

**Hodge numbers of
moduli spaces of stable bundles on
K3 surfaces**

**L. Göttsche
D. Huybrechts**

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

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Very few higher dimensional symplectic manifolds are known. Beauville has shown that the Hilbert schemes of points on a K3 surface are symplectic [B1]. Other examples are provided by the moduli spaces of stable vector bundles on these surfaces [Mu1]. Very often both spaces are closely related by a birational correspondence, but in general they are not isomorphic ([Mu2], p. 167). Beauville also proved that all except one of the deformation directions of the Hilbert scheme are obtained by deforming the underlying K3 surface. It is natural to ask if in fact the moduli spaces are obtained by deforming the Hilbert scheme in this extra direction. The aim of this paper is to show that for some of the moduli spaces an important class of deformation invariants, namely the Hodge numbers, coincide with those of an appropriate Hilbert scheme. The following theorem is proven:

Theorem: *Let X be a K3 surface, L a primitive big and nef line bundle and H a generic polarization. If $\overline{M}_H(L, c_2)$ denotes the moduli space of rank two semi-stable torsion-free sheaves and $\dim \overline{M}_H(L, c_2) > 8$ then its Hodge numbers coincide with the Hodge numbers of the Hilbert scheme of $l := 2c_2 - \frac{L^2}{2} - 3$ points, i.e.*

$$h^{p,q}(\overline{M}_H(L, c_2)) = h^{p,q}(\text{Hilb}^l(X)).$$

By a generic polarization we mean a polarization which does not lie on any wall, i.e. any H -semi-stable sheaf is stable. Note that by the smoothness criterion [Mu1] the moduli space $\overline{M}_H(L, c_2)$ is smooth and projective of dimension $4c_2 - L^2 - 6$ if it is non-empty. Moreover, by dimension counting (cf. [T], lemma 3.1, [O'G], prop. 7.2.1) the moduli space of μ -stable vector bundles is dense in $\overline{M}_H(L, c_2)$. We would like to mention that the Hodge numbers of the Hilbert scheme of points on a surface can be expressed in terms of the Hodge numbers of the surface [GS, Ch]. In fact, even the Hodge structure is known. We would like to compare the Hodge structure of the moduli spaces and of the Hilbert schemes, but even in the special case dealt with in section 1 we cannot prove that they coincide.

Our work was motivated by a talk of J. Le Potier in Lambrecht in May 1994. In this talk he explained how to use moduli spaces of coherent systems (or framed modules) to

compute the $SU(2)$ –Donaldson polynomials for K3 surfaces. This was done before by O’Grady by other methods [O’G1]. We tried to use Le Potier’s approach to compute the $SO(3)$ –polynomials which turned out to be even easier. (In the meantime the computation of them using O’Grady’s method has appeared in [Na]. Therefore our computation is not included.) Then we realized that the approach also works in the context of this paper.

Notations: X will always be a projective K3 surface.

P_E - the Hilbert polynomial of a sheaf E .

$M_H(L, c_2)$ - the moduli space of rank two vector bundles with determinant $L \in \text{Pic}(X)$ and second Chern class c_2 which are μ –stable with respect to a polarization H .

$\overline{M}_H(L, c_2)$ - the moduli space of rank two torsion-free sheaves with determinant L and second Chern class c_2 which are semi-stable with respect to H .

$M_H(L, c_2, D, \delta)$ - moduli space of framed modules $(E, \alpha : E \rightarrow D)$, E locally free with the given invariants, which are μ –stable with respect to δ and H .

$\overline{M}_H(L, c_2, D, \delta)$ - the moduli space of all semi-stable framed modules.

If $H = L$ we drop H in the notation.

1 A special case

In this section we prove the theorem in the case that $\text{Pic}(X) = \mathbb{Z} \cdot L$ and $c_2 = \frac{L^2}{2} + 3$.

1.1 The birational correspondence to the Hilbert scheme

Throughout this section we will assume that the Picard group is generated by an ample line bundle L , i.e. $\text{Pic}(X) = \mathbb{Z} \cdot L$. Under this assumption a torsion-free sheaf with determinant L is μ –stable if and only if it is μ –semi-stable.

For the convenience of the reader we recall the stability condition for framed modules ([HL]): Let $\delta = \delta_1 \cdot n + \delta_0$, $D \in \text{Pic}(X)$ and E be a torsion-free rank two sheaf. A framed module (E, α) consists of E and a non-trivial homomorphism $\alpha : E \rightarrow D$. It is (semi-)stable if $P_{\text{Ker}\alpha}(\leq) P_E/2 - \delta/2$ and for all rank one subsheaves $M \subset E$ the inequality $P_M(\leq) P_E/2 + \delta/2$ holds.

In fact, any semi-stable framed module is torsion-free. In [HL] it was shown that there exists a coarse projective moduli space of semi-stable framed modules.

