The type III_1 factor generated by regular representations of the infinite dimensional nilpotent group $B_0^{\mathbb{Z}}$

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Abstract

We study the von Neumann algebra, generated by the regular representations of the infinite-dimensional nilpotent group $B_0^{\mathbb{Z}}$. In [14] a condition have been found on the measure for the right von Neumann algebra to be the commutant of the left one. In the present article, we prove that, in this case, the von Neumann algebra generated by the regular representations of group $B_0^{\mathbb{Z}}$ is the type III_1 hyperfinite factor.

We use a technique, developed in [20] where a similar result was proved for the group $B_0^{\mathbb{N}}$. The crossed product allows us to remove some technical condition on the measure used in [20].

Key words: von Neumann algebra, modular operator, operator of canonical conjugation, type III_1 factor, unitary representation, infinite-dimensional groups, nilpotent groups, regular representations, irreducibility, infinite tensor products, Gaussian measures, Ismagilov conjecture
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Contents

1 Introduction

We study the von Neumann algebra, generated by the regular representations of the infinite-dimensional nilpotent group $B_0^{\mathbb{Z}}$. The conditions of irreducibility of the regular and quasiregular representations of infinite-dimensional groups (associated with some quasi-invariant Gaussian measures μ_b) are given by the so-called Ismagilov conjecture (see [11–13]). In this case the corresponding von Neumann algebra is a type I_∞ factor. In [14] a condition have been found on the measure for the right von Neumann algebra to be the commutant of the left one.

In the present article, we prove that, in this case, the von Neumann algebra generated by the regular representations of the infinite-dimensional nilpotent group $B_0^{\mathbb{Z}}$ is the type III_1 hyperfinite factor. Moreover this factor is unique.

We recall that the *first examples of a non-type I factor*, namely a type II_1 factor, were also obtained by Murray and von Neumann as von Neumann algebra *generated by the regular representation* of a discrete ICC group (i.e. the group for which all conjugacy classes are infinite, except the trivial one) We shall show that the regular representations of non-discrete infinite-dimensional groups provide examples of non-type I or II, but type III factors, namely the type III_1.

2 Regular representations

Let us consider the group $\tilde{G} = B^{\mathbb{Z}}$ of all upper-triangular real matrices of infinite order with units on the diagonal

$$\tilde{G} = B^{\mathbb{Z}} = \{I + x \mid x = \sum_{k,n\in\mathbb{Z}\, k<n} x_{kn}E_{kn}\},$$

and its subgroup

$$G = B_0^{\mathbb{Z}} = \{I + x \in B^{\mathbb{Z}} \mid \ x \text{ is finite}\},$$

where E_{kn} is an infinite-dimensional matrix with 1 at the place $k, n \in \mathbb{Z}$ and zeros elsewhere, $x = (x_{kn})_{k<n}$ is *finite* means that $x_{kn} = 0$ for all (k, n) except for a finite number of indices $k, n \in \mathbb{Z}$.

Obviously, $B_0^{\mathbb{Z}} = \varinjlim_n B(2n-1,\mathbb{R})$ is the inductive limit of the group $B(2n-1,\mathbb{R})$ of real upper-triangular matrices with units on the principal diagonal realized in the following form

$$B(2n-1,\mathbb{R}) = \{I + \sum_{-n+1\leq k<r\leq n-1} x_{kr}E_{kr} \mid x_{kr} \in \mathbb{R}\}, \quad n \in \mathbb{N},$$

with respect to the symmetric imbedding

$$B(2n-1,\mathbb{R}) \ni x \mapsto i^s(x) = x + E_{-n,-n} + E_{nn} \in B(2n+1,\mathbb{R}).$$

We define the Gaussian measure μ_b on the group $B^{\mathbb{Z}}$ in the following way

$$d\mu_b(x) = \otimes_{k,n\in\mathbb{Z},\, k<n}(b_{kn}/\pi)^{1/2}\exp(-b_{kn}x_{kn}^2)dx_{kn} = \otimes_{k,n\in\mathbb{Z},\, k<n}d\mu_{b_{kn}}(x_{kn}), \tag{1}$$

where $b = (b_{kn})_{k<n}$ is some set of positive numbers $b_{kn} > 0,\ k, n \in \mathbb{Z}$.

Let us denote by R and L the right and the left action of the group $B^{\mathbb{Z}}$ on itself: $R_t(s) = st^{-1},\ \ L_t(s) = ts,\ \ s, t \in B^{\mathbb{Z}}$ and by $\Phi : B^{\mathbb{Z}} \mapsto B^{\mathbb{Z}},\ \Phi(I + x) := (I + x)^{-1}$ the inverse mapping. It is known [17,19] that

Lemma 1 *$\mu_b^{R_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{Z}}$ if and only if $S_{kn}^R(b) < \infty$, $\forall k, n \in \mathbb{Z}$, $k < n$ where*

$$S_{kn}^R(b) = \sum_{r=-\infty}^{k-1} \frac{b_{rn}}{b_{rk}}. \tag{2}$$

Lemma 2 *$\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{Z}}$ if and only if $S_{kn}^L(b) < \infty$, $\forall k, n \in \mathbb{Z}$, $k < n$, where*

$$S_{kn}^L(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{b_{nm}}. \tag{3}$$

Lemma 3 *$\mu_b^{L_{I+tE_{kn}}} \perp \mu_b$ $\forall t \in \mathbb{R}\backslash\{0\} \Leftrightarrow S_{kn}^L(b) = \infty$, $k, n \in \mathbb{Z}$, $k < n$.*

Let us denote

$$E(b) = \sum_{k<n<r} \frac{b_{kr}}{b_{kn} b_{nr}}, \;\; E_m(b) = \sum_{k<n<r\leq m} \frac{b_{kr}}{b_{kn} b_{nr}}, \;\; m \in \mathbb{Z}. \tag{4}$$

Lemma 4 [15] *If $E(b) < \infty$, then $\mu_b^{\Phi} \sim \mu_b$.*

Remark 5 [15] *If $\mu_b^{\Phi} \sim \mu_b$ then $\mu_b^{L_t} \sim \mu_b \Leftrightarrow \mu_b^{R_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{Z}}$.*

PROOF. This follows from the fact that the inversion Φ replace the right and the left action: $R_t \circ \Phi = \Phi \circ L_t$ $\forall t \in B^{\mathbb{Z}}$. Indeed, if we denote $\mu^f(C) = \mu(f^{-1}(C))$ for a measurable set C, we have $(\mu^f)^g = \mu^{f\circ g}$. Hence

$$\mu_b \sim \mu_b^{R_t} \sim (\mu_b^{R_t})^{\Phi} = \mu_b^{R_t \circ \Phi} = \mu_b^{\Phi \circ L_t} = (\mu_b^{\Phi})^{L_t} \sim \mu_b^{L_t}, \;\; \forall t \in B_0^{\mathbb{Z}}.$$

□

Remark 6 *We have*

$$E(b) = \sum_{k<n} \frac{S_{kn}^L(b)}{b_{kn}} = \sum_{k<n} \frac{S_{kn}^R(b)}{b_{kn}}, \;\; E_m(b) = \sum_{k<n\leq m} \frac{S_{kn}^R(b)}{b_{kn}}. \tag{5}$$

Indeed

$$\sum_{k<n} \frac{S_{kn}^L(b)}{b_{kn}} = \sum_{k<n} \sum_{r=n+1}^{\infty} \frac{b_{kr}}{b_{kn} b_{nr}} = \sum_{k<n<r} \frac{b_{kr}}{b_{kn} b_{nr}} = E(b)$$
$$= \sum_{n<r} \frac{1}{b_{nr}} \sum_{k=-\infty}^{n-1} \frac{b_{kr}}{b_{kn}} = \sum_{n<r} \frac{S_{nr}^R(b)}{b_{nr}}.$$

If $\mu_b^{R_t} \sim \mu_b$ and $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{Z}}$, one can define in a natural way (see [11,12]), an analogue of the right $T^{R,b}$ and the left $T^{L,b}$ regular representations of the group $B_0^{\mathbb{Z}}$ in the Hilbert space $H_b = L^2(B^{\mathbb{Z}}, \mu_b)$

$$T^{R,b}, \; T^{L,b} : B_0^{\mathbb{Z}} \to U(H_b = L^2(B^{\mathbb{Z}}, \mu_b)),$$

$$(T_t^{R,b}f)(x) = (d\mu_b(xt)/d\mu_b(x))^{1/2}f(xt),$$
$$(T_s^{L,b}f)(x) = (d\mu_b(s^{-1}x)/d\mu_b(x))^{1/2}f(s^{-1}x).$$

3 Von Neumann algebras

Let $\mathfrak{A}^{R,b} = (T_t^{R,b} \mid t \in B_0^{\mathbb{Z}})''$ (resp. $\mathfrak{A}^{L,b} = (T_s^{L,b} \mid s \in B_0^{\mathbb{Z}})''$) be the von Neumann algebras generated by the right $T^{R,b}$ (resp. the left $T^{L,b}$) regular representation of the group $B_0^{\mathbb{Z}}$.

