Regulators

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To Steve Lichtenbaum for his 65th birthday

Contents

1 Introduction

The ζ–function is one of the most deep and mysterious objects in mathematics. During the last two centuries it has served as a key source of new ideas and concepts in arithmetic algebraic geometry. The ζ–function seems to be created to guide mathematicians into the right directions. To illustrate this, let me recall three themes in the 20th century mathematics which emerged from the study of the most basic properties of ζ–functions: their zeros, analytic properties and special values.

1. Weil's conjectures on ζ–functions of varieties over finite fields inspired Grothendieck's revolution in algebraic geometry and led Grothendieck to the concept of motives, and Deligne to the yoga of weight filtrations. In fact (pure) motives over $\mathbb{Q}$ can be viewed as the simplest pieces of algebraic varieties for which the L-function can be defined. Conjecturally the L-function characterizes a motive.

2. Langlands' conjectures predict that n–dimensional representations of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ correspond to automorphic representations of $GL(n)_{/\mathbb{Q}}$. The relationship between these seemingly unrelated objects was manifested by L-functions: the Artin L-function of the Galois representation coincides with the automorphic L–function of the corresponding representation of $GL(n)$.

3. Investigation of the behavior of L-functions of arithmetic schemes at integer points, culminated in Beilinson's conjectures, led to the discovery of the key principles of the theory of mixed motives.

In this survey we elaborate on a single aspect of the third theme: regulators. We focus on the analytic and geometric aspects of the story, and explore several different approaches to motivic complexes and regulator maps. We neither touch the seminal Birch - Swinnerton-Dyer conjecture and the progress made in its direction nor do we consider the vast generalization of this conjecture, due Bloch and Kato [BK].

1. Special values of the Riemann ζ-function and their motivic nature. Euler proved the famous formula for the special values of the Riemann ζ-function at positive even integers:

$$\zeta(2n) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!}\Big(\frac{-B_{2k}}{2k}\Big)$$

All attempts to find a similar formula expressing $\zeta(3), \zeta(5), ...$ via some known quantities failed. The reason became clear only in the recent time: First, the special values $\zeta(n)$ are periods of certain elements

$$\zeta^{\mathcal{M}}(n) \in \mathrm{Ext}^1_{\mathcal{M}_T(\mathbb{Z})}(\mathbb{Z}(0), \mathbb{Z}(n)), \quad n = 2, 3, 4, ... \tag{1}$$

where on the right stays the extension group in the abelian category $\mathcal{M}_T(\mathbb{Z})$ of mixed Tate motives over $\mathrm{Spec}(\mathbb{Z})$ (it has been defined, see [DG]). Second, the fact that $\zeta^{\mathcal{M}}(2n-1)$ are non torsion elements should imply, according to a version of Grothendieck's conjecture on periods, that π and $\zeta(3), \zeta(5), ...$ are algebraically independent over $\mathbb{Q}$. The analytic manifestation of the motivic nature of the special values is the formula

$$\zeta(n) = \int_{0<t_1<...<t_n<1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2} \wedge ... \wedge \frac{dt_n}{t_n} \tag{2}$$

discovered by Leibniz. This formula presents $\zeta(n)$ as a length n iterated integral. The existence of such a formula seems to be a specific property of the L-values at integer points. A geometric construction of the motivic ζ-element (1) using the moduli space $\mathcal{M}_{0,n+3}$ is given in Chapter 4.5.

Beilinson conjectured a similar picture for special values of L-functions of motives at integer points. In particular, his conjectures imply that these special values should be periods (in fact a very special kind of periods). For the Riemann ζ-function this is given by the formula (2). In general a period is a number given by an integral

$$\int_{\Delta_B} \Omega_A$$

where Ω_A is a differential form on a variety X with singularities at a divisor A, Δ_B is a chain with boundary at a divisor B, and X, A, B are defined over $\mathbb{Q}$. So far we can write L-values at integer points as periods only in a few cases. Nevertheless, in all cases when Beilinson's conjecture was confirmed, we have such a presentation. More specifically, such a presentation for the special values of the Dedekind ζ-function this comes from the Tamagawa Number formula and Borel's work [Bo2], and in the other cases it is given by Rankin-Selberg type formulas. In general the mechanism staying behind this phenomenon remains a mystery.

Let us now turn to another classical example: the residue of the Dedekind ζ-function at $s = 1$.

2. The class number formula and the weight one Arakelov motivic complex. Let F be a number field with r_1 real and r_2 complex places, so that $[F : \mathbb{Q}] = 2r_1 + r_2$. Let $\zeta_F(s)$ be the Dedekind ζ-function of F. Then according to Dirichlet and Dedekind one has

$$\mathrm{Res}_{s=1}\zeta_F(s) = \frac{2^{r_1+r_2}\pi^{r_2} R_F h_F}{w_F |D_F|^{1/2}} \tag{3}$$

Here w_F is the number of roots of unity in F, D_F is the discriminant, h_F is the class number, and R_F is the regulator of F, whose definition we recall below. Using the functional equation for $\zeta_F(s)$, S. Lichtenbaum [Li1] wrote (3) as

$$\lim_{s\to 0} s^{-(r_1+r_2-1)}\zeta_F(s) = -\frac{h_F R_F}{w_F} \tag{4}$$

Let us interpret the right hand side of this formula via the weight one Arakelov motivic complex for $\mathrm{Spec}(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers in F. Let us define the following diagram, where $\mathcal{P}$ runs through all prime ideals of $\mathcal{O}_F$:

$$\begin{array}{ccc} \mathbb{R}^{r_1+r_2} & \stackrel{\Sigma}{\longrightarrow} & \mathbb{R} \\ & & \\ R_1 \uparrow & & \uparrow l \\ & & \\ F^* & \stackrel{\text{div}}{\longrightarrow} & \oplus_{\mathcal{P}}\mathbb{Z} \end{array} \tag{5}$$

In this diagram, the maps are given as follows: If $\text{val}_{\mathcal{P}}$ is the canonical valuation defined on F by $\mathcal{P}$ and $|\mathcal{P}|$ is the norm of $\mathcal{P}$, then

$$\text{div}(x) = \sum \text{val}_{\mathcal{P}}(x)[\mathcal{P}], \qquad l : [\mathcal{P}] \longmapsto -\log|\mathcal{P}|, \qquad \Sigma : (x_1, ..., x_n) \longmapsto \Sigma x_i$$

The regulator map R_1 is defined by $x \in F^* \longmapsto (\log|x|_{\sigma_1}, ..., \log|x|_{\sigma_{r_1+r_2}})$, where $\{\sigma_1, ..., \sigma_{r_1+r_2}\}$ is the set of all archimedean places of F and $| * |_\sigma$ is the valuation defined σ, ($|x|_\sigma := |\sigma(x)|^2$ for a complex place σ).

The product formula tells us $\Sigma \circ R_1 + l \circ \text{div} = 0$. Therefore summing up the groups over the diagonals in (5) we get a complex. The first two groups of this complex, placed in degrees $[1, 2]$, form the *weight one Arakelov motivic complex* $\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)$ of $\mathcal{O}_F$. There is a map

$$H^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1) \longrightarrow \mathbb{R}$$

Let $\widetilde{H}^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)$ be its kernel. Then there is an exact sequence

$$0 \longrightarrow \frac{\text{Ker}(\Sigma)}{R_1(\mathcal{O}_F^*)} \longrightarrow \widetilde{H}^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1) \longrightarrow \text{Cl}_F \longrightarrow 0$$

Further, $R_1(\mathcal{O}_F^*)$ is a lattice in $\text{Ker}(\Sigma) = \mathbb{R}^{r_1+r_2-1}$, its volume with respect to the measure $\delta(\sum x_i)dx_1 \wedge ... \wedge dx_{r_1+r_2}$ is R_F, and $h_F = |\text{Cl}_F|$. Therefore

$$\text{vol}\widetilde{H}^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1) = R_F h_F; \quad H^1\widetilde{\Gamma}_{\mathcal{A}}(\mathcal{O}_F, 1) = \mu_F$$

Now the class number formula (4) reads

$$\lim_{s \to 0} s^{-(r_1+r_2-1)}\zeta_F(s) = -\frac{\text{vol}\widetilde{H}^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)}{H^1\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)} \tag{6}$$

The right hand side is a volume of the determinant of a complex, see Chapter 2.5. I do not know the cohomological origin of the sign in (6).

3. Special values of the Dedekind ζ-functions, Borel regulators and polylogarithms. B. Birch and J. Tate [T] proposed a generalization of the class number formula for totally real fields using Milnor's K_2–group of $\mathcal{O}_F$:

$$\zeta_F(-1) = \pm\frac{|K_2(\mathcal{O}_F)|}{w_2(F)}$$

Here $w_2(F)$ is the largest integer m such that $\text{Gal}(\overline{F}/F)$ acts trivially on $\mu_\infty^{\otimes 2}$. Up to a power of 2, the above formula follows from the Iwasawa main conjecture for totally real fields, proved by B. Mazur and A. Wiles for $\mathbb{Q}$ [MW], and by A. Wiles [W] in general.

S. Lichtenbaum [Li1] suggested that for $\zeta_F(n)$ there should be a formula similar to (4) with a higher regulator defined using Quillen's K-groups $K_*(F)$ of F. Such a formula for $\zeta_F(n)$, considered up to a non zero rational factor, has been established soon after in the fundamental work of A. Borel [Bo1]-[Bo2]. Let us discuss it in more detail. The rational K–groups of a field can be defined as the primitive part in the homology of GL. Even better, one can show that

$$K_{2n-1}(F) \otimes \mathbb{Q} \stackrel{\sim}{=} \text{Prim}H_{2n-1}(GL_{2n-1}(F), \mathbb{Q})$$

Let $\mathbb{R}(n) := (2\pi i)^n\mathbb{R}$. There is a distinguished class, called the Borel class,

$$B_n \in H_c^{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{R}(n-1))$$

in the continuous cohomology of the Lie group $GL_{2n-1}(\mathbb{C})$. Pairing with this class provides the Borel regulator map

$$r_n^{\rm Bo} : K_{2n-1}(\mathbb{C}) \longrightarrow \mathbb{R}(n-1)$$

Let $X_F := \mathbb{Z}^{Hom(F,\mathbb{C})}$. The Borel regulator map on $K_{2n-1}(F)$ is the composition

$$K_{2n-1}(F) \longrightarrow \oplus_{Hom(F,\mathbb{C})}K_{2n-1}(\mathbb{C}) \longrightarrow X_F \otimes \mathbb{R}(n-1)$$

The image of this map is invariant under complex conjugation acting both on $Hom(F, \mathbb{C})$ and $\mathbb{R}(n-1)$. So we get the map

$$R_n^{\rm Bo} : K_{2n-1}(F) \longrightarrow (X_F \otimes \mathbb{R}(n-1))^+ \qquad (7)$$

Here $+$ means the invariants under the complex conjugation. Borel proved that for $n > 1$ the image of this map is a lattice, and the volume $R_n(F)$ of this lattice is related to the Dedekind ζ-function as follows:

$$R_n(F) \sim_{\mathbb{Q}^*} \lim_{s\to 1-n}(s-1+n)^{-d_n}\zeta_F(s),$$

Here $a \sim_{\mathbb{Q}^*} b$ means $a = \lambda b$ for some $\lambda \in \mathbb{Q}^*$, and

$$d_n = \dim\left(X_F \otimes \mathbb{R}(n-1)\right)^+ = \begin{cases} r_1 + r_2 & : n > 1 \text{ odd} \\ r_2 & : n \geq 2 \text{ even} \end{cases}$$

Using the functional equation for $\zeta_F(s)$ it tells us about $\sim_{\mathbb{Q}^*} \zeta_F(n)$. However Lichtenbaum's original conjecture was stronger since it was about $\zeta_F(n)$ itself.

In 1977 S. Bloch discovered [Bl4], [Bl5] that the regulator map on $K_3(\mathbb{C})$ can be explicitly defined using the dilogarithm. Here is how the story looks today. The dilogarithm is a multivalued analytic function on $\mathbb{CP}^1 - \{0, 1, \infty\}$:

$$\mathrm{Li}_2(z) := -\int_0^z \log(1-z)\frac{dz}{z}; \qquad \mathrm{Li}_2(z) = \sum_{k=1}^{\infty}\frac{z^2}{k^2} \quad \text{for } |z| \leq 1$$

The dilogarithm has a single-valued version, called the Bloch-Wigner function:

$$\mathcal{L}_2(z) := \mathrm{Im}\left(\mathrm{Li}_2(z) + \log(1-z)\log|z|\right)$$

It vanishes on the real line. Denote by $r(z_1, ..., z_4)$ the cross-ratio of the four points $z_1, ..., z_4$ on the projective line. The Bloch-Wigner function satisfies Abel's five term relation: for any five points $z_1, ..., z_5$ on $\mathbb{CP}^1$ one has

$$\sum_{i=1}^{5}(-1)^i\mathcal{L}_2(r(z_1, ..., \widehat{z}_i, ..., z_5)) = 0 \qquad (8)$$

Let $\mathcal{H}_3$ be the hyperbolic three space. Its absolute is identified with the Riemann sphere $\mathbb{CP}^1$. Let $I(z_1, ..., z_4)$ be the ideal geodesic simplex with the vertices at the points $z_1, ..., z_4$ at the absolute. Lobachevsky proved that

$$\mathrm{vol}I(z_1, ..., z_4) = \mathcal{L}_2(r(z_1, ..., z_4))$$

(He got this in a different but equivalent form). Lobachevsky's formula makes Abel's equation obvious: the alternating sum of the geodesic simplices with the vertices at $z_1, ..., \widehat{z}_i, ..., z_5$ is empty.