Lemma 1.1 *Let $D = L$ and $0 < \delta_1 < L^2$. Then a framed module (E, α) is μ –stable if and only if E is μ –stable. The moduli space $\overline{M}_H(L, c_2, D, \delta)$ is independent of the specific δ in this range.*

Proof: Let (E, α) be semi-stable, then $\mu(M) \leq L^2/2 + \delta_1/2$ for all $M = L^{\otimes n} \otimes I_Z \subset E$. Thus $nL^2 \leq L^2/2 + \delta_1 < L^2$, i.e. $n < 1$. Hence E is μ –stable. If E is μ –stable, then

(E, α) is μ -stable if $\mu(\text{Ker}\alpha) < L^2/2 - \delta_1/2$. But writing $\text{Ker}(\alpha) = L^{\otimes n} \otimes I_Z$ and using the stability of E we conclude $n < 1$. Hence $\mu(\text{Ker}\alpha) \leq 0 < L^2/2 - \delta_1/2$. The second statement follows immediately. \square

Henceforth δ is chosen as in the lemma. Note that as for sheaves also for framed modules μ -stability is equivalent to μ -semi-stability. It can also be shown that both moduli spaces $M(L, c_2, L, \delta)$ and $\overline{M}(L, c_2)$ are fine. The universality property of the moduli space induces a morphism

$$\overline{\varphi} : \overline{M}(L, c_2, L, \delta) \longrightarrow \overline{M}(L, c_2).$$

Note that by the previous lemma the fibre of $\overline{\varphi}$ over $[E]$ is isomorphic to $\mathbb{P}(\text{Hom}(E, L))$.

Lemma 1.2 *If $c_2 \leq L^2/2 + 3$, then φ is surjective.*

Proof: It suffices to show that for a μ -stable vector bundle E there is always a non-trivial homomorphism $E \rightarrow L$. Since $\text{Hom}(E, L) = H^0(X, E)$ and $H^2(X, E) \cong H^0(X, E^*) = 0$ by the stability of E , the Riemann-Roch-Hirzebruch formula $\chi(E) = L^2/2 - c_2 + 4$ shows that under the assumption $h^0(X, E) > 0$. \square

Lemma 1.3 *Let $N(L, c_2, L, \delta)$ be the set of all $(E, \alpha) \in \overline{M}(L, c_2, L, \delta)$ such that $\text{Ker}(\alpha) \cong \mathcal{O}_X$. It is a closed subset, which contains all stable pairs (E, α) with E locally free.*

Proof: If E is locally free, then $\text{Ker}(\alpha)$ has to be locally free. By stability it is thus isomorphic to \mathcal{O}_X . $N(L, c_2, L, \delta)$ is closed, since $\text{Ker}(\alpha) \cong \mathcal{O}_X$ if and only if $\text{length}(\text{Coker}(\alpha)) = c_2$, i.e. if the length is maximal; this is a closed condition. \square

Under the condition of 1.2 we have a surjective morphism

$$\varphi : N(L, c_2, L, \delta) \longrightarrow \overline{M}(L, c_2).$$

By 1.3 any framed module $(E, \alpha) \in N(L, c_2, L, \delta)$ sits in an extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \xrightarrow{\alpha} I_Z \otimes L \longrightarrow 0,$$

where I_Z is the ideal sheaf of a codimension two cycle of length c_2 . Thus we can define a morphism

$$\psi : N(L, c_2, L, \delta) \longrightarrow \text{Hilb}^{c_2}(X)$$

by mapping (E, α) to $[\text{Coker}(\alpha)]$.

Lemma 1.4 *ψ is surjective, if $c_2 \geq L^2/2 + 3$.*

Proof: It is enough to show that $\text{Ext}^1(I_Z \otimes L, \mathcal{O}_X) \neq 0$ for all $Z \in \text{Hilb}^{c_2}(X)$. By the assumption $h^0(X, L|_Z) - h^0(X, L) \geq 1$. Thus $h^1(L \otimes I_Z) \geq 1$. Now use $\text{Ext}^1(I_Z \otimes L, \mathcal{O}_X) \cong H^1(X, I_Z \otimes L)^*$. \square

We have seen that any $(E, \alpha) \in N(L, c, L, \delta)$ induces an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \xrightarrow{\alpha} I_Z \otimes L \longrightarrow 0.$$

Conversely, any section $s \in H^0(X, E)$ of $E \in \overline{M}(L, c)$ gives a homomorphism $\alpha : E \rightarrow L$ with $\text{Ker}(\alpha) \cong \mathcal{O}_X$. Thus the fibre of $\varphi : N(L, c, L, \delta) \rightarrow \overline{M}(L, c)$ over $[E]$ is isomorphic to $\mathbb{P}(H^0(X, E))$. In fact, $N(L, c, L, \delta)$ can be identified with Le Potier's moduli space of coherent systems of rank one [LP].

The picture we get in the case $c_2 = c := L^2/2 + 3$ is described by the following diagram.

$$\begin{array}{ccc} & N(L, c, L, \delta) & \\ \varphi \swarrow & & \searrow \psi \\ \overline{M}(L, c) & & \text{Hilb}^c(X) \end{array}$$

Both morphisms φ and ψ are birational. This is due to the fact that for the generic $[Z] \in \text{Hilb}^c(X)$ the restriction map $H^0(X, L) \rightarrow H^0(X, L_Z)$ is injective and hence $h^1(X, I_Z \otimes L) = 1$. This shows that ψ is generically an isomorphism. Since the fibres of φ are connected and both spaces are of the same dimension, also φ is birational. Note that in particular the moduli space $\overline{M}(L, c)$ is irreducible.