Theorem 7 [14] *(The commutation theorem) If $E(b) < \infty$ then $\mu_b^{\Phi} \sim \mu_b$. In this case the right and the left regular representations are well defined and the commutation theorem holds:*

$$(\mathfrak{A}^{R,b})' = \mathfrak{A}^{L,b}. \tag{6}$$

Moreover, the operator J_{μ_b} given by

$$(J_{\mu_b}f)(x) = (d\mu_b(x^{-1})/d\mu_b(x))^{1/2}\overline{f(x^{-1})} \tag{7}$$

is an intertwining operator:

$$T_t^{L,b} = J_{\mu_b}T_t^{R,b}J_{\mu_b},\ t \in B_0^{\mathbb{Z}} \quad and \ J_{\mu_b}\mathfrak{A}^{R,b}J_{\mu_b} = \mathfrak{A}^{L,b}.$$

If $\mu_b^{R_t} \sim \mu_b \ \forall t \in B_0^{\mathbb{Z}}$ but $\mu_b^{L_t} \perp \mu_b \ \forall t \in B_0^{\mathbb{Z}}\backslash\{e\}$ one can't define the left regular representation of the group $B_0^{\mathbb{Z}}$. Moreover the following theorem holds

Theorem 8 [17] *The right regular representation $T^{R,b} : B_0^{\mathbb{Z}} \mapsto U(H_b)$ is irreducible if*
(1) $\mu_b^{L_s} \perp \mu_b \ \forall s \in B_0^{\mathbb{Z}}\backslash\{0\}$,
(2) the measure μ_b is $B_0^{\mathbb{Z}}$ right-ergodic,
(3) $\sigma_{kn}(b) = \infty, \ \forall k < n, \ k, n \in \mathbb{Z}$, where

$$\sigma_{kn}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}^2}{[S_{km}^R(b) + b_{km}][S_{nm}^R(b) + b_{nm}]}.$$

Remark 9 *We do not know whether the Ismagilov conjecture holds in this case, namely, whether conditions 1) and 2) of the theorem are the criteria of the irreducibility of the representation $T^{R,b}$ of the group $B_0^{\mathbb{Z}}$ as holds for example for the group $B_0^{\mathbb{N}}$ (see [11,12]).*

Remark 10 *We do not know the criterion of the $B_0^{\mathbb{Z}}$-ergodicity of the measure μ_b on the space $B^{\mathbb{Z}}$. The sufficient conditions are the following $E_m(b) < \infty$ for all $m \in \mathbb{Z}$.*

Remark 11 *The von Neumann algebra $\mathfrak{A}^{R,b}$ is a type I_∞ factor if the conditions of the Theorem 8 are valid.*

Let us assume now that $\mu_b^{L_t} \sim \mu_b \sim \mu_b^{R_t}\ \forall t \in B_0^{\mathbb{Z}}$. In this case the right regular representation and the left regular representation of the group $B_0^{\mathbb{Z}}$ are well defined.

In [15] the condition were studied *when the von Neumann algebra $\mathfrak{A}^{R,b}$ is a factor*, i.e.

$$\mathfrak{A}^{R,b} \cap (\mathfrak{A}^{R,b})' = \{\lambda \mathbf{I} | \lambda \in \mathbb{C}\}.$$

Since $T_t^{L,b} \in (\mathfrak{A}^{R,b})'\ \forall t \in B_0^{\mathbb{Z}}$, we have $\mathfrak{A}^{L,b} \subset (\mathfrak{A}^{R,b})'$, hence

$$\mathfrak{A}^{R,b} \cap (\mathfrak{A}^{R,b})' \subset (\mathfrak{A}^{L,b})' \cap (\mathfrak{A}^{R,b})' = (\mathfrak{A}^{R,b} \cup \mathfrak{A}^{L,b})'. \tag{8}$$

The last relation shows that $\mathfrak{A}^{R,b}$ is factor if the representation

$$B_0^{\mathbb{Z}} \times B_0^{\mathbb{Z}} \ni (t,\, s) \to T_t^{R,b} T_s^{L,b} \in U(H_b)$$

is irreducible. Let us denote

$$S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}^2}{[S_{km}^R(b) + b_{km}][S_{nm}^L(b) + S_{nm}^R(b)]}, \quad k < n. \tag{9}$$

Theorem 12 [15] *The representation*

$$B_0^{\mathbb{Z}} \times B_0^{\mathbb{Z}} \ni (t,\, s) \to T_t^{R,b} T_s^{L,b} \in U(H_b)$$

is irreducible if $S_{kn}^{R,L}(b) = \infty,\ \forall k < n$ and the measure μ_b is $B_0^{\mathbb{Z}}$ right-ergodic.

Corollary 13 *The von Neumann algebra $\mathfrak{A}^{R,b}$ is factor if $S_{kn}^{R,L}(b) = \infty\ \forall k < n$ and the measure μ_b is $B_0^{\mathbb{Z}}$ right-ergodic.*

Remark 14 *In what follows, we shall show that the condition $E(b) < \infty$ is already sufficient for $\mathfrak{A}^{R,b}$ (and $\mathfrak{A}^{L,b}$) to be a factor.*

4 Examples

In this section we give an example of a measure $\mu_b,\ b = (b_{kn})_{k<n}$ for which $E(b) < \infty$, hence the representations $T^{R,b}$ and $T^{L,b}$ are well defined and the commutation theorem (Theorem 7) for von Neumann algebras $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ holds. We show that the set $b = (b_{kn})_{k<n}$ for which

$$E(b) < \infty \ \text{ and hence } S_{kn}^R(b) < \infty,\ S_{kn}^L(b) < \infty \tag{10}$$

defined respectively by (4), (2) and (3), is not empty.

In the example (15) below for the particular case $b_{kn} = (a_k)^n$ we give some sufficient conditions on the sequence a_n implying conditions (10).

Example 15 *Let us take* $b_{kn} = (a_k)^n,\ k, n \in \mathbb{Z}$.

We have

$$S^R_{kn}(b) = \sum_{r=-\infty}^{k-1} \frac{b_{rn}}{b_{rk}} = \sum_{r=-\infty}^{k-1} a_r^{n-k} < \infty \text{ if } \sum_{r=-\infty}^{0} a_r < \infty, \tag{11}$$

$$S^L_{kn}(b) = \sum_{m=n+1}^{\infty} \left(\frac{a_k}{a_n}\right)^m = \left(\frac{a_k}{a_n}\right)^{n+1} \sum_{m=0}^{\infty} \left(\frac{a_k}{a_n}\right)^m = \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} < \infty, \tag{12}$$

iff $a_k < a_{k+1},\ k \in \mathbb{Z}$. Finally we get

$$E(b) = \sum_{k=-\infty}^{\infty} \sum_{n=k+1}^{\infty} \frac{S^L_{kn}(b)}{b_{kn}} = \sum_{k=-\infty}^{\infty} \sum_{n=k+1}^{\infty} \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} \frac{1}{a_k^n} =$$

$$\sum_{k=-\infty}^{\infty} a_k \sum_{n=k+1}^{\infty} \left(\frac{1}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} < \sum_{k=-\infty}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left(\frac{1}{a_n}\right)^{n+1}$$

$$< \sum_{k=-\infty}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left(\frac{1}{a_{k+1}}\right)^{n+1} = \sum_{k=-\infty}^{\infty} \frac{a_k}{1-\frac{a_k}{a_{k+1}}} \left(\frac{1}{a_{k+1}}\right)^{k+2} \frac{1}{1-\frac{1}{a_{k+1}}} =$$

$$\sum_{k=-\infty}^{\infty} \frac{\frac{a_k}{a_{k+1}}}{\left(1-\frac{a_k}{a_{k+1}}\right)^2} \left(\frac{1}{a_{k+1}}\right)^{k+1}.$$

Example 16 *Let us take* $b_{kn} = (a_k)^n,\ k, n \in \mathbb{Z}$ *where* $a_k = s^k,\ k \in \mathbb{Z}$ *with* $s > 1$.

Conditions (10) hold for $a_k = s^k$. By (11) and (12) we have

$$S^L_{kn}(b) = \left(\frac{a_k}{a_n}\right)^{n+1} \frac{1}{1-\frac{a_k}{a_n}} = \left(\frac{1}{s^{n-k}}\right)^{n+1} \frac{1}{1-\frac{1}{s^{n-k}}} \sim \left(\frac{1}{s^{n-k}}\right)^{n+1},$$

$$S^R_{kn}(b) = \sum_{r=-\infty}^{k-1} a_r^{n-k} = \sum_{r=-\infty}^{k-1} s^{r(n-k)} = \sum_{r=1-k}^{\infty} \frac{1}{s^{r(n-k)}} =$$

$$\left(\frac{1}{s^{n-k}}\right)^{1-k} \frac{1}{1-\frac{1}{s^{n-k}}} \sim s^{(n-k)(k-1)},$$

since

$$1 < \frac{1}{1-\frac{1}{s^{n-k}}} < \frac{1}{1-\frac{1}{s}}.$$

Using the latter equivalence we conclude that $E(b) < \infty$. Indeed we have

$$E(b) = \sum_{k=-\infty}^{\infty} \sum_{n=k+1}^{\infty} \frac{S^L_{kn}(b)}{b_{kn}} \sim \sum_{k=-\infty}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{s^{(n-k)(n+1)}} \frac{1}{s^{kn}} = \sum_{k=-\infty}^{\infty} s^k \sum_{n=k+1}^{\infty} \frac{1}{s^{n(n+1)}}$$

$$= \sum_{k=-\infty}^{0} s^k \sum_{n=k+1}^{\infty} \frac{1}{s^{n(n+1)}} + \sum_{k=1}^{\infty} s^k \sum_{n=k+1}^{\infty} \frac{1}{s^{n(n+1)}} < \sum_{k=-\infty}^{0} s^k \sum_{n=-\infty}^{\infty} \frac{1}{s^{n(n+1)}}$$

$$+ \sum_{k=1}^{\infty} \frac{s^k}{s^{(k+1)^2}} \sum_{n=k+1}^{\infty} \frac{1}{s^n} = \sum_{k=-\infty}^{0} s^k \sum_{n=-\infty}^{\infty} \frac{1}{s^{n(n+1)}} + \sum_{k=1}^{\infty} \frac{1}{s^{(k+1)^2+1}} < \infty.$$

5 Cyclycity

We prove that the function $1 \in L^2(B^{\mathbb{Z}}, \mu_b)$ is cyclic and separating for $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ if $E(b) < \infty$. We use a method similar to the proof of ergodicity of μ_b (under the same condition) in [17], Lemma 4 by reducing the situation to the case of $B^{\mathbb{N}}$ ([20]).