Abel's equation can be interpreted as follows: for any $z \in \mathbb{CP}^1$ the function

$$\mathcal{L}_2(r(g_1z, ..., g_4z)), \quad g_i \in GL_2(\mathbb{C}) \qquad (9)$$

is a measurable 3-cocycle of the Lie group $GL_2(\mathbb{C})$. The cohomology class of this cocycle is non trivial. The simplest way to see it is this. The function

$$\mathrm{vol}I(g_1x, ..., g_4x), \quad g_i \in GL_2(\mathbb{C}), x \in \mathcal{H}^3$$

provides a smooth 3-cocycle of $GL_2(\mathbb{C})$. Its cohomology class is nontrivial: indeed, its infinitesimal version is provided by the volume form in $\mathcal{H}_3$. Since the cohomology class does not depend on x, we can take x at the absolute, proving the claim.

Let F be an arbitrary field. Denote by $\mathbb{Z}[F^*]$ the free abelian group generated by the set F^*. Let $R_2(F)$ be the subgroup of $\mathbb{Z}[F^*]$ generated by the elements

$$\sum_{i=1}^{5}(-1)^i\{r(z_1, ..., \widehat{z}_i, ..., z_5)\}$$

Let $B_2(F)$ be the quotient of $\mathbb{Z}[F^*]$ by the subgroup $R_2(F)$. Then one shows that the map

$$\mathbb{Z}[F^*] \longrightarrow \Lambda^2 F^*; \quad \{z\} \longmapsto (1-z) \wedge z$$

kills the subgroup $R_2(F)$, providing a complex (called the Bloch-Suslin complex)

$$\delta_2 : B_2(F) \longrightarrow \Lambda^2 F^* \tag{10}$$

By Matsumoto's theorem $\mathrm{Coker}\delta_2 = K_2(F)$. Let us define the Milnor ring $K_*^M(F)$ of F as the quotient of the tensor algebra of the abelian group F^* by the two sided ideal generated by the Steinberg elements $(1-x) \otimes x$ where $x \in F^* - 1$. The product map in the K-theory provides a map

$$\otimes^n K_1(F) = \otimes^n F^* \longrightarrow K_n(F)$$

It kills the Steinberg elements, and thus provides a map $K_n^M(F) \to K_n(F)$. Set

$$K_3^{\mathrm{ind}}(F) := \mathrm{Coker}(K_3^M(F) \longrightarrow K_3(F))$$

A.A. Suslin [Su] proved that there is an exact sequence

$$0 \longrightarrow \mathrm{Tor}(F^*, F^*)^\sim \longrightarrow K_3^{\mathrm{ind}}(F) \longrightarrow \mathrm{Ker}\delta_2 \longrightarrow 0 \tag{11}$$

where $\mathrm{Tor}(F^*, F^*)^\sim$ is a nontrivial extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathrm{Tor}(F^*, F^*)$.

Abel's relation provides a well defined homomorphism

$$\mathcal{L}_2 : B_2(\mathbb{C}) \longrightarrow \mathbb{R}; \quad \{z\}_2 \longmapsto \mathcal{L}_2(z)$$

Restricting it to the subgroup $\mathrm{Ker}\delta_2 \subset B_2(\mathbb{C})$, and using (11), we get a map $K_3^{\mathrm{ind}}(\mathbb{C}) \longrightarrow \mathbb{R}$. Using the interpretation of the cohomology class of the cocycle (9) as a volume of geodesic simplex, one can show that it is essentially the Borel regulator. Combining this with Borel's theorem we get an explicit formula for $\zeta_F(2)$ for an arbitrary number field F.

How to generalize this beautiful story? Recall the classical polylogarithms

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad |z| \leq 1; \qquad \mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(z)\frac{dz}{z}$$

D. Zagier [Z1] formulated a precise conjecture expressing $\zeta_F(n)$ via classical polylogarithms, see the survey [GaZ]. It was proved for $n = 3$ in [G1]-[G2], but it is not known for $n > 3$, although its easier part has been proved in [dJ3], [BD2] and, in a different way, in Chapter 4.4 below.

Most of the $\zeta_F(2)$ picture has been generalized to the case of $\zeta_F(3)$ in [G1]-[G2] and [G3]. Namely, there is a single valued version of the trilogarithm:

$$\mathcal{L}_3(z) := \mathrm{Re}\left(\mathrm{Li}_3(z) - \mathrm{Li}_2(z)\log|z| + \frac{1}{6}\mathrm{Li}_1(z)\log^2|z|\right)$$

It satisfies the following functional equation which generalizes (8). Let us define the generalized cross-ratio of 6 points $x_0, ..., x_5$ in P^2 as follows. We present P^2 as a projectivization of the three dimensional vector space V_3 and choose the vectors $l_i \in V_3$ projecting to the points x_i. Let us choose a volume form $\omega \in \det V_3^*$ and set $\Delta(a,b,c) :=< a\wedge b\wedge c, \omega >$. Set

$$r_3(x_0, ..., x_5) := \mathrm{Alt}_6\{\frac{\Delta(l_0,l_1,l_3)\Delta(l_1,l_2,l_4)\Delta(l_2,l_0,l_5)}{\Delta(l_0,l_1,l_4)\Delta(l_1,l_2,l_5)\Delta(l_2,l_0,l_3)}\} \in \mathbb{Z}[F^*]$$

Here Alt_6 denotes the alternation of $l_0, ..., l_5$. The function $\mathcal{L}_3$ extends by linearity to a homomorphism $\mathcal{L}_3 : \mathbb{Z}[\mathbb{C}^*] \longrightarrow \mathbb{R}$, and there is a generalization of Abel's equation to the case of the trilogarithm (see [G2] and the appendix to [G3]):

$$\sum_{i=1}^{7}(-1)^i\mathcal{L}_3(r_3(x_1, ..., \widehat{x}_i, ...x_7)) = 0 \tag{12}$$

We define the group $B_3(F)$ as the quotient of $\mathbb{Z}[F^*]$ by the subgroup generated by the functional equations (12) for the trilogarithm. Then there is complex

$$B_3(F) \xrightarrow{\delta_3} B_2(F)\otimes F^* \xrightarrow{\delta_2\wedge \mathrm{Id}} \Lambda^3F^*; \quad \delta_3 : \{x\}_3 \longmapsto \{x\}_2\otimes x \tag{13}$$

There is a map $K_5(F) \longrightarrow \mathrm{Ker}\delta_3$ such that in the case $F = \mathbb{C}$ the composition

$$K_5(\mathbb{C}) \longrightarrow \mathrm{Ker}\delta_3 \hookrightarrow B_3(\mathbb{C}) \xrightarrow{\mathcal{L}_3} \mathbb{R}$$

coincides with the Borel regulator (see [G2] and appendix in [G3]). This plus Borel's theorem leads to an explicit formula expressing $\zeta_F(3)$ via the trilogarithm conjectured by Zagier [Z1].

In Chapter 3 we define, following [G4], the Grassmannian n-logarithm function $\mathcal{L}_n^G$. It is a function on the configurations of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. One of its functional equations generalizes Abel's equation:

$$\sum_{i=1}^{2n+1}(-1)^i\mathcal{L}_n^G(h_1, ..., \widehat{h}_i, ..., h_{2n+1}) = 0$$

It means that for a given hyperplane h the function $\mathcal{L}_n^G(g_1h, ..., g_{2n}h)$, $g_i \in GL_n(\mathbb{C})$, is a measurable cocycle of $GL_n(\mathbb{C})$. Its cohomology class essentially coincides with the restriction of the Borel class B_n to $GL_n(\mathbb{C})$. To prove this we show that $\mathcal{L}_n^G$ is the boundary value of a certain function defined on configurations of $2n$ points in the symmetric space $SL_n(\mathbb{C})/SU(n)$. Using this we express the Borel regulator via the Grassmannian polylogarithm. We show that for $n = 2$ we recover the dilogarithm story: $\mathcal{L}_2^G$ coincides with the Bloch-Wigner function, $SL_2(\mathbb{C})/SU(2)$ is the hyperbolic space, and the extension of $\mathcal{L}_2^G$ is given by the volume of geodesic simplices. The proofs can be found in [G7].

4. Beilinson's conjectures and Arakelov motivic complexes. A conjectural generalization of the class number formula (3) to the case of elliptic curves was suggested in the seminal work of Birch and Swinnerton-Dyer. Several years later J.Tate formulated conjectures relating algebraic cycles to the poles of zeta functions of algebraic varieties.

Let X be a regular algebraic variety over a number field F. Generalizing the previous works of Bloch [Bl4] and P. Deligne [D], A.A. Beilinson [B1] suggested a fantastic picture unifying all the above conjectures. Beilinson defined the rational motivic cohomology of X via the algebraic K-theory of X by

$$H^i_{Mot}(X, \mathbb{Q}(n)) := gr^{\gamma}_n K_{2n-i}(X) \otimes \mathbb{Q}$$

Here γ is the Adams γ-filtration.

Let us assume that X is projective. For schemes which admit regular models over $\mathbb{Z}$, Beilinson [B1] defined a $\mathbb{Q}$-vector subspace

$$H^i_{Mot/\mathbb{Z}}(X, \mathbb{Q}(n)) \subset H^i_{Mot}(X, \mathbb{Q}(n)), \tag{14}$$

called *integral part in the motivic cohomology.* Using alterations, A. Scholl [Sch] extended this definition to arbitrary regular projective schemes over a number field F. For every regular, projective and flat over $\mathbb{Z}$ model X' of X the subspace (14) coincides with the image of the map $H^i_{Mot}(X', \mathbb{Q}(n)) \longrightarrow H^i_{Mot}(X, \mathbb{Q}(n))$.

For a regular complex projective variety X Beilinson constructed in [B1] the regulator map to the Deligne cohomology of X:

$$H^i_{Mot}(X, \mathbb{Q}(n)) \longrightarrow H^i_{\mathcal{D}}(X(\mathbb{C}), \mathbb{Z}(n)) \tag{15}$$

It can be projected to the Deligne cohomology with real coefficients $H^i_{\mathcal{D}}(X(\mathbb{C}), \mathbb{R}(n))$.

Now let X again be a regular projective scheme over a number field F, We will view it as a scheme over $\mathbb{Q}$ via the projection $X \to \mathrm{Spec}(F) \to \mathrm{Spec}(\mathbb{Q})$. We define the *real Deligne cohomology of* X as the following $\mathbb{R}$-vector space:

$$H^i_{\mathcal{D}}\Big((X \otimes_{\mathbb{Q}} \mathbb{R})(\mathbb{C}), \mathbb{R}(n)\Big)^{\overline{F}_\infty}$$

where $\overline{F}_\infty$ is an involution given by the composition of complex conjugation acting on $X(\mathbb{C})$ and on the coefficients. Then, restricting the map (15) to the integral part in the motivic cohomology (14) and projecting the image onto the real Deligne cohomology of X, we get a regulator map

$$r_{Be} : H^i_{Mot/\mathbb{Z}}(X, \mathbb{Q}(n)) \longrightarrow H^i_{\mathcal{D}}\Big((X \otimes_{\mathbb{Q}} \mathbb{R})(\mathbb{C}), \mathbb{R}(n)\Big)^{\overline{F}_\infty}$$

Beilinson formulated a conjecture expressing the special values $\sim_{\mathbb{Q}^*} L(h^{i-1}(X), n)$ of the L-function of Grothendieck's motive $h^{i-1}(X)$ in terms of this regulator map, see [B1], the survey [R] and the book [RSS] for the original version of the conjecture, and the survey by J. Nekovář [N] for a motivic reformulation. For $X = \mathrm{Spec}(F)$, where F is a number field, it boils down to Borel's theorem. A precise Tamagawa Number conjecture about the special values $L(h^i(X), n)$ was suggested by Bloch and Kato [BK].

