NONLINEAR SMOOTH GROUP ACTIONS

ON DISKS, SPHERES, AND EUCLIDEAN SPACES

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Disks D^n, spheres S^n, and Euclidean spaces $\mathbb{R}^n$ are among the most important manifolds to deal with in the study of smooth actions of compact Lie groups. Here are some reasons which make them outstanding.

(i) D^n, S^n, and $\mathbb{R}^n$ are homologically simple. In particular, the celebrated Smith Theory can be applied to smooth actions of a torus, a finite p–group or its extension by a torus.

(ii) D^n, S^n, and $\mathbb{R}^n$ admit linear actions which are among the simplest and most natural examples of smooth actions.

(iii) Unlike manifolds not admitting smooth compact Lie group actions, D^n, S^n and $\mathbb{R}^n$ are among manifolds with the highest degree of symmetry, so that one expects a variety of smooth actions on these manifolds.

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Here, a natural approach is to compare the geometric behavior of general smooth actions with the geometric behavior of linear actions. There is a number of regularity theorems which assert that, to some degree, smooth actions satisfying some regularity conditions resemble linear actions (see, e.g., the paper of Hsiang [H] for an excelent survey of related results). On the other hand, during the past twenty years, a number of authors have constructed many examples of smooth actions showing that the regularity theorems fail in general. Therefore, one may ask to what extend smooth actions can differ from linear actions. The goal of this paper is to deal with some related specific problems.

In Section 1, we state nine related problems. For linear actions, the answers to all quoted problems are affirmative. In Section 2, we collect first examples of smooth actions which provide negative answers to some of the problems. In Section 3, we discuss some results obtained by the author which allow us to give further negative answers. In Section 4, we construct new exampes of smooth actions which give negative answers to all of the problems.

Section 1

Let G be a compact Lie group and let M be a smooth manifold. We are interested in *smooth actions* of G on M ; i.e. smooth maps

$$G \times M \longrightarrow M\,, \quad (g,x) \longmapsto gx$$

fulfilling the following two conditions.

(1) $ex = x$ for all $x \in M$ and the neutral element $e \in G$.

(2) $g(hx) = (gh)x$ for all $x \in M$ and $g,h \in G$.

The simplest and most natural examples of smooth actions of G on $\mathbb{R}^n$ are *linear actions* ; i.e., actions given via linear representations $\rho : G \longrightarrow GL(n,\mathbb{R})$ by the formula

$$G \times \mathbb{R}^n \longrightarrow \mathbb{R}^n , \ (g,x) \longmapsto \rho(g)\cdot x \ .$$

A linear action of G on $\mathbb{R}^n$ is also called a representation of G on $\mathbb{R}^n$. The existence of a positive definite inner product on $\mathbb{R}^n$, invariant under a given linear action of G , allows us to assume that the action of G on $\mathbb{R}^n$ is *orthogonal* ; i.e., it is given via an orthogonal representation $\rho : G \longrightarrow O(n)$. Clearly, such an action restricts to an orthogonal action of G on D^n , as well as on S^{n-1} and on

$$S^{n-1} \times [0,1] \cong \{x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 2\} \ .$$

Assume G acts linearly on $\mathbb{R}^n$. Let H be a subgroup of G occuring as the isotropy subgroup at a point $x \in \mathbb{R}^n$; i.e.,

$$H = \{g \in G \mid gx = x\} \ .$$

If $x \neq 0$, then H occurs also as the isotropy subgroup at any point $y \neq 0$ lying on the line passing throught 0 and x . Moreover, the H–fixed point set

$$F(H,\mathbb{R}^n) = \{x \in \mathbb{R}^n \mid hx = x \quad \text{for all} \quad h \in H\}$$

is a k–dimensional linear subspace of $\mathbb{R}^n$ for $k \leq n$, so that $F(H,\mathbb{R}^n)$ is diffeomorphic to $\mathbb{R}^k$. Clearly, for an orthogonal action of G on D^n , the H–fixed point set $F(H,D^n)$ is diffeomorphic to D^k.

Any smooth action of G on M induces (via the differential of the action) a linear representation of G on the tangent space T_xM at any point $x \in M$ left fixed by the action of G. If $M = \mathbb{R}^n$ and the action is linear, then the representation of G on T_xM is equivalent to the original action on M.

Now we wish to state some specific problems in transformation groups. These problems are interrelated and it follows easily from the above discussion that all of them have affirmative answers in the case of linear actions. Unless otherwise stated, G is a compact Lie group and H is a closed subgroup of G. As usual, $M_{(H)}$ consists of all orbits in M of type G/H.

Problem 1. If G acts on D^n with the origin as a fixed point, then for $H \neq G$ with $D^n_{(H)} \neq \emptyset$, is $S^{n-1}_{(H)} \neq \emptyset$? Is this the case when the action on the boundary $\partial D^n = S^{n-1}$ is orthogonal?

Problem 2. If G acts on D^n and if $F(G,D^n) \subset \operatorname{int} D^n$, does $F(G,D^n)$ contain **at most** one point?

