f_\alpha(t_1)\cdots f_\alpha(t_a)\right)\right) = \\
&\quad = \frac{1}{k!(a-k)!}\ \overline{\mathrm{Sym}}_t^{\,a}\Bigg(P\left(f_\alpha(t_1)\cdots f_\alpha(t_{a-k})f_{\alpha+\beta}(t_{a-k+1})\cdots f_{\alpha+\beta}(t_a)\right) \\
&\quad\times\prod_{i<j}\frac{s_i-s_j}{qs_i-q^{-1}s_j}\frac{q^{-1}\tilde t_i-q\tilde t_j}{\tilde t_i-\tilde t_j}\ \widetilde{\mathrm{Sym}}_{\tilde t}^{\,k}\left(\oint\cdots\oint\prod_{i=1}^{k}\frac{du_i}{u_i}\,\delta\left(\frac{\tilde t_i}{qu_i}\right)Y({}^{\hat\omega}\overline{u};\,{}^{\hat\omega}\overline{s})\right)\Bigg)
\end{aligned}
\tag{5.17}
$$

where $\tilde t_i = t_{a-k+i}$, $i=1,\dots,k$. After performing the integration, the last line in (5.17) reads

$$
\begin{aligned}
&\prod_{i<j}\frac{s_i-s_j}{qs_i-q^{-1}s_j}\frac{q^{-1}\tilde t_i-q\tilde t_j}{\tilde t_i-\tilde t_j}\ \widetilde{\mathrm{Sym}}_{\tilde t}^{\,k}\left(Y(q^{-1}\cdot({}^{\hat\omega}\tilde t);\,{}^{\hat\omega}\overline{s})\right)\ = \\
&= \prod_{i<j}\frac{s_i-s_j}{qs_i-q^{-1}s_j}\frac{q\tilde t_i-q^{-1}\tilde t_j}{\tilde t_i-\tilde t_j}\ \overline{\mathrm{Sym}}_{\tilde t}^{\,k}\left(Y(q^{-1}\cdot\tilde t;\,{}^{\hat\omega}\overline{s})\right)\ =\ \overline{\mathrm{Sym}}_s^{\,k}\left(Y(q^{-1}\cdot\tilde t;\,\overline{s})\right)\ .
\end{aligned}
\tag{5.18}
$$

In the first equality in (5.18) the relation between q- and q^{-1}-symmetrizations is used (3.23), while the second equality is obtained from the combinatorial identity (3.31).

The projection (5.15) reads now as:

$$
\begin{aligned}
P\left(\mathrm{ad}^{(D)}_{P^-\left(f_\beta(s_1)\cdots f_\beta(s_k)\right)} f_\alpha(t_1)\cdots f_\alpha(t_a)\right) &= \frac{1}{k!(a-k)!} \\
\times\ \overline{\mathrm{Sym}}^{\,a}_t\ \overline{\mathrm{Sym}}^{\,k}_s &\Big(Y(q^{-1}t_{a-k+1},\ldots,q^{-1}t_a;s_1,\ldots,s_k) \\
&\times P\left(f_\alpha(t_1)\cdots f_\alpha(t_{a-k})f_{\alpha+\beta}(t_{a-k+1})\cdots f_{\alpha+\beta}(t_a)\right)\Big).
\end{aligned}
\tag{5.19}
$$

Return now to the proof of the Proposition 1. By definition of q-symmetrization (3.6) the r.h.s. of the last formula is q-symmetric over the variables $s_1,\ldots,s_k$. Then additional q-symmetrization cancels unwanted factorial (see (5.6)) and we get the proof of Proposition 1. $\square$

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Appendix A. The projection P^-

In this Appendix we collect the most important formulas for the opposite projection operator P^-.

1. As well as for the projection $P=P^+$, an expression $P^-\left(f_{\iota(i_1)}(t_{i_1})\cdots f_{\iota(i_n)}(t_{i_n})\right)$ is also a series in the domain $|t_{i_1}|\gg\cdots\gg|t_{i_n}|$, which admits an analytical continuation into different asymptotical zones. In a sense of analytical continuation it has the same properties as analogous expressions for the projections P^+.

