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MULTIPLE ELLIPTIC POLYLOGARITHMS

FRANCIS BROWN AND ANDREY LEVIN

Abstract. We study the de Rham fundamental group of the configuration space $\mathcal{E}^{(n)}$ of $n+1$ marked points on an elliptic curve $\mathcal{E}$, and define multiple elliptic polylogarithms. These are multivalued functions on $\mathcal{E}^{(n)}$ with unipotent monodromy, and are constructed by a general averaging procedure. We show that all iterated integrals on $\mathcal{E}^{(n)}$, and in particular the periods of the unipotent fundamental group of the punctured curve $\mathcal{E}\backslash\{0\}$, can be expressed in terms of these functions.

1. Introduction

1.1. **Motivation.** Iterated integrals on the moduli space $\mathfrak{M}_{0,n}$ of curves of genus 0 with n ordered marked points can be expressed in terms of multiple polylogarithms. These are defined for $n_1, \ldots, n_r \in \mathbb{N}$ by

$$\mathrm{Li}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = \sum_{0<k_1<\ldots<k_r} \frac{x_1^{k_1}\ldots x_r^{k_r}}{k_1^{n_1}\ldots k_r^{n_r}} \quad \text{where } |x_i|<1 \ , \tag{1.1}$$

and have many applications from arithmetic geometry to quantum field theory. By specializing (1.1) at $x_i = 1$, one obtains the multiple zeta values

$$\zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1}\ldots k_r^{n_r}} \ , \qquad \text{where } n_r \geq 2 \ , \tag{1.2}$$

which are of particular interest since they are the periods of the fundamental group of $\mathbb{P}^1\backslash\{0,1,\infty\}$, and generate the periods of all mixed Tate motives over $\mathbb{Z}$.

The goal of this paper is to construct the elliptic analogues of the multiple polylogarithms and to set up the necessary algebraic and analytic background required to study multiple elliptic zeta values. The former are iterated integrals on the configuration space $\mathcal{E}^{(n)}$ of $n+1$ marked points on an elliptic curve, i.e., the fiber of the map $\mathfrak{M}_{1,n+1} \to \mathfrak{M}_{1,1}$, where $\mathfrak{M}_{1,m}$ denotes the moduli space of curves of genus 1 with m marked points. They generalize the classical elliptic polylogarithms studied in [13], and are the universal periods of unipotent variations of mixed elliptic Hodge structures.

In a sequel to this paper, we shall study the multiple elliptic zeta values, obtained by specializing multiple elliptic polylogarithms to the zero section of the universal elliptic curve. They define multivalued functions on $\mathfrak{M}_{1,1}$ which degenerate to ordinary multiple zeta values at the cusp. The existence of these functions sheds light on the structural relations between ordinary multiple zeta values, and in particular, the relation between double zetas and period polynomials for cusp forms [9].

1.2. **The rational case.** Firstly we recall the definition of iterated integrals [6]. Let M be a smooth real manifold, and let $\omega_1, \ldots, \omega_n$ denote smooth 1-forms on M. Let $\gamma : [0,1] \to M$ be a smooth path, and write $\gamma^*\omega_i = f_i(t)dt$ for some smooth functions $f_i : [0,1] \to \mathbb{R}$, where $1 \leq i \leq n$. The iterated integral of $\omega_1, \ldots, \omega_n$ is defined by

$$\int_\gamma \omega_1 \ldots \omega_n = \int_{0\leq t_n\leq\ldots\leq t_1\leq 1} f_1(t_1)\ldots f_n(t_n)\, dt_1\ldots dt_n \ . \tag{1.3}$$

Now let $M=\mathbb{P}^1\backslash\{0,1,\infty\}$, and let $\omega_0=\frac{dz}{z}$ and $\omega_1=\frac{dz}{1-z}$. Let $0<z<1$, and denote the straight path from 0 to z by γ_z. The initial point $\gamma(0)$ does not in fact lie in M, but the following iterated integral still makes sense nonetheless, and gives

$$\int_{\gamma_z}\underbrace{\omega_0\ldots\omega_0}_{n_r-1}\omega_1\ldots\underbrace{\omega_0\ldots\omega_0}_{n_1-1}\omega_1=\pm\mathrm{Li}_{n_1,\ldots,n_r}(1,\ldots,1,z)\ . \tag{1.4}$$

This is easily proved by a series expansion of the forms ω_1. The periods of the fundamental groupoid of M from 0 to 1 (with tangential basepoints 1, -1), are obtained by taking the limit as $z\to 1$. In the case $n_r\geq 2$, this yields

$$\int_0^1\underbrace{\omega_0\ldots\omega_0}_{n_r-1}\omega_1\ldots\underbrace{\omega_0\ldots\omega_0}_{n_1-1}\omega_1=\pm\zeta(n_1,\ldots,n_r)\ , \tag{1.5}$$

as first observed by Kontsevich. Similarly, one can define the regularized iterated integral from 0 to 1 of any word in the one-forms ω_0,ω_1, and in every case, it is easy to show that it is a linear combination of multiple zeta values.

To verify that all (homotopy invariant) iterated integrals on M are expressible in terms of multiple polylogarithms requires Chen's reduced bar construction. By Chen's general theory, the iterated integrals on a manifold M are described by the zeroth cohomology of the bar construction on the de Rham complex of M. To write this down explicitly for $M=\mathbb{P}^1\backslash\{0,1,\infty\}$, we can use the rational model

$$\mathbb{Q}\oplus(\mathbb{Q}\,\omega_0\oplus\mathbb{Q}\,\omega_1)\hookrightarrow\Omega^{\cdot}_{DR}(\mathbb{P}^1\backslash\{0,1,\infty\};\mathbb{Q})$$

which is a quasi-isomorphism of differential graded algebras. From this one deduces that $H^0(\mathbb{B}(\Omega^{\cdot}_{DR}(\mathbb{P}^1\backslash\{0,1,\infty\};\mathbb{Q})))\cong T(\mathbb{Q}\,\omega_0\oplus\mathbb{Q}\,\omega_1)$, where $\mathbb{B}$ is the bar complex, and T is the tensor algebra. This is the $\mathbb{Q}$-vector space generated by words in the forms ω_0,ω_1, which leads to integrals of the form (1.5). The upshot is that the periods of the unipotent fundamental groupoid $\pi_1^{un}(\mathbb{P}^1\backslash\{0,1,\infty\},0^+,z)$ are multiple polylogarithms (1.4), and the periods of $\pi_1^{un}(\mathbb{P}^1\backslash\{0,1,\infty\},0^+,1^-)$ are multiple zetas (1.2).

We need to generalize this picture to higher dimensions as follows. For $r\geq 1$, let

$$\mathfrak{M}_{0,r+3}(\mathbb{C})=\{(t_1,\ldots,t_r)\in\mathbb{C}^r:t_i\neq 0,1\ ,t_i\neq t_j\} \tag{1.6}$$

denote the moduli space of genus 0 curves with $r+3$ marked points. One can likewise write down a rational model for the de Rham complex in terms of the one-forms

$$\frac{dt_i-dt_j}{t_i-t_j}\ ,\ \frac{dt_i}{1-t_i}\ ,\ \frac{dt_i}{t_i}$$

which satisfy certain quadratic relations due to Arnold. Forgetting a marked point defines a fibration $\mathfrak{M}_{0,r+3}\to\mathfrak{M}_{0,r+2}$, and by general properties of the bar construction of fibrations, one can likewise write down all homotopy invariant iterated integrals on $\mathfrak{M}_{0,r+3}$ [4] and show that they are expressible in terms of the functions

$$I_{n_1,\ldots,n_r}(t_1,\ldots,t_r)=\mathrm{Li}_{n_1,\ldots,n_r}\big(\frac{t_1}{t_2},\ldots,\frac{t_{r-1}}{t_r},t_r\big)\ . \tag{1.7}$$

The purpose of this paper is to generalize the above picture for $\mathbb{P}^1\backslash\{0,1,\infty\}$ to the case of a punctured elliptic curve $\mathcal{E}^\times$. In particular, we compute the periods of $\pi_1^{un}(\mathcal{E}^\times,\varrho,\xi)$, where $\rho,\xi\in\mathcal{E}^\times$ are finite basepoints (the case of tangential basepoints is similar and will be postponed to a later paper). There are two parts: first, to write down the iterated integrals generalizing the left hand side of (1.4) using Chen's general theory, and the second is to construct multiple elliptic polylogarithm functions which correspond to the right-hand side of (1.4). In so doing, we are forced to consider the higher-dimensional configuration spaces $\mathcal{E}^{(n)}$, even to construct the functions on $\mathcal{E}^\times$.

1.3. **The elliptic case.** Let $\mathcal{E}$ be an elliptic curve, viewed as the analytic manifold $\mathbb{C}/\tau\mathbb{Z}\oplus\mathbb{Z}$, where $\tau\in\mathbb{C}$ satisfies $\mathrm{Im}\,(\tau)>0$. We first require a model for the de Rham complex on $\mathcal{E}^\times$. For this, we construct a universal family of smooth one-forms

$$\nu,\omega^{(0)},\omega^{(1)},\omega^{(2)},\ldots \quad \in \mathcal{A}^1(\mathcal{E}^\times)\ , \tag{1.8}$$

where $\omega^{(0)},\nu$ are closed and form a basis of $H^1(\mathcal{E}^\times)$ which is compatible with its Hodge structure. The forms $\omega^{(i)}$, $i\geq 1$ can be viewed as higher Massey products satisfying

$$d\omega^{(i)}=\nu\wedge\omega^{(i-1)} \qquad \text{for } i\geq 1\ .$$

Note that *a priori* $\mathcal{E}^\times$ has no natural $\mathbb{Q}$-structure on its de Rham complex. However, the forms (1.8) have good modularity and rationality properties as a function of the moduli τ, and there are good reasons to take

$$X = \text{graded } \mathbb{Q}\text{-algebra spanned by the } \nu,\omega^{(i)}, i\geq 0$$

as a $\mathbb{Q}$-model for the C^∞-de Rham complex on $\mathcal{E}^\times$, (indeed, $X\otimes_{\mathbb{Q}}\mathbb{C}\hookrightarrow\mathcal{A}^{\cdot}(\mathcal{E}^\times)$ is a quasi-isomorphism). We also define a higher-dimensional model for the configuration space $\mathcal{E}^{(n)}$ of $n+1$ points on $\mathcal{E}$. It is an elliptic version of Arnold's theorem describing the cohomology of the configuration space of n points in $\mathbb{P}^1$.

The next stage is to write down the bar construction of X, which defines a $\mathbb{Q}$-structure on the iterated integrals on $\mathcal{E}^\times$. The bar construction has a filtration by the length, and the associated graded is just the tensor algebra on $\omega^{(0)}$ and ν:

$$\mathrm{gr}^\ell H^0(\mathbb{B}(X))\cong T(\mathbb{Q}\,\omega^{(0)}\oplus\mathbb{Q}\,\nu)\ . \tag{1.9}$$

The Massey products $\omega^{(i)}$, for $i\geq 1$, give a canonical way to lift an element of (1.9) to $H^0(\mathbb{B}(X))$, and thus enable us to write down explicitly all the iterated integrals on $\mathcal{E}^\times$. These are indexed by any word in the two one-forms $\omega^{(0)}$ and ν. The Hodge filtration on the space of iterated integrals is related to the number of ν's. This completes the algebraic description of the iterated integrals on $\mathcal{E}^\times$.

The main problem is then to write down explicit formulae for these iterated integrals, and for this we write the elliptic curve via its Jacobi uniformization

$$\mathcal{E}\cong\mathbb{C}^\times/q^{\mathbb{Z}}\ ,$$

where $q=\exp(2\pi i\tau)$. In order to construct multivalued functions on $\mathcal{E}^\times$, the basic idea is to average a multivalued function on $\mathbb{C}^\times$ with respect to multiplication by q as was done for the classical polylogarithms [2, 13]. However, applying this idea naively to the multiple polylogarithms in one variable (1.4) does not lead to elliptic functions. Instead, the correct approach is to view the multiple polylogarithms in r variables (1.1), or rather their variants (1.7), as multivalued functions on

$$\mathfrak{M}_{0,r+3}\cong\underbrace{\mathbb{C}^\times\times\ldots\times\mathbb{C}^\times}_{r}\backslash\text{diagonals}$$

and average with respect to the group $q^{\mathbb{Z}}\times\ldots\times q^{\mathbb{Z}}$ (r factors). Since polylogarithms have logarithmic singularities at infinity, the straightforward average diverges, and so instead one must take a weighted average with respect to some auxilliary parameters u_i to dampen the singularities. In short, one considers the functions

$$\sum_{m_1,\ldots,m_r\in\mathbb{Z}} u_1^{m_1}\ldots u_r^{m_r} I_{n_1,\ldots,n_r}(q^{m_1}t_1,\ldots,q^{m_r}t_r) \tag{1.10}$$

which converge uniformly under some conditions on the u_i. A considerable part of this paper is devoted to studying the structure of the poles of (1.10) in the u_i variables, which are related to the geometry of $\overline{\mathfrak{M}}_{0,r+3}$ and the asymptotics of the polylogarithms at infinity. Finally, writing $u_i=\exp(2i\pi\alpha_i)$ for $1\leq i\leq r$, the multiple elliptic polylogarithms are defined to be the coefficients of (the finite part of) (1.10) with respect to the α_i. The analysis involved in this averaging procedure is quite general and should apply to a class of functions of finite determination and moderate growth on certain toric varieties.

The functions obtained in this way are multivalued functions on $\mathcal{E}^{(n)}$. By allowing some of the arguments t_i of (1.10) to degenerate to 1, we obtain multivalued functions on $\mathcal{E}^{\times}$. By computing the differential equations satisfied by these functions, we see that they are iterated integrals in the forms (1.8) and, using the description of the bar construction of X, we deduce that all the iterated integrals on $\mathcal{E}^{\times}$ are of this form.

1.4. **Plan of the paper.** First, §2 consists of reminders on multiple polylogarithms and the moduli space of curves of genus 0 which are used throughout the paper. Thereafter, the exposition splits into two parts - the first part, consisting of sections 3,4, and 5, concerns the de Rham complex of differential forms on $\mathcal{E}^{(n)}$. The second, consisting of sections 6, 7 and 8, concerns the procedure for averaging multiple polylogarithms. Since this is quite involved, we give a separate overview of the method in §6.1.

In §3 we use the Kronecker series (Proposition-Definition 4) to define the fundamental one-forms (1.8). In §4, we define some differential graded algebras X_n and prove by a Leray spectral sequence argument that they are $\mathbb{Q}$-models for the de Rham complex on $\mathcal{E}^{(n)}$. In §5 we use the models X_n to study Chen's reduced bar construction on $\mathcal{E}^{(n)}$, and hence obtain an algebraic description for the iterated integrals on $\mathcal{E}^{(n)}$. Some of the results of this section require some generalities on the bar construction of differential graded algebras which we decided to relegate to a separate paper [3].

In §6, we study the general averaging procedure for functions on $\mathfrak{M}_{0,n}(\mathbb{C})$. This requires constructing a certain partial compactification of $\mathfrak{M}_{0,n}$ and analyzing the asymptotics of series in the neighbourhood of boundary divisors. We apply this formalism to the classical multiple polylogarithms in §7. In §8, which is logically independent from the rest of the paper, we compute the asymptotics of the Debye polylogarithms at infinity in terms of a certain coproduct. The Debye multiple polylogarithms (definition 1) are essentially generating series of multiple polylogarithms and are useful for simplifying many formulae. In §9, we treat the case of the classical elliptic polylogarithms (depth 1) and the double elliptic polylogarithms (depth 2) in detail. The two parts of the story recombine in §10 where we prove that all iterated integrals on $\mathcal{E}^{\times}$, with respect to finite basepoints, are obtained by averaging.

1.5. **Related work.** One of many motivations for this paper is the study of mixed elliptic motives. Since the Beilinson-Soulé conjectures are currently unavailable in this case, our goal was to tease out the elementary consequences of such a theory, and in particular, write down the underlying numbers and functions in the belief that they will find applications in other parts of mathematics. We learned at a conference in Bristol in 2011 that there has been recent progress in constructing categories of mixed elliptic motives [12, 17], and universal elliptic motives [10] with similar goals. In this paper we completely neglected the Lie algebra side, which is dual to the bar construction, and its relation to quantum groups and stable derivations. This is treated in [5, 8, 18]. Somewhat further afield, it may also be helpful to point out related work in the profinite setting [16], and possible diophantine applications [11].

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2. Preliminaries: the rational case

2.1. **Standard coordinates on $\mathfrak{M}_{0,n+3}$.** Let $\mathfrak{M}_{0,n+3}$ denote the moduli space of curves of genus 0 with $n+3$ ordered marked points. By placing three of the marked points at $0, 1, \infty$, it can be identified with an affine hyperplane complement:

$$\mathfrak{M}_{0,n+3} \cong \{(t_1, \dots, t_n) \in \mathbb{P}^1 \backslash \{0, 1, \infty\} : t_i \neq t_j\} \ . \tag{2.1}$$

We refer to the coordinates t_i as simplicial coordinates. We will often write $t_{n+1} = 1$. There is a smooth compactification $\overline{\mathfrak{M}}_{0,n+3}$, such that the complement $\overline{\mathfrak{M}}_{0,n+3} \backslash \mathfrak{M}_{0,n+3}$ is a normal crossing divisor. Its irreducible components are indexed by partitions $S \cup T$ of the set of marked points, which we denote by $S|T$, or, $T|S$, where $S \cap T = \emptyset$ and $|S|, |T| \geq 2$. The corresponding divisor is isomorphic to $\overline{\mathfrak{M}}_{0,|S|+1} \times \overline{\mathfrak{M}}_{0,|T|+1}$.

2.2. **Multiple polylogarithms.** These are defined for $n_1, \dots, n_r \in \mathbb{N}$ by

$$\mathrm{Li}_{n_1,\dots,n_r}(x_1, \dots, x_r) = \sum_{0<k_1<\dots<k_r} \frac{x_1^{k_1} \dots x_r^{k_r}}{k_1^{n_1} \dots k_r^{n_r}} \tag{2.2}$$

which converges absolutely for $|x_i| < 1$, and therefore defines a family of holomorphic functions in the neighbourhood of the origin. We can also write these functions in simplicial coordinates, using standard notations:

$$I_{n_1,\dots,n_r}(t_1, \dots, t_r) = \mathrm{Li}_{n_1,\dots,n_r}\Big(\frac{t_1}{t_2}, \dots, \frac{t_{r-1}}{t_r}, t_r\Big) \ . \tag{2.3}$$

We denote the quantities $N = n_1 + \dots + n_r$ and r by the weight and the depth, respectively. There is an iterated integral representation

$$I_{n_1,\dots,n_r}(t_1, \dots, t_r) = (-1)^r \int_{0 \leq \sigma_1 \leq \dots \sigma_r \leq 1} \frac{d\sigma_1}{\sigma_1 - \rho_1} \dots \frac{d\sigma_N}{\sigma_N - \rho_N} \ , \tag{2.4}$$

where $(\rho_1, \dots, \rho_N) = (t_1^{-1}, 0^{n_1-1}, t_2^{-1}, 0^{n_2-1}, \dots, t_r^{-1}, 0^{n_r-1})$, $N = n_1 + \dots + n_r$, and 0^n denotes a string of n 0's. This is easily proved by expanding the integrand into a geometric series. One can deduce that they extend to multivalued functions on $\mathfrak{M}_{0,n+3}(\mathbb{C}) \subset (\mathbb{C}^\times)^n$. The following differential equation is easily verified from (2.2).

$$dI_{1,\dots,1}(t_1, \dots, t_n) = \sum_{k=1}^{n} \Big[dI_1\Big(\frac{t_k}{t_{k+1}}\Big) - dI_1\Big(\frac{t_k}{t_{k-1}}\Big)\Big] I_{1,\dots,1}(t_1, \dots, \widehat{t_k}, \dots, t_n) \tag{2.5}$$

where by convention we take $dI_1(t_1/0) = 0$, $t_{n+1} = 1$ and $t_0 = 0$. The differential equations for the multiple polylogarithms $I_{n_1,\dots,n_r}(t_1, \dots, t_r)$ in the general case are easily computed and left to the reader, since they are not required in the sequel.

Definition 1. The generating series of multiple Debye polylogarithms is:

$$\Lambda_r(t_1, \dots, t_r; \beta_1, \dots, \beta_r) = t_1^{-\beta_1} \dots t_r^{-\beta_r} \sum_{m_1,\dots,m_r \geq 1} I_{m_1,\dots,m_r}(t_1, \dots, t_r) \beta_1^{m_1-1} \dots \beta_r^{m_r-1}$$

One easily verifies from $d\mathrm{Li}_n(t) = t^{-1}\mathrm{Li}_{n-1}(t)$, valid for $n \geq 2$, that

$$d\Lambda_1(t; \beta) = t^{-\beta} d\mathrm{Li}_1(t) \ .$$

In general, they satisfy a differential equation which is entirely analogous to equation (2.5) for the multiple 1-logarithm. Namely, $d\Lambda_r(t_1, .., t_r; \beta_1, .., \beta_r) =$

$$\begin{aligned} &= \sum_{k=1}^{r} d\Lambda_1\Big(\frac{t_k}{t_{k+1}}, \beta_k\Big) \Lambda_{r-1}(t_1, .., \widehat{t_k}, .., t_r; \beta_1, .., \beta_k + \beta_{k+1}, .., \beta_r) \\ &- \sum_{k=2}^{r} d\Lambda_1\Big(\frac{t_k}{t_{k-1}}, \beta_k\Big) \Lambda_{r-1}(t_1, .., \widehat{t_k}, .., t_r; \beta_1, .., \beta_{k-1} + \beta_k, .., \beta_r) \end{aligned} \tag{2.6}$$

where $\beta_{r+1}=0$, so the last term in the first line is $d\Lambda_1(t_r;\beta_r)\Lambda_{r-1}(t_1,..,t_{r-1};\beta_1,..,\beta_{r-1})$.

2.3. **Asymptotics of regular nilpotent connections.** Let X be a smooth projective complex variety, let $D\subset X$ be a smooth normal crossing divisor, and let $U=X\backslash D$. Let $V\subset X$ be a simply connected open set and let $z_1,\ldots,z_n$ denote local coordinates on V such that $V\cap D=\cup_{i=1}^k\{z_i=0\}$, for some $k\leq n$.

Definition 2. Let f be a multivalued holomorphic function on U (i.e. f is holomorphic on a covering of U). We say that f has locally unipotent mondromy (or is locally unipotent) on V if it admits a finite expansion:

$$f(z_1,\ldots,z_n)=\sum_{I=(i_1,\ldots,i_k)}\log^{i_1}(z_1)\ldots\log^{i_k}(z_k)f_I(z_1,\ldots,z_n)\ , \tag{2.7}$$

where $f_I(z_1,\ldots,z_n)$ is holomorphic on V. We say that a multivalued function f on $X\backslash D$ is unipotent if it is everywhere locally unipotent.

The main class of functions which are studied in this paper, and in particular the multiple polylogarithms, are unipotent. The expansion (2.7) can be characterized by a property that f has 'moderate' growth near D (in particular, no poles), and that for sufficiently large N, $(\mathcal{M}_{i_1}-id)\ldots(\mathcal{M}_{i_N}-id)f=0$ for all $i_1,\ldots,i_N\in\{1,\ldots,k\}$, where $\mathcal{M}_i$ denotes analytic continuation around a small loop encircling $z_i=0$.