Lemma 17 *If $E(b) < \infty$ then the function $1 \in L^2(B^{\mathbb{Z}}, \mu_b)$ is cyclic and separating for $\mathfrak{A}^{R,b}$.*

PROOF. First we prove that 1 is cyclic for $\mathfrak{A}^{R,b}$. For any $m \in \mathbb{Z}$ we define the subgroups B^m and $B_{(m)}$ of the group $B^{\mathbb{Z}}$ as follows:

$$B^m := \{1 + x \in B^{\mathbb{Z}} \mid x = \sum_{k<n\leq m} x_{kn} E_{kn}\},$$

$$B_{(m)} := \{1 + x \in B^{\mathbb{Z}} \mid x = \sum_{k<n, n>m} x_{kn} E_{kn}\}.$$

Obviously, $B^{\mathbb{Z}}$ is a semi-direct product of the two groups above ($B_{(m)}$ is a normal subgroup of $B^{\mathbb{Z}}$) for any m:

$$B^{\mathbb{Z}} = B_{(m)} \rtimes B^m, \quad z = yx,\ z \in B^{\mathbb{Z}},\ y \in B_{(m)},\ x \in B^m.$$

Let $\mu_{b,(m)},\ \mu_b^m$ be the projections of the measure μ_b on the above groups:

$$\mu_b^m := \otimes_{k<n\leq m} \mu_{b_{kn}}, \quad \mu_{b,(m)} := \otimes_{k<n, n>m} \mu_{b_{kn}}.$$

Then we have

$$L^2(B^{\mathbb{Z}}, \mu_b) = L^2(B_{(m)}, \mu_{b,(m)}) \otimes L^2(B^m, \mu_b^m), \quad f(z) = f(yx).$$

Furthermore let $B_{0,(m)},\ B_0^m$ be the intersection of the above groups with $B_0^{\mathbb{Z}}$. Now, fix an $m \in \mathbb{Z}$ and consider a function $f(z) = f(yx) \in L^2(B^{\mathbb{Z}}, \mu_b)$. Further, suppose that

$$(f, a1) = 0, \quad \forall a \in \mathfrak{A}^{R,b}. \tag{13}$$

First we note that the points of $B_{(m)}$ are invariant under the right action R_t for all $t \in B_0^m$. Indeed, we have for $t \in B_0^m$

$$(xt)_{kn} = \sum_{j=k+1}^{n-1} x_{kj} t_{jn} = x_{kn}, \quad n > m,$$

since $t_{kn} = \delta_{kn}$ for $n > m$. We have for $t \in B_0^m$

$$\begin{aligned} 0 =&(f, T_t^{R,b} 1) = \int_{B_{(m)}} \int_{B^m} f(yx) T_t^{R,b} 1(yx) d\mu_{b,(m)}(y) d\mu_b^m(x) \\ =& \int_{B^m} f_m(x) T_t^{R,b} 1(x) d\mu_{b,(m)}(x), \end{aligned}$$

where

$$f_m(x) := \int_{B_{(m)}} f(yx) d\mu_{b,(m)}(y).$$

For the function f_m holds

$$(f_m, T_t^{R,b} 1)_m = 0, \quad \forall t \in B_0^m, \tag{14}$$

where $(.,.)_m$ denotes the restriction of the inner product $(.,.)$ to $L^2(B^m, \mu_b^m)$. Next, we define a bijection $\Psi : B^m \mapsto B_m$, where B_m is the group

$$B_m := \{1 + x; x = \sum_{m \le k < n} x_{kn} E_{kn}\} \cong B^{\mathbb{N}},$$

$$x'_{kn} = (\Psi(x))_{kn} := x_{2m-n2m-k}.$$

Note that Ψ are reflections around the axis $k + n = 2m$ and if $m = 0$, $x'_{kn} = x_{-n-k}$. Now we continue with the equation (14):

$$\begin{aligned} 0 = (f_m, T_t^{R,b} 1)_m =& \int_{B^m} f_m(x) \sqrt{\frac{d\mu_b^m(xt)}{d\mu_b^m(x)}} d\mu_b^m(x) \\ =& \int_{B_m} f_m^\Psi(x') \sqrt{\frac{d\mu_b^{m,\Psi}(t'x')}{d\mu_b^{m,\Psi}(x')}} d\mu_b^{m,\Psi}(x'), \end{aligned}$$

where $f_m^\Psi := f_m \circ \Psi$ and $\mu_b^{m,\Psi}(C) = \mu_b^m(\Psi(C))$ for each Borel set C. Since this holds for all $t \in B_0^m$ and hence all $t' \in B_{0,m}$ and $B_m \cong B^{\mathbb{N}}$,

$$0 = \int_{B^{\mathbb{N}}} f_m^\Psi(x) T_t^{L,b} 1 d\mu_b(x) = (f^\Psi, T_t^{L,b} 1)_{\mathbb{N}},$$

where, more precisely, f_m^Ψ is interpreted as its image under the isomorphism form B_m to $B^{\mathbb{N}}$ and $(.,.)_{\mathbb{N}}$ is the inner product on $L^2(B^{\mathbb{N}}, \mu_b)$. It also follows (after taking the linear span and weak limits) that

$$(f_m^\Psi, a1)_{\mathbb{N}} = 0, \quad \forall a \in \mathfrak{A}^{L,b,\mathbb{N}},$$

where $\mathfrak{A}^{L,b,\mathbb{N}}$ are the algebras generated by the left regular representation $T^{L,b}$ of the group $B_0^{\mathbb{Z}}$. But 1 is cyclic for $\mathfrak{A}^{L,b,\mathbb{N}}$, by [20] and hence $f_m^{\Psi}(x')=0$ for all $x'\in B_m$. Since Ψ is a bijection, hence we get $f_m=0$ for any m.

In addition we have $f_m\to f$, when $m\to\infty$ in $L^2(B^{\mathbb{Z}},\mu_b)$ (see [17], Corollary 1). Thus $f_m=0$ for all $m\in\mathbb{Z}$ implies $f=0$. Since f is arbitrary, using (13) we conclude, that the set $\mathfrak{A}^{R,b}\mathbf{1}$ is dense in $L^2(B^{\mathbb{Z}},\mu_b)$ and hence 1 is cyclic for $\mathfrak{A}^{R,b}$.

Now we turn to the separating property. We know that 1 is cyclic for $\mathfrak{A}^{R,b}$. We prove that the same holds for $(\mathfrak{A}^{R,b})'=\mathfrak{A}^{L,b}$. Thus, again consider $f\in L^2(B^{\mathbb{Z}},\mu_b)$ and assume

$$(f,b1)=0,\forall b\in\mathfrak{A}^{L,b}. \tag{15}$$

Recall that $E(b)<\infty$ implies the existence of the intertwining operator $J=J_{\mu_b}$, which is anti-unitary. Then the following calculation holds:

$$\begin{aligned}(f,T_t^{R,b}1)&=(JT_t^{R,b}1,Jf)\\ &=\int\sqrt{\frac{d\mu_b(x^{-1})}{d\mu_b(x)}}\sqrt{\frac{d\mu_b((xt)^{-1})}{d\mu_b(x^{-1})}}\sqrt{\frac{d\mu_b(x^{-1})}{d\mu_b(x)}}\overline{f(x^{-1})}d\mu_b(x)\\ &=\int\sqrt{\frac{d\mu_b(t^{-1}x^{-1})}{d\mu_b(x^{-1})}}\overline{f(x^{-1})}d\mu(x^{-1}).\end{aligned}$$

If we replace x^{-1} by x in the above integral we obtain $(f,T_t^{R,b}1)=(f,T_t^{L,b}1)$ for all $t\in B_0^{\mathbb{Z}}$. From (1) we know that $(f,T_t^{R,b}1)=0$ for all $t\in B_0^{\mathbb{Z}}$ implies that $f=0$. Hence $(f,T_t^{L,b}1)=0$ for all $t\in B_0^{\mathbb{Z}}$ also implies that $f=0$ and hence 1 is cyclic for $\mathfrak{A}^{L,b}$, since we chose f arbitrarily. $\square$

6 Modular operator

In this section we recall the construction of the modular operator for a locally compact group and generalize it to the infinite-dimensional case. We recall [7] (see also [20]) how to find the modular operator and the operator of canonical conjugation for the von Neumann algebra $\mathfrak{A}_G^{\rho}$, generated by the right regular representation ρ of a locally compact Lie group G. Let h be a right invariant Haar measure on G and

$$\rho,\lambda:G\mapsto U(L^2(G,h))$$

be the right and the left regular representations of the group G defined by

$$(\rho_t f)(x) = f(xt),\ (\lambda_t f)(x) = (dh(t^{-1}x)/dh(x))^{1/2} f(t^{-1}x).$$

To define the *right Hilbert algebra* on G we can proceed as follows. Let $M(G)$ be an algebra of all probability measures on G with convolution $\mu * \nu$ determined by

$$\int_G f(u)d(\mu * \nu)(u) = \int_G \int_G f(st)d\mu(s)d\nu(t).$$

We define the homomorphism

$$M(G) \ni \mu \mapsto \rho^\mu = \int_G \rho_t d\mu(t) \in B(L^2(G, h)).$$

We have $\rho^\mu \rho^\nu = \rho^{\mu*\nu}$, indeed

$$\rho^\mu \rho^\nu = \int_G \rho_t d\mu(t) \int_G \rho_s d\nu(s) = \int_G \int_G \rho_{ts} d\mu(t)\nu(s) = \int_G \rho_t d(\mu * \nu)(t) = \rho^{\mu*\nu}.$$

Let us consider a subalgebra $M_h(G) := \{\mu \in M(G) \mid \nu \sim h\}$ of the algebra $M_h(G)$. In the case when $\mu \in M_h(G)$ we can associate with the measure μ its Rodon-Nikodim derivative $d\mu(t)/dh(t) = f(t)$. When $f \in C_0^\infty(G)$ or $f \in L^1(G)$ we can write

$$\rho^f = \int_G f(t)\rho_t dh(t),$$

hence we can replace the algebra $M_h(G)$ by its subalgebra identified with an algebra of functions $C_0^\infty(G)$ or $L^1(G, h)$ with convolutions.

If we replace the Haar measure h with some measure $\mu \in M_h(G)$ we obtain the isomorphic image $T^{R,\mu}$ of the right regular representation ρ in the space $L^2(G, \mu)$: $T_t^{R,\mu} = U\rho_t U^{-1}$ where $U : L^2(G, h) \mapsto L^2(G, \mu)$ defined by $(Uf)(x) = \left(\frac{dh(x)}{d\mu(x)}\right)^{1/2} f(x)$. We have

$$(T_t^{R,\mu} f)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt),$$

and

$$T^f = \int_G f(t)T_t^{R,\mu} d\mu(t).$$

We have (see [4], p.462) (we shall write T_t instead of $T_t^{R,\mu}$)

$$S(T^f) := (T^f)^* = \int_G \overline{f(t)} T_{t^{-1}} d\mu(t) = \int_G \overline{f(t)} T_{t^{-1}} \frac{d\mu(t)}{d\mu(t^{-1})} d\mu(t^{-1})$$

$$= \int_G \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})} T_t d\mu(t).$$

Hence

$$(Sf)(t) = \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})}. \tag{16}$$

In the *Tomita-Takesaki theory* [22], Chapter VI, Lemma 1.2, the operator S is defined for the von Neumann algebra M of operators on the Hilbert space H by

$$H \ni x\omega \mapsto Sx\omega = x^*\omega \in H,$$

where $x \in M$ and $\omega \in H$ is cyclic (generating) and separating vector.