Problem 3. If G acts on $S^n \times [0,1]$ so that the set F of fixed points touches $S^n \times \{0\}$, does F also touch $S^n \times \{1\}$? Is this the case when the action on both ends is orthogonal?

Problem 4. Let G act on $\mathbb{R}^n$ with a fixed point x. If $\mathbb{R}^n_{(H)} \neq \emptyset$ for $H \neq G$, must x be in the closure of $\mathbb{R}^n_{(H)}$? In the case of smooth actions, by the Slice Theorem, this amounts to asking whether each isotropy subgroup in $\mathbb{R}^n$ occurs also in the representation on the tangent space $T_x\mathbb{R}^n$.

Problem 5. Let G be a torus acting on $\mathbb{R}^n$. For $H \neq G$, is $F(H,\mathbb{R}^n)$ connected? Is it connected when $G = S^1$?

Problem 6. Let G act smoothly on $\mathbb{R}^n$. If $\mathbb{R}^n_{(H)} \neq \emptyset$ for $H \neq G$, is it true that the isotropy subgroup representations on the normal spaces of orbits at two points in $\mathbb{R}^n_{(H)}$ are equivalent?

Problem 7. If G acts on $\mathbb{R}^n$ and if $F(G,\mathbb{R}^n)$ is compact, does $F(G,\mathbb{R}^n)$ contain **at most** one point? If $F(H,\mathbb{R}^n)$ is also compact for $H \neq G$, does $F(H,\mathbb{R}^n) = F(G,\mathbb{R}^n)$?

Problem 8. Let G act on $\mathbb{R}^n$, D^n, or S^n with fixed point set F. Is it true that each connected component of F has the same dimension?

Problem 9. Let G act smoothly on $M = \mathbb{R}^n$ or D^n (resp., S^n) with at least two (resp., three) fixed points. Is it true that for any two fixed points x and y, the representations of G on the tangent spaces $T_x M$ and $T_y M$ are equivalent?

Problems 1, 2, 3, and 4 are listed in the Bredon's book [B; p. 205]. Problem 4 was posed by Raymond for $G = S^1$, and Problem 5 is due to Mostert; see [M; Problems 11 and 12 on p. 353]. Problem 6 goes back to Hsiang and Hsiang [HH; Problem 16]. Problem 7 was posed by Smith [Sm; Question on p. 412] for $G = \mathbb{Z}_{pq}$, the cyclic group of order pq for two relatively prime integers p and q. Problems 8 and 9 are stated in the Bredon's book [B; the second remark on p. 58]. In Problem 9, we excluded the special and important case of smooth actions of G on S^n with exactly two fixed points. In this case, the question of the equivalence of the representations of G at two fixed points goes back to Smith [Sm; the footnote on p. 406]. The complete list of groups G for which the answer to the Smith question is negative, is still unknown, and we will not discuss this question here.

Finally, observe that if G (resp., H) is a torus, a finite p–group or its extension by a torus, then it follows from Smith Theory that the answers to Problems 1–9 all are affirmative.

Section 2

In this section, we collect first examples of smooth actions which provide negative answers to some of the problems stated in Section 1. First recall that a smooth manifold F is called *stably complex* if there exists a smooth embedding of F into some Euclidean space such that the normal bundle of the embedding admits a complex structure. In particular, a stably complex manifold F is orientable and all connected components of F are either even or odd dimensional.

Example 2.1. Let p and q be two relatively prime integers and let F be a closed smooth manifold such that each connected component of F has the same dimension. Edmonds and Lee [EL] showed that there is a smooth action of $\mathbb{Z}_{pq}$ on some $\mathbb{R}^n$ with fixed point set F is the following two cases.

(i) F is stably parallelizable.

(ii) F is stably complex and the integers p and q are sufficiently large with respect to the dimension of F.

This provides negative answers to the first question in Problem 7.

Example 2.2. For two relatively prime integers p and q, Edmonds and Lee [EL] constructed a smooth action of $\mathbb{Z}_{pq}$ on some $\mathbb{R}^n$ with exactly two fixed points and inequivalent respresentations there at, providing a negative answer to Problem 9 in the case $G = \mathbb{Z}_{pq}$ and $M = \mathbb{R}^n$.

Example 2.3. For $G = S^1$ and $H = \mathbb{Z}_6 \subset S^1$, Stein [St] constructed a smooth action of G on S^5 with isotropy subgroups G, H, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and the trivial subgroup 0, such that $F(G,S^5) \cong S^1$ and $S^5_{(H)}$ consists of just one orbit. By taking the equivariant connected sum of k copies of S^5 for any $k \geq 1$, we get a smooth action of G on S^5 with the same isotropy subgroups as before, such that $F(G,S^5) \cong S^1$ and $S^5_{(H)}$ consists of k orbits. By removing from S^5 a sufficiently small open invariant disk around a fixed point, we get a smooth action of G on D^5 (orthogonal on $\partial D^5 = S^4$) with isotropy subgroups G, H, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and 0, such that $F(G,D^5) \cong D^1$ and $D^5_{(H)}$ consists of k orbits. Clearly, $D^5_{(H)} \subset \text{int } D^5$, so that we get a negative answer to Problem 1. By restricting the action to int D^5, we get a smooth action of G on $\mathbb{R}^5$ with isotropy subgroups G, H, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and 0, such that $F(G,\mathbb{R}^5) \cong \mathbb{R}^1$ and $\mathbb{R}^5_{(H)}$ consists of k orbits. This provides negative answers to Problems 4 and 5. However, note that the representations of H on the normal spaces of orbits at any two points in $\mathbb{R}^5_{(H)}$ are equivalent, so that this example does not provide a negative answer to Problem 6.