Beilinson [B2] and Lichtenbaum [Li2] conjectured that the weight n integral motivic cohomology of a scheme X should appear as cohomology of some complexes of abelian groups $\mathbb{Z}^\bullet_X(n)$, called the weight n motivic complexes of X. One must have

$$H^i_{Mot}(X, \mathbb{Q}(n)) = H^i\mathbb{Z}^\bullet_X(n) \otimes \mathbb{Q}$$

Motivic complexes are objects of the derived category. Beilinson conjectured [B2] that there exists an abelian category $\mathcal{MS}_X$ of mixed motivic sheaves on X, and that one should have an isomorphism in the derived category

$$\mathbb{Z}^\bullet_X(n) = RHom_{\mathcal{MS}_X}(\mathbb{Q}(0)_X, \mathbb{Q}(n)_X) \tag{16}$$

Here $\mathbb{Q}(n)_X := p^*\mathbb{Q}(n)$, where $p : X \to \mathrm{Spec}(F)$ is the structure morphism, is a motivic sheaf on X obtained by pull back of the Tate motive $\mathbb{Q}(n)$ over the point $\mathrm{Spec}(F)$. This formula implies the Beilinson-Soulé vanishing conjecture:

$$H^i_{Mot}(X, \mathbb{Q}(n)) = 0 \quad \text{for } i < 0 \text{ and } i = 0, n > 0$$

Indeed, the negative Ext's between objects of an abelian category are zero, and we assume that the objects $\mathbb{Q}(n)_X$ are mutually non-isomorphic. Therefore it is quite natural to look for representatives of motivic complexes which are zero in the negative degrees, as well as in the degree zero for $n = 0$.

Motivic complexes are more fundamental, and in fact simpler objects then rational K–groups. Several constructions of motivic complexes are known.

i) Bloch [Bl1]-[Bl2] suggested a construction of the motivic complexes, called the Higher Chow complexes, using algebraic cycles. The weight n cycle complex appears in a very natural way as a "resolution" for the codimension n Chow groups on X modulo rational equivalence, see s. 2.1 below. These complexes as well as their versions defined by Suslin and Voevodsky played an essential role in the construction of triangulated categories of mixed motives [V], [Lev], [Ha]. However they are unbounded from the left.

ii) Here is a totally different construction of the first few of motivic complexes. One has $\mathbb{Z}^\bullet_X(0) := \mathbb{Z}$. Let $X^{(k)}$ be the set of all irreducible codimension k subschemes of a scheme X. Then $\mathbb{Z}^\bullet_X(1)$ is the complex

$$\mathcal{O}^*_X \xrightarrow{\partial} \oplus_{Y \in X^{(1)}} \mathbb{Z}; \qquad \partial(f) := \mathrm{div}(f)$$

The complex $\mathbb{Q}^\bullet_X(2)$ is defined as follows. First, using the Bloch-Suslin complex (10), we define the following complex

$$B_2(\mathbb{Q}(X)) \xrightarrow{\delta_2} \Lambda^2 \mathbb{Q}(X)^* \xrightarrow{\partial_1} \oplus_{Y \in X^{(1)}} \mathbb{Q}(Y)^* \xrightarrow{\partial_2} \oplus_{Y \in X^{(2)}} \mathbb{Z}$$

Then tensoring it by $\mathbb{Q}$ we get $\mathbb{Q}^\bullet_X(2)$. Here ∂_2 is the tame symbol, and ∂_1 is given by the divisor of a function on Y. Similarly one can define a complex $\mathbb{Q}^\bullet_X(3)$ using the complex (13). Unlike the cycle complexes, these complexes are concentrated exactly in the degrees where they might have nontrivial cohomology. It is amazing that motivic complexes have two so different and beautiful incarnations.

Generalizing this, we introduce in Chapter 4 the *polylogarithmic motivic complexes*, which are conjectured to be the motivic complexes for an arbitrary field ([G1]-[G2]). Then we give a motivic proof of the weak version of Zagier's conjecture for a number field F. In Chapter 5 we discuss how to define motivic complexes for an arbitrary regular variety X using the polylogarithmic motivic complexes of its points.

In Chapters 2 and 5 we discuss constructions of the regulator map on the level of complexes. Precisely, we want to define for a regular complex projective variety X a homomorphisms of complexes of abelian groups

$$\text{a weight } n \text{ motivic complex of } X \quad \longrightarrow \quad \text{a weight } n \text{ Deligne complex of } X(\mathbb{C}) \tag{17}$$

In Chapter 2 we present a construction of a regulator map on the Higher Chow complexes given in [G4], [G7]. In Chapter 5 we define, following [G6], a regulator map on polylogarithmic complexes. It is given explicitly in terms of the classical polylogarithms. Combining this with Beilinson's conjectures we arrive at explicit conjectures expressing the special values of L-functions via classical polylogarithms. If $X = \mathrm{Spec}(F)$, where F is a number field, this boils down to Zagier's conjecture.

The cone of the map (17), shifted by -1, defines the *weight n Arakelov motivic complex*, and so its cohomology are the *weight n Arakelov motivic cohomology* of X.

The weight n Arakelov motivic complex should be considered as an ingredient of a definition of the *weight n arithmetic motivic complex*. Namely, one should exist a complex computing the weight n integral motivic cohomology of X, and a natural map from this complex to the weight n Deligne complex. The cone of this map, shifted by -1, would give the weight n arithmetic motivic complex. The weight one arithmetic motivic complex is the complex $\Gamma_{\mathcal{A}}(\mathcal{O}_F, 1)$.

In Chapter 6 we discuss a yet another approach to motivic complexes of fields: as standard cochain complexes of the motivic Lie algebras. We also discuss a relationship between the motivic Lie algebra of a field and the (motivic) Grassmannian polylogarithms.

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2 Arakelov motivic complexes

In this chapter we define a regulator map from the weight n motivic complex, understood as Bloch's Higher Chow groups complex [Bl1], to the weight n Deligne complex. This map was defined in [G4], and elaborated in detail in [G7]. The construction can be immediately adopted to the Suslin-Voevodsky motivic complexes.

1. Bloch's cycle complex [Bl1]. A non degenerate simplex in $\mathbb{P}^m$ is an ordered collection of hyperplanes $L_0, ..., L_m$ with empty intersection. Let us choose in $\mathbb{P}^m$ a simplex L and a generic hyperplane H. Then L provides a simplex in the affine space $\mathbb{A}^m := \mathbb{P}^m - H$.

Let X be a regular scheme over a field. Let $I = (i_1, ..., i_k)$ and $L_I := L_{i_1} \cap ... \cap L_{i_k}$. Let $\mathcal{Z}_m(X; n)$ be the free abelian group generated by irreducible codimension n algebraic subvarieties in $X \times \mathbb{A}^m$ which intersect properly (i.e. the intersection has the right dimension) all faces $X \times L_I$. Intersection with the codimension 1 face $X \times L_i$ provides a group homomorphism $\partial_i : \mathcal{Z}_m(X; n) \longrightarrow \mathcal{Z}_{m-1}(X; n)$. Set $\partial := \sum_{i=0}^m (-1)^i \partial_i$. Then $\partial^2 = 0$, so $(\mathcal{Z}_\bullet(X; n), \partial)$ is a homological complex. Consider the cohomological complex $\mathcal{Z}^\bullet(X; n) := \mathcal{Z}_{2r-\bullet}(X; n)$. Its cohomology give the motivic cohomology of X:

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := H^i(\mathcal{Z}^\bullet(X; n))$$

According to the fundamental theorem of Bloch ([Bl1], [Bl2])

$$H^i(\mathcal{Z}^\bullet(X; n) \otimes \mathbb{Q}) = gr^\gamma_n K_{2n-i}(X) \otimes \mathbb{Q}$$

2. The Deligne cohomology and Deligne's complex. Let X be a regular projective variety over $\mathbb{C}$. The Beilinson-Deligne complex $\mathbb{R}^\bullet(X; n)_\mathcal{D}$ is the following complex of sheaves in the classical topology on $X(\mathbb{C})$:

$$\underline{\mathbb{R}}(n) \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \Omega^2_X \longrightarrow ... \longrightarrow \Omega^{n-1}_X$$

Here the constant sheaf $\underline{\mathbb{R}}(n) := (2\pi i)^n \underline{\mathbb{R}}$ is in the degree zero. The hypercohomology of this complex of sheaves is called the *weight n Deligne cohomology* of $X(\mathbb{C})$. They are finite dimensional real vector spaces. Beilinson proved [B3] that the truncated weight n Deligne cohomology, which are obtained by putting the weight n Deligne cohomology equal to zero in the degrees $> 2n$, can be interpreted as the absolute Hodge cohomology of $X(\mathbb{C})$.

One can replace the above complex of sheaves by a quasiisomorphic one, defined as the total complex associated with the following bicomplex:

$$\begin{array}{cccccccccccc}
\Big(\underline{\mathcal{D}}^0_X & \xrightarrow{d} & \underline{\mathcal{D}}^1_X & \xrightarrow{d} & \dots & \xrightarrow{d} & \underline{\mathcal{D}}^n_X & \xrightarrow{d} & \underline{\mathcal{D}}^{n+1}_X & \xrightarrow{d} & \dots \Big) & \otimes \mathbb{R}(n-1) \\
 & & & & & & \uparrow \pi_n & & \uparrow \pi_n & & & \\
 & & & & & & \Omega^n_X & \xrightarrow{\partial} & \Omega^{n+1}_X & \xrightarrow{\partial} & &
\end{array}$$

Here $\underline{\mathcal{D}}^k_X$ is the sheaf of real k-distributions on $X(\mathbb{C})$, that is k-forms with the generalized function coefficients. Further,

$$\pi_n : \underline{\mathcal{D}}^p_X \otimes \mathbb{C} \longrightarrow \underline{\mathcal{D}}^p_X \otimes \mathbb{R}(n-1)$$

is the projection induced by the one $\mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n) \longrightarrow \mathbb{R}(n-1)$, the sheaf $\underline{\mathcal{D}}^0_X$ is placed in degree 1, and $(\Omega^\bullet_X, \partial)$ is the De Rham complex of sheaves of holomorphic forms.

To calculate the hypercohomology with coefficients in this complex we replace the holomorphic de Rham complex by its Doulbeut resolution, take the global sections of the obtained complex, and calculate its cohomology. Taking the canonical truncation of this complex in the degrees $[0, 2n]$ we get a complex calculating the absolute Hodge cohomology of $X(\mathbb{C})$. Let us define, following Deligne, yet another complex of abelian groups quasiisomorphic to the latter complex.

Let $\mathcal{D}_X^{p,q} = \mathcal{D}^{p,q}$ be the abelian group of complex valued distributions of type (p,q) on $X(\mathbb{C})$. Consider the following cohomological bicomplex, where $\mathcal{D}_{cl}^{n,n}$ is the subspace of closed distributions, and $\mathcal{D}^{0,0}$ is in degree 1:

$$
\begin{array}{ccccccccc}
 & & & & & & & & \mathcal{D}_{cl}^{n,n} \\
 & & & & & & & 2\overline{\partial}\partial \nearrow & \\
\mathcal{D}^{0,n-1} & \xrightarrow{\partial} & \mathcal{D}^{1,n-1} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \mathcal{D}^{n-1,n-1} & & \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & & & \overline{\partial}\uparrow & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & & & \overline{\partial}\uparrow & & \\
\mathcal{D}^{0,1} & \xrightarrow{\partial} & \mathcal{D}^{1,1} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \mathcal{D}^{n-1,1} & & \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & & & \overline{\partial}\uparrow & & \\
\mathcal{D}^{0,0} & \xrightarrow{\partial} & \mathcal{D}^{1,0} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \mathcal{D}^{n-1,0} & &
\end{array}
$$

The complex $C_{\mathcal{D}}^{\bullet}(X;n)$ is a subcomplex of the total complex of this bicomplex provided by the $\mathbb{R}(n-1)$-valued distributions in the $n \times n$ square of the diagram and the subspace $\mathcal{D}_{\mathbb{R},cl}^{n,n}(n) \subset \mathcal{D}_{cl}^{n,n}$ of the $\mathbb{R}(n)$-valued distributions of type (n,n). Notice that $\overline{\partial}\partial$ sends $\mathbb{R}(n-1)$-valued distributions to $\mathbb{R}(n)$-valued distributions. The cohomology of this complex of abelian groups is the absolute Hodge cohomology of $X(\mathbb{C})$, see Proposition 2.1 of [G7]. Now if X is a variety over $\mathbb{R}$, then

$$C_{\mathcal{D}}^{\bullet}(X_{/\mathbb{R}};n) := C_{\mathcal{D}}^{\bullet}(X;n)^{\overline{F}_\infty}; \quad H_{\mathcal{D}}^{i}(X_{/\mathbb{R}}, \mathbb{R}(n)) = H^i C_{\mathcal{D}}^{\bullet}(X_{/\mathbb{R}};n)$$

where $\overline{F}_\infty$ is the composition of the involution F_∞ on $X(\mathbb{C})$ induced by the complex conjugation with the complex conjugation of coefficients.