To calculate S^* we use the fact that S is antilinear, i.e. $(Sf, g) = (S^*g, f)$. We have

$$(Sf, g) = \int_G \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})g(t)} d\mu(t) = \int_G \overline{f(t^{-1})g(t)} d\mu(t^{-1})$$

$$= \int_G \overline{g(t^{-1})f(t)} d\mu(t) = (S^*g, f),$$

hence $(S^*g)(t) = \overline{g(t^{-1})}$. Finally the modular operator Δ defined by $\Delta = S^*S$ has the following form $(\Delta f)(t) = \frac{d\mu(t)}{d\mu(t^{-1})} f(t)$. Indeed we have

$$f(t) \stackrel{S}{\mapsto} \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})} \stackrel{S^*}{\mapsto} \frac{d\mu(t)}{d\mu(t^{-1})} f(t).$$

Finally, since $J = S\Delta^{-1/2}$ (see [4] p.462) we get

$$f(t) \stackrel{\Delta^{-1/2}}{\mapsto} \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} f(t) \stackrel{S}{\mapsto} \frac{d\mu(t^{-1})}{d\mu(t)} \left(\frac{d\mu(t)}{d\mu(t^{-1})}\right)^{1/2} \overline{f(t^{-1})}$$

$$= \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} \overline{f(t^{-1})},$$

$$(Jf)(t) = \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} \overline{f(t^{-1})}, \text{ and } (\Delta f)(t) = \frac{d\mu(t)}{d\mu(t^{-1})} f(t). \tag{17}$$

To prove that $JT_t^{R,\mu}J = T_t^{L,\mu}$ we get

$$f(t) \stackrel{J}{\mapsto} \left(\frac{d\mu(x^{-1})}{d\mu(x)}\right)^{1/2} \overline{f(x^{-1})} \stackrel{T_t^{R,\mu}}{\mapsto} \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} \left(\frac{d\mu((xt)^{-1})}{d\mu(xt)}\right)^{1/2} \overline{f((xt)^{-1})}$$

$$= \left(\frac{d\mu(t^{-1}x^{-1})}{d\mu(x)}\right)^{1/2} \overline{f(t^{-1}x^{-1})} \stackrel{J}{\mapsto} \left(\frac{d\mu(x^{-1})}{d\mu(x)}\right)^{1/2} \left(\frac{d\mu(t^{-1}x)}{d\mu(x^{-1})}\right)^{1/2} f(t^{-1}x)$$

$$= \left(\frac{d\mu(t^{-1}x)}{d\mu(x)}\right)^{1/2} f(t^{-1}x) = (T_t^{L,\mu} f)(x).$$

Remark 18 *The representation T^{R,μ_b} is the inductive limit of the representations T^{R,μ_b^m} of the group $B(m, \mathbb{R})$ where the measure μ_b^m is the projection of the measure μ_b onto subgroup $B(m, \mathbb{R})$. Obviously μ_b^m is equivalent with the Haar measure h_m on $B(m, \mathbb{R})$.*

7 Structure of von Neumann algebras and the flow of weights invariant of Connes and Takesaki

Let us denote as before $M = \mathfrak{A}^{R,b} = (T_s^{R,b} \mid s \in B^{\mathbb{Z}})''$, $\mathfrak{A}^{L,b} = (T_t^{L,b} \mid t \in B_0^{\mathbb{Z}})''$. We assume that

$$E(b) < \infty, \quad \text{hence} \quad S_{kn}^R(b) < \infty, \ \text{and} \ S_{kn}^L(b) < \infty.$$

We also note that from [17], Lemma 4 (see Remark 10) follows that the condition $E(b) < \infty$ implies the ergodicity of the measure μ_b with respect to the right action.
From the previous section (see (17)) we conclude that the modular operator Δ is defined as follows

$$(\Delta f)(x) = d\mu_b(x)/d\mu_b(x^{-1})f(x). \tag{18}$$

In the next section we shall prove that M (and hence M') is a type III_1 factor, without assuming further conditions on the measure. From our proof immediately follows that M is a factor and we do not use the conditions (9). In [20] the case of the group $B_0^{\mathbb{N}}$ was studied. There the author proved the type III_1 property by directly showing that the fixed point algebra of M w.r.t. the modular group is trivial. The idea was to prove that the one-parameter groups generated by multiplication operators by the independent variables x_{kn} are contained in the commutant of the fixed point algebra. This, together with the fact that also all translations $T_t^{R,b}, t \in B_0^{\mathbb{N}}$ are in the latter commutant, implies the triviality of this fixed point algebra. In the case of $B_0^{\mathbb{Z}}$, however, this method does not directly give the answer, because of the presence of infinite sums (see later). In this paper, we came up with a different approach, by making use of the *flow of weights* of a type III factor.

After the original *classification of type III factors* by Connes in 1973 ([2]), there came an equivalent classification of type III factors using the flow of weights invariant, which was concluded in the joint work of Connes and Takesaki [5]. The definition of the flow of weights relies on the *duality theory for von Neumann algebras*, which was discovered by Connes ([2]) and Takesaki ([21]).

Remark 19 *(See [22], chap.X, §2) To a von Neumann algebra M with an action σ of a locally compact abelian group G one can associate another von Neumann algebra N and an action θ of the dual group $\hat{G}$. The pair (N, θ) is called the* dual *system.*

For the case when M is a type III factor and σ is the *modular automorphism group* of a *faithful semi-finite normal weight*, this becomes an invariant, called *the non-commutative flow of weights.* The pair $(\mathcal{C}_N, \theta)$, where $\mathcal{C}_N$ is the center of N, is called *the flow of weights.* In this section we review some basic facts about the structure of von Neumann algebras and the flow of weights of a type III factor. For more details we refer to e.g. [22].

Definition 20 *A* W**-dynamical system is a triple* (M, α, G)*, where* M *is a von Neumann algebra,* G *a locally compact group and* α *an action of* G *on* M *by automorphisms, i.e. a strongly continuous homomorphism from* G *into* $Aut(M)$*.*

A *crossed product* of (M, α, G) is defined as follows.

Definition 21 *Consider the following representations of* M *and* G *on* $L^2(G, \mathcal{H}) = L^2(G) \otimes \mathcal{H}$*, where* $\mathcal{H}$ *is the Hilbert space* M *acts on:*

$$(\pi_\alpha(x)\xi)(s) = \alpha_s^{-1}(x)\xi(s), \quad x \in M \quad s \in G,$$

$$(\lambda(t)\xi)(s) = \xi(t^{-1}s), \quad s, t \in G \quad \xi \in L^2(G, \mathcal{H}).$$

The representation (π_α, λ) *is covariant, i.e.* $\pi_\alpha \circ \alpha_s(x) = \lambda(s)\pi_\alpha(x)\lambda(s)^*$*. Then*

$$M \rtimes_\alpha G := (\pi_\alpha(M) \cup \lambda(G))''$$

is called the crossed product *of* (M, α, G)*.*

When G is abelian we can define an action of $\hat{G}$ (the dual of G) on the crossed product by the following formulas.

$$(\mu(p)\xi)(s) = \overline{\langle s, p \rangle}\xi(s), \quad p \in \hat{G},$$

$$\hat{\alpha}_p(x) = \mu(p)x\mu(p)^*, \quad x \in M \rtimes_\alpha G.$$

Let us denote $U(s) = \lambda(s)$ and $V(p) = \mu(p)$, $s \in G, p \in \hat{G}$. It follows that U and V satisfy the following relation:

$$U(s)V(p)U(s)^*V(p)^* = \langle s, p \rangle. \tag{19}$$

Definition 22 *In general, a pair of unitary representations* U *of* G *and* V *of* $\hat{G}$ *on the same Hilbert space* $\mathcal{H}$ *is said to be* covariant *if the commutation relation (19) is satisfied. The commutation relation (19) is called the* Weyl-Heisenberg *commutation relation.*

In what follows we will need the following result for covariant representations in a von Neumann algebra.

Proposition 23 ([22], Proposition 2.2) *The covariant representation* $\{\lambda_G, \mu_G\}$ *generates the factor* $B(L^2(G))$ *of all bounded operators on* $L^2(G)$*.*

PROOF. For each $f \in L^1(\hat{G})$, we define

$$V(f) := \int_{\hat{G}} f(p)V(p)dp.$$

Then V is a *-representation of $L^1(\hat{G})$, so that it can be extended to the enveloping C^*-algebra $C_0(G)$ (the algebra of continuous functions vanishing at infinity [1]). We shall denote the extended representation of $C_0(G)$ by V again. In the case when $V = \mu_G$, we have that $\mu_G(f)$ is the multiplication by f on $L^2(G)$ ($f \in C_0(G)$). Hence the von Neumann algebra A generated by $\{\mu_G(f); f \in C_0(G)\}$ is the multiplication algebra $L^\infty(G)$ on $L^2(G)$. So it is maximal abelian (i.e. $L^\infty(G)' = L^\infty(G)$). Now, we have

$$\lambda_G(s)\mu_G(f)\lambda_G(s)^* = \mu_G(\lambda_s f), \quad s \in G, \quad f \in L^\infty(G),$$

where $(\lambda_s f)(r) = f(r - s)$. Hence the operators A commuting with $\lambda_G(G)$ are only scalars (the Haar measure dr is ergodic). Therefore,

$$\{\lambda_G(G), \mu_G(\hat{G})\}' = \mathbb{C},$$

so that $\{\lambda_G, \mu_G\}$ is irreducible. □

The definition of the flow of weight relies on the following duality theorem of Connes and Takesaki.