Example 2.4. For $G = \mathbb{Z}_{pqr}$ and $H = \mathbb{Z}_{pq}$, where p, q, and r are three distinct mutually prime integers, Assadi [A; pp. 91–92] constructed a smooth action of G on some $\mathbb{R}^n$ with isotropy subgroups G, H, $\mathbb{Z}_p$, $\mathbb{Z}_q$, and 0, such that $F(G,\mathbb{R}^n)$ is just one point and $F(H,\mathbb{R}^n)$ consists of r+1 points, so that $\mathbb{R}^n_{(H)}$ is just one orbit. This provides negative answers to Problems 4 and 7.

Example 2.5. For $G = \mathbb{Z}_{pqr} \oplus \mathbb{Z}_{pqr} \oplus \mathbb{Z}_s$ and $H = \mathbb{Z}_{pqr} \oplus \mathbb{Z}_{pqr}$, where p, q, r, and s all are distinct and mutually prime, Assadi [A; pp. 92–94] constructed a smooth action of G on a disk D^n such that $F(G,D^n)$ consists of k points for any $k \geq 1$, and $F(H,D^n)$ consists of $k+s$ points, so that $D^n_{(H)}$ is just one orbit. Clearly, $F(G,D^n) \subset \operatorname{int} D^n$. Hence, for $k > 1$, this provides a negative answer to Problem 2. By setting $k = 1$ and taking the equivariant double of D^n, he obtained a smooth action of G on S^n such that $F(G,S^n)$ consists of two points and $S^n_{(H)}$ consists of two orbits. By removing from S^n a sufficiently small open invariant disk around a fixed point, he obtained a smooth action of G on D^n (orthogonal on $\partial D^n = S^{n-1}$) such that $F(G,D^n)$ is just one point and $D^n_{(H)}$ consists of two orbits lying in the interior of D^n. This provides a negative answer to Problem 1. By restricting the action to $\operatorname{int} D^n$, he obtained a smooth action of G on $\mathbb{R}^n$ such that $F(G,\mathbb{R}^n)$ is just one point and $\mathbb{R}^n_{(H)}$ consists of two orbits. This, in turn, provides negative answers to Problems 4 and 7.

Section 3

In this section, we wish to discuss some results obtained by the author. First, we point out that for a compact Lie group G, the answers to Problems 8 and 9 depend only on the quotient group G/G_0, where G_0 is the identity connected component of G. More specifically, the following two theorems hold.

Theorem 3.1. *Let G be a compact Lie group. Then the following three conditions are equivalent.*

(1) *For any smooth action of G on a disk (resp., Euclidean space), at any two fixed points the representations of G are equivalent.*

(2) *For any smooth action of G on a disk (resp., Euclidean space), each fixed point set connected component has the same dimension.*

(3) *In* G/G_0, *each element has prime power order*.

Theorem 3.2. *Let* G *be a compact Lie group. Then the following three conditions are equivalent.*

(1) *For any smooth action of* G *on a sphere (resp., homotopy sphere) with at least three fixed points, at any two fixed points the representations of* G *are equivalent.*

(2) *For any smooth action of* G *on a sphere (resp., homotopy sphere), each fixed point set connected component has the same dimension.*

(3) *In* G/G_0, *each element has prime power order*.

In Theorems 3.1 and 3.2, (3) implies (1) by [P_2; Propositions 7.1 and 7.2] and (1) implies (2) because, by the Slice Theorem, the trivial summand of the representation of G at a fixed point x has the same dimension as does the fixed point set connected component containing x. In order to show that (2) implies (3), for any compact Lie group G such that G/G_0 has a cyclic subgroup not of prime power order, the author has constructed smooth actions of G on disk, spheres, and Euclidean spaces with fixed point set connected components of different dimensions (see [P_2; Example 6.1], [P_4; Theorems (1) and (2)], and Examples 3.3 and 3.4 below). Therefore, the answers to Problems 8 and 9 are negative if and only if G/G_0 has a cyclic subgroup not of prime power order.

Example 3.3. Let G be a compact Lie group and let F be a smooth manifold without boundary. The author [P_3] proved that there is a smooth action of G on some $\mathbb{R}^n$ with fixed point set F in the following two cases (cf. Example 2.1).

(i) Either G_0 is abelian and G/G_0 is not of prime power order, or G_0 is non-abelian, and F is a stably parallelizable manifold with all connected components of the same dimension.

(ii) G/G_0 has a cyclic subgroup not of prime power order, and F is a stably complex manifold.