3. The regulator map.

Theorem - Construction 2.1 *Let X be a regular complex projective variety. Then there exists* canonical *homomorphism of complexes*

$$\mathcal{P}^{\bullet}(n) : \mathcal{Z}^{\bullet}(X;n) \longrightarrow C_{\mathcal{D}}^{\bullet}(X;n)$$

If X is defined over $\mathbb{R}$ then its image lies in the subcomplex $C_{\mathcal{D}}^{\bullet}(X_{/\mathbb{R}};n)$.

To define this homomorphism we need the following construction. Let X be a variety over $\mathbb{C}$ and $f_1, ..., f_m$ be m rational functions on X. The form

$$\pi_m\Big(d\log f_1 \wedge ... \wedge d\log f_m\Big),$$

where $\pi_n(a+ib) = a$ if n odd, and $\pi_n(a+ib) = ib$ if n even, has zero periods. It has a canonical primitive defined as follows. Consider the following $(m-1)$-form on $X(\mathbb{C})$:

$$r_{m-1}(f_1 \wedge ... \wedge f_m) := \qquad (18)$$

$$\mathrm{Alt}_m \sum_{j \geq 0} c_{j,m} \log|f_1| d\log|f_2| \wedge ... \wedge d\log|f_{2j+1}| \wedge di\arg f_{2j+2} \wedge ... \wedge di\arg f_m$$

Here $c_{j,m} := ((2j+1)!(m-2j-1)!)^{-1}$ and Alt_m is the operation of alternation:

$$\mathrm{Alt}_m F(x_1, ..., x_m) := \sum_{\sigma \in S_m} (-1)^{|\sigma|} F(x_{\sigma(1)}, ..., x_{\sigma(m)})$$

So $r_{m-1}(f_1 \wedge ... \wedge f_m)$ is an $\mathbb{R}(m-1)$-valued $(m-1)$-form. One has

$$dr_{m-1}(f_1 \wedge ... \wedge f_m) = \pi_m \Big(d\log f_1 \wedge ... \wedge d\log f_m \Big)$$

It is sometimes convenient to write the form (18) as a multiple of

$$\mathrm{Alt}_m \sum_{i=1}^{m} (-1)^i \log |f_1| d\log f_2 \wedge ... \wedge d\log f_i \wedge d\log \overline{f}_{i+1} \wedge ... \wedge d\log \overline{f}_m$$

Precisely, let $\mathcal{A}^i(M)$ be the space of smooth i-forms on a real smooth manifold M. Consider the following map

$$\omega_{m-1} : \Lambda^m \mathcal{A}^0(M) \longrightarrow \mathcal{A}^{m-1}(M) \tag{19}$$

$$\omega_{m-1}(\varphi_1 \wedge ... \wedge \varphi_m) :=$$

$$\frac{1}{m!} \mathrm{Alt}_m \Big(\sum_{k=1}^{m} (-1)^{k-1} \varphi_1 \partial\varphi_2 \wedge ... \partial\varphi_k \wedge \overline{\partial}\varphi_{k+1} \wedge ... \wedge \overline{\partial}\varphi_m \Big)$$

For example

$$\omega_0(\varphi_1) = \varphi_1; \qquad \omega_1(\varphi_1 \wedge \varphi_2) = \frac{1}{2} \Big(\varphi_1 \partial\varphi_2 - \varphi_2 \partial\varphi_1 - \varphi_1 \overline{\partial}\varphi_2 + \varphi_2 \overline{\partial}\varphi_1 \Big)$$

Then one easily checks that

$$d\omega_{m-1}(\varphi_1 \wedge ... \wedge \varphi_m) = \partial\varphi_1 \wedge ... \wedge \partial\varphi_m + (-1)^m \overline{\partial}\varphi_1 \wedge ... \wedge \overline{\partial}\varphi_m + \tag{20}$$

$$\sum_{i=1}^{m} (-1)^i \overline{\partial}\partial\varphi_i \wedge \omega_{m-2}(\varphi_1 \wedge ... \wedge \widehat{\varphi}_i \wedge ... \wedge \varphi_m)$$

Now let f_i be rational functions on a complex algebraic variety X. Set $M := X^0(\mathbb{C})$, where X^0 is the open part of X where the functions f_i are regular. Then $\varphi_i := \log |f_i|$ are smooth functions on M, and we have

$$\omega_{m-1}(\log |f_1| \wedge ... \wedge \log |f_m|) = r_{m-1}(f_1 \wedge ... \wedge f_m)$$

Denote by $\mathcal{D}^*_{X,\mathbb{R}}(k) = \mathcal{D}^*_{\mathbb{R}}(k)$ the subspace of $\mathbb{R}(k)$–valued distributions in $\mathcal{D}^*_{X(\mathbb{C})}$.

Let Y^0 be the nonsingular part of Y, and $i^0_Y : Y^0(\mathbb{C}) \hookrightarrow Y(\mathbb{C})$ the canonical embedding.

Proposition 2.2 *Let Y be an arbitrary irreducible subvariety of a smooth complex variety X and $f_1, ..., f_m \in \mathcal{O}^*(Y)$. Then for any smooth differential form ω with compact support on $X(\mathbb{C})$ the following integral is convergent:*

$$\int_{Y^0(\mathbb{C})} r_{m-1}(f_1 \wedge ... \wedge f_m) \wedge i^0_Y \omega$$

Thus there is a distribution $r_{m-1}(f_1 \wedge ... \wedge f_m)\delta_Y$ on $X(\mathbb{C})$:

$$< r_{m-1}(f_1 \wedge ... \wedge f_m)\delta_Y, \omega > := \int_{Y^0(\mathbb{C})} r_{m-1}(f_1 \wedge ... \wedge f_m) \wedge i^0_Y \omega$$

It provides a group homomorphism

$$r_{m-1} : \Lambda^m \mathbb{C}(Y)^* \longrightarrow \mathcal{D}^{m-1}_{X,\mathbb{R}}(m-1) \tag{21}$$

Construction. We have to construct a morphism of complexes

$$\begin{array}{ccccccccc} ... & \longrightarrow & \mathcal{Z}^1(X;n) & \longrightarrow & ... & \longrightarrow & \mathcal{Z}^{2n-1}(X;n) & \longrightarrow & \mathcal{Z}^{2n}(X;n) \\ & & \downarrow \mathcal{P}^1(n) & & ... & & \downarrow \mathcal{P}^{2n-1}(n) & & \downarrow \mathcal{P}^{2n}(n) \\ 0 & \longrightarrow & \mathcal{D}^{0,0}_{\mathbb{R}}(n-1) & \longrightarrow & ... & \longrightarrow & \mathcal{D}^{n-1,n-1}_{\mathbb{R}}(n-1) & \xrightarrow{2\overline{\partial}\partial} & \mathcal{D}^{n,n}_{\mathbb{R}}(n) \end{array}$$

Here at the bottom stays the complex $\mathcal{C}^{\bullet}_{\mathcal{D}}(X;n)$.

Let $Y \in \mathcal{Z}^{2n}(X;n)$ be a codimension n cycle in X. By definition

$$\mathcal{P}^{2n}(n)(Y) := (2\pi i)^n \delta_Y$$

Let us construct homomorphisms

$$\mathcal{P}^{2n-k}(n) : \mathcal{Z}^{2n-k}(X;n) \longrightarrow \mathcal{D}^{2n-k-1}_X, \quad k > 0$$

Denote by $\pi_{\mathbb{A}^k}$ (resp. π_X) the projection of $X \times \mathbb{A}^k$ to $\mathbb{A}^k$ (resp. X), and by $\overline{\pi}_{\mathbb{A}^k}$ (resp. $\overline{\pi}_X$) the projection of $X(\mathbb{C}) \times \mathbb{CP}^k$ to $\mathbb{CP}^k$ (resp. $X(\mathbb{C})$).

The pair (L, H) in $\mathbb{P}^k$ defines uniquely homogeneous coordinates $(z_0 : ... : z_k)$ in $\mathbb{P}^k$ such that the hyperplane L_i is given by equation $\{z_i = 0\}$ and the hyperplane H is $\{\sum_{i=1}^k z_i = z_0\}$. Then there is an element

$$\frac{z_1}{z_0} \wedge ... \wedge \frac{z_k}{z_0} \in \Lambda^{k-1}\mathbb{C}(\mathbb{A}^k)^* \tag{22}$$

Let $Y \in \mathcal{Z}^{2n-k}(X;n)$. Restricting to Y the inverse image of the element (22) by $\pi^*_{\mathbb{A}^k}$ we get an element

$$g_1 \wedge ... \wedge g_k \in \Lambda^k\mathbb{C}(Y)^* \tag{23}$$

Remark. This works if and only if the cycle Y intersects properly all codimension one faces of $X \times L$. Indeed, if Y does not intersect properly one of the faces, then the equation of this face restricts to zero to Y, and so (23) does not make sense.

The element (23) provides, by Proposition 2.2, a distribution on $X(\mathbb{C}) \times \mathbb{CP}^k$. Pushing it down by $(2\pi i)^{n-k} \cdot \overline{\pi}_X$ we get the distribution $\mathcal{P}^{2n-k}(n)(Y)$:

Definition 2.3 $\mathcal{P}^{2n-k}(n)(Y) := (2\pi i)^{n-k} \cdot \overline{\pi}_{X*} r_{k-1}(g_1 \wedge ... \wedge g_k)$.

In other words, the following distribution makes sense:

$$\mathcal{P}^{2n-k}(n)(Y) = \quad (2\pi i)^{n-k}\overline{\pi}_{X*}\left(\delta_Y \wedge \overline{\pi}^*_{\mathbb{A}^k} r_{k-1}\left(\frac{z_1}{z_0} \wedge ... \wedge \frac{z_k}{z_0}\right)\right)$$

One can rewrite definition 2.3 more explicitly as an integral over $Y(\mathbb{C})$. Namely, let ω be a smooth form on $X(\mathbb{C})$ and $i_Y : Y \hookrightarrow X \times \mathbb{P}^k$. Then

$$< \mathcal{P}^{2n-k}(n)(Y), \omega >= (2\pi i)^{n-k} \int_{Y^0(\mathbb{C})} r_{k-1}(g_1 \wedge ... \wedge g_k) \wedge i^*_{Y^0(\mathbb{C})}\overline{\pi}^*_X\omega$$

It is easy to check that $\mathcal{P}^{2n-k}(n)(Y)$ lies in $\mathcal{C}^{2n-k}_{\mathcal{D}}(X;n)$. Therefore we defined the maps $\mathcal{P}^k(n)$. It was proved in chapter [G7] that $\mathcal{P}^{\bullet}(n)$ is a homomorphism of complexes.

4. The Higher Arakelov Chow groups. Let X be a regular complex variety. Denote by $\widetilde{C}^{\bullet}_{\mathcal{D}}(X;n)$ the quotient of the complex $C^{\bullet}_{\mathcal{D}}(X;n)$ along the subgroup $\mathcal{A}^{n,n}_{cl,\mathbb{R}}(n) \subset \mathcal{D}^{n,n}_{cl,\mathbb{R}}(n)$ of closed smooth forms. The cone of the homomorphism $\mathcal{P}^{\bullet}(n)$ shifted by -1 is the Arakelov motivic complex: :

$$\widehat{\mathcal{Z}}^{\bullet}(X;n) := \mathrm{Cone}\Big(\mathcal{Z}^{\bullet}(X;n) \longrightarrow \widetilde{C}^{\bullet}_{\mathcal{D}}(X;n)\Big)[-1]$$

The Higher Arakelov Chow groups are its cohomology:

$$\widehat{CH}^n(X;i) := H^{2n-i}(\widehat{\mathcal{Z}}^\bullet(X;n))$$

Recall the arithmetic Chow groups defined by Gillet-Soulé [GS] as follows:

$$\widehat{CH}^n(X) :=$$

$$\frac{\{(Z,g); \frac{\overline{\partial}\partial}{\pi i}g + \delta_Z \in \mathcal{A}^{n,n}\}}{\{(0, \partial u + \overline{\partial} v); (\mathrm{div} f, -\log|f|), f \in \mathbb{C}(Y), \mathrm{codim}(Y) = n-1\}} \tag{24}$$

Here Z is a divisor in X, f is a rational function on a divisor Y in X,

$$g \in \mathcal{D}_{\mathbb{R}}^{n-1,n-1}(n-1), \quad (u,v) \in C_{\mathcal{D}}^{2n-2}(X;n) = (\mathcal{D}^{n-2,n-1} \oplus \mathcal{D}^{n-1,n-2})_{\mathbb{R}}(n-1)$$

Proposition 2.4 $\widehat{CH}^n(X;0) = \widehat{CH}^n(X)$.