Theorem 24 ([2,21]) *For a W^*-dynamical system (M, α, G), where G is abelian the following isomorphism holds:*

$$(M \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong M \overline{\otimes} B(L^2(G)).$$

Definition 25 *The representation μ of $\hat{G}$ on $L^2(G, \mathcal{H})$ defined above is called* the dual representation *to λ. The action $\hat{\alpha}$ of $\hat{G}$ on the crossed product $\hat{M} = M \rtimes_\alpha G$ is called* the dual action *and the resulting dynamical system $(\hat{M}, \hat{\alpha}, \hat{G})$, we call* the dual system.

Later we will need a convenient description of the commutant of $\hat{M}$. The following theorem gives us the desired answer:

Theorem 26 ([22]) *Consider a W^*-dynamical system (M, G, α), where M acts on $\mathcal{H}$, G a locally compact group and α is implemented by a unitary one parameter group $V(s)$ on $\mathcal{H}$, i.e. $\alpha_s(a) = V(s)aV(s)^*$, for $a \in M, s \in G$. Define*

$$(W\xi)(s) = V(s)^*\xi(s), \quad \xi \in L^2(G, \mathcal{H}).$$

Then the following holds

$$M \rtimes_\alpha G = \left(WMW^* \cup \mathfrak{A}_G^\lambda\right)'',$$

$$(M \rtimes_\alpha G)' = (M' \cup W\mathfrak{A}_G^\rho W^*)'',$$

[1] A function f on G is said to vanish at infinity if given any $\epsilon > 0$, there is a compact subset of G such that $|f(x)| < \epsilon$ for x outside this subset

where $\mathfrak{A}_G^\lambda$ *(resp.* $\mathfrak{A}_G^\rho$*) is the left (resp. the right) von Neumann algebra of* G*.*

The following Theorem describes the so-called *continuous decomposition* of a von Neumann algebra. It is crucial for the definition of flow of weights for type III factors. From now on we set $G = \mathbb{R}$ and denote the W^*-dynamial systen $(N, \theta, \mathbb{R})$ by (N, θ).

Theorem 27 ([21,5]) *(1) Let* (N, θ) *be a* W^**-dynamical system such that*

- N *admits a faithful semi-finite normal trace* τ*;*
- θ *transforms in such a way that*

$$\tau \circ \theta_s = e^{-s}\tau, \quad s \in \mathbb{R}.$$

Then the crossed product $M = N \rtimes_\theta \mathbb{R}$ *is properly infinite and the center* $\mathcal{C}_M$ *is precisely the fixed point algebra* $\mathcal{C}_N^\theta$ *of the center of* N *under the canonical embedding of* N *into* M *(the representation* π_θ*). Furthermore,* M *is of type III (i.e. all the factors in the decomposition of* M *are of type III) if and only if the central dynamical system* $(\mathcal{C}_N, \theta)$ *does not contain an invariant subalgebra* $\mathcal{A}$*, such that the subsystem* $(\mathcal{A}, \theta)$ *is isomorphic to* $L^\infty(\mathbb{R})$ *together with the translation action of* $\mathbb{R}$*. In the case that* M *is of type III,* N *is necessarily of type* II_∞ *(i.e.* $\tau(I) = \infty$*).*

(2) If M *is a von Neumann algebra of type III, then there exists a unique, up to conjugacy, covariant system* (N, θ) *satisfying the conditions of (1).*

Moreover, the above theorem implies that M is a factor if and only if θ is *centrally ergodic*. Now we are ready to introduce the flow of weights. It was discovered, in the context of the duality theory for von Neumann algebras, by Connes in [2] and Takesaki in [21] and was studied in detail in their joint work [5].

Definition 28 *The dynamical system* $(N, \theta, \mathbb{R})$*, such that* $M \cong N \rtimes_\theta \mathbb{R}$ *is called* the non-commutative flow of weights*, whereas* $(\mathcal{C}_N, \theta, \mathbb{R})$ *is called* the flow of weights *associated to* $(M, \sigma, \mathbb{R})$*. By the above theorem it is an invariant for the algebraic type of* M*.*

Recall that in [2] Alain Connes classified type III factors with the following invariant.

$$S(M) = \bigcap_{\phi \in \mathcal{W}} Sp\Delta_\phi, \tag{20}$$

where $\mathcal{W}$ is the set of *faithful normal semi-finite weights* on M. $S(M)\backslash\{0\}$ is a multiplicative subgroup of $\mathbb{R}_+$ and subdivides type III factors into three classes ([2]):

(1) When $S(M) = [0, +\infty)$, M is said to be of type III_1
(2) When $S(M) = \{\lambda^n | n \in \mathbb{Z}\} \cup \{0\}$, where $0 < \lambda < 1$, M is said to be of type III_λ,

(3) When $S(M) = \{0, 1\}$, M is said to be of type III_0.

The following theorem states an equivalent description of the types in terms of the flow of weights.

Theorem 29 **([5])** *Let M be a factor of type III.*

(1) M is of type III_1 if and only if the flow of weights is trivial, i.e. N is a factor.
(2) M is of type III_0 if and only if N is not a factor and the flow of weights has no period.
(3) M is of type III_λ if and only if N is not a factor and $T > 0$ is the period of the flow of weights,where $\lambda = \mathrm{e}^{-T}$.

8 Type III_1 factor

For the von Neumann algebra $M = \mathfrak{A}^{R,b}$ we shall prove that the corresponding flow of weights is trivial , i.e. N is a factor. Using the theorem 29, we conclude that M is then of type III_1. [From Theorem 27 it follows, first of all that M is a factor, since the center of M is contained in the center of its dual. Moreover, by the same theorem, it is of type III, since there can not be a non trivial subsystem isomorphic to $L^2(\mathbb{R})$ with the translation action on $\mathbb{R}$. Furthermore, by theorem 29, M is of type III_1.]
Note that to prove the factor property we do not use the sufficient conditions from Corollary 13. Now we state the main theorem.

Theorem 30 *Consider the von Neumann algebra $\mathfrak{A}^{R,b}$ generated by the right regular representation $T^{R,b}$ of $B_0^{\mathbb{Z}}$. Assume that $E(b) < \infty$. Let $\phi(a) = (1, a1)$ be the faithful normal state on $\mathfrak{A}^{R,b}$, associated to the cyclic and separating vector 1, and σ the corresponding modular automorphism group. Then the dual algebra $N := \mathfrak{A}^{R,b} \rtimes_\sigma \mathbb{R}$ is a factor. Hence, $\mathfrak{A}^{R,b}$ is a type III_1 factor. The same holds for $\mathfrak{A}^{L,b}$.*

PROOF. On the space $L^2(\mathbb{R}, L^2(B^{\mathbb{Z}}, \mu_b)) = L^2(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)$ (m is the Lebesgue measure) we define the operator W as follows

$$(Wf)(x,t) = \Delta^{-it}(x) f(x,t) = \left(\frac{d\mu_b(x^{-1})}{d\mu_b(x)}\right)^{it} f(x,t). \tag{21}$$

Denote $N := \hat{M} = M \rtimes_\sigma \mathbb{R}$. By Theorem 26, we have

$$N = (WMW^* \cup \mathfrak{A}_{\mathbb{R}}^{\lambda}), \quad N' = (M' \cup W\mathfrak{A}_{\mathbb{R}}^{\rho} W^*),$$

hence

$$\mathcal{C}_N = N' \cap N = (N \cup N')' = (WMW^* \cup M' \cup \lambda(\mathbb{R}) \cup W\rho(\mathbb{R})W^*)'. \quad (22)$$

From (22) we see that $\mathcal{C}'_N$ contains the following set of elements:

$$\left(WT_u^{R,b}W^*, T_u^{L,b}, \lambda(s), W\rho(s)W^*; u \in B_0^{\mathbb{Z}},\ s \in \mathbb{R}\right). \quad (23)$$

We would like to prove the triviality of $\mathcal{C}_N$. For this we show that operators of multiplication by the independent variables x_{kn} and t, in the space $L^2(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)$, are affiliated to $\mathcal{C}'_N$ (see Lemma 33). In this case, since $T_u^{L,b} \in \mathcal{C}'_N, \forall u \in B_0^{\mathbb{Z}}$, the two lemmas below would imply that $\mathcal{C}_N$ is trivial.

Lemma 31 ([20]) *Let g be a multiplication on $L^2(B^{\mathbb{Z}}, \mu_b)$ by a measurable function g on $B^{\mathbb{Z}}$, then*

$$(T_t^{R,b} g T_{t^{-1}}^{R,b} f)(x) = g(xt)f(x), \quad \textit{for all } t \in B_0^{\mathbb{Z}}, f \in L^2(B^{\mathbb{Z}}, \mu_b).$$

Lemma 32 *Let M be a von Neumann algebra on $L^2(B^{\mathbb{Z}}, \mu_b) \otimes L^2(\mathbb{R}, m) = L^2(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)$. If $e^{ist}, e^{isx_{kn}} \in M', k < n$, $T_u^{L,b} \in M', \forall u \in B_0^{\mathbb{Z}}, s \in \mathbb{R}$, $\lambda(s) \in M'$ for all $s \in \mathbb{R}$ and the measure μ_b is ergodic, then $M = \mathbb{C}I$.*

PROOF. From proposition 23 follows that the result holds in the one-dimensional case. The space $L^2(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)$ is isomorphic to $L^2(\mathbb{R}^\infty, \mu_b \otimes m) = \otimes_{k<n\in\mathbb{Z}} L^2(\mathbb{R}^1, \mu_{b_{kn}}) \otimes L^2(\mathbb{R}, m)$. Since the variables x_{kn} and t are independent, the condition $e^{its}, e^{ix_{kn}s} \in M'$, for all $k < n \in \mathbb{Z}$ and $s \in \mathbb{R}$, means that $L^\infty(\mathbb{R}, m), L^\infty(\mathbb{R}, \mu_{b_{kn}}) \subset M'$ for all $k < n$. This implies that the von Neumann algebra generated by $(L^\infty(\mathbb{R}, \mu_{b_{kn}}))_{k<n} \cup L^\infty(\mathbb{R}, m)$ is contained in M'. The latter is isomorphic to $L^\infty(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)$, which is maximally abelian. Hence, $M \subset L^\infty(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)' = L^\infty(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)$. Moreover, since we assume that $\lambda(s), T_u^{L,b} \in M'$ for all $u \in B_0^{\mathbb{Z}}, s \in \mathbb{R}$, all functions in M are $B_0^{\mathbb{Z}}$-left invariant, by Lemma 31, and translation invariant in the last argument. By the ergodicity of the measure, they are constant $\mu_b \otimes m$-a.e. □

Thus, Theorem 30 is proved. □

Lemma 33 *Let Q_{kn} and Q_t be the multiplication operators*

$$(Q_{kn}f)(x,t) := x_{kn}f(x,t), \quad (Q_t f)(x,t) := tf(x,t), \quad (24)$$

in $L^2(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)$. Then

$$e^{iQ_{kn}s}, e^{iQ_t s} \in \mathcal{C}'_N,$$

for all $s \in \mathbb{R}$, $k, n \in \mathbb{Z}$, $k < n$.