3. Differential forms on $\mathcal{E}^{(n)}$

3.1. **Basic notations.** Let $\mathbf{e}(z)$ denote the function $\mathbf{e}(z)=\exp(2\pi iz)$. In accordance with [19], Greek letters ξ and η denote coordinates on $\mathbb{C}$, the letter α will denote a formal variable, and τ will denote a point of the upper half-plane $\mathbb{H}=\{\tau\in\mathbb{C}:\mathrm{Im}\,(\tau)>0\}$. Put $z=\mathbf{e}(\xi)$, $w=\mathbf{e}(\eta)$ and $q=\mathbf{e}(\tau)$. Then $z,w\in\mathbb{C}^*$, and $0<|q|<1$.

3.2. **Uniformization.** We represent a complex elliptic curve $\mathcal{E}=\mathbb{C}/(\tau\mathbb{Z}+\mathbb{Z})$, where $\tau\in\mathbb{H}$, and $q=\mathbf{e}(\tau)$, via its Jacobi uniformization

$$\mathcal{E}\xrightarrow{\sim}\mathbb{C}^*/q^{\mathbb{Z}}\ .$$

Let ξ (resp. $z=\mathbf{e}(\xi)$) denote the coordinate on $\mathcal{E}$ (resp. $\mathbb{C}^*$). The punctured curve $\mathcal{E}^\times=\mathcal{E}\backslash\{0\}$ is isomorphic to $\mathbb{C}^*\backslash\{q^{\mathbb{Z}}\}/q^{\mathbb{Z}}$. For $n\geq 1$, let $\mathcal{E}^{(n)}$ denote the configuration space of $n+1$ distinct points on $\mathcal{E}$ modulo translation by $\mathcal{E}$. Thus

$$\mathcal{E}^{(n)}\cong\{(\xi_1,\ldots,\xi_n)\in(\mathcal{E}\backslash\{0\})^n:\xi_i\neq\xi_j\text{ for }i\neq j\}\ ,$$

and has an action of $\mathfrak{S}_{n+1}$ which permutes the marked points. Setting $t_i=\mathbf{e}(\xi_i)$ for $1\leq i\leq n$, and setting $t_{n+1}=1$, gives an isomorphism

$$\mathcal{E}^{(n)}\cong\{(t_1,\ldots,t_n)\in\mathbb{C}^*,t_i\notin q^{\mathbb{Z}}t_j\text{ for }1\leq i<j\leq n+1\}/q^{\mathbb{Z}^n}\ .$$

The set on the right-hand side is the largest open subset of $\mathfrak{M}_{0,n+3}(\mathbb{C})$ stable under translation by $q^{\mathbb{Z}^n}$. The symmetry group $\mathfrak{S}_{n+1}$ of $\mathcal{E}^{(n)}$ can thus be identified with the subgroup of $\mathrm{Aut}(\mathfrak{M}_{0,n+3}(\mathbb{C}))\cong\mathfrak{S}_{n+3}$ which fixes the marked points 0 and ∞.

In order to fix branches when considering multivalued functions, and to ensure convergence when averaging functions on $\mathfrak{M}_{0,n+3}(\mathbb{C})$, we must fix certain domains in $\mathcal{E}^{(n)}$. Let D be the standard open fundamental domain for $\mathbb{Z}+\tau\mathbb{Z}$ (the parallelogram with corners $0,1,\tau,1+\tau$), and let

$$U=\{(\xi_1,\ldots,\xi_n)\in D^n:\mathrm{Im}\,(\xi_n)<\ldots<\mathrm{Im}\,(\xi_1)\} \tag{3.1}$$

3.3. **Elliptic functions.** Let $\theta(\xi,\tau)$ denote "two thirds of the Jacobi triple formula":

$$\theta(\xi,\tau)=q^{1/12}(z^{1/2}-z^{-1/2})\prod_{j=1}^{\infty}(1-q^jz)\prod_{j=1}^{\infty}(1-q^jz^{-1})=\frac{\theta_{11}(\xi,\tau)}{\eta(\tau)}, \tag{3.2}$$

where $\theta_{11}(\xi,\tau)$ is the standard odd elliptic theta function and $\eta(\tau)$ is the Dedekind η-function $q^{1/24}\prod_{j=1}^{\infty}(1-q^j)$. Recall from [19] that the Eisenstein summation of a double series $(a_{m,n})_{m,n\in\mathbb{Z}}$ is defined by:

$$\sum_{m,n}{}^{e}\, a_{m,n}=\lim_{N\to\infty}\lim_{M\to\infty}\sum_{n=-N}^{N}\sum_{m=-M}^{M}a_{m,n}$$

Define the Eisenstein functions $E_j(\xi,\tau)$ and the Eisenstein series $e_j(\tau)$, for $j\geq 1$, by

$$E_j(\xi,\tau)=\sum_{m,n}{}^{e}\frac{1}{(\xi+m+n\tau)^j},\quad e_j(\tau)=\sum_{m,n}{}^{\prime e}\frac{1}{(m+n\tau)^j},$$

where $\prime$ means that we omit $(m,n)=(0,0)$ in the summation.

Lemma 3. *It follows from the definitions that for $j\geq 1$,*

$$\frac{\partial}{\partial\xi}E_j(\xi,\tau)=-jE_{j+1}(\xi,\tau)\ ,\qquad \frac{\partial}{\partial\xi}\log(\theta(\xi,\tau))=E_1(\xi,\tau),$$

and $E_1(\alpha,\tau)=1/\alpha-\sum_{j=0}^{\infty}e_{j+1}(\tau)\alpha^j$. The series $e_j(\tau)$ vanish for odd indices j.

The Weierstrass function $\wp$ is equal to E_2-e_2, and $\wp'=-2E_3$. The coefficients of the Weierstrass equation ${\wp'}^2=4\wp^3-g_2\wp-g_3$ are given by $g_2=60e_4, g_3=140e_6$.

3.4. **The Kronecker function.** See also [13, 14, 20] for further details.

Proposition-Definition 4. *The following three definitions are equivalent:*

$$\begin{aligned}
&i) && F(\xi,\eta,\tau) &&= \frac{\theta'(0)\theta(\xi+\eta)}{\theta(\xi)\theta(\eta)}\ ,\\
&ii) && F(\xi,\eta,\tau) &&= -2\pi i\left(\frac{z}{1-z}+\frac{1}{1-w}+\sum_{m,n>0}(z^mw^n-z^{-m}w^{-n})q^{mn}\right)\ ,\\
&iii) && F(\xi,\alpha,\tau) &&= \frac{1}{\alpha}\exp\left(-\sum_{j\geq 1}\frac{(-\alpha)^j}{j}(E_j(\xi,\tau)-e_j(\tau))\right)\ .
\end{aligned}$$

The equivalence of the (i) and (ii) is proved in [19]. The equivalence of (i) and (iii) follows by computing the logarithmic derivative of F, from the relationship between E_1 and $\log(\theta)$ (lemma 3), and the Taylor expansion of E_1 at a point α. The following properties of the Kronecker function F will be important for the sequel.

Proposition 5. *$F(\xi,\eta,\tau)$ has the following properties:*

i) Quasi-periodicity with respect to $\xi\mapsto\xi+1$ and $\xi\mapsto\xi+\tau$:

$$F(\xi+1,\eta,\tau)=F(\xi,\eta,\tau)\qquad F(\xi+\tau,\eta,\tau)=w^{-1}F(\xi,\eta,\tau)$$

ii) The mixed heat equation:

$$2\pi i\frac{\partial F}{\partial\tau}=\frac{\partial^2F}{\partial\xi\partial\eta}\ .$$

iii) The Fay identity:

$$F(\xi_1,\eta_1,\tau)F(\xi_2,\eta_2,\tau)=F(\xi_1,\eta_1+\eta_2,\tau)F(\xi_2-\xi_1,\eta_2,\tau)+F(\xi_2,\eta_1+\eta_2,\tau)F(\xi_1-\xi_2,\eta_1,\tau)\ .$$

Proof. The quasi-periodicity is immediate from the first definition of F. The mixed heat equation follows from the second definition of F. The last statement is a consequence of the third representation of F, and the Fay trisecant equation (see [15]). $\square$

The following formula is an easy corollary of *iii*):

$$F(\xi,\alpha_1)F_2'(\xi,\alpha_2)-F_2'(\xi,\alpha_1)F(\xi,\alpha_2)=F(\xi,\alpha_1+\alpha_2)(E_2(\alpha_1)-E_2(\alpha_2))\ , \tag{3.3}$$

where F_2' denotes the derivative of F with respect to its second argument.

3.5. **Massey products on $\mathcal{E}^{(n)}$.** We use the Eisenstein-Kronecker series F to write down some explicit one-forms on $\mathcal{E}^{(n)}$. First consider a single elliptic curve $\mathcal{E}^\times$ with coordinate ξ as above. Write $\xi=s+r\tau$, where $r,s\in\mathbb{R}$ and τ is fixed, and let $\omega=d\xi$ and $\nu=2\pi i dr$. The classes $[\omega],[\nu]$ form a basis for $H^1(\mathcal{E}^\times;\mathbb{C})$.

Lemma 6. *The form $\Omega(\xi;\alpha)=\mathbf{e}(\alpha r)F(\xi;\alpha)d\xi$ is invariant under elliptic transformations $\xi\mapsto\xi+\tau$ and $\xi\mapsto\xi+1$, and satisfies $d\,\Omega(\xi;\alpha)=\nu\alpha\wedge\Omega(\xi;\alpha)$.*

Proof. Straightforward calculation using proposition 5i). $\square$

We can view $\Omega(\xi;\alpha)$ as a generating series of one-forms on $\mathcal{E}^\times$. Let $\xi_1,\ldots,\xi_n$ denote the usual holomorphic coordinates on $\mathcal{E}^\times\times\ldots\times\mathcal{E}^\times$ and set $\nu_i=2i\pi dr_i$.

Definition 7. Let $\xi_0=0$ and define holomorphic one forms $\omega^{(k)}_{i,j}\in\mathcal{A}^1(\mathcal{E}^{(n)})$ for all $0\leq i\leq j\leq n$ and $k\geq 0$ by the generating series:

$$\Omega(\xi_i-\xi_j;\alpha)=\sum_{k\geq 0}\omega^{(k)}_{i,j}\alpha^{k-1}\ . \tag{3.4}$$

We clearly have $\omega^{(k)}_{i,i}=0$ and $\omega^{(k)}_{i,j}+(-1)^k\omega^{(k)}_{j,i}=0$ for all i,j,k. The leading terms $\omega^{(0)}_{i,j}$ are equal to $d\xi_i-d\xi_j$ and therefore satisfy the relations:

$$\omega^{(0)}_{i,j}+\omega^{(0)}_{j,k}=\omega^{(0)}_{i,k}\quad\text{for all } i,j,k\ . \tag{3.5}$$

The higher terms $\omega^{(k)}_{i,j}$ can be viewed as Massey products via the equation:

$$d\,\omega^{(k+1)}_{i,j}=(\nu_i-\nu_j)\wedge\omega^{(k)}_{i,j}\quad\text{for } k\geq 0\ , \tag{3.6}$$

which follows from lemma 6. The Fay identity implies that

$$\begin{aligned}\Omega(\xi_i-\xi_\ell;\alpha)\wedge\Omega(\xi_j-\xi_\ell;\beta)\ &+\ \Omega(\xi_j-\xi_i;\beta)\wedge\Omega(\xi_i-\xi_\ell;\alpha+\beta)\\ &+\ \Omega(\xi_j-\xi_\ell;\alpha+\beta)\wedge\Omega(\xi_i-\xi_j;\alpha)\ =\ 0\end{aligned} \tag{3.7}$$

which gives rise to infinitely many quadratic relations between the $\omega^{(k)}_{i,j}$. Finally, the definition of F shows that the residues of these forms are given by

$$\mathrm{Res}_{\xi_i=\xi_j}\,\omega^{(k)}_{i,j}=2i\pi\,\delta_{1k}\ , \tag{3.8}$$

where δ denotes the Kronecker delta. Now consider the projection $\mathcal{E}^{(n)}\rightarrow\mathcal{E}^{(n-1)}$ given by $(\xi_1,\ldots,\xi_n)\rightarrow(\xi_1,\ldots,\xi_{n-1})$. Its fibers $\mathcal{E}_{F_n}$ are isomorphic to the punctured elliptic curve $\mathcal{E}^\times\backslash\{\xi_1,\ldots,\xi_{n-1}\}$ with coordinate ξ_n. Let $\overline{\omega}^{(k)}_{i,j}$ (resp. $\overline{\nu}_n$) denote the relative forms obtained by restricting $\omega^{(k)}_{i,j}$ (resp. ν_n) to the fiber. Clearly $\overline{\omega}^{(0)}_{n,i}=d\xi_n$ for all i.

Lemma 8. *The 1-forms $\{\overline{\nu}_n,\,d\xi_n,\,\overline{\omega}^{(k)}_{n,i}$ for $k\geq 1$, all $i\}\subset\mathcal{A}^1(\mathcal{E}_{F_n})$, and the 2-forms*

$$\{\overline{\nu}_n\wedge d\xi_n\ ,\ \overline{\nu}_n\wedge\overline{\omega}^{(k)}_{n,i}\ \text{for } k\geq 1,\ \text{all } i\}\subset\mathcal{A}^2(\mathcal{E}_{F_n}) \tag{3.9}$$

are linearly independent over $\mathbb{C}$.

Proof. Since the $\overline{\omega}$'s are of type $(1,0)$ and $\overline{\nu}_n$ is not, it follows from (3.8) that the forms $d\xi_n, \overline{\omega}^{(1)}_{n,0}, \dots, \overline{\omega}^{(1)}_{n,n-1}, \overline{\nu}_n$ are linearly independent. Consider a non-trivial relation

$$\sum_{0\leq i<n,\ k\leq w} c_{i,k}\, \overline{\omega}^{(k)}_{n,i} = 0\ ,\ \text{ where } c_{i,k}\in\mathbb{C}\ ,$$

and w is minimal. Differentiating gives $\overline{\nu}_n \wedge \big(\sum_{i,k} c_{i,k}\overline{\omega}^{(k-1)}_{n,i}\big) = 0$, by (3.6). Since $\overline{\nu}_n$ has a non-zero component of type $(0,1)$, the wedge product by $\overline{\nu}_n$ on $(1,0)$-forms is injective, giving a smaller relation $\sum_{i,k\leq w-1} c_{i,k+1}\, \overline{\omega}^{(k)}_{n,i} = 0$, which is a contradiction. The same argument proves that (3.9) are linearly independent. $\square$

4. A rational model for the de Rham complex on $\mathcal{E}^{(n)}$

We construct a differential graded algebra X_n over $\mathbb{Q}$ which is defined by generators and quadratic relations, along with a quasi-isomorphism $X_n\otimes_{\mathbb{Q}}\mathbb{C}\hookrightarrow \mathcal{A}_n$, where $\mathcal{A}_n = \mathcal{A}^\bullet(\mathcal{E}^{(n)})$ is the C^∞-de Rham complex on the configuration space of $n+1$ points on $\mathcal{E}$. We show that X_n carries a mixed Hodge structure and give a presentation for $H^\bullet(\mathcal{E}^{(n)})$ which is an elliptic analogue of Arnold's theorem in the genus 0 case.

4.1. **Differential graded algebras and fibrations.** Let k be a field of characteristic zero. Recall that a (positively-graded) DGA over k is a graded-commutative algebra $A=\bigoplus_{n\geq 0} A^n$ with a differential $d: A\rightarrow A$ of degree $+1$ which satisfies the Leibniz rule. It is said to be connected if $A^0\cong k$. We shall consider algebras A which are either finite-dimensional in each degree, or else carry a second grading (called the weight grading) for which they are finite-dimensional in every bidegree.

Let A_T be such a DGA with differential d_T, and let $A_B\subset A_T$ be a sub-DGA. Define

$$A_F = A_T / A_B^{\geq 1} A_T\ , \tag{4.1}$$

which inherits a differential d_F from d_T. We call the triple A_B, A_T, A_F a *fibration* if A_T is a free A_B-module. The indices T,B,F stand for the total space, base, and fiber. Now suppose that we are given a splitting $i_F: A_F\rightarrow A_T$ of A^0_B-modules. When A_B, A_T, A_F is a fibration, the map i_F defines an isomorphism of A_B-modules:

$$A_T \cong A_B\otimes_{A^0_B} A_F = \textstyle\bigoplus_{i\geq 0} A^i_B\otimes_{A^0_B} A_F\ , \tag{4.2}$$

which does not necessarily respect the differential or algebra structure.

4.2. **The model X_n.** We consider the differential graded algebra X_n generated by symbols corresponding to the forms considered in §3.5. By abuse of notation, we use the same symbol to denote the generators in X_n and their images in $\mathcal{A}_n=\mathcal{A}^\bullet(\mathcal{E}^{(n)})$. This will be justified when we show that $X_n\rightarrow\mathcal{A}_n$ is injective (corollary 16).

Definition 9. Let X_n be the $\mathbb{Q}$-differential graded algebra generated by elements

$$\begin{array}{ll} \omega^{(k)}_{i,j} & \text{for } k\geq 0 \text{ and } 0\leq i\leq j\leq n \\ \nu_i & \text{for } 1\leq i\leq n \end{array}$$

in degree 1, modulo the graded-commutative ideal generated by the relations (3.5) and the coefficients of (3.7). The differential is given by $d\nu_i=0$, $d\omega^{(0)}_{i,j}=0$, and (3.6) in all other cases. It is a simple calculation to check that the differential ideal generated by the Fay identity (3.7) is equal to the (usual) ideal it generates.

There is an obvious map $X_{n-1}\rightarrow X_n$. Let X^+_{n-1} be the ideal in X_n generated by the images of elements of X_{n-1} of positive degree, and let $X_{F_n}=X_n/X^+_{n-1}$. Denote the images of $\omega^{(k)}_{i,j}$ and ν_i under the natural map $X_n\rightarrow X_{F_n}$ by $\overline{\omega}^{(k)}_{i,j}$ and $\overline{\nu}_i$, respectively.

Lemma 10. *X_{F_n} is isomorphic to the $\mathbb{Q}$-differential graded algebra generated by $\overline{\omega}^{(k)}_{n,i}$ and $\overline{\nu}_n$ in degree 1, subject to the relations: $\overline{\omega}^{(0)}_{n,i} = \overline{\omega}^{(0)}_{n,j}$ for all i,j; $\overline{\nu}_n \wedge \overline{\nu}_n = 0$; and*

$$\overline{\omega}^{(k)}_{n,i} \wedge \overline{\omega}^{(\ell)}_{n,j} = 0 \qquad \forall \quad i,j,k,\ell \ . \tag{4.3}$$

The differential is given by $d\overline{\omega}^{(0)}_{n,i} = d\overline{\nu}_n = 0$ and $d\overline{\omega}^{(k+1)}_{n,i} = \overline{\nu}_n \wedge \overline{\omega}^{(k)}_{n,i}$ for $k \geq 1$.

Proof. All the relations are obvious except for (4.3). It follows from the Fay identity (3.7) that $\Omega(\xi_n - \xi_i, \alpha) \wedge \Omega(\xi_n - \xi_j, \beta) = 0 \mod X^+_{n-1}$. □

In particular, X_{F_n} is concentrated in degrees 0, 1, and 2. Let $i_{F_n} : X_{F_n} \rightarrow X_n$ denote the splitting of the quotient map $X_n \rightarrow X_{F_n}$ defined by:

$$i_{F_n}(\overline{\nu}_n) = \nu_n \qquad , \qquad i_{F_n}(\overline{\omega}^{(k)}_{n,i}) = \omega^{(k)}_{n,i} - \omega^{(k)}_{0,i} \ .$$

4.3. **Mixed Hodge structure on X_n.** The complex of C^∞ forms on $\mathcal{E}^{(n)}$ with logarithmic singularities carries a Hodge and weight filtration. The weight filtration on 1-foms is concentrated in degrees 1 and 2. But it turns out that there is a refined weight filtration on X_n, which is in fact a grading. To define it, set $W_0 X^1_n = 0$ and

$$W_\ell X^1_n = \langle \nu_i, \omega^{(k)}_{i,j} : k < \ell \rangle \qquad \text{for all} \quad \ell \geq 1 \ ,$$

and extend it by multiplication to X_n. It is well-defined because the relations implied by (3.7) are homogeneous for the weight. The Hodge filtration is given by

$$F^0 X^1_n = X^1_n \quad \supset \quad F^1 X^1_n = \langle \omega^{(k)}_{i,j} \rangle \quad \supset \quad F^2 X^1_n = 0$$

and extends to X_n in the same way. One easily verifies that this defines a mixed Hodge structure on X_n such that $d : X_n \rightarrow X_n$ is homogeneous for the weight. Likewise X_{F_n} inherits a mixed Hodge structure which is compatible with the map i_{F_n}.

4.4. **Quadratic Algebras.** We give a sufficient criterion for an algebra defined by quadratic relations to be a fibration. We shall only apply this in the case of X_n.

Definition 11. Let V be a finite dimensional vector space over a field k. Let $R \subseteq \bigwedge^2 V$ be a subspace (the space of relations). The associated quadratic algebra is

$$Y^{\cdot} = \bigwedge^{\cdot} V / \langle R \rangle \ ,$$

where $\langle R \rangle \subseteq \bigwedge^{\cdot} V$ is the ideal generated by R. We have $Y^0 = k$, $Y^1 = V$.

Now suppose that $V_B \subseteq V$ is a subspace, and let $V_F = V/V_B$. Choose a splitting

$$V = V_B \oplus V_F \ ,$$

which induces a splitting $\bigwedge^2 V = \bigwedge^2 V_B \oplus (V_B \otimes_k V_F) \oplus \bigwedge^2 V_F$. Let $\pi_F : \bigwedge^2 V \rightarrow \bigwedge^2 V_F$ denote projection onto the last component. Assume that the space of relations splits:

$$R = R_B \oplus R_F \ ,$$

where $R_B \subseteq \bigwedge^2 V_B$, and $R_F \subseteq (V_B \otimes_k V_F) \oplus \bigwedge^2 V_F$. Let $Y^{\cdot}_B = \bigwedge^{\cdot} V_B / \langle R_B \rangle$.