Proof. Let us look at the very right part of the complex $\widehat{\mathcal{Z}}^\bullet(X;n)$:

$$\begin{array}{ccccc}
... & \longrightarrow & \mathcal{Z}^{2n-1}(X;n) & \longrightarrow & \mathcal{Z}^{2n}(X;n) \\
 & & \downarrow \mathcal{P}^{2n-1}(n) & & \downarrow \mathcal{P}^{2n}(n) \\
(\mathcal{D}^{n-2,n-1} \oplus \mathcal{D}^{n-1,n-2})_{\mathbb{R}}(n-1) & \stackrel{(\partial,\overline{\partial})}{\longrightarrow} & \mathcal{D}_{\mathbb{R}}^{n-1,n-1}(n-1) & \stackrel{2\overline{\partial}\partial}{\longrightarrow} & \mathcal{D}_{\mathbb{R}}^{n,n}(n)/\mathcal{A}_{\mathbb{R}}^{n,n}(n)
\end{array}$$

Consider the stupid truncation of the Gersten complex on X:

$$\prod_{Y \in X_{n-1}} \mathbb{C}(Y)^* \longrightarrow \mathcal{Z}_0(X;n) \tag{25}$$

It maps to the stupid truncation $\sigma_{\geq 2n-1}\widehat{\mathcal{Z}}^\bullet(X;n)$ of the cycle complex as follows. The isomorphism $\mathcal{Z}_0(X;n) = \mathcal{Z}^{2n}(X;n)$ provides the right component of the map. A pair $(Y;f)$ where Y is an irreducible codimension $n-1$ subvariety of X maps to the cycle $(y, f(y)) \subset X \times (\mathbb{P}^1 - \{1\})$. It is well known that such cycles $(y, f(y))$ plus $\partial\widehat{\mathcal{Z}}^{2n-2}(X;n)$ generate $\widehat{\mathcal{Z}}^{2n-1}(X;n)$. Computing the composition of this map with the homomorphism $\mathcal{P}^\bullet(n)$ we end up precisely with formula (24). The proposition is proved.

5. Special values of Dedekind ζ-functions and Arakelov motivic complexes. Let $\widetilde{\Gamma}_{\mathcal{A}}(\mathcal{O}_F, 1)$ be the three term complex (5). It consists of locally compact abelian groups. Each of them is equipped with a natural Haar measure. Indeed, the measure of a discrete group is normalized so that the measure of the identity element is 1; the group $\mathbb{R}$ has the canonical measure dx; and we use the product measure for the products. We need the following general observation.

Lemma - Definition 2.5 *Let*

$$A^\bullet = \quad ... \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$$

be a complex of locally compact abelian groups such that

i) Each of the groups A_i is equipped with an invariant Haar measure μ_i.

ii) The cohomology groups are compact.

iii) Only finite number of the cohomology groups are nontrivial, and almost all groups A_i are discrete groups with canonical measures.

Then there is a naturally defined number $\mathrm{R}_\mu A^\bullet$, *and* $\mathrm{R}_\mu A^\bullet[1] = (\mathrm{R}_\mu A^\bullet)^{-1}$.

Construction. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{26}$$

be an exact sequence of locally compact abelian groups. Then a choice of Haar measure for any two of the groups A, B, C determines naturally the third one. For example Haar measures μ_A, μ_C on A and C determine the following Haar measure $\mu_{A,C}$ on B. Take a compact subset of C and its section $K_C \subset B$, and take a compact subset K_A. Then $\mu_{A,C}(K_A \cdot K_C) := \mu_A(K_A)\mu_C(K_C)$.

For the complex (26) placed in degrees $[0,2]$ we put $\mathrm{R}_\mu(26) = \mu_{A,C}/\mu_B$.

Let us treat first the case when $A^\bullet$ is a finite complex. Then we define the invariant $\mathrm{R}_\mu A^\bullet$ by induction. If $A^\bullet = A[-i]$ is concentrated in just one degree, i, that A is compact and equipped with Haar measure μ. We put $\mathrm{R}_\mu A^\bullet := \mu(A)^{(-1)^i}$. Assume $A^\bullet$ starts from A_0. One has

$$\mathrm{Ker} f_0 \longrightarrow A_0 \longrightarrow \mathrm{Im} f_0 \longrightarrow 0; \qquad \mathrm{Im} f_0 \longrightarrow A_1 \longrightarrow A_1/\mathrm{Im} f_0 \longrightarrow 0$$

Since $\mathrm{Ker} f_0$ is compact, we can choose the volume one Haar measure on it. This measure and the measure μ_0 on A_0 provides, via the first short exact sequence, a measure on $\mathrm{Im} f_0$. Similarly using this and the second exact sequence we get a measure on $A_1/\mathrm{Im} f_0$. Therefore we have the measures on the truncated complex $\tau_{\geq 1} A^\bullet$. Now we define

$$\mathrm{R}_\mu A^\bullet := \mu_0(A_0) \cdot \mathrm{R}_\mu \tau_{\geq 1} A^\bullet$$

If $A^\bullet$ is an infinite complex we set $\mathrm{R}_\mu A^\bullet := \mathrm{R}_\mu(\tau_{-N,N} A^\bullet)$ for sufficiently big N. Here $\tau_{-N,N}$ is the canonical truncation functor. Thanks to iii) this does not depend on the choice of N. The lemma is proved.

Now the class number formula (4) reads

$$\lim_{s \to 0} s^{-(r_1+r_2-1)} \zeta_F(s) = -\mathrm{R}_\mu \widetilde{\Gamma}_{\mathcal{A}}(\mathcal{O}_F, 1) \tag{27}$$

Lichtenbaum's conjectures [Li1] on the special values of the Dedekind ζ-functions can be reformulated in a similar way:

$$\zeta_F(1-n) \stackrel{?}{=} \pm \mathrm{R}_\mu \Gamma_{\mathcal{A}}(\mathcal{O}_F, n); \quad n > 1$$

It is not quite clear what is the most natural normalization of the regulator map. In the classical $n = 1$ case this formula needs modification, as was explained in s. 1.1, to take into account the pole of the ζ–function.

Example. The group $H^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2)$ sits in the exact sequence

$$0 \longrightarrow R_2(F) \longrightarrow H^2\Gamma(\mathcal{O}_F, 2) \longrightarrow K_2(\mathcal{O}_F) \longrightarrow 0$$

To calculate it let us define the Bloch-Suslin complex for $\mathrm{Spec}(\mathcal{O}_F)$:

$$B(\mathcal{O}_F, 2): \qquad B_2(F) \longrightarrow \Lambda^2 F^* \longrightarrow \prod_{\mathcal{P}} k_{\mathcal{P}}^* \tag{28}$$

Its Arakelov version is the total complex of the following bicomplex, where the vertical map is given by the dilogarithm: $\{x\}_2 \longmapsto (\mathcal{L}_2(\sigma_1(x), ..., \mathcal{L}_2(\sigma_{r_2}(x))$.

$$\begin{array}{ccccc} \mathbb{R}^{r_2} & & & & \\ \uparrow & & & & \\ B_2(F) & \longrightarrow & \Lambda^2 F^* & \longrightarrow & \prod_{\mathcal{P}} k_{\mathcal{P}}^* \end{array} \tag{29}$$

Then $H^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2) = H^2 B_{\mathcal{A}}(\mathcal{O}_F, 2)$. The second map in (28) is surjective ([Mi], cor. 16.2). Using this one can check that

$$H^i\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2) = 0 \quad \text{for } i \geq 3$$

(Notice that $H^1 B_{\mathcal{A}}(\mathcal{O}_F, 2) \neq H^1\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2)$). Summarizing, we should have

$$\zeta_F(-1) \stackrel{?}{=} \pm \mathrm{R}_\mu \Gamma_{\mathcal{A}}(\mathcal{O}_F, 2) = \pm \frac{\mathrm{vol} H^2\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2)}{|H^1\Gamma_{\mathcal{A}}(\mathcal{O}_F, 2)|}$$

For totally real fields it is a version of the Birch-Tate conjecture. It would be very interesting to compare this with the Bloch–Kato conjecture.

3 Grassmannian polylogarithms and Borel's regulator

1. The Grassmannian polylogarithm [G4]. Let $h_1, ..., h_{2n}$ be arbitrary $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. Choose an additional hyperplane h_0. Let f_i be a rational function on $\mathbb{CP}^{n-1}$ with divisor $h_i - h_0$. It is defined up to a scalar factor. Set

$$\mathcal{L}_n^G(h_1, ..., h_{2n}) := (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-2}(\sum_{j=1}^{2n} (-1)^j f_1 \wedge ... \wedge \widehat{f_j} \wedge ... \wedge f_{2n})$$

It is skew-symmetric by definition. It is easy to see that it does not depend on the choice of scalar in the definition of f_i. To check that it does not depend on the choice of h_0 observe that

$$\sum_{j=1}^{2n} (-1)^j f_1 \wedge ... \wedge \widehat{f_j} \wedge ... \wedge f_{2n} = \frac{f_1}{f_{2n}} \wedge \frac{f_2}{f_{2n}} \wedge ... \wedge \frac{f_{2n-1}}{f_{2n}}$$

So if we choose rational functions $g_1, ..., g_{2n-1}$ such that $\mathrm{div} g_i = h_i - h_{2n}$ then

$$\mathcal{L}_n^G(h_1, ..., h_{2n}) = (2\pi i)^{1-n} \int_{\mathbb{CP}^{n-1}} r_{2n-2}(g_1 \wedge ... \wedge g_{2n-1})$$

Remark. The function $\mathcal{L}_n^G$ is defined on the set of all configurations of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$. However it is not even continuous on this set. It is real analytic on the submanifold of generic configurations.

Theorem 3.1 *The function $\mathcal{L}_n^G$ satisfies the following functional equations:*
a) For any $2n+1$ hyperplanes $h_1, ..., h_{2n+1}$ in $\mathbb{CP}^n$ one has

$$\sum_{j=1}^{2n+1} (-1)^j \mathcal{L}_n^G(h_j \cap h_1, ..., h_j \cap h_{2n+1}) = 0 \tag{30}$$

b) For any $2n+1$ hyperplanes $h_1, ..., h_{2n+1}$ in $\mathbb{CP}^{n-1}$ one has

$$\sum_{j=1}^{2n+1} (-1)^j \mathcal{L}_n^G(h_1, ..., \widehat{h}_j, ..., h_{2n+1}) = 0 \tag{31}$$

Proof. a) Let $f_1, ..., f_{2n+1}$ be rational functions on $\mathbb{CP}^n$ as above. Then

$$dr_{2n-1}\Big(\sum_{j=1}^{2n+1} (-1)^j f_1 \wedge ... \wedge \widehat{f_j} \wedge ... f_{2n+1}\Big) = \tag{32}$$

$$\sum_{j \neq i} (-1)^{j+i-1} 2\pi i \delta(f_j) r_{2n-2}\Big(f_1 \wedge ... \widehat{f_i} \wedge ... \wedge \widehat{f_j} \wedge ... f_{2n+1}\Big)$$

(Notice that $d\log f_1 \wedge ... \wedge \widehat{d\log f_j} \wedge ... \wedge d\log f_{2n+1} = 0$ on $\mathbb{CP}^n$). Integrating (32) over $\mathbb{CP}^n$ we get a).

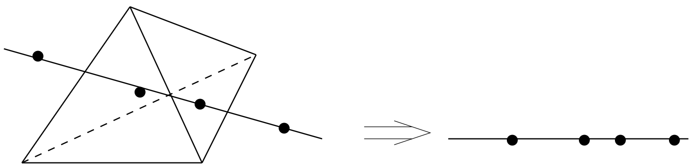
b) is obvious: we apply r_{2n-1} to zero element. Theorem is proved.

There is a natural bijection

$$PG_n^{n-1} := \ \{(n-1)\text{–planes in } \mathbb{P}^{2n-1} \text{ in generic position to a simplex } L\}/(\mathbb{G}_m^*)^{2n-1}$$

$$< - - > \quad \big\{\text{Configurations of} \quad 2n \quad \text{generic hyperplanes in } \mathbb{P}^{n-1}\big\}$$

given by intersecting of an $(n-1)$–plane h with the codimension one faces of L.



Thus $\mathcal{L}_n^G$ is a function on the torus quotient PG_n^{n-1} of the generic part of Grassmannian.

Gelfand and MacPherson [GM] suggested a beautiful construction of the real valued version of $2n$-logarithms on $PG_{2n}^{2n-1}(\mathbb{R})$. The construction uses the Pontryagin form. These functions generalize the Rogers dilogarithm.

The defined above functions $\mathcal{L}_n^G$ on complex Grassmannians generalize the Bloch-Wigner dilogarithm. They are related to the Chern classes. It would be very interesting to find a link between the construction in [GM] with our construction.

The existence of the multivalued analytic Grassmannian n-logarithms on complex Grassmannians was conjectured in [BMS]. They were constructed in [HM1]-[HM2] and, as a particular case of the analytic Chow polylogarithms, in [G4].