By choosing compact F, we get negative answers to the first question in Problem 7. In (ii), by choosing F with fixed point set connected components of different dimensions, we get negative answers to Problems 8 and 9 for $M = \mathbb{R}^n$.

In the case of smooth actions of G on disks for a compact Lie group G such that either G_0 is not of prime power order, or G_0 is nonabelian, there is a restriction on the Euler characteristic of the fixed point set F. Namely, it follows from the work of Oliver $[O_1]$ and $[O_2]$ that

$$\chi(F) \equiv 1 \pmod{n_G},$$

where n_G is the integer defined and calculated by Oliver $[O_1]$, $[O_2]$, and $[O_3]$. Recall that $n_G = n_{G/G_0}$ when G_0 is abelian and $n_G = 1$ when G_0 is nonabelian.

Example 3.4. Let G be a compact Lie group and let F be a compact smooth manifold. The author $[P_3]$ proved that there is a smooth action of G on some D^n with fixed point set F in the following two cases.

(i) Either G_0 is abelian and G/G_0 is not of prime power order, or G_0 is nonabelian, and F is a stably parallelizable manifold with all connected components of the same dimension and with $\chi(F) \equiv 1 \pmod{n_G}$.

(ii) G/G_0 has a cyclic subgroup not of prime power order, and F is a stably complex manifold with $\chi(F) \equiv 1 \pmod{n_G}$.

By choosing closed F, we have $F \subset \operatorname{int} D^n$, so that we get negative answers to Problem 2. In (ii), by choosing F with fixed point set connected components of different dimensions and taking the equivariant double of D^n, we get a smooth action of G on S^n

with fixed point set connected components of different dimension. Thus, we get negative answers to Problems 8 and 9 for $M = D^n$ and S^n.

Example 3.5. In Example 3.4 (ii), choose F so that it contains at least one isolated point and one connected component of positive dimension. By taking the equivariant double of D^n, we get a smooth action of G on S^n with fixed point set containing an isolated point x and a point y in a connected component of positive dimension. By removing from S^n sufficiently small open invariant disks around x and y, we get a smooth action of G on $S^{n-1} \times [0,1]$ which is orthogonal on both ends $S^{n-1} \times \{0\}$ and $S^{n-1} \times \{1\}$, and the fixed point set touches only one end. Thus, we get a negative answer to Problem 3.

Note that if in G/G_0, each element has prime power order, then such an action of G on $S^{n-1} \times [0,1]$ does not exist. In fact, if this can happen, then there would exist a smooth action of G on D^n, as well as on S^n, with fixed point set connected components of different dimensions which is impossible by Theorems 3.1 and 3.2. Therefore, the answer to Problem 3 is negative if and only if G/G_0 has a cyclic subgroup not of prime power order.

Example 3.6. For $G = \mathbb{Z}_{pqr}$ and $H = \mathbb{Z}_{pq}$, where p, q, and r are three distinct mutually prime integers, the author [P_2; Example 6.2] has constructed a smooth action of G on some D^n with isotropy subgroups $G, H, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r$, and 0, such that $F(G,D^n)$ is just one point and $D^n_{(H)}$ consists of r copies of $D^1 \times S^1$. By taking the equivariant double of D^n, we get a smooth action of G on S^n such that $F(G,S^n)$ consists of two points and $S^n_{(H)}$ consists of r copies of the torus $T^2 = S^1 \times S^1$. By removing from S^n a sufficiently small open invariant disk around a fixed point, we get a smooth action of G on D^n (orthogonal on $\partial D^n = S^{n-1}$) such that $F(G,D^n)$ is just one point and $D^n_{(H)}$ consists of r copies of the torus T^2. Clearly, $D^n_{(H)} \subset \operatorname{int} D^n$, so that we get a negative answer to Problem 1. By restricting the action to int D^n, we get a smooth action of G on $\mathbb{R}^n$ such that $F(G,\mathbb{R}^n)$ is just one point and $\mathbb{R}^n_{(H)}$ consists of r copies of the torus. Clearly, $F(H,\mathbb{R}^n) = F(G,\mathbb{R}^n) \cup \mathbb{R}^n_{(H)}$. Thus, we get negative answers to Problems 4 and 7. Compare the results obtained here with those in Examples 2.4 and 2.5.

Section 4

In this section, we construct new examples of smooth actions of G on disks, spheres, and Euclidean spaces with prescribed H–fixed point sets for a proper subgroup H of G. In order to construct these actions, we apply the equivariant thickening procedure obtained by the author $[P_1]$ (see $[P_3]$ for the details). The procedure requires the existence of a suitable G–vector bundle over a G–CW complex X. In order to get such a G–vector bundle, we use the space $\mathrm{Map}_\bullet(G,SU(n))$ of maps $\theta : G \longrightarrow SU(n)$ preserving the neutral elements of G and $SU(n)$, with the action of G given by $g\,\theta(a) = \theta(ag)\theta(g)^{-1}$. This G–space is useful because there is a natural one–one correspondence between special unitary G–vector bundle structures on $X \times \mathbb{C}^n$ over X and equivariant maps from X into $\mathrm{Map}_\bullet(G,SU(n))$. For a given map

$$X \longrightarrow \mathrm{Map}_\bullet(G,SU(n)), \quad x \longmapsto \theta_x ,$$

the corresponding action of G on $X \times \mathbb{C}^n$ is defined by

$$g(x,v) = (gx, \theta_x(g)\cdot v) \ ;$$

cf. [B; Chapter VI, Proposition 11.1] and $[P_2$; Proposition 4.1].