Let us recall the formula for the projection P^+ of the product of currents corresponding to the same root:

$$
P^+\left(f_\alpha(t_1)\cdots f_\alpha(t_a)\right) = f^+_\alpha(t_1)f^+_\alpha(t_2;t_1)\cdots f^+_\alpha(t_{a-1};t_1,\ldots,t_{a-2})f^+_\alpha(t_a;t_1,\ldots,t_{a-1}).
$$

The structure of this formula was explained during the proof of the Theorem 2. From analogous considerations (see next Appendix) one can obtain that similar triangular decomposition is valid for the projection P^- of the product of currents with the same root index:

$$
P^-\left(f_\beta(s_1)\cdots f_\beta(s_b)\right) = (-1)^b f^-_\beta(s_1;s_2,\ldots,s_b)f^-_\beta(s_2;s_3,\ldots,s_b)\cdots f^-_\beta(s_{b-1};s_b)f^-_\beta(s_b),
\tag{A.1}
$$

where currents $f^-_\beta(s_k;s_{k+1},\ldots,s_b)$ are defined as sums

$$
f^-_\beta(s_k;s_{k+1},\ldots,s_b) = f^-_\beta(s_k) - \sum_{m=k+1}^{b}\tilde\varphi_{s_m}(s_k;s_{k+1},\ldots,s_b)f^-_\beta(s_m)\ ,
\tag{A.2}
$$

and rational functions $\tilde\varphi_{s_j}(s;s_1,\dots,s_b)$

$$\tilde\varphi_{s_j}(s;s_1,\dots,s_b)=\prod_{i=1,\ i\neq j}^{b}\frac{s-s_i}{s_j-s_i}\prod_{i=1}^{b}\frac{qs_j-q^{-1}s_i}{qs-q^{-1}s_i} \tag{A.3}$$

as functions of the variable s, have simple poles at the points $s=q^{-2}s_i$, $i=1,\dots,b$, tend to zero when $s\to\infty$, and satisfy the property: $\varphi_{s_j}(s_i;s_1,\dots,s_b)=\delta_{ij}$. These conditions define the set of functions $\tilde\varphi_{s_j}(s;s_1,\dots,s_b)$ in a unique way.

As well as for projection P^+, the projection (A.1) for he product of currents can be written in the inverse order

$$P^-\left(f_\beta(s_1)\cdots f_\beta(s_b)\right)=(-1)^b\prod_{1\le i<j\le b}\frac{q^{-1}s_i-qs_j}{qs_i-q^{-1}s_j}\ f_\beta^-(s_b;s_{b-1},\dots,s_1)\cdots f_\beta^-(s_2;s_1)f_\beta^-(s_1)\ .$$

Expression (A.1) can be also written in the form of the integral transform of the product of the total currents

$$P^-\left(f_\beta(s_1)\cdots f_\beta(s_b)\right)=\prod_{i<j}\frac{s_i-s_j}{qs_i-q^{-1}s_j}\oint\prod_{k=1}^{b}\frac{du_k}{u_k}Y(^{\hat\omega}\overline{u},^{\hat\omega}\overline{s})f_\beta(u_1)\cdots f_\beta(u_b)=$$

$$=\prod_{i<j}\frac{s_i-s_j}{qs_i-q^{-1}s_j}\oint\prod_{k=1}^{b}\frac{du_k}{u_k}\frac{1}{1-s_k/u_k}\prod_{i=k+1}^{b}\frac{q-q^{-1}s_i/u_k}{1-s_i/u_k}f_\beta(u_1)\cdots f_\beta(u_b)\ ,$$

which was already used in the previous Section for the proof of the Theorem 1.

2. The calculation of the projection $P^-(f_\alpha(t_1)\cdots f_\alpha(t_a)f_\beta(s_1)\cdots f_\beta(s_b))$ is reduced as well to the calculations of projections of strings

$$\begin{aligned}&P^-\left(f_\alpha(t_1)\cdots f_\alpha(t_a)f_\beta(s_1)\cdots f_\beta(s_b)\right)=\\&=\sum_{k=0}^{\min\{a,b\}}\frac{1}{k!(a-k)!(b-k)!}\ \overline{\mathrm{Sym}}_t^{\,a}\overline{\mathrm{Sym}}_s^{\,b}\Big(P^-\left(f_\alpha(t_1)\cdots f_\alpha(t_{a-k})\right)\times\\&\quad P^-\left(f_{\alpha+\beta}(qs_1)\cdots f_{\alpha+\beta}(qs_k)f_\beta(s_{k+1})\cdots f_\beta(s_b)\right)\ Z(q^{-1}t_{a-k+1},...,q^{-1}t_a;s_1,\dots,s_k)\Big)\ ,\end{aligned} \tag{A.4}$$