Results about birationality of certain moduli spaces and corresponding Hilbert schemes have been known for some time, e.g. Zuo has shown that $M_H(\mathcal{O}_X, n^2 H^2 + 3)$ is birational to $\text{Hilb}^{2n^2 H^2 + 3}(X)$ (cf. [Z]). The moduli spaces of framed modules make this relation more explicit. They are used in the next section to show that the Hodge numbers of the moduli space and the Hilbert schemes coincide.

1.2 Comparison of the Hodge numbers

First, we recall the notion of virtual Hodge polynomials [D], [Ch].

For any quasi-projective variety X there exists a polynomial $e(X, x, y)$ with the following properties:

i) If X is smooth and projective then

$$e(X, x, y) = h(X, -x, -y) := \sum_{p,q} (-1)^{p+q} h^{p,q}(X) x^p y^q.$$

ii) If $Y \subset X$ is Zariski closed and U its complement then

$$e(X, x, y) = e(Y, x, y) + e(U, x, y).$$

iii) If $X \rightarrow Y$ is a Zariski locally trivial fibre bundle with fibre F then

$$e(X, x, y) = e(Y, x, y) \cdot e(F, x, y).$$

iv) If $X \rightarrow Y$ is a bijective morphism then $e(X, x, y) = e(Y, x, y)$.

In particular, if

$$\begin{array}{ccc} & Z & \\ & \swarrow & \searrow \\ X & & Y \end{array}$$

is a diagram of quasi-projective varieties, where both morphisms are bijective to \mathbb{P}_n -bundles, then $e(X, x, y) \cdot e(\mathbb{P}_n, x, y) = e(Z, x, y) = e(Y, x, y) \cdot e(\mathbb{P}_n, x, y)$. Hence $e(X, x, y) = e(Y, x, y)$.

The idea to prove that $\overline{M}(L, c)$ and $Hilb^c(X)$, with $c := \frac{L^2}{2} + 3$, have the same Hilbert polynomial is to stratify both by locally closed subsets $\overline{M}(L, c)_k$ and $Hilb^c(X)_k$ such that the birational correspondence given by the moduli space of framed modules induces \mathbb{P}_{k-1} -bundles $N(L, c, L, \delta)_k \rightarrow \overline{M}(L, c)_k$ and $N(L, c, L, \delta)_k \rightarrow Hilb^c(X)_k$. One concludes $e(\overline{M}(L, c)_k, x, y) = e(Hilb^c(X)_k, x, y)$ and hence $e(\overline{M}(L, c), x, y) = \sum_k e(\overline{M}(L, c)_k, x, y) = \sum_k e(Hilb^c(X)_k, x, y) = e(Hilb^c(X), x, y)$.

We first define the stratification.

Definition 1.5 $Hilb^c(X)_k := \{[Z] \in Hilb^c(X) \mid h^1(X, I_Z \otimes L) = k\}$

$N(L, c, L, \delta)_k := \psi^{-1}(Hilb^c(X)_k)$

$\overline{M}(L, c)_k := \varphi(N(L, c, L, \delta)_k)$

Using the universal subscheme $\mathcal{Z} \subset X \times Hilb^c(X)$ with the two projections p and q to X and $Hilb^c(X)$, resp., and the semi-continuity applied to the sheaf $I_{\mathcal{Z}} \otimes p^*(L)$ and the projection q it is easy to see that this defines a stratification into locally closed subschemes. All strata are given the reduced induced structure.

We want to show that both morphisms

$$N(L, c, L, \delta)_k \rightarrow Hilb^c(X)_k$$

and

$$N(L, c, L, \delta)_k \rightarrow \overline{M}(L, c)_k$$

are bijective to \mathbb{P}_{k-1} -bundles. In fact, they are \mathbb{P}_{k-1} -bundles, but by property iv) of the virtual Hodge polynomials we only need the bijectivity.

Definition 1.6 Let $\mathcal{A}_k := \pi_*(\mathcal{E}_k)$, where $\pi : N(L, c, L, \delta)_k \times X \rightarrow N(L, c, L, \delta)_k$ denotes the projection and \mathcal{E}_k is the restriction of the universal sheaf \mathcal{E} , and let $\mathcal{B}_k := Ext_q^1((I_{\mathcal{Z}})_k \otimes p^*(L), \mathcal{O}_X)$ be the relative Ext-sheaf, where $(I_{\mathcal{Z}})_k$ denotes the restriction of $I_{\mathcal{Z}}$ to $Hilb^c(X)_k \times X$

Lemma 1.7 \mathcal{A}_k and \mathcal{B}_k are locally free sheaves on $\overline{M}(L, c)_k$ and $\text{Hilb}^c(X)_k$, resp., and compatible with base change, i.e. $\mathcal{A}_k([E]) \cong H^0(X, E)$ and $\mathcal{B}_k([Z]) \cong \text{Ext}^1(I_Z \otimes L, \mathcal{O}_X)$.