Example 4.1. Let H be a finite group not of prime power order, such that any two Sylow subgroups of H intersect trivially. Let $F_1,\dots,F_k$ be parallelizable smooth manifolds all either even or odd dimensional, such that each F_i has the structure of a CW complex containing as a deformation retract a subcomplex L_i which is either a point or a wedge of circles. Assume also that each F_i is compact and

$$\chi(L_1) + \dots + \chi(L_k) \equiv 0 \pmod{n_H} ,$$

where n_H is the Oliver integer of H, (resp., assume that each F_i is without boundary; no restriction on the Euler characteristic). Let F and L be the disjoint unions of all F_i and L_i, respectively. Since L has finitely many cells and $\chi(L) \equiv 0 \pmod{n_H}$ (resp., L has countably many cells), and H is not of prime power order, it follows from the work of Oliver [O_1] (resp., Assadi [A]) that there is a finite (resp., finite dimensional, infinite, countable) contractible H–CW complex Y with fixed point set

$$Y^H = \{b\} \coprod L ,$$

the disjoint union of a point b and L. Moreover, we may assume that for each proper subgroup I of H not of prime power order, each equivariant cell $H/I \times D^m$ in Y has an attching map, defined on $H/I \times S^{m-1}$, that is constant on each copy $\{hI\} \times S^{m-1}$ of the sphere S^{m-1} (resp., there is no equivariant cell of the form $H/I \times D^m$ in Y). Since any two Sylow subgroups of G intersect trivially, we may also assume that for each equivariant cell $G/I \times D^m$ in Y, $m \leq 2$ when I is nontrivial, and $m \leq 3$ when I is trivial; cf. [P_2; Remarks 2.5 and 2.6] (resp., [A; Corollary II.7.3]).

Now, consider $Y \cup_L F$, the sum of Y and F along L, with the obvious action of H (trivial on F). Let C be a finite group. Put

$$G = H \times C$$

and consider $(Y \cup_L F) \times C$ with the product action of G. Let F_0 be a contractible smooth manifold with $\dim F_0 \equiv \dim F_i \pmod 2$ for $i = 1,\dots,k$, and assume F_0 is compact (resp., without boundary). Let X be the G–space obtained from $(Y \cup_L F) \times C$ and F_0 by identifying all points (b,c), $c \in C$, with a point in the interior of F_0. Then X is a finite (resp., finite dimensional, infinite, countable) contractible G–CW complex with fixed point set $X^G = F_0$ and

$$X_{(H)} = F \times C \ .$$

Moreover, the family of isotropy subgroups in $X - X^G$ consists of the subgroups $I \times \{e\}$ of G for all I occuring as the isotropy subgroups in Y. In particular,

$$X^H = F_0 \coprod (F \times C) \ .$$

Let $V_1,\dots,V_k$ be unitary representations of H with $V_i^H = \{0\}$ for $i = 1,\dots,k$. Assume that the following two conditions hold.

Dimension Condition. For all $1 \leq i$, $j \leq k$,

$$\dim F_i + \dim_{\mathbb{R}} V_i = \dim F_j + \dim_{\mathbb{R}} V_j \ .$$

Smith Condition. For each prime power order subgroup P of H and all $1 \leq i$, $j \leq k$, the nontrivil summands of the restricted representations $\mathrm{res}_P^H V_i$ and $\mathrm{res}_P^H V_j$ are equivalent.

For $i = 1,\ldots,k$, put $n_i = [(\dim F_i + 1)/2]$, the greatest integer in $(\dim F_i + 1)/2$. Consider the representation $\mathbb{C}^{n_i} \oplus V_i$ of H , where H acts trivially on $\mathbb{C}^{n_i}$. It follows from the Dimension Condition that the representations $\mathbb{C}^{n_i} \oplus V_i$ all have the same dimension, say n . Let $\sigma_i : H \longrightarrow U(n)$ be the homomorphism corresponding to $\mathbb{C}^{n_i} \oplus V_i$. It follows from the Smith Condition that $\sigma_i | P$ and $\sigma_j | P$ are equivalent for each prime power order subgroup P of H . Therefore, by adding (if necessary) to each σ_i the same 1–dimensional complex representation of H , we may assume that σ_i all are special unitary representations; i.e., $\sigma_i(H) \subset SU(n)$; see $[P_2$; Lemma 7.3].