Proposition 12. *Suppose that*

$$\pi_F : R_F \longrightarrow {\textstyle\bigwedge}^2 V_F \ \textit{is an isomorphism.} \tag{4.4}$$

In this case, the relations R_F define the graph of a map $\alpha : \bigwedge^2 V_F \rightarrow V_B \otimes_k V_F$, where $\alpha = id - \pi_F^{-1}$. Suppose furthermore that the map induced by α:

$$\bigwedge^3 V_F \longrightarrow V_B \otimes_k \bigwedge^2 V_F \longrightarrow Y^2_B \otimes_k V_F$$

is well-defined, i.e., for all $v_1, v_2, v_3 \in V_F$, there is the associativity condition:

$$(id \wedge \alpha)(\alpha(v_1 \wedge v_2) \wedge v_3) - (id \wedge \alpha)(\alpha(v_2 \wedge v_3) \wedge v_1) \in R_B \otimes_k V_F \ . \tag{4.5}$$

Then $Y_B \to Y$ is injective, and a fibration, with fibers Y_F, where $Y_F^0 = k$, and $Y_F^1 \cong V_F$, and $Y_F^k = 0$ for $k \geq 2$. Thus there is an isomorphism of Y_B-modules:

$$Y \cong Y_B \otimes_k Y_F \cong Y_B \oplus (Y_B \otimes_k V_F) \ .$$

Proof. There is an obvious natural map

$$i : Y_B \oplus (Y_B \otimes_k V_F) \longrightarrow Y \ .$$

We construct an inverse to i by defining by induction a sequence of linear maps

$$\alpha_n : \bigwedge\nolimits^n V_F \longrightarrow Y_B^{n-1} \otimes_k V_F \qquad \text{for } n \geq 2 \ ,$$

such that $i \circ \alpha_n(\xi) \equiv \xi \mod \langle R \rangle$. For this, let α_2 be the map $\alpha = id - \pi_F^{-1}$ defined above, and let α_n be the map obtained by composing α with itself $n-1$ times. By the associativity property (4.5), α_n is well-defined. It is clear from the definition that $i \circ \alpha_2 \equiv id \mod R$, and from this we deduce that $i \circ \alpha_n \equiv id \mod \langle R \rangle$ for all n by induction. Now write

$$\bigwedge\nolimits^n V = \bigoplus\nolimits_{i=0}^n \bigwedge\nolimits^i V_B \otimes_k \bigwedge\nolimits^{n-i} V_F \ .$$

If we set $\alpha_0 : k \to k$ and $\alpha_1 : V_F \to V_F$ to be the identity maps, we deduce a map

$$\rho = \bigoplus_{i=0}^n \pi_B^i \otimes \alpha_{n-i} : \bigwedge\nolimits^n V \longrightarrow Y_B \otimes_k (Y_B \otimes_k V_F) \ ,$$

where $\pi_B^i : \bigwedge^i V_B \to Y_B$ is the natural map. Since $\alpha_n(\langle R \rangle) = 0$ for all n, and since $R = R_B \oplus R_F$, the map ρ passes to the quotient to define a map

$$\overline{\rho} : Y \longrightarrow Y_B \oplus (Y_B \otimes_k V_F)$$

which satisfies $\overline{\rho} \circ i = id$ by definition and $i \circ \overline{\rho}$ is an isomorphism since $i \circ \alpha_n \equiv id \mod \langle R \rangle$. Thus i is an isomorphism. □

Remark 13. In the previous discussion, we can also replace V with a graded vector space which is of finite dimension in every degree, and R by a graded subspace.

4.5. **Structure of X_n.** We show that X_{n-1}, X_n, X_{F_n} is a fibration of DGA's.

Lemma 14. *There is an isomorphism of graded-commutative algebras*

$$X_n \cong \bigwedge (\mathbb{Q}\nu_1 \oplus \ldots \oplus \mathbb{Q}\nu_n) \otimes_{\mathbb{Q}} Z_n \ ,$$

where Z_n is the subalgebra of X_n spanned by the elements $\omega_{i,j}^{(k)}$. Likewise,

$$X_{F_n} \cong \bigwedge (\mathbb{Q}\overline{\nu}_n) \otimes_{\mathbb{Q}} Z_{F_n} \ ,$$

where Z_{F_n} is the subalgebra of X_{F_n} spanned by $\overline{\omega}_{n,i}^{(k)}$. Note that these isomorphisms do not respect the differential structures on X_n and X_{F_n}.

Proof. All defining relations of X_n have Hodge filtration ≥ 1, so $X_n/F^1 X_n$ is isomorphic to the free graded-commutative algebra spanned by $\nu_1, \ldots, \nu_n$. An identical argument gives the corresponding isomorphism for X_{F_n}. □

Lemma 15. *The map $X_{n-1} \to X_n$ is injective, and X_n is a free X_{n-1}-module.*

Proof. We must prove that $X_{n-1} \hookrightarrow X_n$ and $X_n \cong X_{n-1} \otimes_{\mathbb{Q}} X_{F_n}$ as X_{n-1}-modules. By lemma 14 this is equivalent to showing that $Z_{n-1} \hookrightarrow Z_n$ and $Z_n \cong Z_{n-1} \otimes_{\mathbb{Q}} Z_{F_n}$ as Z_{n-1}-modules. Since Z_n is quadratic, it is enough to verify the criteria of proposition 12. The quadratic relations R are defined by (3.7) and so R_F is generated by

$$(i, n; \alpha) \wedge (j, n, \beta) + (j, n; \alpha + \beta) \wedge (i, j; \alpha) + (j, i; \beta) \wedge (i, n; \alpha + \beta) = 0 \tag{4.6}$$

where $i, j \leq n-1$ and $(i, n; \alpha)$ denotes $\Omega(\xi_i - \xi_n; \alpha)$, etc. Since every term $\omega_{n,i}^{(k)} \wedge \omega_{n,j}^{(\ell)}$ for $k, \ell \geq 1$ occurs exactly once in the Taylor expansion of the first term of (4.6), the condition (4.4) is verified. The cases where k or $\ell = 0$ are trivial to check. To verify (4.5), apply the identity (4.6) four times to get:

$$\begin{aligned}\big[\big[(i,n;\alpha)\wedge(j,n;\beta)\big]\wedge(k,n;\gamma)\big] &= (j,i;\beta)\wedge(k,i;\gamma)\wedge(i,n;\alpha+\beta+\gamma)\\ &+ (k,j;\gamma)\wedge(i,j;\alpha)\wedge(j,n;\alpha+\beta+\gamma)\\ &+ (i,k;\alpha)\wedge(j,k,\beta)\wedge(k,n;\alpha+\beta+\gamma)\end{aligned}$$

Since the right-hand side is antisymmetric, the left hand side clearly does not depend on the bracketing, and the analogue of proposition 12 holds in the infinite graded case (remark 13), where the grading is given by the weight grading of §4.3. $\square$

Let us write $\mathcal{A}_{F_n} = \mathcal{A}_n/\mathcal{A}_{n-1}^+$ and let ϕ denote the natural map $X_n \to \mathcal{A}_n$. The choice of coordinate ξ_n on the fiber of $\mathcal{E}^{(n)} \to \mathcal{E}^{(n-1)}$ gives an isomorphism

$$\mathcal{A}_{n-1} \otimes_{\mathcal{A}_{n-1}^0} \mathcal{A}_{F_n} \cong \mathcal{A}_n \ . \tag{4.7}$$

Corollary 16. *The map ϕ is injective.*

Proof. By lemma 10, X_{F_n} is concentrated in degrees at most two, so it follows from lemma 8 that $X_{F_n} \to \mathcal{A}_{F_n}$ is injective. The injectivity of $X_1 \to \mathcal{A}_1$ is a special case. The lemma follows by induction on n using the previous lemma and (4.7). $\square$

4.6. **Proof that X_n is a model.** We now show that $\phi : X_n \otimes_{\mathbb{Q}} \mathbb{C} \to \mathcal{A}_n$ is a quasi-isomorphism. First we compute $H^1(X_{F_n})$ and the Gauss-Manin connection on it.

Lemma 17. *We have $H^0(X_{F_n}) = \mathbb{Q}$, $H^k(X_{F_n}) = 0$ if $k \geq 2$, and*

$$\mathrm{gr}_1^W H^1(X_{F_n}) \cong \mathbb{Q}[\overline{\nu}_n] \oplus \mathbb{Q}[\overline{\omega}_{n,0}^{(0)}] \ , \quad \mathrm{gr}_2^W H^1(X_{F_n}) \cong \bigoplus_{1 \leq i \leq n-1} \mathbb{Q}[\overline{\omega}_{n,i}^{(1)} - \overline{\omega}_{n,0}^{(1)}] \ ,$$

where $H^1(X_{F_n}) \cong \mathrm{gr}_1^W H^1(X_{F_n}) \oplus \mathrm{gr}_2^W H^1(X_{F_n})$.

Proof. For all $k \geq 3$, $X_{F_n}^k = 0$ and so $H^k(X_{F_n}) = 0$. By (4.3) and $\overline{\nu}_n \wedge \overline{\nu}_n = 0$, any two-form can be written

$$\sum_{k,i} c_{n,i}^k \, \overline{\nu}_n \wedge \overline{\omega}_{n,i}^{(k)} = d\big(\sum_{k,i} c_{n,i}^k \, \overline{\omega}_{n,i}^{(k+1)}\big) \qquad \text{where } c_{n,i}^k \in \mathbb{Q} \ ,$$

so is exact. Thus $H^2(X_{F_n}) = 0$ and clearly $H^0(X_{F_n}) \cong X_{F_n}^0 = \mathbb{Q}$. Since $\overline{\nu}$ and $\overline{\omega}_{n,0}^{(0)}$ are closed, it suffices by lemma 10 to consider a closed one-form

$$\eta = \sum_{k \geq 1, 0 \leq i < n} c_{n,i}^k \, \overline{\omega}_{n,i}^{(k)} \quad \text{where } c_{n,i}^k \in \mathbb{Q} \ , \quad \text{ such that } d\eta = 0 \ .$$

This implies that $d\eta = \overline{\nu}_n \wedge \big(\sum_{k \geq 1, 0 \leq i < n} c_{n,i}^k \overline{\omega}_{n,i}^{(k-1)}\big) = 0$. By lemma 14 we have

$$\sum_{k \geq 1, 0 \leq i < n} c_{n,i}^k \, \overline{\omega}_{n,i}^{(k-1)} = 0 \ .$$

Since the forms $\overline{\omega}_{n,i}^{(k)}$, $k \geq 1$ and $\overline{\omega}_{n,0}^{(0)}$ are linearly independent in X_{F_n} by lemma 8, and since $\overline{\omega}_{n,n-1}^{(0)} = \ldots = \overline{\omega}_{n,0}^{(0)}$, we conclude that the closed forms in $X_{F_n}^1$ are spanned by

$$\overline{\nu}_n \ , \ \overline{\omega}_{n,0}^{(0)} \ , \qquad \text{and } \ \{\eta = \sum_{0 \leq i < n} c_{n,i}^1 \overline{\omega}_{n,i}^{(1)} \quad \text{such that } \sum_{0 \leq i < n} c_{n,i}^1 = 0\} \ .$$

This implies the result, along with the definition of the mixed Hodge structure §4.3. $\square$

Since X_n is a fibration, we can easily compute the Gauss-Manin connection on $H^1(X_{F_n})$ (see [3] for further details). It is *a priori* nilpotent since the weight filtration is defined on all of X_n, and satisfies $W_0 X_{n-1} = 0$. Explicitly, it is

$$\begin{array}{rcl} H^1(X_{F_n}) & \rightarrow & X^1_{n-1} \otimes_{\mathbb{Q}} H^1(X_{F_n}) \\ \nabla[\overline{\omega}^{(1)}_{n,0} - \overline{\omega}^{(1)}_{n,i}] & = & \nu_i \otimes [\overline{\omega}^{(0)}_{n,0}] + \omega^{(0)}_{i,0} \otimes [\overline{\nu}_n] \\ \nabla[\overline{\omega}^{(0)}_{n,0}] & = & 0 \\ \nabla[\overline{\nu}_n] & = & 0 \end{array} \tag{4.8}$$

Using the fact that $\omega^{(0)}_{n,i} = \omega^{(0)}_{n,0} - \omega^{(0)}_{i,0}$, the first line follows from the calculation

$$\begin{array}{rcl} i_{F_n}(\overline{\omega}^{(1)}_{n,i} - \overline{\omega}^{(1)}_{n,0}) & = & \omega^{(1)}_{n,i} - \omega^{(1)}_{0,i} - \omega^{(1)}_{n,0} \\ d(i_{F_n}(\overline{\omega}^{(1)}_{n,i} - \overline{\omega}^{(1)}_{n,0})) & = & (\nu_n - \nu_i) \wedge \omega^{(0)}_{n,i} - \nu_i \wedge \omega^{(0)}_{i,0} - \nu_n \wedge \omega^{(0)}_{n,0} \ , \\ & = & -\nu_i \wedge \omega^{(0)}_{n,0} - \nu_n \wedge \omega^{(0)}_{i,0} \ . \end{array}$$

The second and third lines of (4.8) follow from the fact that $\omega^{(0)}_{n,0}$ and ν_n are exact.

Lemma 18. *$H^1(\phi) : \mathrm{gr}^W_\cdot H^1(X_{F_n}) \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathrm{gr}^W_\cdot H^1(\mathcal{A}_{F_n})$ is an isomorphism.*

Proof. The differential graded algebra $\mathcal{A}_{F_n}$ computes the de Rham cohomology of the fiber of the map $\mathcal{E}^{(n)} \rightarrow \mathcal{E}^{(n-1)}$, which is isomorphic to $\mathcal{E}$ minus n points. Furthermore, it carries a Hodge and weight filtration which induces the corresponding filtrations on $H^1(\mathcal{E}\backslash\{n \text{ points}\})$. The Gysin sequence gives:

$$0 \rightarrow H^1(\mathcal{E};\mathbb{C}) \rightarrow H^1(\mathcal{E}\backslash\{n \text{ points}\};\mathbb{C}) \rightarrow \mathbb{C}(-1)^{n-1} \rightarrow 0 \ ,$$

where the third map is given by the residue. Therefore $\mathrm{gr}^W_1 H^1(\mathcal{A}_{F_n}) \cong H^1(\mathcal{E})$ and $\mathrm{gr}^W_2 H^1(\mathcal{A}_{F_n}) \cong \mathbb{C}(-1)^{n-1}$. The lemma follows from the fact that $[\phi(\overline{\nu}_n)], [\phi(\overline{\omega}^{(0)}_{n,0})]$ is a basis of $H^1(\mathcal{E})$ and $\phi(\overline{\omega}^{(1)}_{n,i})$ has residue $2\pi i$ at $\xi_n = \xi_i$, by (3.8). □

Theorem 19. *$\phi : X_n \otimes_{\mathbb{Q}} \mathbb{C} \hookrightarrow \mathcal{A}_n$ is a quasi-isomorphism.*

Proof. The case $n = 1$ follows from the previous lemma. The case $n > 1$ follows by induction by a standard Leray spectral sequence argument. Details are given in [3]. □

In conclusion, X_n is a model for the de Rham complex on $\mathcal{E}^{(n)}$ and provides a universal $\mathbb{Q}$-structure on its cohomology.

4.7. **A simplified model.** Although we shall not use it, one can consider a finitely-generated DGA model Y_n for the cohomology of a configuration of elliptic curves.

Definition 20. Let Y_n be the commutative graded $\mathbb{Q}$-algebra defined by generators ω_i, ν_i for $1 \leq i \leq n$ and ω_{ij} for $1 \leq i \leq j \leq n$ in degree one such that

$$\begin{array}{rcl} \omega_{ij} - \omega_{ji} & = & 0 \\ \omega_i \wedge \nu_i & = & 0 \\ \omega_{ij} \wedge \omega_i + \omega_{ji} \wedge \omega_j & = & 0 \\ \omega_{ij} \wedge \nu_i + \omega_{ji} \wedge \nu_j & = & 0 \\ \omega_{i\ell} \wedge \omega_{j\ell} + \omega_{j\ell} \wedge \omega_{ij} + \omega_{ji} \wedge \omega_{il} & = & 0 \end{array} \tag{4.9}$$

and define a differential $d : Y_n \rightarrow Y_n$ by $d\omega_i = d\nu_i = 0$ and

$$d\omega_{ij} = \omega_i \wedge \nu_j + \omega_j \wedge \nu_i \ . \tag{4.10}$$

There is a surjective map $X_n \to Y_n$ which sends $\omega_{i,0}^{(0)}$ to ω_i and $\omega_{i,j}^{(1)}$ to ω_{ij}, ν_i to ν_i and all $\omega_{i,j}^{(k)}$ for $k \geq 2$, to zero. We can define a mixed Hodge structure on Y_n in the same way as for X_n, i.e., ω_i, ν_i have weight one and ω_{ij} weight two.

Theorem 21. *The map $X_n \to Y_n$ is a quasi-isomorphism, i.e., $H^\bullet(Y_n) \cong H^\bullet(\mathcal{E}^{(n)})$.*

Proof. (Sketch) Follow the steps of the proof that $X_n \to \mathcal{A}_n$ is a quasi-isomorphism: first define Y_{F_n} to be Y_n/Y_{n-1}^+ and check that it is a fibration. Then apply the Leray spectral sequence argument, noting that (4.10) exactly corresponds to the Gauss-Manin connection on $H^1(X_{F_n}) \cong H^1(Y_{F_n})$. □

The model Y_n kills all higher Massey products in X_n. Since Y_n is finitely generated it may be useful for explicit implementation of the algebra $H^\bullet(\mathcal{E}^{(n)})$. We have quasi-isomorphisms $Y_n \twoheadleftarrow X_n \hookrightarrow \mathcal{A}_n$ but note there is no map from Y_n to $\mathcal{A}_n$.

5. Bar construction of the de Rham complex of $\mathcal{E}^{(n)}$

The model X_n enables us to put a $\mathbb{Q}$-structure on the bar construction of $\mathcal{A}_n$.

5.1. **Reminders on the bar construction.** See [7], §1 for further details. Let A be a DGA as in §4.1, and further assume that A has an augmentation map $\varepsilon : A \to k$. Let $s : A \to A$ denote the map $s(a) = (-1)^{\deg a} a$. Let $A^+ = \ker \varepsilon$ denote the augmentation ideal of A and let $T(A^+) = k \oplus A^+ \oplus A^{+\otimes 2} \oplus \ldots$ denote the tensor algebra on A^+, where all tensor products are over k. We write the element $a_1 \otimes \ldots \otimes a_n \in A^{\otimes n}$ using the bar notation $[a_1|\ldots|a_n]$. Recall that $T(A^+)$ is a commutative Hopf algebra for the shuffle product ($\Sigma(r,s)$ denotes the set of r,s shuffles):

$$[a_1|\ldots|a_r] ш [a_{r+1}|\ldots|a_{r+s}] = \sum_{\sigma\in\Sigma(r,s)} \varepsilon(\sigma)[a_{\sigma(1)}|\ldots|a_{\sigma(r+s)}] \ ,$$

where the sign $\varepsilon(\sigma)$ also depends on the degrees of the a_i's but is always equal to 1 if all a_i are of degree 1. The coproduct $\Delta : T(A^+) \to T(A^+) \otimes_k T(A^+)$ is given by

$$\Delta([a_1|\ldots|a_r]) = \sum_{i=1}^{r} [a_1|\ldots|a_i] \otimes [a_{i+1}|\ldots|a_r] \ .$$

The *length filtration* is the increasing filtration associated to the tensor grading

$$F_n T(A^+) = \bigoplus_{0\leq i\leq n} A^{+\otimes i} \ .$$

The *bar complex* is the double complex with terms $(A^{+\otimes p})_q$ (elements of total degree q and length p in $T(A^+)$), and with one differential $(-1)^p d_i : (A^{+\otimes p})_q \to (A^{+\otimes p})_{q+1}$:

$$d_i([a_1|\ldots|a_p]) = \sum_{1\leq i\leq p} (-1)^i [sa_1|\ldots|sa_{i-1}|da_i|a_{i+1}|\ldots|a_p]$$

and a second differential $d_e : (A^{+\otimes p})_q \to (A^{+\otimes p-1})_q$ where:

$$d_e([a_1|\ldots|a_p]) = \sum_{1\leq i<p} (-1)^{i+1} [sa_1|\ldots|sa_{i-1}|sa_i \wedge a_{i+1}|a_{i+2}|\ldots|a_p]$$

The bar construction $\mathbb{B}(A)$ is defined to be the total complex $\bigoplus (A^{+\otimes p})_q$ with total differential $D = d_i + d_e$. Note that the total degree of elements in $\mathbb{B}(A)$ is given by

$$\deg([a_1|\ldots|a_p]) = \deg(a_1) + \ldots + \deg(a_p) - p \ . \tag{5.1}$$

Let $V(A) = H^0(\mathbb{B}(A))$ be the zeroth cohomology. It is a commutative Hopf algebra.

5.2. **Connected case.** When A is connected, this construction simplifies. The augmentation ideal $A^+ = \bigoplus_{n\geq 1} A^n$ is the set of elements of positive degree. It is clear from (5.1) that only elements of A of degree 1 contribute to $V(A) = H^0(\mathbb{B}(A))$, so let $T(A^1)$ denote the tensor algebra generated by elements of degree 1. Note that $(A^{+\otimes p})_q = 0$ if $p > q$ and the total degree (5.1) is non-negative. The total differential $-D = d_i + d_e$ reduces to $D : T(A^1) \to T(A)$:

$$D([w_1|\dots|w_r]) = \sum_{i=1}^{r} [w_1|\dots|w_{i-1}|dw_i|w_{i+1}|\dots|w_r] + \sum_{i=1}^{r-1} [w_1|\dots|w_i \wedge w_{i+1}|\dots|w_r]$$

where we changed the overall sign for convenience. We can simply write

$$V(A) = \ker\big(D : T(A^1) \to T(A)\big) \ . \tag{5.2}$$

We say that elements $\xi \in T(A^1)$ satisfying $D\xi = 0$ are *integrable.*

Definition 22. Define the bar construction of $\mathcal{E}^{(n)}$ to be $V(X_n) = H^0(\mathbb{B}(X_n))$, where X_n is our model (§4.2). It is a commutative Hopf algebra over $\mathbb{Q}$, filtered by the length.

By (5.2), $V(X_n)$ is the subalgebra of $T(X_n^1)$ given by the integrable words in X_n^1. Since X_n carries a mixed Hodge structure, so too does $V(X_n)$.

5.3. **Description of $V(X_{F_n})$.** We first give an explicit description of the bar construction of an elliptic curve with punctures. By the general theory, the length-graded

$$\mathrm{gr}^\ell V(X_{F_n}) \cong \bigoplus_{\ell\geq 0} H^1(\mathcal{E}_{F_n})^{\otimes \ell} \ ,$$

since there is no integrability condition for one-dimensional varieties. The right-hand side is just the set of words in the generators of $H^1(\mathcal{E}_{F_n})$. These generators are represented by the closed one-forms $\overline{\omega}_n^{(0)}$, $\overline{\nu}_n$ and $\eta_i := \overline{\omega}_{n,i}^{(1)} - \overline{\omega}_{n,0}^{(1)}$, for $1 \leq i < n$.