where the series $Z(\overline{t},\overline{s})$ is defined in (3.9) and the projection of a string is given by the formula

$$\begin{aligned}&P^-\left(f_{\alpha+\beta}(qs_1)\cdots f_{\alpha+\beta}(qs_k)f_\beta(s_{k+1})\cdots f_\beta(s_b)\right)=\\&=\prod_{1\le i<k+1\le j\le b}\frac{qs_i-q^{-1}s_j}{s_i-s_j}\prod_{1\le i<j\le b}\frac{q^{-1}s_i-qs_j}{qs_i-q^{-1}s_j}\\&\quad\times P^-\left(f_\beta(s_b;s_{b-1},\dots,s_1)\right)\cdots P^-\left(f_\beta(s_{k+1};s_k,\dots,s_1)\right)\\&\quad\times P^-\left(f_{\alpha+\beta}(qs_k;qs_{k-1},\dots,qs_1)\right)\cdots P^-\left(f_{\alpha+\beta}(qs_1)\right)\ .\end{aligned} \tag{A.5}$$

The currents $f_\gamma(t_i;t_{i-1},\dots,t_1)$ in this formula for the roots $\gamma=\beta,\alpha+\beta$, are defined by the relations (A.2) with coefficient functions (A.3), which are invariant with respect to the simultaneous scaling of all variables. Single currents projections are defined by the formulas (3.13), such that

$P^-\left(f_\alpha(t)\right) = -f^-_\alpha(t)\,,\;\; P^-\left(f_{\alpha+\beta}(qs)\right) = -(f_\alpha[0]f^-_\beta(s) - q^{-1}f^-_\beta(s)f_\alpha[0])\,,\;\; P^-\left(f_\beta(s)\right) = -f^-_\beta(s)\,.$ We have, in particular,

$$\begin{aligned}
P^-\left(f_\alpha(t_1)f_\alpha(t_2)\right) &= \left(f^-_\alpha(t_1) - \frac{(q-q^{-1})t_2}{qt_1-q^{-1}t_2}f^-_\alpha(t_2)\right)f^-_\alpha(t_2) = \\
&= \frac{q^{-1}t_1-qt_2}{qt_1-q^{-1}t_2}\left(f^-_\alpha(t_2) - \frac{(q-q^{-1})t_1}{qt_2-q^{-1}t_1}f^-_\alpha(t_1)\right)f^-_\alpha(t_1),\\
P^-\left(f_{\alpha+\beta}(qs_1)f_\beta(s_2)\right) &= \frac{q^{-1}s_1-qs_2}{s_1-s_2}\left(f^-_\beta(s_2) - \frac{(q-q^{-1})s_1}{qs_2-q^{-1}s_1}f^-_\beta(s_1)\right)P^-(f_{\alpha+\beta}(qs_1))\;.
\end{aligned}$$

3. The proof of the formula (A.4) is based on one more reformulation of the Serre relations which uses the right adjoint action $\tilde{\mathrm{ad}}^{(D)}_x(y)$, see (4.2)

$$\tilde{\mathrm{ad}}^{(D)}_{f_\alpha(t_1)f_\alpha(t_2)}\left(f_\beta(s)\right) = 0 \tag{A.6}$$

and on the identity, analogous to the one, proved in the Proposition 5.4. Namely, for $k\le b$

$$\begin{aligned}
P^-\left(\tilde{\mathrm{ad}}^{(D)}_{f_\alpha(t_1)\cdots f_\alpha(t_k)}\left(f_\beta(s_1)\cdots f_\beta(s_b)\right)\right) &= \frac{1}{k!(b-k)!}\overline{\mathrm{Sym}}^{\,b}_s\overline{\mathrm{Sym}}^{\,k}_t\left(\prod_{i=1}^k\prod_{\ell=i+1}^k\frac{qt_\ell-s_i}{t_\ell-qs_i}\right.\times\\
\times\; P^-\left(\tilde{\mathrm{ad}}^{(D)}_{f_\alpha(t_1)}\left(f_\beta(s_1)\right)\cdots\tilde{\mathrm{ad}}^{(D)}_{f_\alpha(t_k)}\left(f_\beta(s_k)\right)f_\beta(s_{k+1})\cdots f_\beta(s_b)\right)&\left.\vphantom{\prod_{i=1}^k}\right)\;.
\end{aligned}\tag{A.7}$$

Appendix B. Commutation relations with projections of currents

The commutation relations (2.21) imply the rules how to move half-currents $f^-(z)$ to the left and half-currents $f^+(z)$ to the right through the total crrents.