Recall the following general construction. Let X be a G-set and F a G-invariant function on X^n satisfying

$$\sum_{i=1}^{n}(-1)^i F(x_1, ..., \widehat{x}_i, ..., x_n) = 0$$

Choose a point $x \in X$. Then there is an $(n-1)$-cocycle of the group G:

$$f_x(g_1, ..., g_n) := F(g_1 x, ..., g_n x)$$

Lemma 3.2 *The cohomology class of the cocycle f_x does not depend on x.*

Thus thanks to (31) the function $\mathcal{L}_n^G$ provides a measurable cocycle of $GL_n(\mathbb{C})$. We want to determine its cohomology class, but a priori it is not even clear that it is non zero. To handle this problem we will show below that the function $\mathcal{L}_n^G$ is a boundary value of a certain function ψ_n defined on the configurations of $2n$ points inside of the symmetric space $SL_m(\mathbb{C})/SU(n)$. The cohomology class of $SL_n(\mathbb{C})$ provided by this function is obviously related to the so-called Borel class. Using this we will show that the Grassmannian n-logarithm function $\mathcal{L}_n^G$ provides the Borel class, and moreover can be used to define the Borel regulator.

Finally, the restriction of the function $\mathcal{L}_n^G$ to certain special stratum in the configuration space of $2n$ hyperplanes in $\mathbb{CP}^{n-1}$ provides a single valued version of the classical n–logarithm function, see sections 4-5 below.

2. The function ψ_n. Let V_n be an n-dimensional complex vector space. Let

$$\mathbb{H}_n := \{ \text{ positive definite Hermitian forms in } V_n \ \}/\mathbb{R}_+^* = SL_n(\mathbb{C})/SU(n)$$

$$= \{ \text{ positive definite Hermitian forms in } V_n \text{ with determinant } = 1\}$$

It is a symmetric space of rank $n-1$. For example $\mathbb{H}_2 = \mathcal{H}_3$ is the hyperbolic 3-space. Replacing positive definite by non negative definite Hermitian forms we get a compactification $\overline{\mathbb{H}}_n$ of the symmetric space $\mathbb{H}_n$.

Let G_x be the subgroup of $SL_N(\mathbb{C})$ stabilizing the point $x \in \mathbb{H}_n$. A point x defines a one dimensional vector space M_x:

$$x \in \mathbb{H}_n \longmapsto M_x := \{\text{measures on } \mathbb{CP}^{n-1} \text{ invariant under } G_x\}$$

Namely, a point x corresponds to a hermitian metric in V_n. This metric provides the Fubini-Studi Kahler form on $\mathbb{CP}^{n-1} = P(V_n)$. Its imaginary part is a symplectic form. Raising it to $(n-1)$-th power we get the Fubini-Studi volume form. The elements of M_x are its multiples.

Let $x_0, ..., x_{2n-1}$ be points of the symmetric space $SL_n(\mathbb{C})/SU(n)$. Consider the following function

$$\psi_n(x_0, ..., x_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log |\frac{\mu_{x_1}}{\mu_{x_0}}| d\log |\frac{\mu_{x_2}}{\mu_{x_0}}| \wedge ... \wedge d\log |\frac{\mu_{x_{2n-1}}}{\mu_{x_0}}|$$

More generally, let X be an m-dimensional manifold. For any $m+2$ measures $\mu_0, ..., \mu_{m+1}$ on X such that $\frac{\mu_i}{\mu_j}$ are smooth functions consider the following differential m-form on X:

$$\overline{r}_m(\mu_0 : ... : \mu_{m+1}) := \log |\frac{\mu_1}{\mu_0}| d\log |\frac{\mu_2}{\mu_0}| \wedge ... \wedge d\log |\frac{\mu_{m+1}}{\mu_0}|$$

Proposition 3.3 *The integral* $\int_X \overline{r}_m(\mu_0 : ... : \mu_{m+1})$ *satisfies the following properties:*

1) Skew symmetry with respect to the permutations of μ_i.

2) Homogeneity: if $\lambda_i \in \mathbb{R}^*$ *then*

$$\int_X \overline{r}_m(\lambda_0\mu_0 : ... : \lambda_{m+1}\mu_{m+1}) = \int_X \overline{r}_m(\mu_0 : ... : \mu_{m+1})$$

3)Additivity: for any $m+3$ *measures* μ_i *on* X *one has*

$$\sum_{i=0}^{m+2} (-1)^i \int_X \overline{r}_m(\mu_0 : ... : \widehat{\mu}_i : ... : \mu_{m+2}) = 0$$

4) Let g *be a diffeomorphism of* X. *Then*

$$\int_X \overline{r}_m(g^*\mu_0 : ... : g^*\mu_{m+1}) = \int_X \overline{r}_m(\mu_0 : ... : \mu_{m+1})$$

3. The Grassmannian polylogarithm $\mathcal{L}_n^G$ is the boundary value of the function ψ_n [G7]. Let $(z_0 : ... : z_{n-1})$ be homogeneous coordinates in $\mathbb{P}^{n-1}$. Let

$$\sigma_n(z, dz) := \sum_{i=0}^{n-1} (-1)^i z_i dz_0 \wedge ... \wedge \widehat{dz_i} \wedge ... \wedge dz_{n-1}$$

$$\omega_{FS}(H) := \frac{1}{(2\pi i)^{n-1}} \frac{\sigma_n(z, dz) \wedge \sigma_n(\overline{z}, d\overline{z})}{H(z, \overline{z})^n} \tag{33}$$

This form is clearly invariant under the group preserving the Hermitian form H. In fact it is the Fubini-Studi volume form.

Take any $2n$ non zero nonnegative definite Hermitian forms $H_0, ..., H_{2n-1}$, possibly degenerate. For each of the forms H_i choose a multiple μ_{H_i} of the Fubini-Studi form given by formula (33). It is a volume form with singularities along the projectivization of kernel of H_i.

Lemma 3.4 *The following integral is convergent*

$$\psi_n(H_0, ..., H_{2n-1}) := \int_{\mathbb{CP}^{n-1}} \log|\frac{\mu_{H_1}}{\mu_{H_0}}| d\log|\frac{\mu_{H_2}}{\mu_{H_0}}| \wedge ... \wedge d\log|\frac{\mu_{H_{2n-1}}}{\mu_{H_0}}|$$

This integral does not change if we multiply one of the Hermitian forms by a positive scalar. Therefore we can extend ψ_n to a function on the configuration space of $2n$ points in the compactification $\overline{\mathbb{H}}_{n-1}$. This function is discontinuous.

One can realize $\mathbb{CP}^{n-1}$ as the smallest stratum of the boundary of $\mathbb{H}_n$. Indeed, let $\mathbb{CP}^{n-1} = P(V_n)$. For a hyperplane $h \in V_n$ let

$$F_h := \{ \text{ nonnegative definite hermitian forms in } V_n \text{ with kernel } h\} / \mathbb{R}^*_+$$

The set of hermitian forms in V_n with the kernel h is isomorphic to $\mathbb{R}^*_+$, so F_h defines a point on the boundary of $\overline{\mathbb{H}}_n$. Therefore Lemma 3.4 provides a function

$$\psi_n(h_0, ..., h_{2n-1}) := \psi_n(F_{h_0}, ..., F_{h_{2n-1}}) \tag{34}$$

Applying Lemma 3.2 to the case when X is $\overline{\mathbb{H}}_n$ and using only the fact that the function $\psi_n(x_0, ..., x_{2n-1})$ is well defined for *any* $2n$ points in $\overline{\mathbb{H}}_n$ and satisfies the cocycle condition for any $2n+1$ of them we get

Proposition 3.5 *Let $x \in \mathbb{H}_n$ and h is a hyperplane in $\mathbb{CP}^{n-1}$. Then the cohomology classes of the following cocycles coincide:*

$$\psi_n(g_0x, ..., g_{2n-1}x) \quad \textit{and} \quad \psi_n(g_0h, ..., g_{2n-1}h)$$

The Fubini-Studi volume form corresponding to a hermitian form from the set F_h is a Lebesgue measure on the affine space $\mathbb{CP}^{n-1} - h$. Indeed, if $h_0 = \{z_0 = 0\}$ then (33) specializes to

$$\frac{1}{(2\pi i)^{n-1}} d\frac{z_1}{z_0} \wedge ... \wedge d\frac{z_{n-1}}{z_0} \wedge d\frac{\overline{z}_1}{\overline{z}_0} \wedge ... \wedge d\frac{\overline{z}_{n-1}}{\overline{z}_0}$$

Using this it is easy to prove the following proposition.

Proposition 3.6 *For any $2n$ hyperplanes $h_0, ..., h_{2n-1}$ in $\mathbb{CP}^{n-1}$ one has*

$$\psi_n(h_0, ..., h_{2n-1}) = (-4)^{-n} \cdot (2\pi i)^{n-1}(2n)^{2n-1}\binom{2n-2}{n-1} \cdot \mathcal{L}^G_n(h_0, ..., h_{2n-1})$$

4. A normalization of the Borel class b_n. Denote by $H^*_c(G, \mathbb{R})$ the continuous cohomology of a Lie group G. Let us define an isomorphism

$$\gamma_{\mathrm{DR}} : H^k_{\mathrm{DR}}(SL_n(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} H^k_c(SL_n(\mathbb{C}), \mathbb{C})$$

We do it in two steps. First, let us define an isomorphism

$$\alpha : H^k_{\mathrm{DR}}(SL_n(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} \mathcal{A}^k(SL_n(\mathbb{C})/SU(n))^{SL_n(\mathbb{C})} \otimes \mathbb{C}$$

It is well known that any cohomology class on the left is represented by a biinvariant, and hence closed, differential k–form Ω on $SL_n(\mathbb{C})$. Let us restrict it first to the Lie algebra, and then to the orthogonal complement $su(n)^\perp$ to the Lie subalgebra $su(n) \subset sl_n(\mathbb{C})$. Let e be the point of $\mathbb{H}_n$ corresponding to the subgroup $SU(n)$. We identify the $\mathbb{R}$–vector spaces $T_e\mathbb{H}_n$ and $su(n)^\perp$. The obtained exterior form on $T_e\mathbb{H}_n$ is restriction of an invariant closed differential form ω on the symmetric space $\mathbb{H}_n$.

Now let us construct, following J. Dupont [Du], an isomorphism

$$\beta : \mathcal{A}^k(SL_n(\mathbb{C})/SU(n))^{SL_n(\mathbb{C})} \xrightarrow{\sim} H_c^k(SL_n(\mathbb{C}), \mathbb{R})$$

For any ordered $m+1$ points $x_1, ..., x_{m+1}$ in $\mathbb{H}_n$ there is a geodesic simplex $I(x_1, ..., x_{m+1})$ in $\mathbb{H}_n$. It is constructed inductively as follows. Let $I(x_1, x_2)$ be the geodesic from x_1 to x_2. The geodesics from x_3 to the points of $I(x_1, x_2)$ form a geodesic triangle $I(x_1, x_2, x_3)$, and so on. If $n > 2$ the geodesic simplex $I(x_1, ..., x_k)$ depends on the order of vertices.

Let ω be an invariant differential m-form on $SL_n(\mathbb{C})/SU(n)$. Then it is closed, and provides a volume of the geodesic simplex:

$$\mathrm{vol}_\omega I(x_1, ..., x_{m+1}) := \int_{I(x_1,...,x_{m+1})} \omega$$

The boundary of the simplex $I(x_1, ..., x_{m+2})$ is the alternated sum of simplices $I(x_1, ..., \widehat{x}_i, ..., x_{m+2})$. Since the form ω is closed, the Stokes theorem yields

$$\sum_{i=1}^{m+2} (-1)^i \int_{I(x_1,...,\widehat{x}_i,...,x_{m+2})} \omega = \int_{I(x_1,...,x_{m+2})} d\omega = 0$$

This just means that for a given point x the function $\mathrm{vol}_\omega I(g_1 x, ..., g_{m+1} x)$ is a smooth m-cocycle of the Lie group $SL_n(\mathbb{C})$. By Lemma 3.2 cocycles corresponding to different points x are canonically cohomologous. The obtained cohomology class is the class $\beta(\omega)$. Set $\gamma_{\mathrm{DR}} := \beta \circ \alpha$.

It is known that

$$H^*_{\mathrm{DR}}(SL_n(\mathbb{C}), \mathbb{Q}) = \Lambda^*_{\mathbb{Q}}(C_3, ..., C_{2n-1})$$

where

$$C_{2n-1} := \mathrm{tr}(g^{-1}dg)^{2n-1} \in \Omega^{2n-1}(SL)$$

The Hodge considerations shows that $[C_{2n-1}] \in H^{2n-1}_{\mathrm{Betti}}(SL_n(\mathbb{C}), \mathbb{Q}(n))$.

Lemma 3.7 $\alpha(C_{2n-1})$ *is an* $\mathbb{R}(n-1)$*–valued differential form. So it provides a cohomology class*

$$b_n := \gamma_{\mathrm{DR}}(C_{2n-1}) \in H_c^{2n-1}(SL_n(\mathbb{C}), \mathbb{R}(n-1))$$

We call the cohomology class provided by this lemma the Borel class, and use it below to construct the Borel regulator.