Proposition 23. *Any word in the letters $\overline{\omega}_n^{(0)}, \overline{\nu}_n, \eta_i$ of length ℓ can be canonically lifted to an integrable element in $V(X_{F_n})$ using the forms $\overline{\omega}_{n,i}^{(k)}$ for $k < \ell$.*

Proof. Use Chen's formal power series connections [6]. Let $S = \mathbb{Q}\langle\langle \mathsf{x}_0, \mathsf{x}_1, \mathsf{y}_1, \dots, \mathsf{y}_{n-1}\rangle\rangle$ denote the ring of non-commutative formal power series in symbols $\mathsf{x}_0, \mathsf{x}_1, \mathsf{y}_1, \dots, \mathsf{y}_{n-1}$ and let $\alpha = -\mathrm{ad}(\mathsf{x}_0)$. Consider the formal 1-form [5, 14]:

$$J = \overline{\nu}\,\mathsf{x}_0 + \alpha\,\overline{\Omega}(\xi_n;\alpha)\mathsf{x}_1 + \sum_{i=1}^{n-1} \big(\overline{\Omega}(\xi_n - \xi_i;\alpha) - \overline{\Omega}(\xi_n;\alpha)\big)\mathsf{y}_i \in X_{F_n} \otimes_{\mathbb{Q}} S$$

It is well-defined since all polar terms in α cancel, and is of the form

$$J = \overline{\nu}_n\,\mathsf{x}_0 + \overline{\omega}_n^{(0)}\mathsf{x}_1 + \eta_1 \mathsf{y}_1 + \ldots + \eta_{n-1}\mathsf{y}_{n-1} + \text{ higher order terms in x's,y's}$$

One easily checks from lemma 10 that $dJ = -J \wedge J$. It follows that the formal element $\Xi = [J] + [J|J] + [J|J|J] + \ldots$ lies in $V(X_{F_n}) \otimes_{\mathbb{Q}} S$, and by duality defines a map from $T(\langle \mathsf{x}_0, \mathsf{x}_1, \mathsf{y}_1, \dots, \mathsf{y}_{n-1}\rangle) \cong \mathrm{gr}^\ell V(X_{F_n})$ to $V(X_{F_n})$. Concretely, let w be any word of length ℓ in the alphabet $\{\overline{\omega}_n^{(0)}, \overline{\nu}_n, \eta_i\}$. Mapping $\overline{\nu}_n$ to x_0, $\overline{\omega}_n^{(0)}$ to x_1, η_i to y_i gives a word w' in $\{\mathsf{x}_i, \mathsf{y}_j\}$. The coefficient of w' in Ξ is a finite integrable word in the symbols $\overline{\nu}_n, \overline{\omega}_{n,i}^{(k)}, 0 \leq k \leq \ell - 1$ whose longest term is w. This gives a canonical splitting:

$$\mathrm{gr}^\ell V(X_{F_n}) \longrightarrow V(X_{F_n})$$

Note that it follows from the proof that this lifting has integral coefficients. $\square$

It follows that $V(X_{F_n})$ is canonically isomorphic to the tensor coalgebra spanned by $[\overline{\omega}]$, $[\overline{\nu}]$, $[\eta_i]$, equipped with the shuffle product. The Hodge and weight filtrations are induced from the corresponding filtrations on X_n. More precisely we have

$$[x_1|\dots|x_n] \in F^r V(X_{F_n}) \quad \text{if} \quad \big|\{i : x_i = \overline{\nu}\}\big| \leq n - r \ . \tag{5.3}$$

The weight comes from a grading $w : \mathrm{gr}^\ell_\cdot V(X_{F_n}) \rightarrow \mathbb{N}$ which is defined by

$$w([x_1|\dots|x_n]) = n + \big|\{i : x_i \in \{\eta_j\}\}\big| \ , \tag{5.4}$$

and is obtained by giving $\overline{\omega}^{(0)}$ and $\overline{\nu}$ weight 1, and the η_i's weight 2.

5.3.1. *Example: the bar construction on $\mathcal{E}^\times$.* In the case of a single puncture:

$$\mathrm{gr}^\ell_\cdot V(X_1) \cong T(\mathbb{Q}[\omega^{(0)}] \oplus \mathbb{Q}[\nu]) \ .$$

The formal power series connection J reduces in this case to

$$\begin{array}{rcl} J & = & \nu\,\mathsf{x}_0 + \Omega(\xi, -\mathrm{ad}(\mathsf{x}_0))\,\mathsf{x}_1 \quad \in \quad X^1\langle\langle \mathsf{x}_0, \mathsf{x}_1 \rangle\rangle \\ & = & \nu\,\mathsf{x}_0 + \mathsf{x}_1\omega^{(0)} - [\mathsf{x}_0, \mathsf{x}_1]\omega^{(1)} + [\mathsf{x}_0, [\mathsf{x}_0, \mathsf{x}_1]]\omega^{(2)} + \dots \end{array}$$

and gives an explicit way to lift any word in the letters $[\omega^{(0)}], [\nu]$ to $V(X_1)$.

Corollary 24. *The weight and the length filtrations on $V(X_1)$ coincide.*

Examples 25. The elements of $V(X_1)$ of length at most one are $1, [\omega^{(0)}], [\nu]$. In length ≤ 2 we also have: $[\omega^{(0)}|\omega^{(0)}]$, $[\omega^{(0)}|\nu] + [\omega^{(1)}]$, $[\nu|\omega^{(0)}] - [\omega^{(1)}]$, and $[\nu|\nu]$.

5.4. **Structure of $V(X_n)$.** One of the main results of [3] implies:

Theorem 26. *There is an isomorphism of algebras*

$$V(X_n) \cong \bigotimes_{i=1}^{n} V(X_{F_i}) \ . \tag{5.5}$$

The length and weight filtrations on $V(X_n)$ coincide.

Proof. The bar Gauss-Manin connection $\nabla_{\mathbb{B}} : V(X_{F_n}) \rightarrow X^1_{n-1} \otimes_{\mathbb{Q}} V(X_{F_n})$, which is defined in [3], is nilpotent with respect to the weight grading. It implies the existence of a map $V(X_{F_n}) \rightarrow V(X_n)$ and an isomorphism $V(X_n) \cong V(X_{n-1}) \otimes_{\mathbb{Q}} V(X_{F_n})$ of algebras, from which the statement follows by induction (see [3] for the proofs). $\square$

Note that (5.5) does not respect the Hopf algebra, or differential structures on X_n. It is however a complete algebraic description of all iterated integrals on $\mathcal{E}^{(n)}$, and in particular, enables one to write down a basis for them.

Remark 27. Theorem 26 is proved in [3] by first showing that the bar-de Rham cohomology of X_n is trivial. This is equivalent to the exactness of the sequence:

$$0 \longrightarrow \mathbb{Q} \longrightarrow X^0_n \otimes_{\mathbb{Q}} V(X_n) \longrightarrow X^1_n \otimes_{\mathbb{Q}} V(X_n) \longrightarrow \dots \longrightarrow X^n_n \otimes_{\mathbb{Q}} V(X_n) \longrightarrow 0$$

which establishes a duality between $V(X_n)$ and X_n. One way to understand the refined mixed Hodge structure on X_n (§4.3) is to say that it is induced from the mixed Hodge structure on $V(X_n)$ by this duality. Alternatively, it follows from the decomposition $X_n \cong X_{n-1} \otimes_{\mathbb{Q}} X_{F_n}$ and the refined mixed Hodge structure on X_{F_n} by induction.

* * *

6. Averaging unipotent functions

6.1. **Introduction.** The main idea for constructing multivalued functions on an elliptic curve is to use the Jacobi uniformization

$$\mathcal{E} = \mathbb{C}^\times / q^{\mathbb{Z}}$$

and average a function on $\mathbb{C}^\times$ with respect to multiplication by q. Consider the example of the multivalued function $\mathrm{Li}_1(z) = -\log(1-z)$. Let $q \in \mathbb{C}^\times$ such that $|q| < 1$ and $z \in \mathbb{C}^\times$ such that $1 \notin q^{\mathbb{R}} z$. The spiral $q^{\mathbb{R}} z$ can be lifted to a universal covering space of $\mathbb{C}\backslash\{0,1\}$, and the function $\mathrm{Li}_1(z)$ has a well-defined analytic continuation along it. Near the origin, the function $\mathrm{Li}_1(z)$ vanishes, but at the point $z = \infty$ it has a logarithmic singularity, so the naive average diverges. One way to ensure convergence is to consider the generating series

$$E(z;u) = \sum_{m \in \mathbb{Z}} u^m \mathrm{Li}_1(q^m z)$$

where u^{-1} is chosen small enough to dampen the logarithmic singularity at infinity, but not so small as to wreck the convergence at the origin. For $m \ll 0$, the asymptotic is $u^m \log(q^m z)$, which is bounded if $u > 1$. For $m \gg 0$ the terms are asymptotically $u^m q^m z$, which is bounded if $u < |q|^{-1}$. Thus for $1 < u < |q|^{-1}$ the series $E(z;u)$ converges absolutely, and is almost periodic with respect to multiplication by q.

From this one can easily show that $E(z;u)$ has a simple pole at $u = 1$. Now one must view $E(z;u)$ as a function of ξ and α, where $u = \mathbf{e}(\alpha)$ and $z = \mathbf{e}(\xi)$. Thus the pole at $u = 1$ contributes a pole at $\alpha = 0$ which can be removed to obtain

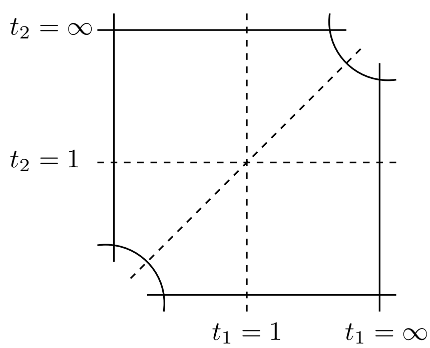
$$E^{\mathrm{reg}}(\xi;\alpha) = E(\xi;\alpha) - \frac{1}{\alpha}\ .$$

This function now admits a Taylor expansion at the point $\alpha = 0$. The procedure for constructing multivalued functions on the elliptic curve $\mathcal{E}^\times$ is to take the coefficients of α^i, $i \geq 0$ in this Taylor expansion.

The situation is more complicated in the case of several complex variables. Suppose that we have a function $f(t_1,t_2)$ on $\mathfrak{M}_{0,5}(\mathbb{C}) = \{(t_1,t_2) : t_1, t_2 \neq 0, 1, t_1 \neq t_2\}$, with singularities along the removed hyperplanes. We wish to average the function

$$\sum_{m_1, m_2 \in \mathbb{Z}} u_1^{m_1} u_2^{m_2} f(q^{m_1} t_1, q^{m_2} t_2) \tag{6.1}$$

The first problem that we encounter is that the function $f(t_1,t_2)$ is simply not well-defined as $t_1, t_2 \to \infty$ since its singularities $t_i = \infty$, $t_1 = t_2$ do not cross normally at that point, and so the limit depends on the direction in which it is approached. The standard solution is to blow-up the points $(0,0)$, (∞,∞), as below:



Note that we do not need to blow up the point $t_1 = t_2 = 1$, which is also a non-normal crossing point, because there are only finitely many lattice points $\{(q^{\mathbb{Z}}t_1, q^{\mathbb{Z}}t_2)\}$ in its neighbourhood. The domain of summation naturally decomposes into the six sectors pictured above, each of which is homeomorphic to a square $[0,1] \times [0,1]$. By analysing the behaviour of $f(t_1, t_2)$ in the neighbourhood of each sector, one finds necessary and sufficient conditions on u_1, u_2 to ensure the absolute convergence of (6.1). It turns out that the poles in the u_1, u_2 plane are in one-to-one correspondence with the boundary divisors in the figure, and depend on the local asymptotic behaviour of f. One can then remove poles in the α_i plane (where $u_i = \mathbf{e}(\alpha_i)$) and compute Taylor expansions to extract multivalued functions on $\mathcal{E}^{(2)}$ with unipotent monodromy.

The plan of the second, analytic, part of this paper is as follows.

(1) First we construct an explicit partial compactification of $\mathfrak{M}_{0,n}$ which is adapted to this averaging procedure.
(2) By studying the analytic properties of the multiple polylogarithms (2.3) we find necessary and sufficient conditions on the dual variables u_i to ensure absolute convergence of the averaging function:

$$\sum_{m_1,\ldots,m_r\in\mathbb{Z}} u_1^{m_1}\ldots u_r^{m_r} I_{n_1,\ldots,n_r}(q^{m_1}t_1,\ldots,q^{m_r}t_r)$$

(3) From its differential equation, we compute the pole structure in the u_i coordinates. The multiple elliptic polylogarithms can be defined as the coefficients in its regularized Taylor expansion at $\alpha_1 = \ldots = \alpha_r = 0$, where $u_i = \mathbf{e}(\alpha_i)$. Note that, since the singularities in the space of α_i parameters are not normal crossing, the regularization must be performed with some care. We do not address the question of explicit regularization in the present paper.

The entire procedure from (1) to (3) will work more generally for any functions satisfying some growth conditions on certain toric varieties. Section §6 covers the general steps (1) and (2). The definition of the multiple elliptic polylogarithms is completed in §7, with examples given in §9. In §8, which is independent from the rest of the paper, we show how to compute the asymptotics of the Debye polylogarithms explicitly at infinity using a certain coproduct. Finally, in §10 we prove that all iterated integrals on a punctured elliptic curve can be obtained by this averaging procedure.

6.2. **Preliminaries.** Let $q = \mathbf{e}(\tau)$, where τ is in the upper-half plane, and let $t_i = \mathbf{e}(\xi_i)$, for $i = 1, \ldots, r$, where $\xi_1, \ldots, \xi_r$ are in the domain U defined by (3.1), in §3.2.

Consider the following preparation map to a universal covering of $\mathfrak{M}_{0,r+3}(\mathbb{C})$:

$$\begin{array}{rcl}\sigma : \mathbb{R}^n & \longrightarrow & \widetilde{\mathfrak{M}}_{0,r+3}(\mathbb{C}) \\ (s_1,\ldots,s_r) & \mapsto & (q^{s_1}t_1,\ldots,q^{s_r}t_r)\ ,\end{array}$$

where we view $\mathfrak{M}_{0,r+3}(\mathbb{C}) \subset \mathbb{C}^r$ in simplicial coordinates. Suppose that we have a multivalued function $f(t_1, \ldots, t_r)$ on $\mathfrak{M}_{0,r+3}(\mathbb{C})$, with a fixed branch in the neighbourhood of some $(t_1, \ldots, t_r)$ such that $t_i t_j^{-1} \notin q^{\mathbb{R}}$ for $i \neq j$ and $t_i \notin q^{\mathbb{R}}$. Then $f(t_1, \ldots, t_r)$ admits a unique analytic continuation to the image of $\sigma(\mathbb{R}^n)$, and in particular, the values $f(q^{n_1}t_1, \ldots, q^{n_r}t_r)$ are well-defined for all $(n_1, \ldots, n_r) \in \mathbb{Z}^r$.

We apply this to the multiple polylogarithm functions

$$I_{m_1,\ldots,m_n}(t_1,\ldots,t_n) = \sum_{a_1,\ldots,a_n\geq 1} \frac{t_1^{a_1}\ldots t_n^{a_n}}{a_1^{m_1}(a_1+a_2)^{m_2}\ldots(a_1+\ldots+a_n)^{m_n}}\ , \qquad (6.2)$$

which give rise to multivalued unipotent functions on $\mathfrak{M}_{0,r+3}(\mathbb{C}) \subset \mathbb{C}^r$, and vanish along the divisors $t_i = 0$. The power series expansion above defines a canonical branch in the neighbourhood of the origin.

6.3. **Compactification of the hypercube.** Let $S = \{1, \ldots, n\}$ and let us write $\mathbb{P}^1_S$ for $\mathrm{Hom}(S, \mathbb{P}^1) \cong (\mathbb{P}^1)^n$. Let $\square_n \subset \mathbb{P}^1_S$ denote the real hypercube $[0, \infty]^n$. For any disjoint pair of subsets $I, J \subset S$, let

$$F_I^J = \bigcap_{i \in I} \{z_i = 0\} \cap \bigcap_{j \in J} \{z_j = \infty\} \subseteq \mathbb{P}^1_S$$

denote the corresponding coordinate linear subspace. The sets $F_I^J \cap \square_n$ give the standard stratification of the hypercube by its faces.

Working first in simplicial coordinates, consider the set of divisors

$$X = \bigcup_{1 \leq i < j \leq n} \{t_i - t_j = 0\} \cup \bigcup_{1 \leq i \leq n} \{t_i = 1\} \ ,$$

which meets the set of faces F_I^J non-normally. Let us write $F_I = F_I^\emptyset$, $F^J = F_\emptyset^J$ and call such divisors of type 0 or type ∞, respectively. Consider the sets of faces:

$$\mathcal{F}^0 = \{F_I : |I| > 1\} \ , \quad \mathcal{F}^\infty = \{F^J : |J| > 1\} \ ,$$

of codimension ≥ 2. Following the standard practice for blowing up linear subspaces, we blow up the set of faces in $\mathcal{F}^0$ of smallest dimension, followed by the strict transforms of faces F_I where $|I| = n - 1$, and so on, in increasing order of dimension, until the strict transforms of all elements in $\mathcal{F}^0$ have been blown up. Now repeat the same procedure with $\mathcal{F}^\infty$, and denote the corresponding space by P_S, with

$$\pi : P_S \longrightarrow \mathbb{P}^1_S \tag{6.3}$$

It does not depend on the chosen order of blowing-up.

Let us denote the strict transform of any face F_I (resp. F^J) by D_I (resp. D^J), for all $|I|, |J| \geq 2$. Let D_i (resp. D^j) denote the strict transform of the divisor $F_i^\emptyset$ (resp. $F_\emptyset^j$) which corresponds to a facet of the original hypercube, and let us denote by $D = \bigcup_{|K| \geq 1} D_K \cup D^K$, the union of all the above. The strict transform of $\square_n$ is a certain polytope $\mathcal{C}_n$, whose facets are in bijection with the irreducible components of D, which number $2 \times (2^n - 1)$. Let $X' \subset P_S$ denote the strict transform of X.

Proposition 28. *The divisor $D \cup X' \subset P_S$ is locally normal crossing near D.*

The proof will be given by computing explicit normal coordinates in every local neighbourhood of D, using a decomposition into sectors.

6.4. **Sector decomposition.** Let us view $\mathfrak{M}_{0,n}(\mathbb{R})$ in simplicial coordinates as the complement of divisors of the form $t_i = t_j$ and $t_i = 1$ in $(\mathbb{R}^\times)^n$. Then $\square_n \cap \mathfrak{M}_{0,n+3}(\mathbb{R})$ admits a decomposition into $(n+1)!$ connected components:

$$\Delta_\pi = \{0 < t_{\pi(1)} < \ldots < t_{\pi(n+1)} < \infty\} \ ,$$

where π is a permutation of $(1, \ldots, n+1)$, the t_i are simplicial coordinates on each component of $(\mathbb{P}^1)^n$, and where $t_{n+1} = 1$. The permutation π should be viewed as a dihedral ordering of the $n+3$ marked points $0, 1, \infty, t_1, \ldots, t_n$ on $\mathbb{P}^1(\mathbb{R})$. To every such π we associate local 'sector' coordinates on $\mathfrak{M}_{0,n+3}(\mathbb{C})$ as follows:

$$s_1^\pi = \frac{t_{\pi(1)}}{t_{\pi(2)}} \ , \ldots \ , s_n^\pi = \frac{t_{\pi(n)}}{t_{\pi(n+1)}} \ . \tag{6.4}$$

The coordinates s_i^π give a homeomorphism from Δ_π to the unit cube $(0,1)^n$. When π is the trivial permutation, the coordinates s_i^π are the same coordinates x_i used to define the multiple polylogarithms in §2.2. For each π, we define the open affine scheme

$$U_\pi = \mathrm{Spec}\, \mathbb{Z}[s_1^\pi, \ldots, s_n^\pi, \{(\prod_{i\leq k\leq j} s_k^\pi - 1)^{-1}\}_{1\leq i\leq j\leq n}] \ .$$

Note that the U_π are all canonically isomorphic, and $(0,1)^n \subset U_\pi(\mathbb{R})$.

Lemma 29. *For every π, U_π defines an affine chart on $\overline{\mathfrak{M}}_{0,n+3}$:*

$$\mathfrak{M}_{0,n+3} \subset U_\pi \subset \overline{\mathfrak{M}}_{0,n+3}$$

Proof. Consider the set of forgetful maps (or 'cross-ratios') $f_T : \overline{\mathfrak{M}}_{0,n+3} \rightarrow \overline{\mathfrak{M}}_{0,4} \cong \mathbb{P}^1$, where T is a subset of any 4 of the $n+3$ marked points. Then $\mathfrak{M}_{0,n+3} \subset \overline{\mathfrak{M}}_{0,n+3}$ is the open subscheme where all f_T's are non-zero. It suffices to check that U_π is isomorphic to the open subscheme of $\overline{\mathfrak{M}}_{0,n+3}$ where some of the f_T's are non-zero. For this, one can write each s_k^π and $\prod_{i\leq k\leq j} s_k^\pi - 1$ as cross-ratios, and conversely every cross-ratio as a function of the s_i^π. We omit the details. □

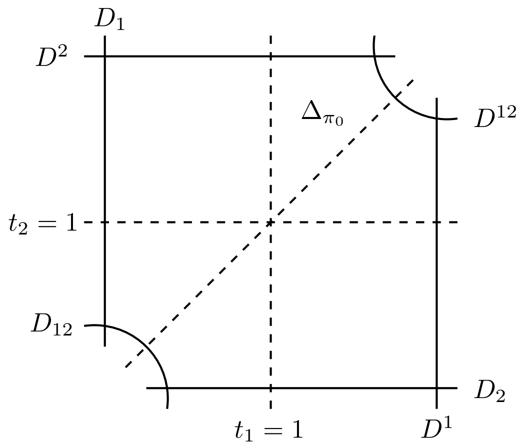
Definition 30. Let $U_n = \bigcup_\pi U_\pi \subset \overline{\mathfrak{M}}_{0,n+3}$ be the scheme obtained by gluing all charts U_π together. Viewing $\mathfrak{S}_{n+1}$ as the stabilizer of $0, \infty$ in $\mathrm{Aut}(\overline{\mathfrak{M}}_{0,n+3}) \cong \mathfrak{S}_{n+3}$, we have

$$U_n = \bigcup_{\pi \in \mathfrak{S}_{n+1}} \pi(U_{id}) \ ,$$

where U_{id} corresponds to the trivial permutation.

The smooth scheme $U_n \subset \overline{\mathfrak{M}}_{0,n+3}$ is equipped with a set of normal crossing divisors defined in each chart by the vanishing of the s_k^π.

Example 31. Let $n = 2$. The coordinate square $(0,\infty) \times (0,\infty) \subset \mathfrak{M}_{0,5}(\mathbb{R})$ is covered by six sectors Δ_π as shown below after blowing up $F_{12} = (0,0)$, $F^{12} = (\infty,\infty)$.



Consider the sector denoted Δ_{π_0}, where $\pi_0 = (1, t_1, t_2)$, which corresponds to the dihedral ordering $0 < 1 < t_1 < t_2 < \infty$ on the marked points of $\mathfrak{M}_{0,5}(\mathbb{R})$. Its sector coordinates are $s_1^{\pi_0} = t_1^{-1}, s_2^{\pi_0} = t_1/t_2$. The divisor $s_2^{\pi_0} = 0$ corresponds to the partition $\{0, 1, t_1\}|\{t_2, \infty\}$ on $\overline{\mathfrak{M}}_{0,5}$, and the divisor $s_1^{\pi_0} = 0$ corresponds to $\{0, 1\}|\{t_1, t_2, \infty\}$.