Let V_0 be a special unitary representation of G with $V_0^G = \{0\}$, and assume that $\dim F_0 + \dim_{\mathbb{R}} V_0 = \dim F_i + \dim_{\mathbb{R}} V_i$ and for each prime power order subgroup P of H , the nontrivial summands of the restricted representations $\mathrm{res}_P^G V_0$ and $\mathrm{res}_P^H V_i$ are equivalent. Put $n_0 = [(\dim F_0 + 1)/2]$ and consider the representation $\mathbb{C}^{n_0} \oplus V_0$ of G , where G acts trivially on $\mathbb{C}^{n_0}$. Let $\rho_0 : G \longrightarrow SU(n)$ be the homomorphism corresponding to $\mathbb{C}^{n_0} \oplus V_0$.

According to $[P_2$; Proposition 4.2], the map

$$f : \mathrm{Map}_\bullet(G,SU(n))^H \longrightarrow \mathrm{Hom}(H,SU(n)) \times \mathrm{Map}_\bullet(C,SU(n))$$

$$\theta \longmapsto (\theta | H, \theta | C)$$

is a homeomorphism. Hereafter, $H = H \times \{e\}$ and $C = \{e\} \times C$. For $i = 1,\ldots,k$, let $\rho_i : G \longrightarrow SU(n)$ be given by $\rho_i = f^{-1}(\sigma_i, \rho_0 | C)$. Explicitely,

$$\rho_i(g) = \rho_0(c)\sigma_i(h)$$

for $g = (h,c)$, $h \in H$, $c \in C$.

Recall that $X^H = F_0 \coprod (F \times C)$ and consider the G–map

$$\eta : X^H \longrightarrow \mathrm{Map}_\bullet(G,SU(n))^H \subset \mathrm{Map}_\bullet(G,SU(n))$$

defined by mapping all points in F_i to ρ_i for $i = 0,1,\dots,k$, and extending thus obtained map on $F_0 \coprod F$ to the unique G–map on X^H . We claim that η extends to a G–map

$$\xi : X \longrightarrow \mathrm{Map}_\bullet(G,SU(n)) .$$

Recall that the family of isotropy subgroups in $X-X^H$ consists of the subgroups $I = I \times \{e\}$ of G for all proper subgroups I of H occuring as the isotropy subgroups in Y . Let B be the disjoint union of X^H and all equivariant 0–cells G/I in $X-X^H$. Extend η on B by mapping each G/I into ρ_0 .

If I is not of prime power order, η extends on cells $G/I \times D^m$ in $X-X^H$ because the attaching maps are constant on each copy $\{gI\} \times S^{m-1}$ of the sphere S^{m-1} .

If I is of prime power order, $\rho_i|I \cong \rho_j|I$ and this amounts to saying that ρ_i and ρ_j lie in the same connected component of the fixed point set $\mathrm{Map}_\bullet(G,SU(n))^I$; cf. [P_2; Corollary 4.3]. Therefore, η extends on cells $G/I \times D^1$ in $X-X^H$. Since each connected component of $\mathrm{Map}_\bullet(G,SU(n))^I$ is 1–connected; cf. [P_2; Corollary 4.5], η extends also on cells $G/I \times D^2$ in $X-X^H$. Finally, η extends on cells $G \times D^1$, $G \times D^2$, and $G \times D^3$ in $X-X^H$ because $SU(n)$ is 2–connected, and thus, so is $\mathrm{Map}_\bullet(G,SU(n))$.

Let E be the G–vector bundle over X corresponding to the G–map $\xi : X \longrightarrow \mathrm{Map}_\bullet(G,SU(n))$. For $i = 1,\dots,k$, put $M_i = F_i \times C$. Since $\xi|X^H = \eta$, the restricted G–vector bundle $E|M_i$ splits into the product bundles $M_i \times \mathbb{C}^{n_i}$ and $M_i \times V_i$ over M_i . Similarly, as G–vector bundles,

$$E\,|\,F_0 \cong (F_0 \times \mathbb{C}^{n_0}) \oplus (F_0 \times V_0) \ .$$

Since each F_i is parallelizable, the tangent bundle TM_i (resp., $TM_i \oplus (M_i \times \mathbb{R}^1)$) admits the structure of the product bundle $M_i \times \mathbb{C}^{n_i}$ over M_i when $\dim F_i$ is even (resp., odd). Similarly, TF_0 (resp., $TF_0 \oplus (F_0 \times \mathbb{R}^1)$) admits the structure of the product bundle $F_0 \times \mathbb{C}^{n_0}$, when $\dim F_0$ is even (resp., odd). Therefore,

$$E\,|\,B \cong TB \oplus U \quad (\text{resp., } TB \oplus U \oplus (B \times \mathbb{R}^1)) \ ,$$

where U is the G–vector bundle over B with $U\,|\,F_0 = F_0 \times V_0$, $U\,|\,M_i = M_i \times V_i$ and $U\,|\,(B–X^H) = E\,|\,(B–X^H)$.