It is not hard to show that the cohomology class of the cocycle $\psi_n(g_0 x, ..., g_{2n-1} x)$ is a non zero multiple of the Borel class. So thanks to propositions 3.5 and 3.6 the same is true for the cohomology class provided by the Grassmannian n–logarithm. The final result will be stated in Theorem 3.8 below.

5. Construction of the Borel regulator via Grassmannian polylogarithms. Let G be a group. The diagonal map $\Delta : G \longrightarrow G \times G$ provides a homomorphism $\Delta_* : H_n(G) \longrightarrow H_n(G \times G)$. Recall that

$$\mathrm{Prim} H_n G := \{x \in H_n(G) | \Delta_*(x) = x \otimes 1 + 1 \otimes x\}$$

Set $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. One has

$$K_n(F)_{\mathbb{Q}} = \mathrm{Prim} H_n GL(F)_{\mathbb{Q}} = \mathrm{Prim} H_n GL_n(F)_{\mathbb{Q}}$$

where the second isomorphism is provided by Suslin's stabilization theorem. Let

$$B_n \in H_c^{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{R}(n-1))$$

be a cohomology which goes to b_n under the restriction map to GL_n. We define the Borel regulator map by restricting the class B_n to the subspace $K_{2n-1}(\mathbb{C})_{\mathbb{Q}}$ of $H_{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{Q})$:

$$r_n^{\mathrm{Bo}}(b_n) :=< B_n, * >: K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \longrightarrow \mathbb{R}(n-1)$$

It does not depend on the choice of B_n.

Recall the Grassmannian complex $C_*(n)$

$$... \xrightarrow{d} C_{2n-1}(n) \xrightarrow{d} C_{2n-2}(n) \xrightarrow{d} ... \xrightarrow{d} C_0(n)$$

where $C_k(n)$ is the free abelian group generated by configurations, i.e. $GL(V)$-coinvariants, of $k+1$ vectors $(l_0, ..., l_k)$ in generic position in an n–dimensional vector space V over a field F, and d is given by the standard formula

$$(l_0, ..., l_k) \longmapsto \sum_{i=0}^{k} (-1)^i (l_0, ..., \widehat{l_i}, ..., l_k) \tag{35}$$

The group $C_k(n)$ is in degree k. Since it is a homological resolution of the trivial $GL_n(F)$–module $\mathbb{Z}$ (see Lemma 3.1 in [G2]), there is canonical homomorphism

$$\varphi^n_{2n-1} : H_{2n-1}(GL_n(F)) \longrightarrow H_{2n-1}(C_*(n))$$

Thanks to Lemma 3.7 the Grassmannian n–logarithm function provides a homomorphism

$$\mathcal{L}_n^G : C_{2n-1}(n) \longrightarrow \mathbb{R}(n-1); \quad (l_0, ..., l_{2n-1}) \longmapsto \mathcal{L}_n^G(l_0, ..., l_{2n-1}) \tag{36}$$

Thanks to the functional equation (30) for $\mathcal{L}_n^G$ it is zero on the subgroup $dC_{2n}(n)$. So it induces a homomorphism

$$\mathcal{L}_n^G : H_{2n-1}(C_*(n)) \longrightarrow \mathbb{R}(n-1);$$

Let us extend the map $\varphi^n_{2n-1} \circ \mathcal{L}_n^G$ to a homomorphism from $H_{2n-1}(GL_{2n-1}(\mathbb{C}))$ to $\mathbb{R}(n-1)$ following s. 3.10 of [G2], Consider the following bicomplex:

$$\begin{array}{ccccccccc}
 & & & & & & ... & \xrightarrow{d} & C_{2n-1}(2n-1) \\
 & & & & & & & & \downarrow \\
 & & & & ... & & ... & & ... \\
 & & & & \downarrow & & & & \downarrow \\
 & & ... & \xrightarrow{d} & C_{2n-1}(n+1) & \xrightarrow{d} & ... & \xrightarrow{d} & C_{n+1}(n) \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
... & \xrightarrow{d} & C_{2n-1}(n) & \xrightarrow{d} & C_{2n-2}(n) & \xrightarrow{d} & ... & \xrightarrow{d} & C_n(n)
\end{array}$$

The horizontal differentials are given by formula (35), and the vertical by

$$(l_0, ..., l_k) \longmapsto \sum_{i=0}^{k} (-1)^i (l_i | l_1, ..., \widehat{l_i}, ..., l_k)$$

Here $(l_i | l_1, ..., \widehat{l_i}, ..., l_k)$ means projection of the configuration $(l_1, ..., \widehat{l_i}, ..., l_k)$ to the quotient $V / < l_i >$. The total complex of this bicomplex is called the weight n bi–Grassmannian complex $BC_*(n)$.

Let us extend homomorphism (36) to a homomorphism

$$\mathcal{L}_n^G : BC_{2n-1}(n) \longrightarrow \mathbb{R}(n-1)$$

by setting it zero on the groups $C_{2n-1}(n+i)$ for $i > 0$. The functional equation (31) for the Grassmannian n–logarithm just means that the composition

$$C_{2n}(n+1) \longrightarrow C_{2n-1}(n) \xrightarrow{\mathcal{L}_n^G} \mathbb{R}(n-1),$$

where the first map is a vertical arrow in $BC_*(n)$, is zero. Therefore we get a homomorphism

$$\mathcal{L}_n^G : H_{2n-1}(BC_*(n)) \longrightarrow \mathbb{R}(n-1)$$

The bottom row of the Grassmannian bicomplex is the stupid truncation of the Grassmannian complex at the group $C_n(n)$. So there is a homomorphism

$$H_{2n-1}(C_*(n)) \longrightarrow H_{2n-1}(BC_*(n)) \tag{37}$$

In [G1]-[G2] we proved that there are homomorphisms

$$\varphi_{2n-1}^m : H_{2n-1}(GL_m(F)) \longrightarrow H_{2n-1}(BC_*(n)), \quad m \geq n$$

whose restriction to the subgroup $GL_n(F)$ coincides with the composition

$$H_{2n-1}(GL_n(F)) \stackrel{\varphi_{2n-1}^n}{\longrightarrow} H_{2n-1}(C_*(n)) \stackrel{(37)}{\longrightarrow} H_{2n-1}(BC_*(n)),$$

Theorem 3.8 *The composition*

$$K_{2n-1}(\mathbb{C}) \stackrel{\sim}{\longrightarrow} \mathrm{Prim}H_{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{Q}) \stackrel{\varphi_{2n-1}^{2n-1}}{\longrightarrow} H_{2n-1}(BC_*(n)_{\mathbb{Q}}) \stackrel{\mathcal{L}_n^G}{\longrightarrow} \mathbb{R}(n-1)$$

equals to

$$-(-1)^{n(n+1)/2} \cdot \frac{(n-1)!^2}{n(2n-2)!(2n-1)!} r_n^{\mathrm{Bo}}(b_n)$$

6. $\mathbb{P}^1 - \{0, \infty\}$ as a special stratum in the configuration space of $2n$ points in $\mathbb{P}^{n-1}$ [G4]. A *special configuration* is a configuration of $2n$ points

$$(l_0, ..., l_{n-1}, m_0, ..., m_{n-1}) \tag{38}$$

in $\mathbb{P}^{n-1}$ such that $l_0, ..., l_{n-1}$ are vertices of a simplex in $\mathbb{P}^{n-1}$ and m_i is a point on the edge $l_i l_{i+1}$ of the simplex different from l_i and l_{i+1}, as on Figure 1.

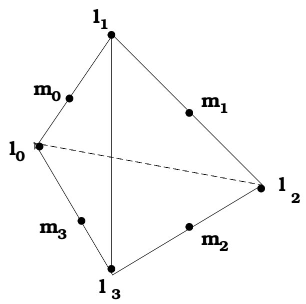


Figure 1: A special configuration of 8 points in $\mathbb{P}^3$.

Proposition 3.9 *The set of special configurations of* $2n$ *points in* $\mathbb{P}^{n-1}$ *is canonically identified with* $\mathbb{P}^1 \backslash \{0, \infty\}$.

Construction. Let $\widehat{m}_i$ be the point of intersection of the line $l_i l_{i+1}$ with the hyperplane passing through all the points m_j except m_i. Let $r(x_1, ..., x_4)$ be the cross-ratio of the four points on $\mathbb{P}^1$. Let us define the generalized cross-ratio by

$$r(l_0, ..., l_{n-1}, m_0, ..., m_{n-1}) := r(l_i, l_{i+1}, m_i, \widehat{m}_{i+1}) \in F^*$$

It does not depend on i, and provides the desired isomorphism. Here is a different definition, which makes obvious the fact that the generalized cross-ratio is cyclically invariant. Consider the one dimensional subspaces L_i, M_j in the $(n+1)$–dimensional vector space projecting to l_i, m_j. Then L_i, M_i, L_{i+1} belong to a two dimensional subspace. The subspace M_i provides a linear map $L_i \longrightarrow L_{i+1}$. The composition of these maps is a linear map $L_0 \longrightarrow L_0$. The element of F^* describing this map is the generalized cross-ratio.

7. Restriction of the Grassmannian n–logarithm to the special stratum. The n–logarithm function $Li_n(z)$ has a single-valued version ([Z1])

$$\mathcal{L}_n(z) \quad := \quad \begin{matrix} \mathrm{Re} & (n: \text{ odd}) \\ \mathrm{Im} & (n: \text{ even}) \end{matrix} \left(\sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z) \right) , \qquad n \geq 2$$

It is continuous on $\mathbb{CP}^1$. Here $\frac{2x}{e^{2x}-1} = \sum_{k=0}^{\infty} \beta_k x^k$, so $\beta_k = \frac{2^k B_k}{k!}$ where B_k are Bernoulli numbers. For example $\mathcal{L}_2(z)$ is the Bloch - Wigner function.

Let us consider the following modification of the function $\mathcal{L}_n(z)$ proposed by A. M. Levin in [Le]:

$$\widetilde{\mathcal{L}}_n(x) := \tag{39}$$

$$\frac{(2n-3)}{(2n-2)} \sum_{k \text{ even};\ 0 \leq k \leq n-2} \frac{2^k (n-2)!(2n-k-3)!}{(2n-3)!(k+1)!(n-k-2)!} \mathcal{L}_{n-k}(x) \log^k |x|$$

For example $\widetilde{\mathcal{L}}_n(x) = \mathcal{L}_n(x)$ for $n \leq 3$, but already $\widetilde{\mathcal{L}}_4(x)$ is different from $\mathcal{L}_4(x)$. A direct integration carried out in Proposition 4.4.1 of [Le] shows that

$$-(2\pi i)^{n-1}(-1)^{(n-1)(n-2)/2} \widetilde{\mathcal{L}}_n(x) =$$

$$\int_{\mathbb{CP}^{n-1}} \log |1 - z_1| \prod_{i=1}^{n-1} d \log |z_i| \wedge \prod_{i=1}^{n-2} d \log |z_i - z_{i+1}| \wedge d \log |z_{n-1} - a|$$

This combined with Proposition 3.1 below implies

Theorem 3.10 *The value of function $\mathcal{L}_n^G$ on special configuration (38) equals*

$$-(-1)^{n(n-1)/2} 4^{n-1} \binom{2n-2}{n-1}^{-1} \widetilde{\mathcal{L}}_n(a); \qquad a = r(l_0, ..., l_{n-1}, m_0, ..., m_{n-1})$$

8. Computation of the Grassmannian n–logarithm. It follows from Theorem 3.10 that $\mathcal{L}_2^G(l_1, ..., l_4) = -2\mathcal{L}_2(r(l_1, ..., l_4))$.

It was proved in Theorem 1.3 of [GZ] that

$$\mathcal{L}_3^G(l_0, ..., l_5) = \frac{1}{90} \mathrm{Alt}_6 \mathcal{L}_3(r_3(l_0, ..., l_5)) +$$

$$\frac{1}{9} \mathrm{Alt}_6 \Big(\log |\Delta(l_0, l_1, l_2)| \log |\Delta(l_1, l_2, l_3)| \Delta(l_2, l_3, l_4)| \Big) \tag{40}$$

We will continue this discussion in chapter 5.

The functions $\mathcal{L}_n^G$ for $n > 3$ can not be expressed via classical polylogarithms.