Proof: By definition and using Serre-duality we see that $\text{Hilb}^c(X)_k = \{Z \mid \dim \text{Ext}^1(I_Z \otimes L, \mathcal{O}_X) = k\}$ and that it is reduced. Thus the claim for \mathcal{B}_k follows immediately from the base change theorem for global Ext-groups [BPS]. In order to prove the assertion for \mathcal{A}_k it suffices to show that $\overline{M}(L, c)_k = \{E \mid h^0(X, E) = k\}$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow I_Z \otimes L \longrightarrow 0.$$

Then $E \in \overline{M}(L, c)_k$ if and only if $h^1(X, I_Z \otimes L) = k$ if and only if $h^0(X, I_Z \otimes L) = k - 1$ if and only if $h^0(X, E) = k$. \square

The kernel of the universal framed module on $N(L, c, L, \delta) \times X$ restricted to $N(L, c, L, \delta)_k$ induces a morphism to $\mathbb{P}(\mathcal{A}_k)$ which is obviously bijective. Analogously, by the universality of $\mathbb{P}(\mathcal{B}_k)$ (cf. [La]) the universal framed module over $N(L, c, L, \delta) \times X$ completed to an exact sequence and restricted to the stratum induces a bijective morphism of $N(L, c, L, \delta)_k$ to $\mathbb{P}(\mathcal{B}_k)$.

We summarize:

Proposition 1.8 *If X is a K3 surface with $\text{Pic}(X) = \mathbb{Z} \cdot L$ and $c_2 = L^2/2 + 3 > 3$ then $h^{p,q}(\overline{M}(L, c_2)) = h^{p,q}(\text{Hilb}^{c_2}(X))$.* \square

Both manifolds $\overline{M}(L, c_2)$ and $\text{Hilb}^{c_2}(X)$ are symplectic. One might conjecture that in general two birational symplectic manifolds have the same Hodge numbers or even isomorphic Hodge structures, but we don't know how to prove this.

2 The general case

By deforming the underlying K3 surface the proof of the theorem is reduced to the case considered in section 1.

2.1 Deformation of K3 surfaces

The following two statements about the existence of certain deformations of a given K3 surface will be needed.

2.1.1 *Let X be a K3 surface, $L \in \text{Pic}(X)$ a primitive nef and big line bundle. Then there exists a smooth connected family $\mathcal{X} \longrightarrow S$ of K3 surfaces and a line bundle \mathcal{L} on \mathcal{X} such that:*

$$\cdot \mathcal{X}_0 \cong X \text{ and } \mathcal{L}_0 \cong L.$$

· $\text{Pic}(\mathcal{X}_t) = \mathbb{Z} \cdot \mathcal{L}_t$ for all $t \neq 0$ (\mathcal{L}_t is automatically ample).

Proof: The moduli space of primitive pseudo-polarized K3 surfaces is irreducible ([B2]). Since any even lattice of index $(1, \rho - 1)$ with $\rho \leq 10$ can be realized as a Picard group of a K3 surface ([Ni],[Mor]) the generic pseudo-polarized K3 surface has Picard group \mathbb{Z} . \square

2.1.2 Let X be a K3 surface whose Picard group is generated by an ample line bundle L , i.e. $\text{Pic}(X) = \mathbb{Z} \cdot L$. Furthermore, let $d \geq 5$ be an integer. Then there exists a smooth connected family $\mathcal{X} \rightarrow S$ of K3 surfaces and a line bundle \mathcal{L} on \mathcal{X} such that:

· $(\mathcal{X}_{t_0}, \mathcal{L}_{t_0}) \cong (X, L)$ for some point $t_0 \in S \setminus \{0\}$.

· $\text{Pic}(\mathcal{X}_t) \cong \mathbb{Z} \cdot \mathcal{L}_t$ for all $t \neq 0$.

· $\text{Pic}(\mathcal{X}_0) = \mathbb{Z} \cdot \mathcal{L}_0 \oplus \mathbb{Z} \cdot D$, where D is represented by a smooth rational curve, both line bundles \mathcal{L}_0 and $\mathcal{L}_0(2D)$ are ample and primitive and the intersection matrix is

$$\begin{pmatrix} L^2 & d \\ d & -2 \end{pmatrix}$$

Proof: Again we use the irreducibility of the moduli space of primitive polarized K3 surfaces. The existence of a triple $(\mathcal{X}_0, \mathcal{L}_0, D)$ with ample \mathcal{L}_0 , smooth rational D and the given intersection form was shown by Oguiso [Og]. It remains to show that $\mathcal{L}_0(2D)$ is ample. Obviously, $\mathcal{L}_0(2D)$ is big and for any irreducible curve $C \neq D$ the strict inequality $(\mathcal{L}_0(2D)) \cdot C > 0$ holds. The assumption on d implies $(\mathcal{L}_0(2D)) \cdot D > 0$. Note that the extra assumption $L^2 \geq 4$ in [Og] is only needed for the very ampleness of \mathcal{L}_0 which we will not use. \square

2.1.3(a) Let X be a K3 surface whose Picard group is generated by an ample line bundle L , i.e. $\text{Pic}(X) = \mathbb{Z} \cdot L$. If $L^2 > 2$ there exists a smooth connected family $\mathcal{X} \rightarrow S$ of K3 surfaces and a line bundle \mathcal{L} on \mathcal{X} such that:

· $(\mathcal{X}_{t_0}, \mathcal{L}_{t_0}) \cong (X, L)$ for some point $t_0 \in S \setminus \{0\}$.