Now, take the disk bundle of U over B, and then replace inductively equivariant cells in $X–B$ by equivariant handles in a way prescribed by E. This converts X into a smooth G–manifold M of dimension $2n$ (resp., $2n–1$), where n is the fiber dimension of E, such that M contains B as a smooth G–invariant submanifold with equivariant normal bundle U, $M–B$ and $U–B$ have the same isotropy subgroups and TM (resp., $TM \oplus (M \times \mathbb{R}^1)$) is induced from E via a G–homotopy equivalence $f : M \longrightarrow X$ coinciding with the identity on B (see [P_3; § 2] for the details of the equivariant thickening procedure that we use here, and observe that in order to apply the procedure we add, if necessary, to E and U the product bundles $X \times W$ over X and $B \times W$ over B, respectively, for a suitable complex representation W of G). In particular, TM (resp. $TM \oplus (M \times \mathbb{R}^1)$) admits the structure of a complex G–vector bundle. Since neither G nor H occurs as an isotropy subgroup in $U–B$, thus

$$M^G = F_0 \quad \text{and} \quad M_{(H)} = M_1 \coprod \cdots \coprod M_k \ .$$

Clearly, $M^H = F_0 \coprod M_{(H)}$. Moreover, V_i (or $V_i \oplus \mathrm{res}_H^G W$) occurs as the normal representation at any point in M_i for $i = 1,\dots,k$, and V_0 (or $V_0 \oplus W$) occurs as the normal representatiom at any point in F_0.

Since the finite (resp., infinite) complex X is contractible, so is M, and it follows from the construction that M is diffeomorphic to either D^{2n} or D^{2n-1} (resp., $\mathbb{R}^{2n}$ or $\mathbb{R}^{2n-1}$).

Using the actions just constructed, it is easy to get negative answers to Problem 1–9 (except for Problem 5) by taking the equivariant doubles of D^{2n} and D^{2n-1}, removing sufficiently small open invariant disks around fixed points in S^{2n} and S^{2n-1}, and restricting the actions to int D^{2n} and int D^{2n-1}. However, in order to get negative answers to Problems 6, 8, and 9, it is necessary to assume that H has a cyclic subgroup not of prime power order (otherwise the representations of H on the normal spaces of orbits at two points in $M_{(H)}$ are equivalent; cf. Theorem 3.1). Now, if H has a cyclic subgroup not of prime power order, then there are unitary representations $V_1,\dots,V_k$ of H with $V_i^H = \{0\}$ fulfilling both the Dimension Condition and Smith Condition, such that V_i and V_j are inequivalent when $i \neq j$ (see, e.g., [P_4; Comments (1) and (2)].

Example 4.2. Let $G = S^1$, the group of complex numbers of absolute value 1, and let $H = \mathbb{Z}_{pq}$, the cyclic subgroup of order pq generated by the primitive pq–th root of unity. For any integer n, write $t^n : G \longrightarrow U(1)$ for the unitary representation of G defined by $t^n(z) = z^n$.

Consider the action of G on $\mathbb{C}$ given by the representation t^1, and the trivial action of G on $\mathbb{R}$. This yields an orthogonal action of G on $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$. Take the closed unit disk D^3 in $\mathbb{R}^3$ with the action of G, and the decomposition of the boundary $\partial D^3 = S^2$ into the closed upper hemisphere S^2_+, the closed lower hemisphere S^2_-, and the equator S^1. Let Y be the quotient space obtained from D^3 by taking the following quotients: $S^2_+/\mathbb{Z}_p$, $S^2_-/\mathbb{Z}_q$, and S^1/H . Then Y admits the structure of a finite G–CW complex whose fixed point set Y^G is a line segment. It follows from the Van Kampen Theorem and the Mayer–Vietoris exact sequence that Y is contractible. By contracting Y^G into a point, we may assume that $Y^G = pt$. Note that Y is built up from pt by adding one equivariant 0–cell G/H , attaching one equivariant 1–cell $G/\mathbb{Z}_p \times D^1$ and one equivariant 1–cell $G/\mathbb{Z}_q \times D^1$, and finally attaching one equivariant 2–cell $G \times D^2$.

Choose a sequence of integers $n_1,\dots,n_k$ with $n_i \geq 1$ for $i = 1,\dots,k$, consider D^{2n_i-1} with the trivial action of G , and put

$$X_i = Y \cup_{G/H} (G/H \times D^{2n_i-1}) \ ,$$

the sum of Y and $G/H \times D^{2n_i-1}$ along $G/H \equiv G/H \times \{0\}$. Then X_i^G is just one point x_i . Now, choose an integer $n_0 \geq 0$, consider D^{2n_0} with the trivial action of G , and take the space X obtained from $X_1,\dots,X_k$ and D^{2n_0} by identifying all isolated fixed points $x_1,\dots,x_k$ with the origin in D^{2n_0} . Then X is a finite contractible G–CW complex with isotropy subgroups $G, H, \mathbb{Z}_p, \mathbb{Z}_q$, and the trivial subgroup $\{1\}$, such that

$$X^G = B_0 \quad \text{and} \quad X_{(H)} = B_1 \coprod \cdots \coprod B_k \ ,$$

where $B_0 = D^{2n_0}$ and $B_i = G/H \times D^{2n_i-1}$ for $i = 1,\dots,k$. Put