In the general case, we have:

Lemma 32. *The divisor $s_k^\pi = 0$ corresponds on $U_\pi \subset \overline{\mathfrak{M}}_{0,n+3}$ to the partition (§2.1)*

$$\{0, t_{\pi(1)}, \dots, t_{\pi(k)}\}\big|\{t_{\pi(k+1)}, \dots, t_{\pi(n+1)}, \infty\} \ .$$

Therefore the scheme U_n is the complement in $\overline{\mathfrak{M}}_{0,n+3}$ of the set of divisors A corresponding to partitions in which the marked points 0 and ∞ lie in the same component.

The simplicial coordinates $t_1, \dots, t_n$ give a canonical map $U_n \to (\mathbb{P}^1)^n$, which identifies U_n with the blow-up P_S of $(\mathbb{P}^1)^n$ defined in §6.3. Thus, we have

$$P_S \backslash X' \cong \overline{\mathfrak{M}}_{0,n+3} \backslash A \cong U_n \ .$$

This identifies the following divisors on $P_S\backslash X'$, $\overline{\mathfrak{M}}_{0,n+3}\backslash A$, and U_n, respectively:

$$D_I \ \leftrightarrow \ \{0, t_k : k \in I\}\big|\{\infty, 1, t_k : k \notin I\} \ \leftrightarrow \ s^\pi_{|I|} = 0 \text{ on every } U_\pi \text{ st } \pi(I) = I$$

$$D^J \ \leftrightarrow \ \{0, 1, t_k : k \notin J\}\big|\{\infty, t_k : k \in J\} \ \leftrightarrow \ s^\pi_{n+1-|J|} = 0 \text{ on every } U_\pi \text{ st } \pi(J) = J$$

We deduce that $D_I \cap D_{I'} \neq \emptyset$ if and only if $I \subseteq I'$ or $I' \subseteq I$ (and likewise for $D^J, D^{J'}$) and $D_I \cap D^J \neq \emptyset$ if and only if $I \cap J = \emptyset$.

Proof. Straightforward. One must only verify that the sector coordinates s_k^π are precisely the local coordinates that one obtains when one blows up $(\mathbb{P}^1)^n$ along divisors F_I and F^J in order of increasing dimension. □

In conclusion, we have three descriptions of the space U_n: first, as a certain blow-up of $(\mathbb{P}^1)^n$ along the boundary of the hypercube; second, as the gluing together of affine schemes U_π; and third, as the complement in $\overline{\mathfrak{M}}_{0,n+3}$ of a certain family of divisors.

Corollary 33. *Let $D \subset U_n$ be an irreducible divisor defined locally by the vanishing of an s_k^π. Then $D \cong U_{k-1} \times U_{n-k}$, where U_0 is a point. It follows that the polytope $\mathcal{C}_n$ (which was defined to be the strict transform $\pi^{-1}(\square_n)$) has the following product structure on its facets: $\mathcal{C}_n \cap D_I \cong \mathcal{C}_{|I|-1} \times \mathcal{C}_{n-|I|}$ and similarly for $\mathcal{C}_n \cap D^J$.*

6.5. **Absolute convergence of multivalued series.** Let

$$\mathcal{S} = \{(q, t_1, \dots, t_n) : 0 < |q| < |t_1| < \dots < |t_n| < 1, \qquad t_i t_j^{-1} \notin q^{\mathbb{R}} \ , t_i \notin q^{\mathbb{R}}\} \ , \tag{6.5}$$

and let f be a unipotent function on $(\mathbb{C}^\times)^n \backslash \{t_i = t_j\}$, i.e., a multivalued function on $\mathfrak{M}_{0,n+3}(\mathbb{C}) \subset \overline{\mathfrak{M}}_{0,n+3}(\mathbb{C})$ with everywhere local unipotent monodromy around boundary divisors (definition 2). Consider a sum

$$F(t_1, \dots, t_n; q) = \sum_{m_1, \dots, m_n \in \mathbb{Z}} u_1^{m_1} \dots u_n^{m_n} f(q^{m_1} t_1, \dots, q^{m_n} t_n) \ , \tag{6.6}$$

By §6.2, the values $f(q^{m_1} t_1, \dots, q^{m_n} t_n)$ are well-defined if we fix a branch of f near some point $(t_1, \dots, t_n)$, where $(q, t_1, \dots, t_n) \in \mathcal{S}$. We give sufficient conditions on the auxilliary variables u_i to ensure the absolute convergence of such a series, by bounding the terms of F in different sectors.

Definition 34. For $0 < \varepsilon \ll 1$, let $U_\pi^\varepsilon \subset U_\pi$ denote the open set of points

$$\{(s_1^\pi, \dots, s_n^\pi) \in U_\pi : |s_i^\pi| < 1 \text{ for all } i, \text{ and } |s_i^\pi| < \varepsilon \text{ for some } 1 \leq i \leq n\} \ ,$$

and let $U^\varepsilon = \bigcup_{\pi \in \mathfrak{S}_{n+1}} U_\pi^\varepsilon$.

Let K be a compact subset of $\mathcal{S}$ (6.5).

Lemma 35. *For $(q, t_1, \dots, t_n)$ in K, there are only finitely many $\underline{m} = (m_1, \dots, m_n)$ such that $(q^{m_1} t_1, \dots, q^{m_n} t_n)$ lies in the complement of U_ε, and $f(q^{m_1} t_1, \dots, q^{m_n} t_n)$ is uniformly bounded for such $\underline{m}$.*

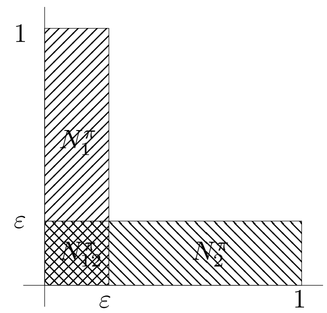
Proof. The first part follows since the complement of U^ε in $\overline{\mathfrak{M}}_{0,n+3}$ is compact, and does not contain the total transform of any divisors $t_i = 0, \infty$. The definition of $\mathcal{S}$ ensures that $q^{m_i} t_i \neq q^{m_j} t_j$ and $q^{m_i} t_i \neq 1$ for all m_i, m_j, $i \neq j$, and since these are the possible singularities of $f(q^{m_1} t_1, \dots, q^{m_n} t_n)$, it is uniformly bounded on K. □

All the remaining terms of (6.6) lie in some sector U^ε_π. Let us fix one such permutation π and work in local sector coordinates $s^\pi_1, \dots, s^\pi_n$. Then U^ε_π can be further decomposed into smaller pieces as follows. For any non-empty $A \subseteq \{1, \dots, n\}$, let

$$N^\pi_A = \{(s^\pi_1, \dots, s^\pi_n) : \ |s^\pi_i| < \varepsilon \text{ for } i \in A, \quad 1 > |s^\pi_i| \geq \varepsilon \text{ for } i \notin A\} \ .$$

We clearly have:

$$U^\varepsilon_\pi = \bigcup_{\emptyset \neq A \subseteq \{1,\dots,n\}} N^\pi_A \ .$$



Proposition 36. *There is a constant $C \in \mathbb{R}$ such that for all $(s^\pi_1, \dots, s^\pi_n) \in N^\pi_A$,*

$$|f(s^\pi_1, \dots, s^\pi_n)| \leq C \,\Big(\prod_{i \in A} \kappa_i(s^\pi_i)\Big)\, f_A((s^\pi_j)_{j \notin A}) \ ,$$

where $f_A(s^\pi_j)_{j\notin A}$ is a unipotent function on $\bigcap_{i\in A}\{s^\pi_i = 0\}$, and

$$\kappa_i(s) = |s|^{M_i} \log^w |s| \ ,$$

where f vanishes along $s^\pi_i = 0$ to order $M_i \geq 0$, and w is some integer ≥ 0.

Proof. This follows immediately from the local expansion of a unipotent multivalued function in the neighbourhood of a normal crossing divisor (definition (2.7)). □

Recall from definition (6.4) that $s^\pi_i = t_{\pi(i)} t^{-1}_{\pi(i+1)}$. Thus the action of the summation index $(m_1, \dots, m_n) \in \mathbb{Z}^n$ on the sector coordinate s^π_i is given by $s^\pi_i \mapsto q^{p^\pi_i} s^\pi_i$, where

$$p^\pi_i = m_{\pi(i)} - m_{\pi(i+1)} \ . \tag{6.7}$$

By lemma 32, the divisor $s^\pi_k = 0$ corresponds to a divisor D_I or D^J. Let us define

$$v^\pi_k = \prod_{i\in I} u_i \qquad \text{or} \qquad v^\pi_k = \prod_{j \in J} u_j^{-1} \tag{6.8}$$

accordingly, where u_i are the parameters in (6.6). One verifies that

$$(v^\pi_1)^{p^\pi_1} \dots (v^\pi_n)^{p^\pi_n} = u_1^{m_1} \dots u_n^{m_n} \ .$$

Thus, for the terms of (6.6) which lie in the sector U_π, we can write

$$u_1^{m_1} \dots u_n^{m_n} f(q^{m_1} t_1, \dots, q^{m_n} t_n) = (v^\pi_1)^{p^\pi_1} \dots (v^\pi_n)^{p^\pi_n} f(q^{p^\pi_1} s^\pi_1, \dots, q^{p^\pi_n} s^\pi_n)$$

in local coordinates. Dropping cluttersome π's from the notation, we have:

Corollary 37. *For all $(q, t_1, \ldots, t_n) \in K$, there exists a constant C such that*

$$\left|v_1^{p_1} \ldots v_n^{p_n} f(q^{p_1} s_1^\pi, \ldots, q^{p_n} s_n^\pi)\right| \leq C \prod_{i \in A} |v_i q^{M_i}|^{p_i} |p_i|^w$$

for all $(p_1, \ldots, p_n) \in \mathbb{Z}^n$ such that $(q^{p_1} s_1^\pi, \ldots, q^{p_n} s_n^\pi) \in N_A^\pi$.

Proof. Consider the bound in proposition 36. The function f_A on the right-hand side can be uniformly bounded by a version of lemma 35, applied to $\bigcap_{i \in A} \{s_i^\pi = 0\}$, which is isomorphic to a product of U_k's. This gives the above bound. □

Theorem 38. *Suppose that the chosen branch of f vanishes along all divisors D_I of type* 0 *with multiplicity $|I|$. Then* (6.6) *converges absolutely on compacta of the polydisc*

$$1 < u_1, \ldots, u_n < |q|^{-1}$$

Proof. Consider first a divisor D_I of type 0. Then, in the previous corollary, the divisor $D_I = \{s_k^\pi = 0\}$ in some chart U_π will correspond in the right-hand side to terms of the form $|v_k^\pi q^{|I|}|^p p^w$, where p is large and positive. The assumptions on u_i imply that

$$|v_k^\pi q^{|I|}|^p p^w = \Big(\prod_{i \in I} |u_i q|\Big)^p p^w < 1 \ ,$$

and therefore $|v_k^\pi q^{|I|}|^p p^w$ tends to zero exponentially fast in p. Now consider a divisor D^J of type ∞. It corresponds to terms of the form $|v_k^\pi q^{M_k}|^p p^w$, where p is large and positive and $M_k, w \geq 0$. But by the assumptions on u_i, we have

$$|v_k^\pi| = \Big(\prod_{j \in J} u_j^{-1}\Big) < 1 \ ,$$

and so once again, $|v_k^\pi q^{M_k}|^p p^w$ tends to zero exponentially fast in p. □

6.6. **Structure of the poles.** We first make some general remarks about the pole structure as follows from the proof of theorem 38. In §7.4 we shall refine this result in the case of the multiple elliptic polylogarithms by exploiting their differential equation.

Corollary 39. *Let f satisfy the conditions of theorem* 38*. For every $I \neq \emptyset$, let w_I denote the order of the logarithmic singularity of f along D^I. Every codimension h face of the polytope $\mathcal{C}_n$ corresponds to a flag $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_h$, where $I_1, \ldots, I_h \subseteq \{1, \ldots, n\}$, and is contained in the intersection $E = D^{I_1} \cap \ldots \cap D^{I_h}$. Let $s_1, \ldots, s_h, s_{h+1}, \ldots, s_n$ denote local normal coordinates in which D^{I_k} is given by $s_k = 0$ for $1 \leq k \leq h$. Then in the neighbourhood of E, the function f has an expansion of the form*

$$f = \sum_{i_1 \leq w_{I_1}, \ldots, i_h \leq w_{I_h}} f_{i_1, \ldots, i_h}(s_{h+1}, \ldots, s_n) \log^{i_1}(s_1) \ldots \log^{i_h}(s_h) \ ,$$

where $(s_{h+1}, \ldots, s_n)$ are coordinates on E. After averaging, each term in the sum which is indexed by $i_1, \ldots, i_h$ contributes singularities to (6.6) *of the form*

$$(\prod_{k \in I_1} u_k - 1)^{-j_1} \ldots (\prod_{k \in I_h} u_k - 1)^{-j_h} \ ,$$

with $j_1 \leq i_1 + 1, \ldots, j_h \leq i_h + 1$. In particular, the term $f_{0,\ldots,0}(s_{h+1}, \ldots, s_n)$ which is constant in $s_1, \ldots, s_h$, contributes a simple pole of the form

$$(\prod_{k \in I_1} u_k - 1)^{-1} \ldots (\prod_{k \in I_h} u_k - 1)^{-1} \ .$$

Proof. Following the method of proof of the previous theorem, one sees that the statement reduces to a local computation in the one-dimensional situation. In this case, it is clear that the averaging procedure applied to $\log^i z$, for $i \geq 0$, gives

$$\sum_{m \geq 0} u^m \log^i q^m z = \sum_{m \geq 0} u^m (m \log q + \log z)^i \ ,$$

which has a pole at $u = 1$ of order at most $i+1$. □

Remark 40. Corollary 39 gives an upper bound on the singularities which occur: it can happen that summing over one sector gives rise to spurious poles in the u_i's, which cancel on taking the total contribution over all sectors.

The upshot of the previous corollary is that if we know the differential equations satisfied by f, then we can deduce the pole structure of F completely from these differential equations, up to constants of integration (see §9 below). The corollary states that the constants of integration necessarily contribute simple poles, and these are in bijection with the $n!$ maximal flags $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_{n-1} \subsetneq I_n$.

Example 41. In the case $n = 2$, we have poles along $u_i = 1$ coming from logarithmic singularites along D^i, for $i = 1, 2$, and along $u_1 u_2 = 1$ coming from D^{12}. The typical contribution from D^1, for example, is of the form

$$\sum_{i=0}^{w} \frac{R_i(\xi_2; u_2)}{(u_1 - 1)^{i+1}}$$

where $R_i(\xi_2; u_2)$ is the result of averaging a function of t_2. The two maximal flags $\{1\} \subset \{1, 2\}$ and $\{2\} \subset \{1, 2\}$ which correspond to the corners $D^1 \cap D^{12}$ and $D^2 \cap D^{12}$, give constant contributions of the form

$$\frac{c_{1,12}}{(u_1 - 1)(u_1 u_2 - 1)} \quad \text{and} \quad \frac{c_{2,12}}{(u_2 - 1)(u_1 u_2 - 1)}$$

where $c_{1,12}$ (resp. $c_{2,12}$) is the regularized limit of f at $D^1 \cap D^{12}$ (resp. $D^2 \cap D^{12}$). For an explicit computation of such a pole structure, see §9.2.

7. Elliptic multiple polylogarithms

In this section, we apply the results of §6 to prove the following theorem.

Theorem 42. *The series obtained by averaging the classical multiple polylogarithm*

$$E_{n_1,\ldots,n_r}(\xi_1, \ldots, \xi_r; u_1, \ldots, u_r) = \sum_{m_1,\ldots,m_r \in \mathbb{Z}} u_1^{m_1} \ldots u_r^{m_r} I_{n_1,\ldots,n_r}(q^{m_1} t_1, \ldots, q^{m_r} t_r)$$

converges for $1 < u_1, \ldots, u_r < |q|^{-1}$, *and* $(q, t_1, \ldots, t_r) \in \mathcal{S}$. *It defines a (generating series) of functions on* $\mathcal{E}^{(r)}$ *with poles given by products of consecutive* u_i*'s only:*

$$u_i = |q|^{-1} \ \textit{for} \ 1 \leq i \leq r \ , \qquad \textit{and} \ \prod_{i \leq k \leq j} u_k = 1 \ \textit{for all} \ 1 \leq i \leq j \leq r \ .$$

In order to extract a convergent Taylor expansion in the variables α_i, where $u_i = \mathbf{e}(\alpha_i)$, it suffices to know the exact asymptotic behaviour of $I_{n_1,\ldots,n_r}(t_1, \ldots, t_r)$ at infinity. This is carried out for the Debye polylogarithms in §7.

7.1. **Analytic continuation of polylogarithms.** The function $I_{n_1,\ldots,n_r}(t_1,\ldots,t_r)$ has a convergent Taylor expansion at the origin, and so defines the germ of a multivalued analytic function on $\mathfrak{M}_{0,n+3}(\mathbb{C}) \subset \mathbb{C}^n$. As is well-known, it is unipotent by (2.5), has a canonical branch at the origin, and vanishes along the divisors $t_i = 0$.

It therefore extends by analytic continuation to a multivalued function on the blow-up $U_n \backslash X(\mathbb{C})$, and can have at most logarithmic divergences along the boundary components D^J and D_I of X. It turns out that for some of these components D, there is no logarithmic divergence and we can speak of the continuation $I_{m_1,\ldots,m_n}(t_1,\ldots,t_n)\big|_D$ of $I_{m_1,\ldots,m_n}(t_1,\ldots,t_n)$ to D. This is not well-defined, since the function is multivalued. However, if D also meets the strict transform of a divisor $t_i = 0$, for some i, there is a canonical branch which vanishes at $\{t_i = 0\} \cap D$, and defined in a neighbourhood of $\{t_i = 0\}$. The following lemma is the key to the absolute convergence of (7.1).

Lemma 43. *Let D_I be of type* 0. *Then $I_{m_1,\ldots,m_n}(t_1,\ldots,t_n)\big|_{D_I}$ vanishes to order $|I|$.*

Proof. The sum (6.2) converges in the neighbourhood of the origin and locally defines a holomorphic function which vanishes along the divisors $t_i = 0$. It therefore vanishes on any exceptional divisor D_I lying above the origin to order $|I|$. □

Setting $f(t_1,\ldots,t_n) = I_{m_1,\ldots,m_n}(t_1,\ldots,t_n)$ proves the first part of theorem 42.

Lemma 44. *Let $J \subset \{1,\ldots,n\}$ be non-empty, and $1 \notin J$. Let $J^c = \{1,\ldots,n\}\backslash J$ and write $J^c = \{i_1,\ldots,i_k\}$ where $1 = i_1 < \ldots < i_k$. Then*

$$I_{m_1,\ldots,m_n}(t_1,\ldots,t_n)\big|_{D^J} = (-1)^{|J|} I_{m'_1,\ldots,m'_k}(t_{i_1},\ldots,t_{i_k}) \ ,$$

where $m'_1 = m_{i_1} + \ldots + m_{i_2-1}$, $m'_2 = m_{i_2} + \ldots + m_{i_3-1}, \ldots,$ $m'_k = m_{i_k} + \ldots + m_n$.

Proof. This follows immediately from the iterated integral representation (2.4). □

Corollary 45. *Every multiple polylogarithm $I_{m_1,\ldots,m_n}(t_1,\ldots,t_n)$ is the analytic continuation to some exceptional divisor of a multiple logarithm $I_{1,\ldots,1}(t_1,\ldots,t_N)$.*

Thus we can restrict ourselves to considering only multiple logarithms if we wish. Another way to interpret lemma 44 is to notice that the terms $I\big|_{D^J}$ are in one-to-one correspondence with the terms in the so-called stuffle product formula.

7.2. **Elliptic Multiple Polylogarithms.** The functions obtained by averaging multiple polylogarithms satisfy differential equations which are easily deduced from (2.5).

Lemma 46. *The function $F(\xi;u)$ is the averaged weighted generating series for $\frac{z}{z-1}$:*

$$F(\xi;u) = -2\pi i \sum_{n\in\mathbb{Z}} \frac{q^n z}{1-q^n z} u^n \ .$$

Proof. By decomposing the domain of summation into various parts we obtain:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{q^n z}{1-q^n z}u^n &= \sum_{n<0} \frac{-q^{-n}z^{-1}}{1-q^{-n}z^{-1}}u^n - \sum_{n<0} u^n + \frac{z}{1-z} + \sum_{n>0}\frac{q^n z}{1-q^n z}u^n \\ &= \sum_{n>0}\sum_{m>0} q^{mn}(-z^{-m}u^{-n} + z^m u^n) + \frac{z}{1-z} + \frac{1}{1-u}\end{aligned}$$

which is the definition of the Eisenstein-Kronecker series $-(2i\pi)^{-1}F(\xi;u)$. □

Lemma 47. *The averaged weighted generating series for $d\mathrm{Li}_1(z)$ is:*

$$\sum_{n\in\mathbb{Z}} d\mathrm{Li}_1(zq^n)u^n = F(\xi;u)d\xi \ .$$

Likewise, the result of averaging $z^{-s}d\mathrm{Li}_1(z)$ is $\mathbf{e}(-\xi s)F(\xi;u)d\xi$.

Proof. Follows from the previous lemma using the fact that $d\xi = \frac{1}{2i\pi}\frac{dz}{z}$. □

Let us define the (unregularized) multiple elliptic polylogarithm to be:

$$E_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;\alpha_1,\ldots,\alpha_r) = \sum_{m_1,\ldots,m_r\in\mathbb{Z}} u_1^{m_1}\ldots u_r^{m_r} I_{n_1,\ldots,n_r}(q^{m_1}t_1,\ldots,q^{m_r}t_r)\ .$$

where $u_i = \mathbf{e}(\alpha_i)$ for $1\leq i\leq r$.

Theorem 48. *The total derivative* $dE_{1,\ldots,1}(\xi_1,\ldots,\xi_n;\alpha_1,\ldots,\alpha_n)$ *equals*

$$\begin{aligned} =\ & \sum_{k=1}^{n} dE_1(\xi_k-\xi_{k+1};\alpha_k)E_{1,\ldots,1}(\xi_1,\ldots,\widehat{\xi_k},\ldots,\xi_n;\alpha_1,\ldots,\alpha_k+\alpha_{k+1},\ldots,\alpha_n) \\ -\ & \sum_{k=2}^{n} dE_1(\xi_k-\xi_{k-1};\alpha_k)E_{1,\ldots,1}(\xi_1,\ldots,\widehat{\xi_k},\ldots,\xi_n;\alpha_1,\ldots,\alpha_{k-1}+\alpha_k,\ldots,\alpha_n) \end{aligned}$$

where $\xi_{n+1}=0$, $\alpha_{n+1}=0$, *and* $dE_1(\xi;\alpha) = F(\xi;\alpha)d\xi$.