· $\text{Pic}(\mathcal{X}_t) \cong \mathbb{Z} \cdot \mathcal{L}_t$ for all $t \neq 0$.

· $\text{Pic}(\mathcal{X}_0) = \mathbb{Z} \cdot \mathcal{L}_0 \oplus \mathbb{Z} \cdot D$, where both line bundles \mathcal{L}_0 and $\mathcal{L}_0(2D)$ are ample and primitive and the intersection matrix is

$$\begin{pmatrix} L^2 & 1 \\ 1 & 0 \end{pmatrix}$$

2.1.3(b) If we assume that $L^2 > 6$ we have the same result as in (a) with “ $\mathcal{L}_0(2D)$ is ample” replaced by “ $\mathcal{L}_0(-2D)$ is ample”.

Proof: For both parts we need to prove the existence of a triple (X_0, H, D) with ample and primitive H and $H(2D)$, such that $D^2 = 0$, $H \cdot D = 1$ and $H^2 = 2n > 2$ for given n . By the results of Nikulin we can find a K3 surface with this intersection

form. It remains to show that H and $H(2D)$ can be chosen ample. We can assume that $H \in C^+$, i.e. H is in the positive component of the positive cone (if necessary change (H, D) to $(-H, -D)$). We check that H is not orthogonal to any (-2) class, i.e. for any $\delta := aH + bD$ ($a, b \in \mathbb{Z}$) with $\delta^2 = a^2H^2 + 2ab = -2$ we have $H \cdot \delta \neq 0$. If H were orthogonal to δ this would imply that $aH^2 + b = 0$. Hence $-a^2H^2 = -2$ which contradicts $H^2 > 2$. Thus H is contained in a chamber. Since the Weyl group W_{X_0} , which is generated by the reflection on the walls, acts transitively on the set of chambers, we find $\sigma \in W_{X_0}$ such that $\sigma(H)$ is contained in the chamber $\{w \in C^+ | w\delta > 0 \text{ for all effective } (-2) \text{ classes } \delta\}$. Applying σ to (H, D) we can in fact assume that H is contained in this chamber. On a K3 surface the effective divisors are generated by the effective (-2) classes and $\overline{C^+} \setminus \{0\}$. On both sets H is positive. Thus H is ample. In order to prove that also $H(2D)$ is ample we show that D is effective and irreducible. This follows from the Riemann-Roch-Hirzebruch formula $\chi(\mathcal{O}(D)) = 2$, which implies D or $-D$ effective, and $H \cdot D = 1$. Thus $C \cdot D \geq 0$ for any curve C . Thus $H(2D) \cdot C > 0$. Since $H(2D)$ is big we conclude that $H(2D)$ is ample. To prove (a) we choose $H^2 := L^2$ and use the irreducibility of the moduli space to show that (X, L) degenerates to (X_0, H) . Defining $\mathcal{L}_0 := H$ this proves (a). In order to prove (b) we fix $(H(2D))^2 := L^2$ and let (X, L) degenerate to $(X_0, H(2D))$. The assumption on H translates to $L^2 > 6$. With $\mathcal{L}_0 := H(2D)$ we obtain (b). \square

2.2 Deformation of the moduli space

We start out with the following

Lemma 2.1 *Let E be a simple vector bundle on a K3 surface such that $L := \det(E)$ is big. The joint deformations of E and X are unobstructed, i.e. $\text{Def}(E, X)$ is smooth. Moreover, $\text{Def}(E, X) \rightarrow \text{Def}(X)$ and $\text{Def}(L, X) \rightarrow \text{Def}(X)$ have the same image.*

Proof: The infinitesimal deformations of a bundle E together with its underlying manifold X are parametrized by $H^1(X, \mathcal{D}_0^1(E))$, where $\mathcal{D}_0^1(E)$ is the sheaf of differential operators of order ≤ 1 with scalar symbol. The obstructions are elements in the second cohomology of this sheaf. Using the symbol map we have a short exact sequence

$$0 \longrightarrow \mathcal{E}nd(E) \longrightarrow \mathcal{D}_0^1(E) \longrightarrow \mathcal{T}_X \longrightarrow 0.$$

Its long exact cohomology sequence

$$H^1(X, \mathcal{D}_0^1(E)) \rightarrow H^1(X, \mathcal{T}_X) \rightarrow H^2(X, \mathcal{E}nd(E)) \rightarrow H^2(X, \mathcal{D}_0^1(E)) \rightarrow 0$$

compares the deformations of E , X , and (E, X) . In particular, if E is simple the trace homomorphism $H^2(X, \mathcal{E}nd(E)) \rightarrow H^2(X, \mathcal{O}_X)$ is bijective and the composition with the boundary map $H^1(X, \mathcal{T}_X) \rightarrow H^2(X, \mathcal{E}nd(E))$ is the cup-product with $c_1(E)$. Since

there is exactly one direction in which a big and nef line bundle L cannot be deformed with X the cup-product with $c_1(L) = c_1(E)$ is surjective. Thus $H^1(X, \mathcal{D}_0^1(E)) \rightarrow H^1(X, \mathcal{T}_X)$ is onto the algebraic deformations of X and $H^2(X, \mathcal{D}_0^1(E))$ vanishes. \square

The following lemma will be needed for the next proposition. Its proof is quite similar to what we will use to prove the theorem.