$$B = B_0 \coprod B_1 \coprod \cdots \coprod B_k \ .$$

Choose special unitary representation $V_0, V_1,\dots,V_k$ of G without trivial summands, such that the representations $\mathbb{C}^{n_i} \oplus V_i$ all have the same dimension, say n , where G acts trivially on $\mathbb{C}^{n_i}$, and for $P = \mathbb{Z}_p$ and $\mathbb{Z}_q$, the restricted representations

$$\mathrm{res}_P^G(\mathbb{C}^{n_i} \oplus V_i) \quad \text{and} \quad \mathrm{res}_P^G(\mathbb{C}^{n_j} \oplus V_j)$$

are equivalent for all $0 \le i,\ j \le k$. Let $\rho_i : G \longrightarrow SU(n)$ be the homomorphism corresponding to $\mathbb{C}^{n_i} \oplus V_i$.

Now, consider the map

$$B \longrightarrow \mathrm{Hom}(G,SU(n)) \subset \mathrm{Map}_\bullet(G,SU(n))$$

which maps all points in B_i into ρ_i for $i = 0,1,\ldots,k$. Clearly, its restriction to B_i corresponds to the product bundle $B_i \times (\mathbb{C}^{n_i} \oplus V_i)$ over B_i.

We claim that the map defined on B extends to a G–map

$$X \longrightarrow \mathrm{Map}_\bullet(G,SU(n)) \ .$$

First, recall that X is built up from B by attaching equivariant cells of the form $G/\mathbb{Z}_p \times D^1$, $G/\mathbb{Z}_q \times D^1$, and $G \times D^2$. The extension on cells $G/\mathbb{Z}_p \times D^1$ and $G/\mathbb{Z}_q \times D^1$ exists because $\rho_i | P \cong \rho_j | P$, so that ρ_i and ρ_j lie in the same connected component of $\mathrm{Map}_\bullet(G,SU(n))^P$ for $P = \mathbb{Z}_p$ and $\mathbb{Z}_q$.

The extension on cells $G \times D^2$ exists as well because the space

$$\mathrm{Map}_\bullet(G,SU(n)) \cong \Omega\, SU(n)$$

is 1–connected, proving the claim.

Let E be the resulting G–vector bundle over X. Clearly,

$$E\,|\,B_i \cong TB_i \oplus (B_i \times V_i) \ .$$

Let U be the G–vector bundle over B defined by $U\,|\,B_i = B_i \times V_i$. Then $E\,|\,B \cong TB \oplus U$. By adding (if necessary) to E and U the product bundles $X \times W$ over X and $B \times W$ over B , respectively, for a suitable representation W of G , we may apply the equivariant thickening procedure described in $[P_3; \S\, 2]$, so that X converts into a smooth G–manifold M diffeomorphic to D^{2n} , with isotropy subgroups G, H, $\mathbb{Z}_p$, $\mathbb{Z}_q$, and $\{1\}$, such that

$$M^G = B_0 \ \text{ and } \ M_{(H)} = B_1 \coprod \cdots \coprod B_k \ .$$

By taking the equivariant double of D^{2n} , we get a smooth action of G on S^{2n} such that

$$F(G,S^{2n}) = S^{2n_0} \ \text{ and } \ S^{2n}_{(H)} = M_1 \coprod \cdots \coprod M_k \ ,$$

where $M_i = S^1 \times S^{2n_i-1}$ for $i = 1,\ldots,k$. By removing from S^{2n} a sufficiently small open invariant disk around a fixed point, we get a smooth action of G on D^{2n} (orthogonal on the boundary) such that

$$F(G,D^{2n}) = D^{2n_0} \ \text{ and } \ D^{2n}_{(H)} = M_1 \coprod \cdots \coprod M_k \ .$$

Finally, by restricting the action to $\operatorname{int} D^{2n}$, we get a smooth action of G on $\mathbb{R}^{2n}$ such that

$$F(G,\mathbb{R}^{2n}) = \mathbb{R}^{2n_0} \ \text{ and } \ \mathbb{R}^{2n}_{(H)} = M_1 \coprod \cdots \coprod M_k \ .$$

In all cases, $\operatorname{res}_H^G(V_i \oplus W)$ occurs as the normal representation at any point in M_i for $i = 1,\ldots,k$.

These actions provide negative answers to Problems 1, 4, 5, 6, and 7. However, in order to get such answers to Problems 6 and 7, we need to put $n_0 = 0$ and choose special unitary representations $V_0, V_1,\ldots,V_k$ of G fulfilling the required conditions, such that $\operatorname{res}_H^G V_i$ and $\operatorname{res}_H^G V_j$ are inequivalent when $i \neq j$. For example, assume that $0 = n_0 < n_1 < \ldots < n_k$, take

$$\theta_i : G \longrightarrow SU(2n_k - n_i + 1)$$

defined by

$$\theta_i = n_i\, t^{p+q} \oplus (n_k - n_i)(t^p \oplus t^q) \oplus t^{-n_k(p+q)} \quad ,$$

and consider the representation V_i of G on $\mathbb{C}^{2n_k - n_i + 1}$ given via θ_i for $i = 0,1,\ldots,k$.

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