Proof. Since it converges uniformly, we can differentiate term by term in the definition of $E_{1,\ldots,1}$. The differential equation then follows from the corresponding differential equation (2.5) for $I_{1,\ldots,1}(t_1,\ldots,t_n)$. The key observation is that a term such as

$$\sum_{m_1,\ldots,m_n\in\mathbb{Z}} dI_1\Big(\frac{q^{m_k}t_k}{q^{m_{k+1}}t_{k+1}}\Big) I_{1,\ldots,1}(q^{m_1}t_1,\ldots,\widehat{q^{m_k}t_k},\ldots,q^{m_n}t_n)\, u_1^{m_1}\ldots u_n^{m_n}$$

can be rewritten in the form

$$\begin{aligned} \sum_{m_1,\ldots,m_n\in\mathbb{Z}} & dI_1\Big(q^{m_k-m_{k+1}}\frac{t_k}{t_{k+1}}\Big)\, u_k^{m_k-m_{k+1}} \\ \times\ & I_{1,\ldots,1}(q^{m_1}t_1,\ldots,\widehat{q^{m_k}t_k},\ldots,q^{m_n}t_n)\, u_1^{m_1}\ldots(u_ku_{k+1})^{m_{k+1}}\ldots u_n^{m_n} \end{aligned}$$

and the region of summation decomposes into a product after a triangular change of basis of the summation variables $(m_1,\ldots,m_n)\mapsto(m_1,\ldots,m_k-m_{k+1},\ldots,m_n)$. □

7.3. **Elliptic Debye polylogarithms.** Recall the definition of the classical Debye polylogarithms (definition 1). Let us write $\underline{\alpha} = (\alpha_1,\ldots,\alpha_r)$ and likewise for β.

Definition 49. The generating series of elliptic Debye polylogarithms is:

$$\mathtt{E}_r(\xi_1,..,\xi_r;\underline{\alpha},\underline{\beta}) = \sum_{m_1,..,m_r\in\mathbb{Z}} \mathbf{e}(m_1\alpha_1+\cdots+m_r\alpha_r)\Lambda_r(q_1^{m_1}t_1,..,q_r^{m_r}t_r;\beta_1,..,\beta_r)$$

The absolute convergence of the series is guaranteed by theorem 38.

One of the main reasons for considering such a generating series is because of a mysterious modularity property relating the parameters α and β (see [13] when $r=1$).

Proposition 50. *Let* $r\geq 2$. *The differential* $d\,\mathtt{E}_r(\xi_1,..,\xi_r;\underline{\alpha},\underline{\beta})$ *is equal to*

$$\begin{aligned} =\ & \sum_{k=1}^{n} d\,\mathtt{E}_1(\xi_k-\xi_{k+1};\alpha_k,\beta_k)\,\mathtt{E}_{r-1}(\xi_1,..,\widehat{\xi_k},..,\xi_r;\alpha_1,..,\alpha_k+\alpha_{k+1},..,\alpha_n,\beta_1,..) \\ -\ & \sum_{k=2}^{n} d\,\mathtt{E}_1(\xi_k-\xi_{k-1};\alpha_k,\beta_k)\,\mathtt{E}_{r-1}(\xi_1,..,\widehat{\xi_k},..,\xi_r;\alpha_1,..,\alpha_{k-1}+\alpha_k,..,\alpha_n,\beta_1,..) \end{aligned}$$

where $\xi_{n+1}=\alpha_{n+1}=\beta_{n+1}=0$, *and in the right-hand side, the arguments in the* β*'s are of the same form as those for the* α*'s. In the case* $r=1$, *we have*

$$d\,\mathtt{E}_1(\xi;\alpha;\beta) = \mathbf{e}(-\beta\xi)\,F(\xi;\alpha-\tau\beta)d\xi\ . \tag{7.1}$$

The proof follows immediately from theorem 48.

7.4. **The structure of the poles of elliptic polylogarithms.** Let us write

$$\gamma_i = \alpha_i - \tau\,\beta_i \quad \text{for } 1 \leq i \leq r\ . \tag{7.2}$$

Proposition 51. *The Debye elliptic polylogarithms (definition 49) have at most simple poles along the divisors which have consecutive indices only:*

$$\sum_{i\leq j\leq k} \gamma_i = 0 \qquad \textit{and} \qquad \sum_{i\leq j\leq k} \alpha_j = 0\ .$$

The multiple elliptic polylogarithm $E_{m_1,..,m_n}(\xi_1,..,\xi_n;\alpha_1,..,\alpha_n)$ has poles along divisors of the form $\sum_{i\leq j\leq k}\alpha_j = 0$ of order at most $m_1+\ldots+m_n+1$.

Proof. By induction. Suppose that $\mathtt{E}_r$ has simple poles along $\sum_{i\leq j\leq k}\alpha_j = 0$, and $\sum_{i\leq j\leq k}\gamma_j = 0$ with consecutive indices only. This is automatically true for $r \leq 2$. It follows from the shape of the differential equation (proposition 50), that $d\,\mathtt{E}_{r+1}$ only has simple poles along $\sum_{i\leq j\leq k}\alpha_j = 0$ and $\sum_{i\leq j\leq k}\gamma_j = 0$. Thus the same conclusion also holds for $\mathtt{E}_{r+1}$, except that the constants of integration might give rise to supplementary poles. To see that such constants of integration must be zero, let I be a set of non-consecutive indices. The divisor D^I meets a divisor of the form $t_j = 0$, for $j \notin I$, along which the function $\Lambda(t_1,\ldots,t_n;s_1,\ldots,s_n)$ vanishes. It follows from the discussion above that Λ has no divergence in the neighbourhood of $D^I \cap \{t_j = 0\}$, and hence no pole in either $\sum_{i\in I}\alpha_i = 0$ or $\sum_{i\in I}\gamma_i = 0$. This proves the result for the generating series $\mathtt{E}_r$. The corresponding statement for its coefficients $E_{m_1,..,m_r}(\xi_1,..,\xi_r;\alpha_1,..,\alpha_r)$ follows on taking a series expansion in the β_i. □

This method for computing the pole structure is illustrated below (§9.2).

8. Asymptotics of Debye polylogarithms.

In the previous section we showed that the polar contributions in the averaging process come from the asymptotic expansion of polylogarithms at infinity. This expansion can be computed explicitly in terms of a combinatorially defined coproduct.

8.1. **The coproduct for Debye polylogarithms.** The Debye multiple polylogarithms are defined by iterated integrals, and so by the general theory [6] admit a coproduct which is dual to the composition of paths. We describe it explicitly below.

Definition 52. Let $n \geq 1$. Let $I = \{1,\ldots,n\}$ be an ordered set of indices and let $\beta_1,\ldots,\beta_n$ be formal variables satisfying $\sum_{i=1}^n \beta_i = 0$. Define a *string* in I to be a consecutive subsequence $S = (i_1,i_2,\ldots,i_l)$ of length $2 \leq l < n$, which is either in increasing or decreasing order and such that $i_1 \neq n$. Let

$$\beta_S = \beta_{i_1} + \beta_{i_2} + \ldots + \beta_{i_{l-1}}\ .$$

For any such string $S = (i_1,\ldots,i_l)$, let A_S denote the symbol

$$A_S = (t_{i_1} : t_{i_2} : \ldots : t_{i_l}; \beta_{i_1},\beta_{i_2},\ldots,\beta_{i_{l-1}},-\beta_S)\ . \tag{8.1}$$

Let H_n denote the commutative ring over $\mathbb{Z}$ generated by all symbols A_S as S ranges over the set of strings in $1,2,\ldots,n$. The length of a string defines a grading on H_n.

The Debye polylogarithm defines a map from H_n to generating series of multivalued functions. If a string S is given by (8.1) then we have

$$\Lambda(A_S) = \Lambda_{l-1}\Big(\frac{t_{i_1}}{t_{i_\ell}},\frac{t_{i_2}}{t_{i_\ell}},\ldots,\frac{t_{i_{l-1}}}{t_{i_\ell}};\beta_{i_1},\beta_{i_2},\ldots,\beta_{i_{l-1}}\Big)\ . \tag{8.2}$$

Denote the last element of a string by $\ell(i_1,i_2,\ldots,i_l) = i_l$, and define the sign $\varepsilon(S)$ of S to be 1 if S is in increasing order, or $(-1)^{l-1}$ if it is in decreasing order.

Definition 53. A finite collection $\mathcal{S} = \{S_\alpha\}$ of strings is *admissible* if the strings intersect at most in their last indices, i.e., if S_α, S_β are in $\mathcal{S}$ and $\alpha \neq \beta$ then either

$$\begin{array}{ll} (1) & S_\alpha \cap S_\beta = \emptyset \\ \text{or} \quad (2) & S_\alpha \cap S_\beta = \{\ell\} \ , \quad \text{where } \ell = \ell(S_\alpha) = \ell(S_\beta) \ . \end{array}$$

Given an admissible set of strings $\mathcal{S} = \{S_\alpha\}$, define the set of remaining indices

$$R_{\mathcal{S}} = \big(I \setminus \bigcup S_\alpha\big) \cup \bigcup \ell(S_\alpha)$$

with the ordering induced from I, and define the corresponding quotient sequence

$$Q_{\mathcal{S}} = (t_{j_1} : t_{j_2} : \ldots : t_{j_m}; \tilde{\beta}_{j_1}, \tilde{\beta}_{j_2}, \ldots, \tilde{\beta}_{j_m}) \ ,$$

where $(j_1, j_2, \ldots, j_m) = R_{\mathcal{S}}$ and $\tilde{\beta}_j = \beta_j + \sum_{\alpha, \ell(S_\alpha)=j} \beta_{S_\alpha}$.

Definition 54. Define a map $\Delta' : H_n \longrightarrow H_n \otimes H_n$ by

$$\Delta' A_J = \sum_{\mathcal{S}=\{S_1,\ldots,S_k\}} \varepsilon(S_1) A_{S_1} \ldots \varepsilon(S_k) A_{S_k} \otimes Q_{\mathcal{S}} \tag{8.3}$$

where $J \subseteq \{1, \ldots, n\}$, and the sum is over all non-empty admissible collections of strings $\mathcal{S}$ in J such that $Q_{\mathcal{S}}$ has at least two elements.

Consider the map which sends A_S of (8.1) to $t_{i_1}^{\beta_{i_1}} \ldots t_{i_l}^{\beta_{i_l}}$, with $\beta_{i_l} = -\beta_S$, and extend by multiplicativity. Then each term in (8.3) maps to $\pm t_1^{\beta_1} \ldots t_n^{\beta_n}$.

Example 55. Writing β_{ij} for $\beta_{\{i,j\}} = \beta_i + \beta_j$, and so on, formula (8.3) gives:

$$\begin{aligned} \Delta'(t_1 : t_2 : t_3; \beta_1, \beta_2, \beta_3) \ &= \ (t_1 : t_2; \beta_1, -\beta_1) \otimes (t_2 : t_3; \beta_{12}, \beta_3) \\ &- \ (t_2 : t_1; \beta_2, -\beta_2) \otimes (t_1 : t_3; \beta_{12}, \beta_3) \\ &+ \ (t_2 : t_3; \beta_2, -\beta_2) \otimes (t_1 : t_3; \beta_1, \beta_{23}) \ . \end{aligned} \tag{8.4}$$

In general, a typical term in $\Delta'(t_1 : \ldots : t_6; \beta_1, \ldots; \beta_6)$ is

$$(t_1 : t_2 : t_3; \beta_1, \beta_2, -\beta_{12})(t_4 : t_3; \beta_4, -\beta_4)(t_5 : t_6; \beta_5, -\beta_5) \otimes (t_3 : t_6; \beta_{1234}, \beta_{56})$$

Proposition 56. *Let $\Delta : H_n \to H_n \otimes H_n$ be $\Delta = 1 \otimes id + id \otimes 1 + \Delta'$. Then H_n, equipped with Δ, is a commutative graded Hopf algebra.*

Proof. We omit the proof. In fact it suffices to show that the 1-part of the coproduct coincides with the differential for the Debye polylogarithms (lemma 57 below). □

Let $\Delta^{(m+1)} : H_n \to H_n^{\otimes m+1}$ denote the m-fold iteration of Δ. Let $\Delta^{(m+1)}_{\star,\ldots,\star,1,\star,\ldots,\star}$ denote its component whose corresponding tensor factor contains only strings of length two. Thus $\Delta_{1,\star}$ extracts all ordered pairs of neighbouring indices except $(n, n-1)$.

Lemma 57. *The differential equation for Λ can be rewritten as*

$$d\,\Lambda = \mu \circ (d\Lambda \otimes \Lambda) \circ \Delta_{1,\star} \ ,$$

where μ denotes the multiplication map.

Proof. Follows from the definition of Λ together with the differential equation (2.6). □

Example 58. For any $i \neq j$, let $t_{[i,j]} = (t_i : \ldots : t_j)$ denote the tuple of consecutive elements. Contributions to $\Delta_{1,\star}$ are of the following kinds (omitting indices β_i):

$$(t_{i-1} : t_i) \quad \otimes \quad (t_1 : \ldots : \widehat{t}_{i-1} : \ldots : t_n) \qquad 1 < i \leq n \tag{8.5}$$

$$(t_i : t_{i-1}) \quad \otimes \quad (t_1 : \ldots : \widehat{t}_i : \ldots : t_n) \qquad 1 < i < n \tag{8.6}$$

Contributions to $\Delta_{\star,1}$ are of the following kinds:

$$t_{[1,n-1]} \otimes (t_{n-1} : t_n) \tag{8.7}$$

$$t_{[2,n]} \otimes (t_1 : t_n) \tag{8.8}$$

$$t_{[n-1,1]} \otimes (t_1 : t_n) \tag{8.9}$$

$$t_{[i,1]}t_{[i+1,n]} \otimes (t_1 : t_n) \qquad 1 < i < n-1 \tag{8.10}$$

$$t_{[1,k]}t_{[i,k]}t_{[i+1,n]} \otimes (t_k : t_n) \qquad 1 < k < i < n-1 \tag{8.11}$$

8.2. **Asymptotic of the Debye polylogarithms.** Let J be a subset of $\{1, 2, \dots, n\}$. We study the asymptotics of $\Lambda(A_{1,\dots,n})$, as defined by (8.2) when t_j for $j \in J$ simultaneously tend to infinity: i.e., for some finite values $t_1^0, \dots, t_n^0$, we set:

$$t_j = Tt_j^0 \ , \quad \text{for } j \in J \ , \quad t_k = t_k^0 \quad \text{for } k \notin J \ , \text{ and let } T \to \infty \ . \tag{8.12}$$

Caveat 59. *The Debye polylogarithms are multivalued, and so their asymptotics are only well-defined up to monodromy. For divisors of the form D^I, where $I \subsetneq \{1, \dots, n\}$, and $i \neq I$, there is a canonical branch in the neighbourhood of $t_i = 0$, where it vanishes (see § 7.1). Only for the divisor $D^{\{1,\dots,n\}}$ must one make some choice. In the following theorem, this ambiguity is contained in the constant C in equation* (8.13).

We call a string $S = (i_1, i_2, \dots, i_l)$ *essential* if $i_l \notin J$ and $i_1, \dots, i_{l-1} \in J$, and *regular* if: either all indices belong to J, or none of its indices belongs to J. Set

$$\Lambda^{\mathrm{reg}}(A_S) = \begin{cases} \Lambda(A_S) & \text{if } S \text{ is regular} \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

and likewise define $\Phi(A_S)$ to be 0 if S is non-essential and

$$\Phi(A_S) = \frac{t_{i_1}^{-\beta_{i_1}} t_{i_2}^{-\beta_{i_2}} \dots t_{i_n}^{-\beta_{i_n}}}{\beta_{i_1}\beta_{i_1,i_2} \dots \beta_{i_1,i_2,\dots,i_{n-1}}} \qquad \text{if } S \text{ is essential} \ .$$

Theorem 60. *With the assumptions* (8.12) *above, for any $0 < \varepsilon \ll 1$ we have*

$$\Lambda(t_1 : t_2 : \dots : t_n; \beta_1, \beta_2, \dots, \beta_n) = \mu_3 \circ (\Phi \otimes \Lambda^{\mathrm{reg}} \otimes C) \circ \Delta^{(3)} + O(T^{\varepsilon-1}) \tag{8.13}$$

for some functions $C(t_1 : t_2 : \dots : t_n; \beta_1, \beta_2, \dots, \beta_n) = C(\beta_1, \beta_2, \dots, \beta_n)$ which are constant in the t's, and where μ_3 denotes the triple product.

Proof. Induction on the depth n. For $n = 2$ this theorem reduces to the well-known asymptotics of the classical Debye polylogarithm. For strings of length two

$$d\Lambda(t_1 : t_2; \beta_1, \beta_2) = t_1^{-\beta_1} t_2^{-\beta_2} d\,\mathrm{Li}_1(t_1/t_2)$$

where $\beta_1 + \beta_2 = 0$. It follows from this that for $S = (1, 2)$, $A_S = (t_1 : t_2; \beta_1, \beta_2)$,

$$d\Lambda(A_S) = \begin{cases} d\Phi(A_S) + O(T^{\varepsilon-1}) & \text{if } S \text{ is essential} \ , \\ d\Lambda^{\mathrm{reg}}(A_S) & \text{if } S \text{ is regular} \ , \\ O(T^{\varepsilon-1}) & \text{otherwise} \ . \end{cases}$$

This follows from the fact that Λ diverges at most logarithmically at infinity, and $\log(T)^a O(T^{\varepsilon-1}) = O(T^{\varepsilon-1})$. Hence, by lemma 57 it follows that asymptotically

$$d\Lambda \sim \mu \circ (d\,\Phi \otimes \Lambda) \circ \Delta_{1,\star} + \mu \circ (d\,\Lambda^{\mathrm{reg}} \otimes \Lambda) \circ \Delta_{1,\star} \ , \tag{8.14}$$

where $a \sim b$ means that $a - b = O(T^{\varepsilon-1})$. For the induction step, we first check that the differential of the difference between both sides of (8.13) vanishes. By induction hypothesis, we replace Λ in (8.14) by (8.13):

$$d\Lambda \sim \mu_4 \circ (d\,\Phi \otimes \Phi \otimes \Lambda^{\mathrm{reg}} \otimes C + d\,\Lambda^{\mathrm{reg}} \otimes \Phi \otimes \Lambda^{\mathrm{reg}} \otimes C) \circ \Delta^{(4)}_{1,\star,\star,\star} \ . \tag{8.15}$$

Now compute the differential of the right-hand side of (8.13),

$$\mu_3 \circ (d\,\Phi \otimes \Lambda^{\mathrm{reg}} \otimes C) \circ \Delta^{(3)} + \mu_4 \circ (\Phi \otimes d\,\Lambda^{\mathrm{reg}} \otimes \Lambda^{\mathrm{reg}} \otimes C) \circ \Delta^{(4)}_{\star,1,\star,\star}\ . \tag{8.16}$$

where in the second term we used lemma 57 applied to $d\Lambda^{\mathrm{reg}}$. In order to show that (8.16) and the right-hand side of (8.15) coincide, it suffices to use the coassociativity of the coproduct and to show that the following expression vanishes:

$$\Omega = d\,\Phi + \mu_2 \circ (\Phi \otimes d\,\Lambda^{\mathrm{reg}}) \circ \Delta_{\star,1} - \mu_2 \circ (d\,\Phi \otimes \Phi + d\,\Lambda^{\mathrm{reg}} \otimes \Phi) \circ \Delta_{1,\star}$$

as the difference of (8.16) and (8.15) is $\mu_3 \circ (\Omega \otimes \Lambda^{\mathrm{reg}} \otimes C) \circ \Delta^{(3)}$. We will prove that

$$d\,\Phi = \mu_2 \circ \big[-(\Phi \otimes d\,\Lambda^{\mathrm{reg}}) \circ \Delta_{\star,1} + (d\,\Phi \otimes \Phi) \circ \Delta_{1,\star} + (d\,\Lambda^{\mathrm{reg}} \otimes \Phi) \circ \Delta_{1,\star} \big] \tag{8.17}$$

applied to ξ, where $\xi = (t_1 : t_2 : \ldots : t_n; \beta_1, \beta_2, \ldots, \beta_n)$.

Case when ξ is essential. Then $J = \{1, \ldots, n-1\}$. It follows from example 58 that the only quotient sequences arising from $\Delta_{1,\star}$ are of the form $(t_i : t_n)$ for some $i < n$, and are therefore not regular. Thus the first term in the right-hand side of (8.17) vanishes. The only contributions to the second summand come from the string (t_{n-1}, t_n); only the strings (t_{i-1}, t_i) and (t_i, t_{i-1}), for $1 < i < n$, contribute to the last summand. For such a string a, let ξ/a denote its quotient. From the definitions:

$$\begin{aligned}(d\,\Lambda^{\mathrm{reg}} \otimes \Phi)(a \otimes \xi/a) &= \beta_{1,2,\ldots,i-1}\, d\,\mathrm{Li}(t_{i-1}t_i^{-1})\,\Phi(\xi) \quad \text{if } a = (t_{i-1}, t_i)\\ (d\,\Lambda^{\mathrm{reg}} \otimes \Phi)(a \otimes \xi/a) &= \beta_{1,2,\ldots,i-1}\, d\,\mathrm{Li}(t_i t_{i-1}^{-1})\,\Phi(\xi) \quad \text{if } a = (t_i, t_{i-1})\end{aligned}$$

Using the fact that $d\,\mathrm{Li}(t_{i-1}t_i^{-1}) - d\,\mathrm{Li}(t_i t_{i-1}^{-1}) = d\,\log(t_i) - d\log(t_{i-1})$ a straightforward calculation shows that both sides of (8.17) agree on ξ.

Case when ξ is non-essential. Either $n \in J$ or some $i < n$ is not in J. Suppose first that $n \in J$. The first term of (8.17) vanishes as either the argument of Φ is not essential, or the argument of Λ^{reg} is not regular. The second and third summands vanish since the arguments of Φ are non-essential. Hereafter, we assume $n \notin J$.

Now suppose that J^c contains at least 3 elements $J^c \supseteq \{i, j, n\}$. Then the entire right-hand side of (8.17) vanishes, since every argument of Φ is always non-essential.

It only remains to check the equality of (8.17) when J^c consists of two elements $\{k, n\}$ for some $k < n$. Consider the second and third terms on the right-hand side of (8.17). The quotient sequences of (8.5) and (8.6) are essential only for the strings $(t_k : t_{k+1})$ and $(t_k : t_{k-1})$. These are non-essential so the second factor $d\Phi \otimes \Phi$ vanishes for all possible values of k. The third factor $d\Lambda^{\mathrm{reg}} \otimes \Phi$ is non-trivial only when $k = n-1$ on the term $(t_{n-1} : t_n)$. In fact, in the case $k = n-1$, we have contributions from $\Phi \otimes d\Lambda^{\mathrm{reg}}$(8.7) and $d\Lambda^{\mathrm{reg}} \otimes \Phi$ applied to (8.5), for $i = n-1$. They cancel.

Thus in all remaining cases $k < n$ only the first term $\Phi \otimes d\Lambda^{\mathrm{reg}}$ of (8.17) can be non-zero. If $k = 1$ then we get terms in the first summand corresponding to (8.8), (8.9), (8.10). The cancellation of these terms follows from the equality

$$\sum_{i=1}^{n-1} (-1)^{i-1} a_{[i,2]}^{-1} b_{[i+1,n-1]}^{-1} = 0 \tag{8.18}$$

where $a_{[i,j]} = \beta_i(\beta_i + \beta_{i-1}) \ldots (\beta_i + \ldots + \beta_j)$, if $i \geq j$ and is equal to 1 otherwise, and $b_{[i,j]} = \beta_i(\beta_i + \beta_{i+1}) \ldots (\beta_i + \ldots + \beta_j)$, if $i \leq j$ and is equal to 1 otherwise. The general case $1 < k < n-1$ is similar, and equivalent to

$$\sum_{1<k<i<n-1} (-1)^{k-i-1} b_{[1,k-1]}^{-1} a_{[i,k+1]}^{-1} b_{[i+1,n-1]}^{-1} = 0 \tag{8.19}$$

Both identities (8.18) and (8.19) are easily checked by taking the residues along the divisors $\beta_i + \ldots + \beta_{n-1} = 0$ and induction.