Lemma 2.2 *If $\text{Pic}(X) = \mathbb{Z} \cdot L$, then $\overline{M}(L, c_2)$ is irreducible for $\dim \overline{M}(L, c_2) = 4c_2 - L^2 - 6 > 8$.*

Proof: 1st step: First, we show that $\overline{M}_H(L, \frac{L^2}{2} + 3)$ is irreducible whenever L is an ample line bundle on a K3 surface.

By a result of [Q] the moduli spaces $\overline{M}_H(L, \frac{L^2}{2} + 3)$ and $\overline{M}_L(L, \frac{L^2}{2} + 3)$ are birational. In particular, the number of irreducible components is the same. We consider a deformation as in 2.1.1. The corresponding family of moduli spaces $\overline{M}_{\mathcal{L}_t}(\mathcal{L}_t, \frac{L^2}{2} + 3)$ is proper and by lemma 2.1 every stable bundle on X can be deformed to a stable bundle on any nearby fibre. This shows that $\overline{M}_{\mathcal{L}_0}(\mathcal{L}_0, \frac{L^2}{2} + 3)$ has as many irreducible components as $\overline{M}_{\mathcal{L}_{t \neq 0}}(\mathcal{L}_{t \neq 0}, \frac{L^2}{2} + 3)$, which is irreducible.

2nd step: Assume $e := c_2 - \frac{L^2}{2} - 3 > 0$ and $L^2 > 2$. We apply 2.1.3(a). By the same arguments as above we obtain that the number of irreducible components of $\overline{M}(L, c_2)$ is at most the number of irreducible components of $\overline{M}_{\mathcal{L}_0}(\mathcal{L}_0, c_2)$. Again using [Q] we know that $\overline{M}_{\mathcal{L}_0}(\mathcal{L}_0, c_2)$ is birational to $\overline{M}_{\mathcal{L}_0(2D)}(\mathcal{L}_0, c_2)$. The μ -stable part of the latter is isomorphic to the μ -stable part of $\overline{M}_{\mathcal{L}_0(2D)}(\mathcal{L}_0(2D), c_2 + 1)$. We have $(\mathcal{L}_0(2D))^2 = L^2 + 4 > 2$ and $c_2 + 1 - \frac{(\mathcal{L}_0(2D))^2}{2} - 3 = e - 1$. Therefore we obtain by induction over e and step 1 that $\overline{M}_{\mathcal{L}_0(2D)}(\mathcal{L}_0(2D), c_2 + 1)$ is irreducible. Since the locally free μ -stable sheaves are dense in the moduli spaces, this accomplishes the proof in this case.

3rd step: Here we assume that $e := c_2 - \frac{L^2}{2} - 3 < 0$. By assumption $4c_2 - L^2 - 6 \geq 10$. Hence $c_2 \geq 6$ and $L^2 > 6$. Now we apply 2.1.3(b). The same arguments as in the previous step show that the number of irreducible components of $\overline{M}(L, c_2)$ is at most that of $\overline{M}_{\mathcal{L}_0(-2D)}(\mathcal{L}_0(-2D), c_2 - 1)$. Since $c_2 - 1 - \frac{(\mathcal{L}_0(-2D))^2}{2} - 3 = e + 1$, we can use induction over $-e$ and step 1 to show the irreducibility in this case.

4th step: It remains to consider the case $L^2 = 2$. Here we apply 2.1.2 with $d = 5$. As above we conclude that the number of irreducible components of $\overline{M}(L, c_2)$ is at most that of $\overline{M}_{\mathcal{L}_0(2D)}(\mathcal{L}_0(2D), c_2 + 3)$. Since $(\mathcal{L}_0(2D))^2 = L^2 + 20 - 8 = 14$ we can conclude by step 2 or 3. \square

Mukai seems to know that all moduli spaces of rank two bundles on a K3 surface are irreducible ([Mu2], p. 157). Since we could not find a proof of this in the literature we decided to include the above lemma.

Let X be a K3 surface and L a line bundle on X . For any c_2 there exists a coarse moduli space $\overline{M}_s(L, c_2)$ of simple sheaves of rank two with determinant L and second

Chern class c_2 . $\overline{M}_s(L, c_2)$ can be realized as a non-separated algebraic space ([AK], [KO]). For any polarization H such that H -semi-stability implies H -stability the projective manifold $\overline{M}_H(L, c_2)$ is an open subset of $\overline{M}_s(L, c_2)$.