Formal Computations: The following method uses calculations with unbounded generators of one parameter groups, using commutator identities. In such a way we want to obtain the variables x_{kn} and t. However, these computations are formal and we would have to verify conditions on the domains of the operators in question, to justify these identities rigorously.

Consider generators of the following one-parameter groups in $\mathcal{C}_N'$:

$$(WT^{R,b}_{1+sE_{pq}}W^*, T^{L,b}_{1+sE_{pq}}, \lambda(t) \mid s, t \in \mathbb{R}). \tag{25}$$

Note that we left out one group, namely $W\rho(\mathbb{R})W^*$. However, the same result can be obtained by replacing $\lambda(\mathbb{R})$ by the latter group.
The generator of $(\lambda(-t))$ is $D_t := \frac{d}{dt}$. From [16] follows that the generators of $T^L_{pq}(s) := T^{L,b}_{1+sE_{pq}}$ are

$$A^L_{pq} := \frac{d}{ds}T^{L,b}_{1+sE_{pq}}|_{s=0} = \Sigma^{\infty}_{m=q+1}x_{qm}D_{pm} + D_{pq}, \tag{26}$$

where $D_{pq} = \frac{\partial}{\partial x_{pq}} - b_{pq}x_{pq}$. Finally we need to calculate the generators of $V_{pq}(s) := WT^{R,b}_{1+sE_{pq}}W^*$. We have

$$(V_{pq}f)(x,t) := \frac{d}{ds}(V_{pq}(s)f)(x,t)|_{s=0} = \frac{d}{ds}(WT^R_{pq}(s)\Delta W^*f)(x,t)|_{s=0}$$

$$= \Delta(x)^{-it}(\frac{d}{ds}\Delta(xs)^{-it}.\mathbf{1})_{|s=0}f(x,t) + \frac{d}{ds}(T^R_{pq}(s)f)(x,t)|_{s=0}.$$

The last term is nothing else than A^R_{pq} defined by (see e.g. [15])

$$A^R_{kn} = \sum_{r=-\infty}^{k-1} x_{kr}D_{rn} + D_{kn}, \quad 1 \le k < n.$$

Set $B_{pq} = V_{pq} - \frac{d}{ds}T^R_{pq}(s)$. We have

$$B_{pq} := \frac{d}{ds}(\Delta(x)^{-it}\exp(sA^R_{pq})\Delta^{it}\exp(-sA^R_{pq})\mathbf{1})(x)_{|s=0}$$

$$= \Delta^{-it}([A^R_{pq}, \Delta^{it}]).$$

The nth term in the Taylor expansion of Δ^{it} is equal to $\frac{(it)^n}{n!}\ln\Delta^n$. Since $[A^R_{pq}, \ln\Delta]$ is a function (see later), it commutes with $\ln\Delta$. Hence one has the following formula:

$$[A^R_{pq}, \frac{(it)^n}{n!}\ln\Delta^n] = n\ln\Delta^{n-1}[A^R_{pq}, \ln\Delta].$$

applying this to the Taylor expansion of Δ^{it} we obtain

$$[A^R_{pq}, \Delta^{it}] = it\Delta^{it}[A^R_{pq}, \ln\Delta].$$

In this manner we obtain

$$B_{pq} = it[A^R_{pq}, \ln \Delta(x)]$$

and hence $V_{pq} = it[A^R_{pq}, \ln \Delta(x)] + A^R_{pq}$. Thus the generator in (25) are as follows:

$$V_{pq} := it[A^R_{pq}, \ln \Delta(x)] + A^R_{pq}, \;\; A^L_{pq}, \;\; D_t := \frac{d}{dt}.$$

Some useful formulas (see [12]).
Let us denote by X^{-1} the inverse matrix to the upper triangular matrix $X = I + x = I + \sum_{k<n} x_{kn} E_{kn} \in B^{\mathbb{Z}}$

$$X^{-1} = (I+x)^{-1} := I + \sum_{k<n} x^{-1}_{kn} E_{kn} \in B^{\mathbb{Z}}.$$

We have by definition $X^{-1}X = XX^{-1} = I$, hence

$$\left(XX^{-1}\right)_{kn} = \sum_{r=k}^{n} x_{kr} x^{-1}_{rn} = \delta_{kn} = \sum_{r=k}^{n} x^{-1}_{kr} x_{rn} = \left(X^{-1}X\right)_{kn}, \quad k \leq n, \tag{27}$$

thus

$$x^{-1}_{kn} + \sum_{r=k+1}^{n-1} x_{kr} x^{-1}_{rn} + x_{kn} = 0 = x_{kn} + \sum_{r=k+1}^{n-1} x_{kr} x^{-1}_{rn} + x^{-1}_{kn}, \quad k < n,$$

and

$$x^{-1}_{kn} = -x_{kn} - \sum_{r=k+1}^{n-1} x_{kr} x^{-1}_{rn} = -x_{kn} - \sum_{r=k+1}^{n-1} x^{-1}_{kr} x_{rn}. \tag{28}$$

We can write also

$$x^{-1}_{kn} = -\sum_{r=k+1}^{n} x_{kr} x^{-1}_{rn} = -\sum_{r=k}^{n-1} x^{-1}_{kr} x_{rn}. \tag{29}$$

There is also the explicit formula for x^{-1}_{kn} (see [10] formula (4.4))
$x^{-1}_{kk+1} = -x_{kk+1}$ and

$$x^{-1}_{kn} = -x_{kn} + \sum_{r=1}^{n-k-1} (-1)^{r+1} \sum_{k \leq i_1 < i_2 < \ldots < i_r \leq n} x_{ki_1} x_{i_1 i_2} \ldots x_{i_r n}, \quad k < n-1. \tag{30}$$

Remark 34 *Using (30) we see that x^{-1}_{kn} depends only on x_{rs} with $k \leq r < s \leq n$.*

Using (29) we have

$$x^{-1}_{kn} + x_{kn} = -\sum_{r=k+1}^{n-1} x_{kr} x^{-1}_{rn}, \quad x^{-1}_{kn} - x_{kn} = 2x_{kn} - \sum_{r=k+1}^{n-1} x_{kr} x^{-1}_{rn}. \tag{31}$$

Let us denote

$$w_{kn} := w_{kn}(x) := (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1}).$$

Using (1) and (18) we get

$$\Delta(x) = \frac{d\mu_b(x)}{d\mu_b(x^{-1})} = \exp\left[\sum_{k,n\in\mathbb{Z},\, k<n} b_{kn}\left((x_{kn}^{-1})^2 - x_{kn}^2\right)\right]. \tag{32}$$

$$-\ln\Delta(x) = \sum_{k,n\in\mathbb{Z},\, k<n} b_{kn}\left[x_{kn}^2 - (x_{kn}^{-1})^2\right] = \sum_{k,n\in\mathbb{Z},\, k<n} b_{kn}(x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1})$$
$$\sum_{k,n\in\mathbb{Z},\, k<n} b_{kn}(x_{kn} + x_{kn}^{-1})[2x_{kn} - (x_{kn} + x_{kn}^{-1})] = \sum_{k,n\in\mathbb{Z},\, k<n} b_{kn}w_{kn}(x).$$

To study the action of the operators

$$A_{kn}^R := \frac{d}{ds}T_{1+sE_{pq}}^{R,b}|_{s=0} = \sum_{r=-\infty}^{k-1} x_{rk}D_{rn} + D_{kn}$$

on the function $\ln\Delta(x)$ we need to know the action of D_{pq} on x_{kn}^{-1}.

Lemma 35 ([20]) *We have*

$$[D_{pq}, x_{kn}^{-1}] = \begin{cases} -x_{kp}^{-1}x_{qn}^{-1}, & \text{if } k \le p < q \le n, \\ 0, & \text{otherwise}. \end{cases} \tag{33}$$

For proof see [20].
Using (33) we get

$$[D_{pq}, x_{kn} + x_{kn}^{-1}] = \begin{cases} -x_{kp}^{-1}x_{qn}^{-1} - \delta_{kp}x_{qn}^{-1} - \delta_{qn}x_{kp}^{-1}, & \text{if } k \le p < q \le n, \\ 0, & \text{otherwise}. \end{cases} \tag{34}$$

Using (34) we have $[D_{pq}, (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1})] =$

$$\begin{cases} 2\,x_{kp}^{-1}x_{qn}^{-1}x_{kn}^{-1} + 2\,\delta_{kp}x_{qn}^{-1}x_{kn}^{-1} + 2\,\delta_{qn}x_{kp}^{-1}x_{kn}^{-1}, & \text{if } k \le p < q \le n,\ (p,q) \ne (k,n), \\ 2(x_{kn} + x_{kn}^{-1}), & \text{if } (p,q) = (k,n), \\ 0, & \text{otherwise}. \end{cases} \tag{35}$$

Indeed, if $k \le p < q \le n$, $(p,q) \ne (k,n)$ we have

$$[D_{pq}, (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1})] = [D_{pq}, (x_{kn} + x_{kn}^{-1})(2x_{kn} - (x_{kn} + x_{kn}^{-1}))]$$
$$= [D_{pq}, (x_{kn} + x_{kn}^{-1})](2x_{kn} - (x_{kn} + x_{kn}^{-1})) - (x_{kn} + x_{kn}^{-1})[D_{pq}, (x_{kn} + x_{kn}^{-1})] =$$
$$-2x_{kn}^{-1}[D_{pq}, (x_{kn} + x_{kn}^{-1})] \overset{(34)}{=} 2x_{kp}^{-1}x_{qn}^{-1}x_{kn}^{-1} + 2\delta_{kp}x_{qn}^{-1}x_{kn}^{-1} + 2\delta_{qn}x_{kp}^{-1}x_{kn}^{-1}.$$

Lemma 36 ([20]) *We have*

$$[A^R_{pq},\, w_{kn}] = \begin{cases} 2x_{kp}x_{kq}, & \textit{if} \;\; n=q,\;\; k\le p-1, \\ 2x^{-1}_{pn}x^{-1}_{qn}, & \textit{if} \;\; k=p,\;\; n\ge q+1, \\ 2(x_{pq}+x^{-1}_{pq}), & \textit{if} \quad k=p, q=n, \\ 0, & \textit{if} \;\; k\neq p \;\; \textit{or} \; n\neq q, \end{cases} \tag{36}$$

hence

$$-[A^R_{pq},\, \ln\Delta(x)] = 2\sum_{r=-\infty}^{p-1} b_{rq}x_{rp}x_{rq} + 2\sum_{n=q+1}^{\infty} b_{pn}x^{-1}_{pn}x^{-1}_{qn} + 2(x_{pq}+x^{-1}_{pq}). \tag{37}$$

Next, we consider the action of A^L_{ij} on (37), where $i<p<j<q$.