In conclusion, we have proved that Ω vanishes, and hence, by induction hypothesis, the differential of the difference between both sides of (8.13) is $O(T^{\varepsilon-1})$. Thus the difference between both sides is a constant plus $O(T^{\varepsilon-1})$, which proves the theorem. □

8.3. **Asymptotics in depths 1 and 2.** In depth 1 we have,

$$\Lambda(t;\beta) \sim \beta^{-1}t^{-\beta} + C(\beta) \quad \text{as } t \to \infty \ ,$$

where $C(\beta) = 2i\pi\,(1-\mathbf{e}(\beta))^{-1}$ (see lemma 62 below).

Let $\beta_{12} = \beta_1 + \beta_2$, $t_{12} = t_{21}^{-1} = t_1 t_2^{-1}$. In depth two, the coproduct (8.4) yields

$$\begin{aligned}\Lambda(t_1,t_2;\beta_1,\beta_2) &\sim \frac{t_{12}^{-\beta_1}}{\beta_1}\Lambda(t_2;\beta_{12}) + \Lambda(t_2;\beta_2)\,C(\beta_1) + C_1 \quad \text{as } t_1 \to \infty \\ &\sim \frac{t_2^{-\beta_2}}{\beta_2}\big[\Lambda(t_1;\beta_1) - t_1^{\beta_2}\Lambda(t_1;\beta_{12})\big] + C_2 \quad \text{as } t_2 \to \infty \\ &\sim \frac{t_1^{-\beta_1}t_2^{-\beta_2}}{\beta_1\beta_{12}} + \big[\Lambda(t_{12};\beta_1) - \Lambda(t_{21};\beta_2)\big]C(\beta_{12}) \\ &\qquad + \frac{t_2^{-\beta_2}}{\beta_2}C(\beta_1) + C_{12} \qquad \text{as } t_1,t_2 \to \infty\end{aligned} \tag{8.20}$$

where C_1, C_2, C_{12} are constant power series in β_1, β_2 to be determined. The constant C_1 is clearly zero as can be seen by letting $t_2 \to 0$ in the first equation of (8.20). The same holds for C_2 (let $t_1 \to 0$). The constant C_{12} can be computed as follows.

8.3.1. *Limit at $D^2 \cap D^{12}$.* From the second line of (8.20), we deduce that

$$\text{constant part of } \lim_{t_1\to\infty}\lim_{t_2\to\infty}\Lambda(t_1,t_2;\beta_1,\beta_2) = 0 \ , \tag{8.21}$$

since $C_2 = 0$. Now let $t_1, t_2 \to \infty$ and then let $t_2/t_1 \to \infty$. The third line gives a constant contribution $C_{12}(\beta_1,\beta_2) - C(\beta_2)C(\beta_{12})$. It follows that

$$C_{12}(\beta_1,\beta_2) = C(\beta_2)C(\beta_{12}) \ . \tag{8.22}$$

8.3.2. *Limit at $D^1 \cap D^{12}$.* From the first line of (8.20), we deduce that

$$\text{constant term of } \lim_{t_2\to\infty}\lim_{t_1\to\infty}\Lambda(t_1,t_2;\beta_1,\beta_2) = C(\beta_1)C(\beta_2) \ . \tag{8.23}$$

Now let $t_1, t_2 \to \infty$ and then let $t_1/t_2 \to \infty$. The third line gives the constant contribution $C(\beta_1)C(\beta_{12}) + C_{12}(\beta_1,\beta_2)$, which yields a second equation for $C_{12}(\beta_1,\beta_2)$. Note however, that the two limit computations are for different branches (see caveat 59), and differ by the monodromy of the third line of (8.20) around the point $t_1 = t_2$ on D^{12}. The monodromy of $\Lambda(t;\beta)$ (resp. $\Lambda(t^{-1};\beta)$) is πi (resp. $-\pi i$) around a positive upper semi-circle from 1^- to 1^+. Therefore, by equating the two different formulae for $C_{12}(\beta_1,\beta_2)$ gives rise to an associator, or pentagon, equation:

$$\big(C(\beta_1) + C(\beta_2) - 2i\pi\big)C(\beta_{12}) = C(\beta_1)C(\beta_2) \tag{8.24}$$

which is indeed satisfied by $C(\beta) = \frac{2\pi i}{1-\mathbf{e}(\beta)}$.

8.4. **Rationality of the constants.** The argument above generalizes:

Proposition 61. *Let $v \in U_n$ be a vertex of U_n, i.e., v is an intersection of boundary divisors D^I of dimension 0. Then there is a branch of the Debye polylogarithm $\Lambda(t_1,\ldots,t_n;\beta_1,\ldots,\beta_n)$ in a neighbourhood of v which is locally of the form*

$$\sum_{I=(i_1,\ldots,i_n)} f_I(s_1,\ldots,s_n)\log^{i_1}s_1\ldots\log^{i_n}s_n$$

where $f_I(0,\ldots,0) \in \mathbb{Q}[\pi i]$, and $s_1,\ldots,s_n$ are local sector coordinates at $v = (0,\ldots,0)$.

Proof. It suffices to show that the constant coefficients lie in $\mathbb{Q}[\pi i]$. But this follows from a standard associator argument: the 1-skeleton of the polytope $\mathcal{C}_n \subset U_n(\mathbb{R})$ is connected, and the restriction of Λ to a one-dimensional stratum is a depth one Debye polyogarithm, whose limiting values at infinity have the desired property, by (9.4). By analytic continuation around the one-dimensional edges of $\mathcal{C}_n$, we deduce that the constants at v are expressible as sums and products of the coefficients of $C(\beta)$. $\square$

9. Examples in depths 1,2

9.1. **Depth 1: the classical elliptic polylogarithms.** Let $q = \mathbf{e}(\tau)$ with $\mathrm{Im}\,(\tau) > 0$, and let $z = \mathbf{e}(\xi)$ with ξ in the fundamental domain D (§3.2). Consider the multivalued generating series of polylogarithms of depth one:

$$L(z;\beta) = \sum_{n\geq 1} \mathrm{Li}_n(z)\beta^{n-1} \ ,$$

which we wish to average over the spiral $(0,\infty) \cong q^{\mathbb{R}}z$ in the universal covering space of $\mathfrak{M}_{0,4}(\mathbb{C})$. The calculations are simplified if one considers the Debye generating series $\Lambda(z;\beta) = z^{-\beta}L(z;\beta)$. Since $d\,\mathrm{Li}_n(z) = z^{-1}\mathrm{Li}_{n-1}(z)dz$ for $n \geq 2$, we have:

$$d\Lambda(z;\beta) = z^{-\beta} d\mathrm{Li}_1(z) \ . \tag{9.1}$$

Note that $\Lambda(z;\beta)$ vanishes at $z=0$, and so by theorem 38 the series

$$\mathtt{E}(z;u,\beta) = \sum_{n\in\mathbb{Z}} u^n \Lambda(q^n z;\beta) \tag{9.2}$$

converges absolutely for $1 < u < |q|^{-1}$, and may have poles at $u=1$, which are given by the asymptotics of $\Lambda(z;\beta)$ at $z=\infty$. Since $d\,\mathrm{Li}(z)$ is asymptotically $-d\log(z)$ at infinity, we deduce from (9.1) that there is some constant $C(\beta)$ such that:

$$\Lambda(z;\beta) \sim \beta^{-1}z^{-\beta} + C(\beta) \ . \tag{9.3}$$

Lemma 62. *The constant at infinity is given by*

$$C(\beta) = -\beta^{-1} + i\pi + \sum_{n\geq 1} 2\,\zeta(2n)\beta^{2n-1} = \frac{2i\pi}{1-\mathbf{e}(\beta)} \ . \tag{9.4}$$

Proof. The following functional equation follows from (9.1) and differentiating:

$$\Lambda(z;\beta) + \Lambda(z^{-1};-\beta) = \beta^{-1}z^{-\beta} + C(\beta) \tag{9.5}$$

Evaluating at $z=1$ gives the expression for $C(\beta)$, since $\mathrm{Li}_n(1) = \zeta(n)$, for $n \geq 2$. $\square$

The corresponding constants in all higher dimensions are explicitly computable from $C(\beta)$. It follows from (9.3) that the singular part of $\mathtt{E}(z;u,\beta)$ comes from:

$$\frac{1}{\beta}\sum_{n<0} u^n\big((q^n z)^{-\beta} + C(\beta)\big) = \frac{z^{-\beta}}{\beta(q^{-\beta}u-1)} + \frac{C(\beta)}{u-1} = \frac{\mathbf{e}(-\beta\xi)}{\beta(\mathbf{e}(\gamma)-1)} + \frac{C(\beta)}{\mathbf{e}(\alpha)-1} \ .$$

where $u = \mathbf{e}(\alpha)$, and $\gamma = \alpha - \beta\tau$. The second expression defines a Taylor series in β with coefficients in $\mathbb{Q}[u,(1-u)^{-1},\log q, i\pi]$. Thus the singular part of $\mathtt{E}(\xi;\alpha,\beta)$ is

$$\mathtt{E}^{\mathrm{sing}}(\xi;\alpha;\beta) = \frac{\mathbf{e}(-\xi\beta)}{\beta\gamma} + \frac{C(\beta)}{\alpha} \ . \tag{9.6}$$

In conclusion, the regularized generating series for the classical elliptic polylog is:

$$\mathtt{E}^{\mathrm{reg}}(\xi;\alpha;\beta) = \sum_{n\in\mathbb{Z}} \mathbf{e}(\alpha n)\Lambda(\mathbf{e}(\xi+n\tau),\beta) - \frac{\mathbf{e}(-\xi\beta)}{\beta\gamma} - \frac{C(\beta)}{\alpha} \ , \tag{9.7}$$

which admits a Taylor expansion in α, β at the origin. Thus we write

$$\mathtt{E}^{\mathrm{reg}}(\xi; \alpha; \beta) = \sum_{m,n \geq 0} \Lambda^E_{m,n}(\xi; \tau) \alpha^m \beta^n \ ,$$

where $\Lambda^E_{m,n}(\xi;\tau)$ are the classical elliptic polylogarithms of [13], and equal to the functions denoted $(-1)^n \Lambda_{m,n}(\xi;\tau)$ in *loc. cit.*, Definition 2.1.

Remark 63. In order to retrieve the explicit formula of [13], Definition 2.1, one can write $\Lambda^E_{m,n}(\xi;\tau)$ as an average of certain modified (and regularized) Debye polylogarithms. For this, one simply replaces the term α^{-1} in (9.6) by the expression

$$\frac{1}{\alpha} = P(\alpha) - \sum_{m>0} \mathbf{e}(m\alpha) \tag{9.8}$$

where P is a power series whose coefficients are related to Bernoulli numbers. Replacing the term in γ^{-1} by a similar expression to (9.8) leads to the required result.

9.2. **Depth 2: the double elliptic polylogarithms.** Consider the generating series of depth two Debye multiple polylogarithms:

$$\Lambda(t_1, t_2; \beta_1, \beta_2) = t_1^{-\beta_1} t_2^{-\beta_2} \sum_{m_1, m_2 \geq 1} I_{m_1,m_2}(t_1,t_2) \beta_1^{m_1-1} \beta_2^{m_2-1} \ .$$

The generating series of elliptic multiple polylogarithms is:

$$\mathtt{E}_2(\xi_1, \xi_2; u_1, u_2, \beta_1, \beta_2) = \sum_{m_1, m_2 \in \mathbb{Z}} u_1^{m_1} u_2^{m_2} \Lambda(q^{m_1} t_1, q^{m_2} t_2; \beta_1, \beta_2) \tag{9.9}$$

which converges absolutely for $1 < u_1, u_2 < |q|^{-1}$ by theorem 38, and has poles along $u_1 = 1, u_2 = 1, u_1 u_2 = 1$ corresponding to logarithmic singularities of $\Lambda(t_1,t_2)$ along D^1, D^2, D^{12}. Let $\gamma_i = \alpha_i - \tau\beta_i$, where $\mathbf{e}(\alpha_i) = u_i$, and $q = \mathbf{e}(\tau)$.

Lemma 64. *The singular part of* $\mathtt{E}_2(\xi_1, \xi_2; \alpha_1, \alpha_2, \beta_1, \beta_2)$ *is* $\mathtt{E}_2^{\mathrm{sing}} = \mathtt{E}_2^{\mathrm{sing}(1)} + \mathtt{E}_2^{\mathrm{sing}(2)}$, *where* $\mathtt{E}_2^{\mathrm{sing}(i)}$ *comes from singularities along divisors of codimension* i. *We have*

$$\mathtt{E}_2^{\mathrm{sing}(2)} = \frac{\mathbf{e}^{-\beta_1\xi_1 - \beta_2\xi_2}}{\beta_1\beta_{12}\gamma_1\gamma_2} + \frac{\mathbf{e}^{-\beta_1\xi_{12}}\, C(\beta_{12})}{\beta_1\gamma_1\, \alpha_{12}} - \frac{\mathbf{e}^{-\beta_2\xi_{21}}\, C(\beta_{12})}{\beta_2\gamma_2\, \alpha_{12}} + \frac{\mathbf{e}^{-\beta_2\xi_2}\, C(\beta_1)}{\beta_2\gamma_2\, \alpha_1} + \frac{C_{1,12}}{\alpha_1(\alpha_1+\alpha_2)}$$

where $C_{1,12} = C(\beta_1)C(\beta_2)$ *and* $C(\beta)$ *is the power series defined by* (9.4), *and*

$$\mathtt{E}_2^{\mathrm{sing}(1)} = \frac{R_1}{\beta_1\gamma_1} + \frac{R_2}{\beta_2\gamma_2} + \frac{A_1}{\alpha_1} + \frac{A_{12}}{\alpha_{12}} \ ,$$

where

$$\begin{aligned}
A_1 &= \mathtt{E}^{\mathrm{reg}}(\xi_2; \alpha_{12}, \beta_2)\, C(\beta_1)\big|_{\alpha_1=0} \\
A_{12} &= \big(\mathtt{E}^{\mathrm{reg}}(\xi_{12}; \alpha_1, \beta_1) - \mathtt{E}^{\mathrm{reg}}(\xi_{21}; -\alpha_1, \beta_2)\big)\, C(\beta_{12})\big|_{\alpha_{12}=0} \\
R_1 &= \mathbf{e}^{-\beta_1\xi_{12}}\ \mathtt{E}^{\mathrm{reg}}(\xi_2; \alpha_{12}; \beta_{12})\big|_{\gamma_1=0} \\
R_2 &= \mathbf{e}^{-\beta_2\xi_2}\ \mathtt{E}^{\mathrm{reg}}(\xi_1; \alpha_1, \beta_1) - \mathbf{e}^{-\beta_2\xi_{21}}\ \mathtt{E}^{\mathrm{reg}}(\xi_1; \alpha_{12}, \beta_{12})\big|_{\gamma_2=0} \ .
\end{aligned}$$

Here, $\mathbf{e}^a = \mathbf{e}(a)$, $\alpha_{12} = \alpha_1 + \alpha_2$, $\beta_{12} = \beta_1 + \beta_2$, *and* $\xi_{12} = -\xi_{21} = \xi_1 - \xi_2$.

Proof. We can compute the singularities of $\mathtt{E}_2$ from the differential equation

$$\begin{aligned}
d\,\mathtt{E}_2(\xi_1, \xi_2; \alpha_1, \alpha_2, \beta_1, \beta_2) &= d\,\mathtt{E}_1(\xi_1 - \xi_2; \alpha_1, \beta_1)\ \mathtt{E}_1(\xi_2; \alpha_1 + \alpha_2, \beta_1 + \beta_2) \\
&- d\,\mathtt{E}_1(\xi_2 - \xi_1; \alpha_2, \beta_2)\ \mathtt{E}_1(\xi_1; \alpha_1 + \alpha_2, \beta_1 + \beta_2) \\
&+ d\,\mathtt{E}_1(\xi_2; \alpha_2, \beta_2)\ \mathtt{E}_1(\xi_1; \alpha_1, \beta_1)
\end{aligned} \tag{9.10}$$

The fact that $d^2\,\mathtt{E}_2(\xi_1,\xi_2,\alpha_1,\alpha_2;\beta_1,\beta_2)=0$ reduces to the Fay identity. Substituting the singular parts $\mathtt{E}^{\mathrm{sing}}$ (given by (9.6)) into (9.10) yields the pole structure of $\mathtt{E}_2$. Using $(fg)^{\mathrm{sing}}=(fg^{\mathrm{sing}}+f^{\mathrm{sing}}g)-f^{\mathrm{sing}}g^{\mathrm{sing}}$, the differential equation for $\mathtt{E}_1$ (7.1) and the additivity of the exponential function, we have $\mathtt{E}_2^{\mathrm{sing}}=\mathtt{E}_2^{\mathrm{sing}(1)}-\mathtt{E}_2^{\mathrm{sing}(2)}$, where

$$
\begin{aligned}
d\,\mathtt{E}^{\mathrm{sing}(2)} &= d\Big(\frac{\mathbf{e}^{-\xi_1\beta_1-\xi_2\beta_2}}{\beta_1\beta_{12}\gamma_1\gamma_2}+\frac{\mathbf{e}^{-\xi_{12}\beta_1}\,C(\beta_{12})}{\beta_1\gamma_1\,\alpha_{12}}-\frac{\mathbf{e}^{-\xi_{21}\beta_2}\,C(\beta_{12})}{\beta_2\gamma_2\,\alpha_{12}}+\frac{\mathbf{e}^{-\beta_2\xi_2}\,C(\beta_1)}{\beta_2\gamma_2\,\alpha_1}\Big)\\
d\,\mathtt{E}_2^{\mathrm{sing}(1)} &= \Big[d\,\mathtt{E}_1(\xi_{12};\alpha_1,\beta_1)-d\,\mathtt{E}_1(\xi_{21};\alpha_2,\beta_2)\Big]\frac{C(\beta_{12})}{\alpha_{12}}+d\,\mathtt{E}_1(\xi_2,\alpha_2,\beta_2)\frac{C(\beta_1)}{\alpha_1}\\
&\quad+\frac{\kappa_{12}|_{\gamma_1+\gamma_2=0}}{\beta_{12}\gamma_{12}}+\frac{\kappa_1|_{\gamma_1=0}}{\beta_1\gamma_1}+\frac{\kappa_2|_{\gamma_2=0}}{\beta_2\gamma_2}
\end{aligned}
$$

where $\gamma_{12}=\gamma_1+\gamma_2$ and

$$
\begin{aligned}
\kappa_{12} &= \big(F(\xi_{12},\gamma_1)+F(\xi_{21},\gamma_2)\big)\mathbf{e}^{-\beta_1\xi_1-\beta_2\xi_2}\,d\xi_{12}\\
\kappa_1 &= \mathbf{e}^{-\beta_1\xi_1-\beta_2\xi_2}F(\xi_2,\gamma_2)\,d\xi_2-\beta_1\,\mathbf{e}^{-\beta_1\xi_{12}}\,\mathtt{E}_1(\xi_2;\alpha_{12};\beta_{12})\,d\xi_{12}\\
\kappa_2 &= \beta_2\,\mathbf{e}^{-\beta_2\xi_{21}}\,\mathtt{E}_1(\xi_1;\alpha_{12},\beta_{12})\,d\xi_{12}-\beta_2\,\mathbf{e}^{-\beta_2\xi_2}\,\mathtt{E}_1(\xi_1;\alpha_1;\beta_1)\,d\xi_2
\end{aligned}
$$

It follows from the expansion (iii) of Proposition-Definition 4 plus the fact that $E_1(\xi,\tau)$ is an odd function of ξ that $\kappa_{12}|_{\gamma_1+\gamma_2}=0$, and therefore does not contribute. There is an obvious of $d\,\mathtt{E}_2^{\mathrm{sing}(2)}$. By integrating, we deduce that:

$$
\mathtt{E}_2^{\mathrm{sing}(1)}=\frac{R_1}{\beta_1\gamma_1}+\frac{R_2}{\beta_2\gamma_2}+\frac{A_1}{\alpha_1}+\frac{A_{12}}{\alpha_1+\alpha_2}\ ,
$$

since $d\,\mathtt{E}^{\mathrm{reg}}$ and $d\,\mathtt{E}$ are equal up to higher order poles. It remains to add the constants of integration. Since these give at most simple poles in the α's and correspond to the limiting values in the corners, they contribute

$$
\frac{C_{1,12}}{\alpha_1(\alpha_1+\alpha_2)}+\frac{C_{2,12}}{\alpha_2(\alpha_1+\alpha_2)}\ ,
$$

where $C_{1,12}$ and $C_{2,12}$ are the constant part of the asymptotic of the Debye double polylogarithm near $D^1\cap D^{12}$ and $D^2\cap D^{12}$ given by (8.21) and (8.23). □

As in the depth 1 case, we therefore define the depth 2 multiple elliptic polylogarithms to be the coefficients in the Taylor expansion:

$$
\mathtt{E}_2-\mathtt{E}_2^{\mathrm{sing}}=\sum_{m_i,n_j\geq 0}\Lambda^E_{(m_1,m_2),(n_1,n_2)}(\xi_1,\xi_2;\tau)\,\alpha_1^{m_1}\alpha_2^{m_2}\beta_1^{n_1}\beta_2^{n_2}\ ,
$$

where $\mathtt{E}_2$ is given by (9.9), and $\mathtt{E}_2^{\mathrm{sing}}$ by the previous lemma.

9.3. **Singular part computed from the coproduct.** Another way to arrive at lemma 64 is from the computation of the asymptotic of the depth 2 Debye polylogarithms given in example 8.3. In general, we have:

Corollary 65. *The singular structure of the elliptic Debye polylogarithm* $\mathtt{E}_n$ *is obtained by averaging the asymptotic of the ordinary Debye polylogarithms. In particular, it is explicitly computable from the coproduct* (8.3) *and the constant terms* C.

In fact, the asymptotic of the Debye polylogarithms in the neighbourhood of boundary divisors of all codimensions can be computed from the coproduct in two different ways. The first, via theorem 60, is to compute the asymptotic in the neighbourhood of codimension 1 divisors, and by induction apply the theorem to the arguments of Λ to obtain the asymptotic in all codimensions. The other, is directly from formula (8.13) which immediately gives the asymptotic in all codimensions, provided that the definition of 'essential', 'regular', and the constants C are modified accordingly.

10. Iterated integrals on $\mathcal{E}^\times$

We compute the integrable words corresponding to the elliptic Debye polylogarithms viewed as functions of one variable, and compare with the bar construction. From this we deduce that all iterated integrals on $\mathcal{E}^\times$ are obtained by averaging.