Proposition 5.6

$$\begin{aligned}
f^+_\alpha(z)f_\beta(w) &= \frac{qz-w}{z-qw}f_\beta(w)f^+_\alpha(z)+\\
&+\;\frac{qw(q^{-1}-q)}{z-qw}f_\beta(w)f^+_\alpha(qw) + \frac{qw}{z-qw}f_{\alpha+\beta}(qw)\,,
\end{aligned}\tag{B.1}$$

$$\begin{aligned}
f_\alpha(z)f^-_\beta(w) &= \frac{qz-w}{z-qw}f^-_\beta(w)f_\alpha(z)+\\
&+\;\frac{z(q^{-1}-q)}{z-qw}f^-_\beta(q^{-1}z)f_\alpha(z) - \frac{z}{z-qw}f_{\alpha+\beta}(z)\,.
\end{aligned}\tag{B.2}$$

Proof, for example, of the relation (B.1) is based on the decomposition of the kernel

$$\frac{qu-w}{(u-qw)(z-u)} = \frac{qz-w}{z-qw}\,\frac{1}{z-u} + \frac{qw(q^{-1}-q)}{z-qw}\,\frac{1}{qw-u}$$

into sum of two kernels. □

Analogously from the relations (4.6), we have

Proposition 5.7 *Nontrivial commutation relations between total and half-currents are*

$$\begin{aligned} f^+_\alpha(z)f_{\alpha+\beta}(w) &= \frac{q^{-1}z-qw}{z-w}f_{\alpha+\beta}(w)f^+_\alpha(z)+\frac{(q-q^{-1})w}{z-w}f_{\alpha+\beta}(w)f^+_\alpha(w)\,, \\ f_\alpha(z)f^-_{\alpha+\beta}(w) &= \frac{q^{-1}z-qw}{z-w}f^-_{\alpha+\beta}(w)f_\alpha(z)+\frac{(q-q^{-1})z}{z-w}f^-_{\alpha+\beta}(z)f_\alpha(z)\,, \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} f^+_{\alpha+\beta}(qz)f_\beta(w) &= \frac{q^{-1}z-qw}{z-w}f_\beta(w)f^+_{\alpha+\beta}(qz)+\frac{(q-q^{-1})w}{z-w}f_\beta(w)f^+_{\alpha+\beta}(q^{p-i}w)\,, \\ f_{\alpha+\beta}(qz)f^-_\beta(w) &= \frac{q^{-1}z-qw}{z-w}f^-_\beta(w)f_{\alpha+\beta}(qz)+\frac{(q-q^{-1})z}{z-w}f^-_\beta(z)f_{\alpha+\beta}(qz)\,, \\ f_{\alpha+\beta}(z)f^-_{\alpha+\beta}(w) &= \frac{q^{-1}z-qw}{qz-q^{-1}w}f^-_{\alpha+\beta}(w)f_{\alpha+\beta}(z)+\frac{z(q-q^{-1})}{qz-q^{-1}w}(1+q^2)f^+_{\alpha+\beta}(q^2z)f_{\alpha+\beta}(z)\,. \end{aligned} \tag{B.4}$$

Remark. The meaning of the commutation relations in the Proposition 5.7 is that one can move half-currents $f^-_\gamma(w)$ to the left and half-currents $f^+_\gamma(z)$ to the right through total currents such that total currents are not changing and this exchange produces only rational functions and sum of the corresponding half-currents in the shifted points. In this paper we used often these properties of exchange between total and half-currents.

The following proposition describes the commutation relations between projections of composed root current and projection of simple root currents.