Lemma 37 ([20]) *One has*

$$[A^L_{ij}, [A^R_{pq}, \ln\Delta(x)]] = -2b_{iq}x_{jq}x_{ip}, \quad \textit{for } i<p<j<q$$

and hence

$$[A^L_{ip}, [A^L_{ij}, [A^R_{pq}, \ln\Delta(x)]]] = 2b_{iq}x_{jq}, \tag{38}$$

which immediately gives us the variables $x_{jq},\; j,q\in\mathbb{Z},\;\; q-j\ge 1$.

PROOF. Recall (see (26)) that $A^L_{ij} = \Sigma^{\infty}_{m=j+1}x_{jm}D_{im} + D_{ij}$. Then

$$\begin{aligned} &[A^L_{ij}, x^{-1}_{kn}] = \Sigma^{\infty}_{m=j+1}x_{jm}[D_{im}, x^{-1}_{kn}] + [D_{ij}, x^{-1}_{kn}] \\ &\overset{(33)}{=} -\textstyle\sum^{n}_{r=j+1} x_{jr}(x^{-1}_{ki}x^{-1}_{rn} + \delta_{ki}x^{-1}_{rn} + \delta_{rn}x^{-1}_{ki} + \delta_{ki}\delta_{rn}) \\ &= -\delta_{jn}(x^{-1}_{ki} + \delta_{ki}). \end{aligned} \tag{39}$$

Moreover we also need the formula

$$\begin{aligned} &[A^L_{ij}, x_{kn}] = \Sigma^{\infty}_{m=i+1}x_{jm}[D_{im}, x_{kn}] + [D_{ij}, x_{kn}] \\ &= -\delta_{ki}(x_{jn} + \delta_{jn}). \end{aligned} \tag{40}$$

By our choice of i and j, $[A^L_{ij}, [A^R_{pq}, \ln\Delta(x)]]$ will only depend on $x_{kn},\, n<p$. Hence

$$[A^L_{ij}, [A^R_{pq}, \ln\Delta(x)]] = -2\sum_{r=-\infty}^{p-1} b_{rq}x_{rp}[A^L_{ij}, x_{rq}] = -2b_{iq}x_{jq}x_{ip}.$$

The last formula of the Lemma follows trivially. □

By applying $[A^L_{jq}, .]$ to (38), we obtain a constant $2b_{iq}$. The direct computation, based on the above formulas gives us the operator of multiplication by x_{kn} and t:

$$[A^L_{jq}, [A^L_{ip}, [A^L_{ij}, C^R_{pq}]]] = 2itb_{iq},$$
$$[D_t, [A^L_{ip}, [A^L_{ij}, C^R_{pq}]]] = 2ib_{iq}x_{jq},$$

since A^R commutes with A^L, t is of course the variable in $L^2(\mathbb{R})$.

Remark 38 *The previous manipulations with the unbounded operators, were formal. Nevertheless they indicate us the form of the expressions in terms of the unitary one-parameter groups generated by A^L_{kn} and A^R_{kn}, we should take, to obtain the desired answer. We should replace the commutator $[x, y]$, $x, y \in L(G)$ in the Lie algebra $L(G)$ by the group commutator $\{a, b\} := aba^{-1}b^{-1}$ where $a, b \in G$.*

Again, denote

$$T^{L,b}_{pq}(s) = T^{L,b}_{I+sE_{pq}}, \quad T^{R,b}_{pq}(s) = T^{R,b}_{I+sE_{pq}},$$
$$V_{pq}(s) = WT^{R,b}_{I+sE_{pq}}W^*, \quad Wf(x, t) = \Delta^{-it}(x)f(x, t).$$

Lemma 39 *Denote*

$$U(\tau, s) = \{T^{L,b}_\tau, V^{R,b}_s\}, \ \tau, s \in B^{\mathbb{Z}}_0, \tag{41}$$

then we have

$$U(\tau, s) = \Delta^{-it}(\tau^{-1}x)\Delta^{it}(\tau^{-1}xs)\Delta^{-it}(xs)\Delta^{it}(x). \tag{42}$$

PROOF. Since

$$U(\tau, s) = \{T^{L,b}_\tau, V^{R,b}_s\} = T^{L,b}_\tau V^{R,b}_s T^{L,b}_{\tau^{-1}} V^{R,b}_{s^{-1}} = T^{L,b}_\tau WT^{R,b}_s W^* T^{L,b}_{\tau^{-1}} WT^{R,b}_{s^{-1}} W^*,$$

we have

$$f(x, t) \overset{W^*}{\to} \Delta^{it}(x)f(x, t) \overset{WT^{R,b}_{s^{-1}}}{\to} \left(\frac{d\mu(xs^{-1})}{d\mu(x)}\right)^{1/2} \Delta^{it}(xs^{-1})f(xs^{-1}, t) \overset{W^*T^{L,b}_{\tau^{-1}}}{\to}$$

$$\Delta^{it}(x)\left(\frac{d\mu(\tau x)}{d\mu(x)}\right)^{1/2} \Delta^{-it}(\tau x)\left(\frac{d\mu(\tau xs^{-1})}{d\mu(\tau x)}\right)^{1/2} \Delta^{it}(\tau xs^{-1})f(\tau xs^{-1}, t)$$
$$= \Delta^{it}(x)\Delta^{-it}(\tau x)\left(\frac{d\mu(\tau xs^{-1})}{d\mu(x)}\right)^{1/2} \Delta^{it}(\tau xs^{-1})f(\tau xs^{-1}, t)$$
$$\overset{WT^{R,b}_s}{\to} \Delta^{-it}(x)\left(\frac{d\mu(\tau x)}{d\mu(x)}\right)^{1/2} \Delta^{-it}(xs)\Delta^{-it}(\tau xs)\Delta^{it}(\tau x)f(\tau x, t) \overset{T^{L,b}_{s\tau}}{\to}$$

$$\left(\frac{d\mu(\tau^{-1}x)}{d\mu(x)}\right)^{1/2} \Delta^{-it}(\tau^{-1}x)\left(\frac{d\mu(x)}{d\mu(\tau^{-1}x)}\right)^{1/2} \Delta^{it}(\tau^{-1}xs)\Delta^{-it}(xs)\Delta^{-it}(x)f(x, t)$$

$$= \Delta^{-it}(\tau^{-1}x)\Delta^{it}(\tau^{-1}xs)\Delta^{-it}(xs)\Delta^{it}(x)f(x,t).$$

□

Consider the following one-parameter groups in $B_0^{\mathbb{Z}}$:

$$E_{pq}(s) := \{1 + sE_{pq}; s \in \mathbb{R}\}, \ p, q \in \mathbb{Z}, \ p < q. \tag{43}$$

We calculate $U(E_{rm+1}(t), E_{mm+1}(s))$ for $t = -1$.

Lemma 40 *Let $U_{rm}(s) \in \mathcal{C}'_N$ be the operators defined by*

$$U_{rm}(s) := U(I - E_{rm+1}, I + sE_{mm+1}),$$

where $s \in \mathbb{R}$. Then for $f \in L^2(B^{\mathbb{Z}} \times \mathbb{R}, \mu_b \otimes m)$ holds

$$(U_{rm}(s)f)(x,t) = \exp(-2ib_{rm+1}stx_{rm}) f(x,t), \quad \forall t, s \in \mathbb{R}. \tag{44}$$

Then $\{U_{rm}(s), \lambda(1)\}$ and $\{U_{rm}(s), T^{L,b}_{rm}(1)\}$ are the one parameter groups generated by the multiplications by x_{rm} and t.

PROOF. Fix $s \in \mathbb{R}$ and define:

$$X' := XE_{mm+1}(s), \quad Y := E_{rm+1}(1)X, \quad Y' := YE_{mm+1}(s),$$

where $X = 1 + x, Y = 1 + y \in B^{\mathbb{Z}}$. First we note that $E_{rm+1}(-1)Y = X$ and $E_{rm+1}(-1)Y' = X'$, since $E_{rm+1}(s)$ are one-parameter groups. By Lemma 39 we get

$$U_{rm}(s) = \frac{\Delta^{it}((I + E_{rm+1})x(I + sE_{mm+1}))\Delta^{it}(x)}{\Delta^{it}((I + E_{rm+1})x)\Delta^{it}(x(I + sE_{mm+1}))}, \quad \text{or}$$
$$U_{rm}(s) = \Delta^{-it}(y)\Delta^{it}(y')\Delta^{-it}(x')\Delta^{it}(x).$$

To obtain (44) we proceed in two steps. First of all we compute $\Delta(x')^{-it}\Delta(x)^{it}$. Secondly, we replace x by y and t by $-t$ in the latter expression, to obtain $\Delta(y')^{it}\Delta(y)^{-it}$. Finally, we combine the two expressions above to obtain the desired result. Recall that

$$-\ln \Delta(x) = \sum_{k,n \in \mathbb{Z}, k<n} b_{kn} w_{kn}(x).$$

We show (44) in two steps.
Step 1: The first step is to compute $\Delta(x')^{-it}\Delta(x)^{it}$.