10.1. **Projective coordinates and degeneration.** Let $\mathbb{G}_m = \mathbb{P}^1 \backslash \{0, \infty\}$, let $n \geq 1$, and write $\Delta \subset \mathbb{G}_m^{n+1}$ for the union of all the diagonals. There is an isomorphism

$$(\mathbb{G}_m^{n+1} \backslash \Delta)/\mathbb{G}_m \xrightarrow{\sim} \mathfrak{M}_{0,n+3} \ . \tag{10.1}$$

If we write homogeneous coordinates on the left-hand side as $(t_1 : \ldots : t_{n+1})$, then the isomorphism is given by $(t_1 : \ldots : t_{n+1}) \mapsto (t_1 t_{n+1}^{-1}, \ldots, t_n t_{n+1}^{-1})$. Let $\beta_1, \ldots, \beta_{n+1}$ be formal parameters satisfying $\beta_1 + \ldots + \beta_{n+1} = 0$. Recall that we set:

$$\Lambda(t_1 : \ldots : t_{n+1}; \beta_1, \ldots, \beta_{n+1}) = \Lambda(t_1 t_{n+1}^{-1}, \ldots, t_n t_{n+1}^{-1}; \beta_1, \ldots, \beta_n) \tag{10.2}$$

Forgetting the marked point t_{n+1} gives rise to a fibration

$$\begin{array}{ccc} \mathfrak{M}_{0,n+3} & \rightarrow & \mathfrak{M}_{0,n+2} \\ (t_1 : \ldots : t_{n+1}) & \mapsto & (t_1 : \ldots : t_n) \end{array} \tag{10.3}$$

whose fiber over the point $(t_1 : \ldots : t_n)$ of $\mathfrak{M}_{0,n}$ is isomorphic to $\mathbb{G}_m \backslash \{t_1, \ldots, t_n\}$. The functions (10.2), when restricted to each fiber, have a particularly simple description.

Lemma 66. *For constant $t_1, \ldots, t_n$ (i.e., $dt_i = 0$ for $i \leq n$), we have*

$$\begin{aligned} d\Lambda(t_1 : \ldots : t_{n+1}; \beta_1, \ldots, \beta_{n+1}) = d\Lambda(t_n : t_{n+1}; \beta_n, -\beta_n) \quad \times \\ \Lambda(t_1 : \ldots : t_{n-1} : t_{n+1}; \beta_1, \ldots, \beta_{n-1}, \beta_n + \beta_{n+1}) \end{aligned} \tag{10.4}$$

For $1 \leq i < j \leq n$, let $\mathcal{M}_{ij}$ denote analytic continuation along a small loop around $t_i = t_j$. Then the functions (10.2) are single-valued around $t_i = t_j$ for $i, j \geq 1$:

$$(\mathcal{M}_{ij} - id)\, \Lambda(t_1 : \ldots : t_{n+1}; \beta_1, \ldots, \beta_{n+1}) = 0 \quad \textit{if} \quad i, j \leq n \ .$$

Proof. The differential equation (10.4) follows from the differential equation for Λ. To prove the singlevaluedness, note that $\mathcal{M}_{ij}$ commutes with $\partial/\partial t_{n+1}$ for $i, j \leq n$. From (10.4) the result follows by induction plus the fact that $\Lambda(t_1 : \ldots : t_{n+1})$ vanishes as t_{n+1} tends to ∞. Alternatively, via (2.3), the Taylor expansion (2.2) of the function $I_{m_1,\ldots,m_n}(t_1, \ldots, t_n)$ at the origin shows that it has trivial monodromy around $t_i = t_j$ for $i < j \leq n$. Thus the same is true of $\Lambda(t_1 : \ldots : t_{n+1})$ by definition. $\square$

The elliptic analogue of (10.1) is as follows. Letting $\Delta \subset \mathcal{E}^{n+1}$ denote the union of all diagonals, and using the notation $(\xi_1 : \ldots : \xi_{n+1})$ for coordinates on $\mathcal{E}^n/\mathcal{E}$, we have:

$$\begin{array}{ccc} (\mathcal{E}^{n+1} \backslash \Delta)/\mathcal{E} & \xrightarrow{\sim} & \mathcal{E}^{(n)} \\ (\xi_1 : \ldots : \xi_{n+1}) & \mapsto & (\xi_1 - \xi_{n+1}, \ldots, \xi_n - \xi_{n+1}) \ . \end{array} \tag{10.5}$$

Again, forgetting the marked point ξ_{n+1} gives rise to a fibration

$$\begin{array}{ccc} \mathcal{E}^{(n)} & \rightarrow & \mathcal{E}^{(n-1)} \\ (\xi_1 : \ldots : \xi_{n+1}) & \mapsto & (\xi_1 : \ldots : \xi_n) \end{array} \tag{10.6}$$

whose fiber over the point $(\xi_1 : \ldots : \xi_n)$ is isomorphic to $\mathcal{E} \backslash \{\xi_1, \ldots, \xi_n\}$.

Definition 67. Define the elliptic Debye hyperlogarithm to be the generating series:

$$G_n(\xi; \xi_1, \ldots, \xi_n, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = \mathtt{E}_n(\xi_1 - \xi, \ldots, \xi_n - \xi; \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$$

viewed as a multivalued function of the single variable $\xi \in \mathcal{E} \backslash \{\xi_1, \ldots, \xi_n\}$.

It follows from equation (10.4) that, for constant $\xi_1, \ldots, \xi_n$ (i.e., $d\xi_i = 0$),

$$dG_n(\xi; \xi_1, \ldots, \xi_n; \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = d\,\mathtt{E}_1(\xi_n - \xi; \alpha_n, \beta_n) \times \quad G_{n-1}(\xi; \xi_1, \ldots, \xi_{n-1}; \alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}) \tag{10.7}$$

Remark 68. There is no obvious way to determine the constant of integration in (10.7) since the averaging process introduces constant terms related to Bernoulli numbers. On the other hand, one natural normalization for an iterated integral on $\mathcal{E}^{(n)}$ is for it to vanish along a tangential base point at 1 on the Tate curve at infinity, which is not the case for the averaged functions $\mathtt{E}_n$. Thus, the comparison between the averaged functions $\mathtt{E}_n$ and such iterated integrals must take into account the constants.

In order to circumvent this issue, let $\varrho \in \mathcal{E}\backslash\{\xi_1, \ldots, \xi_n\}$ be any point. Consider the $n+1$ square matrix M_{ij} with 1's along the diagonal, 0's below the diagonal, and

$$M_{ij} = G_{j-i}(\xi; \xi_i, \ldots, \xi_{j-1}; \alpha_i, \ldots, \alpha_{j-1}, \beta_i, \ldots, \beta_{j-1}) \quad \text{for} \quad 1 \leq i < j \leq n+1 \ .$$

Denote this matrix by M_ξ, viewed as a function of ξ. The differential equation (10.7) translates into an equation of the form $dM_\xi = M_\xi \Omega$ for some square matrix Ω of 1-forms. It follows that $M_\varrho^{-1} M_\xi$ satisfies the same equation. Therefore, we define

$$G_i^\varrho(\xi; \xi_1, \ldots, \xi_i; \alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i) = (M_\varrho^{-1} M_\xi)_{1,i+1} \quad \text{for } 0 \leq i \leq n$$

These functions satisfy the differential equation (10.7) and vanish at $\xi = \varrho$ if $i \geq 1$.

10.2. **Reminders on iterated integrals.** Given a smooth manifold M over $\mathbb{R}$, a smooth path $\gamma : [0,1] \to M$, and smooth one-forms $\omega_1, \ldots, \omega_n$ on M, the iterated integral of $\omega_1, \ldots, \omega_n$ along γ is defined to be 1 if $n = 0$, and for $n \geq 1$:

$$\int_\gamma \omega_1 \ldots \omega_n = \int_{0 \leq t_n \leq \ldots \leq t_1 \leq 1} \gamma^*(\omega_1)(t_1) \ldots \gamma^*(\omega_n)(t_n) \ .$$

Let A be the C^∞ de Rham complex on M, and let $V(A)$ denote the zeroth cohomology of the reduced bar complex of A. Choose a basepoint $\varrho \in M$, and let I_M denote the differential $\mathbb{R}$-algebra of multivalued holomorphic functions on M with global unipotent monodromy. A theorem due to Chen [6] states that the map $V(A) \to V(M)$ given by

$$\sum_{I=(i_1,\ldots,i_n)} c_I [\omega_{i_1}| \ldots |\omega_{i_n}] \quad \mapsto \quad \sum_I c_I \int_\gamma \omega_{i_1} \ldots \omega_{i_n} \tag{10.8}$$

is an isomorphism, where γ is any path from ϱ to z, and the iterated integrals are viewed as functions of the endpoint z. In particular, they only depend on the homotopy class of γ relative to its endpoints. The differential with respect to z is

$$\frac{\partial}{\partial z} \sum_I c_I \int_\gamma \omega_{i_1} \ldots \omega_{i_n} = \sum_I c_I \, \omega_{i_1} \wedge \int_\gamma \omega_{i_2} \ldots \omega_{i_n} \ . \tag{10.9}$$

By successive differentiation, and using formula (10.9), we can reconstruct a bar element in $V(A)$ which corresponds via (10.8) to any given function in $I(M)$. We shall apply this in the following situation. Suppose that $X \hookrightarrow A$ is a connected $\mathbb{Q}$-model for A, so we have an isomorphism $V(X) \otimes_\mathbb{Q} \mathbb{R} \cong V(A)$. Denote the image of the map $V(X)$ in $I(M)$ by $I(M)_\mathbb{Q}$. It defines a $\mathbb{Q}$-structure on the algebra $I(M)$. If $F \in I(M)_\mathbb{Q}$, its bar element in $V(X)$ will be a unique element of $T(X^1)$ by (5.2) (since X is connected). In the sequel, M will be a single elliptic curve with several punctures. We have a family of functions $\mathcal{F} \subset I(M)$, which are the functions obtained by averaging, and want to show that $\mathcal{F} = I(M)_\mathbb{Q}$. For this we shall write the elements of $\mathcal{F}$ as elements of $V(X) \otimes_\mathbb{Q} \mathbb{R}$ by computing their differential equations, and check that: firstly they lie in $V(X)$, and secondly, using our explicit description of $V(X)$, that they span $V(X)$.

10.3. **Integrable words corresponding to the elliptic polylogarithms.**

Definition 69. Define the shuffle exponential to be the formal power series:

$$e_{ш}(\alpha\nu) = \sum_{n\geq 0} \frac{\alpha^n}{n!}\nu^{ш n} = \sum_{n\geq 0} \alpha^n \underbrace{[\nu|\ldots|\nu]}_{n} \in T(\mathbb{Q}[\nu])[[\alpha]] \ .$$

The leading term in the series ($n=0$) is the empty word. Note that if $w_0,\ldots,w_n$ are symbols and $x = e_{ш}(\alpha w_0)$ then we have

$$x \, ш \, [w_1|w_2|\ldots|w_n] = [x|w_1|x|w_2|\ldots|x|w_n|x] \tag{10.10}$$

as an equality of power series in α with coefficients in $T(\mathbb{Q}w_0 \oplus \mathbb{Q}w_1 \oplus \ldots \oplus \mathbb{Q}w_n)$.

Lemma 70. *Let $\varrho, \xi \in \mathcal{E}$, and let α, β be formal parameters, and $\gamma = \alpha - \tau\beta$. Then we have the following equality of generating series of multivalued functions:*

$$\mathbf{e}(\beta\varrho + \gamma r_\varrho)\big(\mathtt{E}_1(\xi;\alpha,\beta) - \mathtt{E}_1(\varrho;\alpha,\beta)\big) = \int_\varrho^\xi [\Omega(\xi;\gamma)|e_{ш}(-\beta'\omega^{(0)} - \gamma\nu)] \tag{10.11}$$

where we write $\varrho = s_\varrho + r_\varrho\tau$ (recall that $\xi = s + r\,\tau$).

Proof. Recall that $\omega^{(0)} = d\xi$ and $\nu = 2\pi i dr$. Let $\beta' = 2\pi i\beta$. It therefore follows immediately from definition 69 and the shuffle product for iterated integrals that

$$\mathbf{e}(-\beta\xi - \gamma r + \beta\varrho + \gamma r_\varrho) = \int_\varrho^\xi e_{ш}(-\beta'\omega^{(0)} - \gamma\nu) \ .$$

We have $d\,\mathtt{E}_1(\xi;\alpha,\beta) = \mathbf{e}(-\beta\xi)F(\xi;\gamma)d\xi$ and $\Omega(\xi;\gamma) = \mathbf{e}(\gamma r)F(\xi;\gamma)d\xi$. Hence

$$d\,\mathtt{E}_1(\xi;\alpha,\beta) = \mathbf{e}(-\beta\xi - \gamma r)\,\Omega(\xi;\gamma) \ .$$

Combining these two facts, we see that, by (10.9),

$$d(\text{LHS of (10.11)}) = \Omega(\xi;\gamma)\int_\varrho^\xi e_{ш}(-\beta'\omega^{(0)} - \gamma\nu) \ ,$$

so the differentials of both sides of (10.11) agree, and both vanish at $\xi = \varrho$. □

It is straightforward to verify that the coefficients in the right-hand side of (10.11) are integrable words in $V(X_1) \otimes_{\mathbb{Q}} \mathbb{C}$. For $n \geq 1$, let us define

$$H_n(\xi;\underline{\alpha};\underline{\beta}) = \mathbf{e}(\beta_{1,\ldots,n}\varrho + \gamma_{1,\ldots,n}r_\varrho)\, G_n^\varrho(\xi;\xi_1,\ldots,\xi_n;\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n) \tag{10.12}$$

where we recall that $\beta_{1,\ldots,n} = \beta_1 + \ldots + \beta_n$, and likewise for γ. By construction, the coefficients of H are combinations of elliptic multiple polylogs.

Recall that X_n is our rational model for the de Rham complex of $\mathcal{E}^{(n)}$, and $a \mapsto \overline{a} : X_n \to X_{F_n}$ is the restriction to the fiber. For any $a_1,\ldots,a_k \in X_n$, let us write

$$\overline{[a_1|\ldots|a_k]} = [\overline{a}_1|\ldots|\overline{a}_k] \in X_{F_n}^{\otimes k} \ ,$$

and extend this definition in the obvious way for formal power series in $T(X_n)$.

Proposition 71. *The generating series of functions H_n is the iterated integral:*

$$H_n(\xi;\underline{\alpha},\underline{\beta}) = \int_\varrho^\xi \overline{W}_n(\xi) \ , \tag{10.13}$$

on the fiber $\mathcal{E}\backslash\{\xi_1,\ldots,\xi_n\}$ of (10.6), where $W_1(\xi) = [\Omega(\xi_1 - \xi;\gamma_1)|e_{ш}(-\beta_1'\omega^{(0)} - \gamma_1\nu)]$, and W_n is defined inductively for $n \geq 2$ by

$$W_n(\xi) = [\Omega(\xi_n - \xi;\gamma_n)|e_{ш}(-\beta_n'\omega^{(0)} - \gamma_n\nu)\, ш\, W_{n-1}(\xi)] \ .$$

Formula (10.13) also remains valid in the case when the marked points ξ_i, $1 \leq i \leq n$, are not necessarily distinct.

Proof. The case $n = 1$ is essentially equation (10.11). For $n > 1$, we have by (10.7):

$$dH_n(\xi) = \mathbf{e}(\beta_n \varrho + \gamma_n r_\varrho) d\,\mathtt{E}_1(\xi_n - \xi; \alpha_n, \beta_n)\, H_{n-1}(\xi)$$

and furthermore, $H_n(\varrho) = 0$. The proof of the proposition in the generic case, i.e., when all ξ_i are distinct follows by induction just as lemma 70. Finally, it follows from lemma 66 that $H_n(\xi; \underline{\alpha}, \underline{\beta})$ has no singularities along $\xi_i = \xi_j$ for $1 \leq i < j \leq n$, and so equation (10.13) remains true after degeneration of the arguments ξ_i. □

Note that both sides of (10.13) have simple poles in the variables $\gamma_1, \ldots, \gamma_n$. Hereafter, extracting the coefficients of a generating series such as either side of (10.13) will mean multiplying by $\gamma_1 \ldots \gamma_n$ and taking the Taylor expansion in α_i, β_i.

10.4. **Comparison theorem.** Let $\Sigma = \{\sigma_0, \ldots, \sigma_m\}$ be distinct points on $\mathcal{E}$, where $\sigma_0 = 0$. Fix a basepoint $\varrho \in \mathcal{E}\backslash\Sigma$. Define $\mathcal{F}_\varrho(\mathcal{E}\backslash\Sigma)$ to be the $\mathbb{Q}$-algebra of multivalued functions spanned by the function $r - r_\varrho$, and the coefficients of the functions

$$H_n(\xi; \xi_1, \ldots, \xi_n; \underline{\alpha}, 0), \text{ for all } n \geq 1, \text{ where } \xi_1, \ldots, \xi_n \in \Sigma\ . \tag{10.14}$$

For every $\sigma \in \Sigma$, let us write $\omega_\sigma^{(i)}$ for the coefficients of

$$\Omega(\xi - \sigma; \alpha) = \sum_{n \geq 0} \omega_\sigma^{(i)} \alpha^{i-1}\ ,$$

and set $\eta_\sigma = \omega_\sigma^{(1)} - \omega_{\sigma_0}^{(1)}$, for $\sigma \neq 0$. Recall that our model X_{F_n} for the punctured elliptic curve $\mathcal{E}\backslash\Sigma$ is generated by $\overline{\nu}$ and the $\omega_\sigma^{(i)}$ for $i \geq 0, \sigma \in \Sigma$ (lemma 10).

Theorem 72. *The map* $\int_\varrho^\xi : V(X_{F_n}) \rightarrow \mathcal{F}_\varrho(\mathcal{E}\backslash\Sigma)$ *is an isomorphism.*

Proof. By (10.10), the integrand $\overline{W}_n(\xi)$ of proposition 71 can also be written:

$$[\overline{\Omega}(\xi_n - \xi; \gamma_n)|P_n|\overline{\Omega}(\xi_{n-1} - \xi; \gamma_{n,n-1})|P_{n-1}| \ldots |\overline{\Omega}(\xi_1 - \xi; \gamma_{n,\ldots,1})|P_1] \tag{10.15}$$

where $P_i = e_{\text{Ш}}(-\beta'_{n,\ldots,i}\overline{\omega}^{(0)} - \gamma_{n,\ldots,i}\overline{\nu})$ for $1 \leq i \leq n$. The functions (10.14) correspond to the constant terms in (10.15) with respect to β_i, namely the iterated integrals:

$$\int_\varrho^\xi [\overline{\Omega}(\xi_n - \xi; \alpha_n)|e_{\text{Ш}}(-\alpha_n\overline{\nu})| \ldots |\overline{\Omega}(\xi_1 - \xi; \alpha_{n,\ldots,1})|e_{\text{Ш}}(-\alpha_{n,\ldots,1}\overline{\nu})]\ . \tag{10.16}$$

One easily checks that (10.16) is integrable, but this also follows from equation (10.13), since the iterated integral only depends on the endpoint ξ, and not the path of integration chosen. Thus the coefficients of (10.16) with respect to α lie in $V(X_{F_n})$.

It suffices to show that the iterated integral of every element of $V(X_{F_n})$ arises in this way. For this, choose any numbers $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$. By the multilinearity of bar elements, the iterated integral from ϱ to ξ of any integrable word of the form

$$\begin{aligned} &[\overline{\Omega}(\xi_n - \xi; \alpha_n) - \varepsilon_n\overline{\Omega}(\xi; \alpha_n)|e_{\text{Ш}}(-\alpha_n\overline{\nu})| \ldots \\ &\qquad \ldots |\overline{\Omega}(\xi_1 - \xi; \alpha_{n,\ldots,1}) - \varepsilon_1\overline{\Omega}(\xi; \alpha_{n,\ldots,1})|e_{\text{Ш}}(-\alpha_{n,\ldots,1}\overline{\nu})] \end{aligned} \tag{10.17}$$

also lies in $\mathcal{F}_\varrho(\mathcal{E}\backslash\Sigma)$. Now let $\pi_\ell : V(X_{F_n}) \rightarrow \mathrm{gr}^\ell V(X_{F_n})$ be the map which projects onto the associated graded for the length filtration, and extended to power series in the obvious way. It kills all Massey products of weight ≥ 2. In particular,

$$\begin{aligned} \pi_\ell\big(\overline{\Omega}(\sigma - \xi; \alpha) - \overline{\Omega}(\xi; \alpha)\big) &= -\overline{\eta}_\sigma \\ \pi_\ell\big(\overline{\Omega}(\sigma - \xi; \alpha)\big) &= \overline{\omega}^{(0)}\alpha^{-1} \end{aligned}$$

Applying π_ℓ to (10.17) (multiplied by $\alpha_1 \ldots \alpha_n$ to clear the poles in α_i) gives a generating series in $\alpha_1, \ldots, \alpha_n$ whose coefficients are all words of the form

$$m_k\overline{\nu}^{i_k} \ldots m_2\overline{\nu}^{i_2} m_1\overline{\nu}^{i_1} \tag{10.18}$$

where $i_1, \ldots, i_k$ are any non-negative integers, and

$$m_i = \begin{cases} \overline{\eta}_{\xi_i} & \text{if } \varepsilon_i = 1 \\ \overline{\omega}^{(0)} & \text{if } \varepsilon_i = 0 \end{cases}$$

It is easy to verify that every word in $\{\overline{\nu}, \overline{\omega}^{(0)}, \overline{\eta}_{\sigma_1}, \ldots, \overline{\eta}_{\sigma_m}\}^\times$ is a linear combination of shuffle products of $\overline{\nu} \ldots \overline{\nu}$ with elements (10.18). It follows from the description (see proposition 23 and preceding discussion):

$$\mathrm{gr}^\ell V(X_{F_n}) \cong T(\mathbb{Q}\overline{\nu} \oplus \mathbb{Q}\overline{\omega}^{(0)} \oplus \mathbb{Q}\overline{\eta}_{\sigma_1} \oplus \ldots \oplus \mathbb{Q}\overline{\eta}_{\sigma_m}) \ ,$$

that the iterated integral of every element in $V(X_{F_n})$ appears a linear combination of products of the function

$$r - r_\varrho = \int_\varrho^\xi \overline{\nu}$$

with coefficients of (10.17). This completes the proof. □

In particular, every iterated integral on $\mathcal{E}^\times$ can be obtained in this way. Our model $V(X_1)$ defines a $\mathbb{Q}$-structure on the de Rham fundamental groupoid of $\mathcal{E}^\times$, hence:

Corollary 73. *The periods of the prounipotent fundamental groupoid ${}_\varrho\Pi_\xi(\mathcal{E}^\times)$ for any initial point $\varrho \in \mathcal{E}^\times$ and endpoint $\xi \in \mathcal{E}^\times$, lie in the $\mathbb{Q}$-algebra generated by $r - r_\varrho$, and the coefficients of* (10.14) *with respect to the α_i's.*

10.5. **Generalizations.** One can extend theorem 72 to the case where ϱ is a tangential basepoint at one of the points $\sigma \in \Sigma$. As a result, a higher-dimensional version of theorem 72 can also be deduced from theorem 26, which states that the iterated integrals on the configuration space $\mathcal{E}^{(n)}$ are products of iterated integrals on the fibers of the map $\mathcal{E}^{(n)} \to \mathcal{E}^{(n-1)}$, which is the one-dimensional case treated above. Therefore all iterated integrals on $\mathcal{E}^{(n)}$ can be obtained from our averaging procedure.

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