Note that in the case that $\text{Pic}(X) = \mathbb{Z} \cdot L$ and $H = L$ any simple vector bundle is in fact slope stable. For sheaves the situation is more complicated.

Now let $(\mathcal{X}, \mathcal{L}) \rightarrow S$ be a family of K3 surfaces with a line bundle \mathcal{L} on \mathcal{X} over a smooth curve S . By [AK], [KO] there exists a relative moduli space of simple sheaves, i.e. there exists an algebraic space $\overline{M}_s(\mathcal{L}, c_2)$ and a morphism from it to S such that the fibre over a point $t \in S$ is isomorphic to $\overline{M}_s(\mathcal{L}_t, c_2)$. By a result of Mukai the fibres are smooth [Mu1]. Lemma 2.2 shows that for a family $(\mathcal{X}, \mathcal{L}) \rightarrow S$ both $\overline{M}_s(\mathcal{L}, c_2)$ and $\overline{M}_s(\mathcal{L}, c_2) \rightarrow S$ are smooth (at least over the locally free sheaves).

For the following we want to assume that $\text{Pic}(\mathcal{X}_t) \cong \mathbb{Z} \cdot \mathcal{L}_t$ for $t \neq 0$ and $\mathcal{L}_t^2 > 0$. To shorten notation we denote by $Z^* \rightarrow S^*$ the restriction of a family $Z \rightarrow S$ to $S^* := S \setminus \{0\}$.

Proposition 2.3 *Assume that $\overline{M}_{\mathcal{L}_t}(\mathcal{L}_t, c_2)$ is irreducible for $t \neq 0$. Then for any generic ample $H \in \text{Pic}(\mathcal{X}_0)$ there exists a smooth proper family $Z \rightarrow S$ of projective manifolds such that $Z^* \rightarrow S^*$ has fibres $\overline{M}_{\mathcal{L}_t}(\mathcal{L}_t, c_2)$ and the fibre over 0 is isomorphic to $\overline{M}_H(\mathcal{L}_0, c_2)$. (“The moduli spaces for different H cannot be separated”)*

Proof: By $\overline{M}(\mathcal{L}, c_2)^* \rightarrow S^*$ we denote the family of the moduli spaces $\overline{M}(\mathcal{L}_t, c_2)$. It is proper over S^* and the fibres are smooth and irreducible.

Claim: If $[E] \in \overline{M}_s(\mathcal{L}_0, c_2)$ is a point in the closure T_0 of $\overline{M}_s(\mathcal{L}, c_2)^* \setminus \overline{M}(\mathcal{L}, c_2)^*$ in $\overline{M}_s(\mathcal{L}, c_2)$, then E is not semi-stable with respect to any polarization H : Semi-continuity shows that a point E in the closure has a subsheaf of rank one with determinant $\mathcal{L}_0^{\otimes n}$ with $n > 0$. Hence it is not semi-stable with respect to any polarization. The set T_1 of simple sheaves $[E] \in \overline{M}_s(\mathcal{L}_0, c_2)$ which are not stable with respect to H is a closed subset of $\overline{M}_s(\mathcal{L}, c_2)$. We define Z to be the complement of the union of T_0 and T_1 in $\overline{M}_s(\mathcal{L}, c_2)$. It is an open subset of $\overline{M}_s(\mathcal{L}, c_2)$. The fibres meet the requirements of the assertion.

Claim: Z is separated: Any simple sheaf on any of the fibres \mathcal{X}_t can also be regarded as a simple coherent sheaf on the complex space \mathcal{X} . Thus Z is a subspace of the space of all simple sheaves on \mathcal{X} . In order to show that two points are separated in Z it suffices to separate them in the bigger space. Now we apply the criterion of [KO] which says that if two simple coherent sheaves are not separated then there exists a non-trivial homomorphism between them. Since any two sheaves parametrized by Z are either supported on different fibres or stable with respect to the same polarization, this is excluded.

Thus Z is a separated with compact irreducible fibres over S^* . Take a locally free $E \in \overline{M}_H(L, c_2)$ and consider a neighbourhood of it in $\overline{M}_s(\mathcal{L}, c_2)$. By the arguments

above this neighbourhood contains locally free simple sheaves on all the nearby fibres. Hence we can assume that all these sheaves on $\mathcal{X}_{t \neq 0}$ are stable, since $\text{Pic}(\mathcal{X}_t) = \mathbb{Z} \cdot \mathcal{L}_t$ for $t \neq 0$. This implies the connectedness of Z . Thus $Z \rightarrow S$ is proper and smooth. \square

Proof of the theorem: *i)* We first show that the result of section 1 generalizes to the case where we drop the assumption that L generates $\text{Pic}(X)$. This is done as follows. By applying 2.3 to a deformation of the type 2.1.1 one sees that $\overline{M}_{\mathcal{L}_t}(\mathcal{L}_t, \frac{\mathcal{L}_t^2}{2} + 3)$ is a deformation of $\overline{M}_H(L, \frac{L^2}{2} + 3)$ for generic H . Since Hodge numbers are invariant under deformations, both spaces have the same Hodge numbers. Those of the second were compared in section 1 with the Hodge numbers of the appropriate Hilbert scheme. By the same trick we can always reduce to the case where the Picard group is generated by L , in particular we can assume that L is ample.