Proposition 5.8

$$\begin{aligned} &P^+\left(f_{\alpha+\beta}(t_1)\right)f^+_\alpha(t_2)-qf^+_\alpha(t_2)P^+\left(f_{\alpha+\beta}(t_1)\right)= \\ &\quad=\frac{q-q^{-1}}{t_1-t_2}\left(f^+_\alpha(t_1)-f^+_\alpha(t_2)\right)\big(t_1P^+\left(f_{\alpha+\beta}(t_1)\right)-t_2P^+\left(f_{\alpha+\beta}(t_2)\right)\big)\,, \\ &f^-_\beta(s_1)P^-\left(f_{\alpha+\beta}(qs_2)\right)-q^{-1}P^-\left(f_{\alpha+\beta}(qs_2)\right)f^-_\beta(s_1)= \\ &\quad=\frac{q-q^{-1}}{s_1-s_2}\big(s_1P^-\left(f_{\alpha+\beta}(qs_1)\right)-s_2P^-\left(f_{\alpha+\beta}(qs_2)\right)\big)\left(f^-_\beta(s_1)-f^-_\beta(s_2)\right). \end{aligned}$$

Proof consists in the application of the screening operator $S_{f_\beta[0]}$ to the equality (5.5) and of $\tilde{S}_{f_\alpha[0]}$ to the analogous equality, where projection of currents $f^+_\alpha(t_i)$ are replaced by $f^-_\beta(s_i)$. It also uses the Serre relations written in terms of the projections of currents in the form

$$\begin{aligned} &qf^+_\alpha(t_1)P^+\left(f_{\alpha+\beta}(t_2)\right)+qf^+_\alpha(t_2)P^+\left(f_{\alpha+\beta}(t_1)\right)= \\ &\quad= P^+\left(f_{\alpha+\beta}(t_1)\right)f^+_\alpha(t_2)+P^+\left(f_{\alpha+\beta}(t_2)\right)f^+_\alpha(t_1)\,, \\ &qf^-_\beta(s_1)P^-\left(f_{\alpha+\beta}(s_2)\right)+qf^-_\beta(s_2)P^-\left(f_{\alpha+\beta}(s_1)\right)= \\ &\quad= P^-\left(f_{\alpha+\beta}(s_1)\right)f^-_\beta(s_2)+P^-\left(f_{\alpha+\beta}(s_2)\right)f^-_\beta(s_1)\,. \end{aligned}$$

□

Appendix C. A direct derivation of Theorem 1

First step of this derivation is the same as in (5.12). Second, we observe that first projection P^+ in (5.12) vanishes for $k>a$. This explains the upper limit of summation in (3.10). To prove this formula we will calculate the projection

$$P^+\left(f_\alpha(t_1)\cdots f_\alpha(t_a)P^-\left(f_\beta(s_1)\cdots f_\beta(s_k)\right)\right)\ ,$$

or

$$P^+\left(f_\alpha(t_1)\cdots f_\alpha(t_a)f^-_\beta(s_1;s_2,\ldots,s_k)\cdots f^-_\beta(s_{k-1};s_k)f^-_\beta(s_k)\right)\ ,$$

moving half-currents $f^-_\beta(s_m;s_{m+1},\cdots,s_k)$, $m=1,\ldots,k$ to the left, using commutation relations (B.2). According to this commutation relation the composed currents, $f_{\alpha+\beta}(t_j)$ will be created at the position $j=1,\ldots,a$. Next we will move these composed currents to the right, using the commutation relation (4.6). Here we use again the assertions of the Proposition 5.1. After these commutations we will obtain the sum over all non-ordered subset $J=\{j_1,\ldots,j_k\}\in\{1,\ldots,a\}$

$$\prod_{i<j}^k\frac{s_i-s_j}{qs_i-q^{-1}s_j}\ \sum_J\prod_{m=1}^k\left(\prod_{\ell_m=1}^{j_m-1}\frac{qt_{j_m}-q^{-1}t_{\ell_m}}{t_{j_m}-t_{\ell_m}}\prod_{\ell_m=j_m+1}^{a}\frac{q^{-1}t_{j_m}-qt_{\ell_m}}{t_{j_m}-t_{\ell_m}}\right)\times$$
$$\times\prod_{m=1}^k\left(\frac{t_{j_m}}{t_{j_m}-qs_m}\prod_{i=m+1}^k\frac{qt_{j_m}-s_i}{t_{j_m}-qs_i}\right)P^+\left(f_{\alpha+\beta}(t_{j_1})\cdots f_{\alpha+\beta}(t_{j_k})\ \underbrace{f_\alpha(t_1)f_\alpha(t_2)\cdots f_\alpha(t_{a-1})f_\alpha(t_a)}_{\text{currents depending on } t_{j_1},\ldots,t_{j_k}\text{ omitted}}\right).$$