Remark 41 *The computations below are analogous to those in [20], which were carried out to obtain the variables x_{kn}, in order to prove the triviality of the fixed point algebra of $\mathfrak{A}^{L,b}$ w.r.t. the modular group. However, in the case of $B_0^{\mathbb{Z}}$ the fixed point algebra method does not work, since all sums are over* *an infinite number of indices. The* flow of weights *allows us to* overcome the problems.

First of all, we would like to know the right action of $E_{mm+1}(s)$ on $X \in B^{\mathbb{Z}}$, where $X = 1 + x$ and on X^{-1}. For $k < n$ we have: $x'_{kn} =$

$$(XE_{mm+1}(s))_{kn} = \sum_{i=k}^{n} X_{ki}(\delta_{in} + s\delta_{mi}\delta_{m+1n}) = x_{kn} + \delta_{m+1n}(sx_{km} + s\delta_{km}). \quad (45)$$

Note that only the $m + 1$st column of x is affected by this transformation.

$$XE_{mm+1}(s) = \begin{pmatrix} \ddots & & & & & & & \\ & 1 & x_{12} & x_{13} & x_{14} & \cdots & x_{1m+1} + sx_{1m} & \cdots \\ & 0 & 1 & x_{23} & x_{24} & \cdots & x_{2m+1} + sx_{2m} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & 0 & 0 & \cdots & \cdots & 1 & x_{mm+1} + s & \cdots \\ & & & & & & & \ddots \end{pmatrix}.$$

From (45) we now can deduce the inverse of x'. In order to do this we note the following:

$$x'^{-1}_{kn} = (XE_{mm+1}(s))^{-1}_{kn} = (E_{mm+1}(-s)X^{-1})_{kn},$$

where $k < n$. Hence we get: $x'^{-1}_{kn} = \sum_{j=k}^{n}(E_{mm+1}(-s))_{kj}X^{-1}_{jn} =$

$$\sum_{j=k}^{n}(\delta_{kj} - s\delta_{mk}\delta_{jm+1})(x^{-1}_{jn} + \delta_{jn}) = x^{-1}_{kn} - s\delta_{km}(x^{-1}_{m+1n} + \delta_{m+1n}),$$

where of course x' depends on $s \in \mathbb{R}$. Since only the rows with number m of x^{-1} and the columns with number $m+1$ of x are affected by the transformation $x \to x'$, it follows that $\ln \Delta(x')$ can be written as the sum of three different terms:

$$-\ln \Delta(x') = \sum_{k<n, k\neq m, n\neq m+1} b_{kn}w_{kn}(x) + \sum_{k<m} b_{km+1}w_{km+1}(x') + \sum_{n>m+1} b_{mn}w_{mn}(x').$$

Note that the term with $k = m, n = m + 1$ vanishes, because $w_{mm+1}(x) = 0$. First we consider the second term: $\sum_{k<m} b_{km+1}w_{km+1}(x') =$

$$\begin{aligned} &\textstyle\sum_{k=-\infty}^{m-1} b_{km+1}(x'_{km+1} - x'^{-1}_{km+1})(x'_{km+1} + x'^{-1}_{km+1}) = \\ &\quad \textstyle\sum_{k=-\infty}^{m-1} b_{km+1}(x_{km+1} + sx_{km} - x^{-1}_{km+1})(x_{km+1} + sx_{km} + x^{-1}_{km+1}) = \\ &\quad \textstyle\sum_{k=-\infty}^{m-1} b_{km+1}w_{km+1}(x) + \sum_{k<m} b_{km+1}\left(2sx_{km}x_{km+1} + s^2(x_{km})^2\right). \end{aligned} \quad (46)$$

The third term is as follows: $\sum_{n>m+1} b_{mn}w_{mn}(x') =$

$$\begin{array}{l}\sum_{n=m+2}^{\infty} b_{mn}(x'_{mn}-x'^{-1}_{mn})(x'_{mn}+x'^{-1}_{mn}) = \\ \quad \sum_{n=m+2}^{\infty} b_{mn}(x_{mn}-x^{-1}_{mn}+sx^{-1}_{m+1n})(x_{mn}+x^{-1}_{mn}-sx^{-1}_{m+1n}) = \\ \quad \sum_{n=m+2}^{\infty} b_{mn}w_{mn}(x)+\sum_{n>m+1} b_{mn}\left(2sx^{-1}_{mn}x^{-1}_{m+1n}-s^2(x^{-1}_{m+1n})^2\right).\end{array} \tag{47}$$

After adding up the terms we get: $\ln\Delta(x)-\ln\Delta(x') =$

$$\sum_{k-\infty}^{m-1} b_{km+1}\left(2sx_{km}x_{km+1}+s^2(x_{km})^2\right)+\sum_{n=m+2}^{\infty} b_{mn}\left(2sx^{-1}_{mn}x^{-1}_{m+1n}-s^2(x^{-1}_{m+1n})^2\right).$$

Hence we get:

$$\begin{array}{l}\Delta^{it}(x)\Delta^{-it}(x') = \exp\left(it\sum_{k=-\infty}^{m-1} b_{km+1}(2sx_{km}x_{km+1}+s^2(x_{km})^2)\right. \\ \left.+it\sum_{n=m+2}^{\infty} b_{mn}\left(2sx^{-1}_{mn}x^{-1}_{m+1n}-s^2(x^{-1}_{m+1n})^2\right)\right).\end{array} \tag{48}$$

Step 2: The next step is to compute $\Delta^{-it}(y)\Delta^{it}(y')$. To obtain the latter we have to perform the following operations in formula (48)

$$x \to E_{rm+1}(1)x, \quad t \to -t.$$

First we have to compute: $y_{kn} = (E_{rm+1}(1)X)_{kn}$

$$= \sum_{i=k+1}^{n-1}(\delta_{ki}+\delta_{kr}\delta_{im+1})(x_{in}+\delta_{in}) = x_{kn}+\delta_{kr}(x_{m+1n}+\delta_{m+1n}). \tag{49}$$

In order to calculate y^{-1} we note that $(E_{rm+1}(s)X)^{-1} = X^{-1}E_{rm+1}(-s)$ ($E_{rm+1}(s)$ are one-parameter groups). Thus

$$y^{-1}_{kn} = x^{-1}_{kn}-\delta_{m+1n}(x^{-1}_{km}+\delta_{km}). \tag{50}$$

According to equation (49) only the row with number r of x is affected by the left $E_{rm+1}(1)$-action. Similarly, by (50), only the column with number $m+1$ of y^{-1} is affected. Moreover, $y_{rm} = x_{rm}$, since $x_{m+1m} = 0$. Hence, in the calculation below we have:

$$y_{km}y_{km+1} = \begin{cases} x_{km}x_{km+1}, & \text{if } k \neq r \\ x_{rm}x_{rm+1}+x_{rm}, & \text{if } k = r, \end{cases}$$

and other terms are unaffected by the transformation $x \to y$. We obtain

$$\Delta^{-it}(y)\Delta^{it}(y') =$$
$$\exp\{-it\sum_{k=-\infty}^{m-1} b_{km+1}(2sy_{km}y_{km+1}+s^2(y_{km})^2)-$$
$$it\sum_{n=m+2}^{\infty} b_{mn}(2sy^{-1}_{mn}y^{-1}_{m+1n}-s^2(y^{-1}_{m+1n})^2\} =$$
$$\exp\{-it\sum_{k=-\infty}^{m-1} b_{km+1}(2sx_{km}x_{km+1}+s^2(x_{km})^2)-$$
$$it\sum_{n=m+2}^{\infty} b_{mn}(2sx^{-1}_{mn}x^{-1}_{m+1n}-s^2(x^{-1}_{m+1n})^2)-2isb_{rm+1}tx_{rm}\}.$$

Therefore

$$(U_{rm}(s)f)\,(x,t) = e^{-2ib_{rm+1}stx_{rm}}f(x,t).$$

Since $T^{L,b}_{rm}(s)$, $\lambda(s) \in \mathcal{C}'_N$ for all $s \in \mathbb{R}$, we conclude that $\{U_{rm}(s),\lambda(1)\}$, $\{U_{rm}(s),\, T^{L,b}_{rm}(1)\} \in \mathcal{C}'_N$. The explicit calculation gives us:

$$(\{U_{rm}(s),\lambda(1)\}f)\,(x,t) = e^{isb_{rm+1}x_{rm}}f(x,t),$$

$$\left(\{U_{rm}(s),T^{L,b}_{rm}(1)\}f\right)(x,t) = e^{isb_{rm+1}t}f(x,t).$$

□

PROOF. Now we finish the proof of Lemma 33. From the equations above we see that the unitary one-parameters groups $\exp(isQ_{rm})$ and $\exp(isQ_t)$, $s \in \mathbb{R}$, generated by the self-adjoint operators Q_{rm} and Q_t, defined by (24), are contained in $\mathcal{C}'_N$, which proves Lemma 33. □

9 Uniqueness of the constructed factor

Theorem 42 *The von Neumann algebras $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are hyperfinite type III_1 factors and hence isomorphic to the factor R_∞ of Araki and Woods.*

PROOF. Let G be a solvable separable locally compact group or a connected locally compact group. Then any representation π of G in a Hilbert space generates a hyperfinite von Neumann algebra ([3]).
The group $B_0^{\mathbb{Z}}$ is the inductive limit of groups of finite dimensional upper-triangular matrices (with units on the diagonal), which are of course solvable, connected locally compact groups. Hence their group algebras are hyperfinite

(and by a theorem of Dixmier ([6]) even type I algebras). Thus the von Neumann algebra $\mathfrak{A}^{R,b}$ is the inductive limit of hyperfinite von Neumann algebras and hence itself hyperfinite. From the theorem of Haagerup ([9]) follows that $\mathfrak{A}^{R,b}$ and $\mathfrak{A}^{L,b}$ are all isomorphic to the Araki-Woods factor R_∞ ([1]). □

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