ii) By applying 2.3 to a deformation of type 2.1.2, 2.1.3(a) or 2.1.3(b) we see that $\overline{M}(L, c_2)$ is a deformation of $\overline{M}_H(\mathcal{L}_0, c_2)$ for generic H . Since μ -stability does not change under twisting by line bundles we have $\overline{M}_H(\mathcal{L}_0, c_2) \cong \overline{M}_H(\mathcal{L}_0(2D), c_2 + \mathcal{L}_0 \cdot D + D^2)$ (or $\overline{M}_H(\mathcal{L}_0(-2D), c_2 - \mathcal{L}_0 \cdot D + D^2)$ in case 2.1.3(b)). The proof of lemma 2.2 shows that by applying 2.1.2, 2.1.3(a) and 2.1.3(b) repeatedly we can reduce to the situation of *i)*, i.e. $c_2 = \frac{L^2}{2} + 3$. \square

Corollary 2.4 *Let X be an arbitrary K3 surface and L a primitive big and nef line bundle. As long as a polarization H does not lie on any wall, all deformation invariants, e.g. Hodge- and Betti numbers, of $\overline{M}_H(L, c_2)$ are independent of H .* \square

For similar results compare [G].

References

- [AK] Altman, A., Kleiman, S. *Compactifying the Picard Scheme*. Adv. Math. 35 (1980), 50-112.
- [B1] Beauville, A. *Variétés Kähleriennes dont la première classe de Chern est nulle*. J. Diff. Geom. 18 (1983), 755-782.
- [B2] Beauville, A. *Application aux espaces de modules*. in Géométrie des surfaces K3 et périodes. Asterisque 126 (1985).
- [BPS] Banica, C., Putinar, M., Schumacher, G. *Variation der globalen Ext in Deformationen kompakter komplexer Räume*. Math. Ann. 250 (1980), 135-155.
- [Ch] Cheah, J. *The Hodge numbers of smooth nests of Hilbert schemes of points on a smooth projective surface*. preprint (1993).
- [D] Danilov, V.I., Khovanskij, A.G. *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*. Math. USSR Izvestija 29. 2 (1987), 279-298.
- [GS] Göttsche, L., Soergel, W. *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*. Math. Ann. 296 (1993), 235-245.
- [G] Göttsche, L. *Change of polarization and Hodge numbers of moduli spaces of torsion free sheaves on surfaces*. MPI 94-9 (1994).

- [HL] Huybrechts, D., Lehn, M. *Stable pairs on curves and surfaces*. to appear in J. Alg. Geom.
- [KO] Kosarew, S., Okonek, C. *Global moduli spaces and simple holomorphic bundles*. Publ. RIMS, Kyoto Univ. 25 (1989), 1-19.
- [La] Lange, H. *Universal Families of Extensions*. J. of Alg. 83 (1983), 101-112.
- [LP] Le Potier, J. *Faisceaux semi-stables et systèmes cohérents*. preprint
- [Mor] Morrison, D. *On K3 surfaces with large Picard number*. Inv. Math. 75 (1984), 105-121.
- [Mu1] Mukai, S. *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*. Inv. Math. 77 (1984), 101-116.
- [Mu2] Mukai, S. *Moduli of vector bundles on K3 surfaces, and symplectic manifolds*. Sugaku Expositions 1(2) (1988), 138-174.
- [Na] Nakashima, T. *A note on Donaldson polynomials for K3 surfaces*. Forum Math. 6 (1994), 385-390.
- [Ni] Nikulin, V. *Integral symmetric bilinear forms and some of their applications*. Izv. Akad. Nauk SSSR 43 (1979), 111-177; Math. USSR Izvestija 14 (1980), 103-167.
- [O'G] O'Grady, K. *The irreducible components of moduli spaces of vector bundles on surfaces*. Inv. Math. 112 (1993), 585-613.
- [O'G1] O'Grady, K. *Donaldson's polynomials for K3 surfaces*. J. Diff. Geom. 35 (1992), 415-427.
- [Og] Oguiso, K. *Two remarks on Calabi-Yau Moishezon threefolds*. J. reine angew. Math. 452 (1994), 153-161.
- [Q] Qin, Z. *Chamber structure of algebraic surfaces with Kodaira dimension zero and moduli spaces of stable rank two bundles*. Math. Z. 207 (1991), 121-136.
- [T] Tyurin, A. N. *Special 0-cycles on a polarized K3-surface*. Math. USSR Izvestija 30(1) (1988), 123-143.
- [Z] Zuo, K. *The moduli spaces of some rank-2 stable vector bundles over algebraic K3 surfaces*. Duke Math J. 64 (1991), 403-408.

Max-Planck-Institut für Mathematik
 Gottfried-Claren-Str. 26
 53225 Bonn
 Germany
 lothar@mpim-bonn.mpg.de
 huybrech@mpim-bonn.mpg.de