Next step is to use the commutation relations (4.6) to move the group of the composed currents $f_{\alpha+\beta}(t_{j_1})\cdots f_{\alpha+\beta}(t_{j_k})$ under projection from left to the right

$$\prod_{i<j}^k\frac{s_i-s_j}{qs_i-q^{-1}s_j}\ \sum_J\prod_{\substack{i<j\\ i,j\in J}}\frac{q^{-1}t_i-qt_j}{t_i-t_j}\prod_{\substack{i<j\\ i\in J,\ j\notin J}}\frac{q^{-1}t_i-qt_j}{qt_i-q^{-1}t_j}\prod_{m=1}^k\left(\frac{t_{j_m}}{t_{j_m}-qs_m}\prod_{i=m+1}^k\frac{qt_{j_m}-s_i}{t_{j_m}-qs_i}\right)\times$$
$$\times\ P^+\left(\underbrace{f_\alpha(t_1)f_\alpha(t_2)\cdots f_\alpha(t_{a-1})f_\alpha(t_a)}_{\text{currents depending on } t_{j_1},\ldots,t_{j_k}\text{ omitted}}\ f_{\alpha+\beta}(t_{j_1})\cdots f_{\alpha+\beta}(t_{j_k})\right).\tag{C.1}$$

Now we use the fact that the summation in (C.1) runs over non-ordered set J of the size k. This means that this summation can be decomposed into two summations: first, over all different, but ordered choices $\{j_1<j_2<\cdots<j_k\}$ from the set $\{1,2,\ldots,a\}$ and second, over all permutations among fixed $\{j_1,j_2,\ldots,j_k\}$. Due to the commutation relations (4.6) between composed currents this second summation can be written as q^{-1}-symmetrization over fixed subset $\{j_1,j_2,\ldots,j_k\}$ of the function

$$\prod_{m=1}^k\left(\frac{1}{t_{j_m}-qs_k}\prod_{i=m+1}^k\frac{qt_{j_m}-s_i}{t_{j_m}-qs_i}\right)$$

since the rest ingredients of the formula (C.1) are stable under permutation of $\{j_1,j_2,\ldots,j_k\}$. Using now combinatorial identity (3.31), which can be written in the form

$$\prod_{i<j}^k\frac{q^{-1}t_i-qt_j}{t_i-t_j}\ \widetilde{\mathrm{Sym}}_t^{\,k}\left(\prod_{m=1}^k\left(\frac{1}{t_m-qs_m}\prod_{i=m+1}^k\frac{qt_m-s_i}{t_m-qs_i}\right)\right)=$$
$$=\prod_{i<j}^k\frac{qs_i-q^{-1}s_j}{s_i-s_j}\ \overline{\mathrm{Sym}}_s^{\,k}\left(\prod_{m=1}^k\left(\frac{1}{t_m-qs_m}\prod_{i=1}^{m-1}\frac{qt_m-s_i}{t_m-qs_i}\right)\right)$$

one may rewrite formula (C.1) as follows

$$
\begin{aligned}
&\sum_{\substack{J\\ j_1<\cdots<j_k}} \prod_{\substack{i<j\\ i\in J,\ j\notin J}} \frac{q^{-1}t_i-qt_j}{qt_i-q^{-1}t_j}\ \overline{\mathrm{Sym}}_s^{\,k}\left(\prod_{m=1}^{k}\frac{t_{j_m}}{t_{j_m}-qs_k}\prod_{i=1}^{m-1}\frac{qt_{j_m}-s_i}{t_{j_m}-qs_i}\right)\times\\
&\times P^+\left(\underbrace{f_\alpha(t_1)f_\alpha(t_2)\cdots f_\alpha(t_{a-1})f_\alpha(t_a)}_{\text{currents depending on } t_{j_1},\ldots,t_{j_k} \text{ omitted}}\ f_{\alpha+\beta}(t_{j_1})\cdots f_{\alpha+\beta}(t_{j_k})\right).
\end{aligned}
\tag{C.2}
$$

Now the summation over all ordered sets J in (C.2) together with the first product in the first line can be written as q-symmetrization over all variables $t_1,\cdots,t_a$ and we obtain (5.19). Repeating arguments given at the end of Subsection 5.2 we finish the direct proof of the Theorem 1.

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