4 Polylogarithmic motivic complexes

1. The groups $\mathcal{B}_n(F)$ and polylogarithmic motivic complexes ([G1]-[G2]). For a set X denote by $\mathbb{Z}[X]$ the free abelian group generated by symbols $\{x\}$ where x run through all elements of the set X. Let F be an arbitrary field. We define inductively subgroups $\mathcal{R}_n(F)$ of $\mathbb{Z}[P^1_F]$, $n \geq 1$ and set

$$\mathcal{B}_n(F) := \mathbb{Z}[P^1_F]/\mathcal{R}_n(F)$$

One has

$$\mathcal{R}_1(F) := (\{x\} + \{y\} - \{xy\}, (x, y \in F^*); \{0\}; \{\infty\}) \; ; \qquad \mathcal{B}_1(F) = F^*$$

Let $\{x\}_n$ be the image of $\{x\}$ in $\mathcal{B}_n(F)$. Consider homomorphisms

$$\mathbb{Z}[P^1_F] \xrightarrow{\delta_n} \begin{cases} \mathcal{B}_{n-1}(F) \otimes F^* & : \quad n \geq 3 \\ \Lambda^2 F^* & : \quad n = 2 \end{cases} \tag{41}$$

$$\delta_n : \{x\} \mapsto \begin{cases} \{x\}_{n-1} \otimes x & : \quad n \geq 3 \\ (1-x) \wedge x & : \quad n = 2 \end{cases} \qquad \delta_n : \{\infty\}, \{0\}, \{1\} \mapsto 0 \tag{42}$$

Set $\mathcal{A}_n(F) :=$ Ker δ_n . Any element $\alpha(t) = \Sigma n_i\{f_i(t)\} \in \mathbb{Z}[P^1_{F(t)}]$ has a specialization $\alpha(t_0) := \Sigma n_i\{f_i(t_0)\} \in \mathbb{Z}[P^1_F]$ at each point $t_0 \in P^1_F$.

Definition 4.1 *$\mathcal{R}_n(F)$ is generated by elements $\{\infty\}, \{0\}$ and $\alpha(0) - \alpha(1)$ where $\alpha(t)$ runs through all elements of $\mathcal{A}_n(F(t))$.*

Then $\delta_n\Big(\mathcal{R}_n(F)\Big) = 0$ ([G1], 1.16). So we get homomorphisms

$$\delta_n : \mathcal{B}_n(F) \longrightarrow \mathcal{B}_{n-1}(F) \otimes F^*, \quad n \geq 3; \quad \delta_2 : \mathcal{B}_2(F) \longrightarrow \Lambda^2 F^*$$

and finally the *polylogarithmic motivic complex* $\Gamma(F, n)$:

$$\mathcal{B}_n \xrightarrow{\delta} \mathcal{B}_{n-1} \otimes F^* \xrightarrow{\delta} \mathcal{B}_{n-2} \otimes \Lambda^2 F^* \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2 \otimes \Lambda^{n-2} F^* \xrightarrow{\delta} \Lambda^n F^*$$

where $\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \to \delta_p(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i$ and $\mathcal{B}_n$ is in degree 1.

Conjecture 4.2 $H^i\Gamma(F, n) \otimes \mathbb{Q} = \mathrm{gr}^{\gamma}_n K_{2n-i}(F) \otimes \mathbb{Q}$.

Denote by $\widehat{\mathcal{L}}_n$ the function $\mathcal{L}_n$, multiplied by i for even n and unchanged for odd n. There is a well defined homomorphism ([G2], Theorem 1.13):

$$\widehat{\mathcal{L}}_n : \mathcal{B}_n(\mathbb{C})) \longrightarrow \mathbb{R}(n-1); \qquad \widehat{\mathcal{L}}_n(\sum m_i\{z_i\}_n) := \sum m_i \widehat{\mathcal{L}}_n(z_i)$$

There are canonical homomorphisms

$$B_n(F) \longrightarrow \mathcal{B}_n(F); \quad \{x\}_n \longmapsto \{x\}_n, \quad n = 1, 2, 3. \tag{43}$$

They are isomorphisms for $n = 1, 2$ and expect to be an isomorphism for $n = 3$, at least modulo torsion.

2. The residue homomorphism for complexes $\Gamma(F, n)$ (1.14, [G1]). Let $F = K$ be a field with a discrete valuation v, the residue field k_v and the group of units U. Let $u \to \bar{u}$ be the projection $U \to k_v^*$. Choose a uniformizer π. There is a homomorphism $\theta : \Lambda^n K^* \longrightarrow \Lambda^{n-1} k_v^*$ uniquely defined by the following properties ($u_i \in U$):

$$\theta\,(\pi \wedge u_1 \wedge \dots \wedge u_{n-1}) = \bar{u}_1 \wedge \dots \wedge \bar{u}_{n-1}; \qquad \theta\,(u_1 \wedge \dots \wedge u_n) = 0$$

It is clearly independent of π. Define a homomorphism $s_v : \mathbb{Z}[P^1_K] \longrightarrow \mathbb{Z}[P^1_{k_v}]$ by setting $s_v\{x\} = \{\bar{x}\}$ if x is a unit and 0 otherwise. It induces a homomorphism $s_v : \mathcal{B}_m(K) \longrightarrow \mathcal{B}_m(k_v)$. Put

$$\partial_v := s_v \otimes \theta : \quad \mathcal{B}_m(K) \otimes \Lambda^{n-m} K^* \quad \longrightarrow \quad \mathcal{B}_m(k_v) \otimes \Lambda^{n-m-1} k_v^*$$

It defines a morphism of complexes $\partial_v : \Gamma(K, n) \longrightarrow \Gamma(k_v, n-1)[-1]$.

3. A variation of mixed Hodge structures on $P^1(\mathbb{C}) - \{0, 1, \infty\}$ corresponding to the classical polylogarithm $\mathrm{Li}_n(z)$ ([D2]). Its fiber $H(z)$ over a point z is described via the period matrix

$$\begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ \mathrm{Li}_1(z) & 2\pi i & 0 & ... & 0 \\ \mathrm{Li}_2(z) & 2\pi i \log z & (2\pi i)^2 & ... & 0 \\ ... & ... & ... & ... & ... \\ \mathrm{Li}_n(z) & 2\pi i \frac{\log^{n-1} z}{(n-1)!} & (2\pi i)^2 \frac{\log^{n-2} z}{(n-2)!} & ... & (2\pi i)^n \end{pmatrix} \tag{44}$$

Its entries are defined using analytic continuation to the point z along a path γ from a given point in $\mathbb{C}$ where all the entries are defined by power series expansions, say the point $1/2$.

Here is a more natural way to define the entries. Consider the following regularized iterated integrals along a certain fixed path γ between 0 to z:

$$\mathrm{Li}_n(z) = \int_0^z \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ ... \circ \frac{dt}{t}}_{n-1 \quad \text{times}}; \qquad \frac{\log^n z}{n!} = \int_0^z \underbrace{\frac{dt}{t} \circ ... \circ \frac{dt}{t}}_{n \quad \text{times}}$$

To regularize the divergent integrals we take the lower limit of integration to be ε. Then it is easy to show that the integral has an asymptotic expansion of type $I_0(\varepsilon) + I_1(\varepsilon)\log\varepsilon + ... + I_k(\varepsilon)\log^k \varepsilon$, where all the functions $I_i(\varepsilon)$ are smooth at $\varepsilon = 0$. Then we take $I_0(0)$ to be the regularized value.

Now let us define a mixed Hodge structure $H(z)$. Let $\mathbb{C}^{n+1}$ be the standard vector space with basis $(e_0, ..., e_n)$, and V_{n+1} the $\mathbb{Q}$-vector subspace spanned by the columns of the matrix (44). Let $W_{-2(n+1-k)}V_{n+1}$ be the subspace spanned by the first k columns, counted from the right to the left. One shows that these subspaces do not depend on the choice of the path γ, i.e. they are well-defined inspite of the multivalued nature of the entries of the matrix. Then $W_\bullet V_{n+1}$ is the weight filtration. We define the Hodge filtration $F^\bullet \mathbb{C}^{n+1}$ by setting $F^{-k}\mathbb{C}^{n+1} := \langle e_0, ..., e_k \rangle_{\mathbb{C}}$. It is opposite to the weight filtration. We get a Hodge-Tate structure, i.e. $h^{pq} = 0$ unless $p \neq q$. One checks that the family of Hodge-Tate structures $H(z)$ forms a unipotent variation of Hodge-Tate structures on $P^1(\mathbb{C}) - \{0, 1, \infty\}$.

Let $n \geq 0$. An n-framed Hodge-Tate structure H is a triple (H, v_0, f_n), where $v_0 : \mathbb{Q}(0) \longrightarrow gr^W_0 H$ and $f_n : gr^W_{-2n} H \longrightarrow \mathbb{Q}(n)$ are nonzero morphisms. A framing plus a choice of a splitting of the weight filtration determines a period of a Hodge-Tate structure. Consider the coarsest equivalence relation on the set of all n-framed Hodge-Tate structures for which $M_1 \sim M_2$ if there is a map $M_1 \to M_2$ respecting the frames. Tnen the set $\mathcal{H}_n$ of the equivalence classes has a natural abelian group structure. Moreover

$$\mathcal{H}_\bullet := \oplus_{n \geq 0} \mathcal{H}_n \tag{45}$$

has a natural Hopf algebra structure with a coproduct Δ, see the Appendix of [G12].

Observe that $\mathrm{Gr}^W_{-2k} H_n(z) = \mathbb{Q}(k)$ for $-n \leq k \leq 0$. Therefore $H_n(z)$ has a natural framing such that the corresponding period is given by the function $\mathrm{Li}_n(z)$. The obtained framed object is denoted by $\mathrm{Li}^{\mathcal{H}}_n(z)$. To define it for $z = 0, 1, \infty$ we use the specialization functor to the punctured tangent space at z, and then take the fiber over the tangent vector corresponding to the parameter z on P^1. It is straitforward to see that the coproduct $\Delta \mathrm{Li}^{\mathcal{H}}_n(z)$ is computed by the formula

$$\Delta \mathrm{Li}^{\mathcal{H}}_n(z) = \sum_{k=0}^{n} \mathrm{Li}^{\mathcal{H}}_{n-k}(z) \otimes \frac{\log^{\mathcal{H}}(z)^k}{k!} \tag{46}$$

where $\log^{\mathcal{H}}(z)$ is the 1-framed Hodge-Tate structure corresponding to $\log(z)$.

4. A motivic proof of the weak version of Zagier's conjecture. Our goal is the following result, which was proved in [dJ3] and in the unfinished manuscript [BD2]. The proof below uses a different set of ideas. It follows the framework described in Chapter 13 of [G8], and quite close to the approach outlined in [BD1], although it is formulated a bit differently, using the polylogarithmic motivic complexes.

Theorem 4.3 *Let F be a number field. Then there exists a homomorphism*

$$l_n : H^1(\Gamma(F,n) \otimes \mathbb{Q}) \longrightarrow K_{2n-1}(F) \otimes \mathbb{Q}$$

such that for any embedding $\sigma : F \hookrightarrow \mathbb{C}$ one has the following commutative diagram

$$\begin{array}{ccccc} H^1(\Gamma(F,n) \otimes \mathbb{Q}) & \xrightarrow{\sigma_*} & H^1(\Gamma(\mathbb{C},n) \otimes \mathbb{Q}) & \xrightarrow{\mathcal{L}_n} & \mathbb{R}(n-1) \\ \downarrow l_n & & & & \downarrow = \\ K_{2n-1}(F) \otimes \mathbb{Q} & \xrightarrow{\sigma_*} & K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q} & \xrightarrow{r_{Bo}} & \mathbb{R}(n-1) \end{array} \quad (47)$$

Proof. We will use the following background material and facts:

i) The existence of the abelian tensor category $\mathcal{M}_T(F)$ of mixed Tate motives over a number field F, satisfying all the desired properties, including Beilinson's formula expressing the Ext groups via the rational K-theory of F and the Hodge realization functor. See [DG] and the references there.

ii) The formalism of mixed Tate categories, including the description of the fundamental Hopf algebra $\mathcal{A}_\bullet(\mathcal{M})$ of a mixed Tate category via framed objects in $\mathcal{M}$. See [G12], Section 8. The fundamental Hopf algebra of the category $\mathcal{M}_T(F)$ is denoted $\mathcal{A}_\bullet(F)$. For example for the category of mixed Hodge-Tate structures fundamental Hopf algebra is the one $\mathcal{H}_\bullet$ from (45). Let

$$\Delta : \mathcal{A}_\bullet(F) \longrightarrow \mathcal{A}_\bullet(F)^{\otimes 2}$$

be the coproduct in the Hopf algebra $\mathcal{A}_\bullet(F)$, and $\Delta' := \Delta - \mathrm{Id} \otimes 1 + 1 \otimes \mathrm{Id}$ is the restricted coproduct. The key fact is a canonical isomorphism

$$\mathrm{Ker}\Delta' \cap \mathcal{A}_n(F) \stackrel{\sim}{=} K_{2n-1}(F) \otimes \mathbb{Q} \quad (48)$$

Since $\mathcal{A}_\bullet(F)$ is graded by ≥ 0 integers, and $\mathcal{A}_0(F) = \mathbb{Q}$, formula (48) for $n = 1$ reduces to an isomorphism

$$\mathcal{A}_1(F) \stackrel{\sim}{=} F^* \otimes \mathbb{Q} \quad (49)$$

iii) The existence of the motivic classical polylogarithms

$$\mathrm{Li}_n^{\mathcal{M}}(z) \in \mathcal{A}_n(F), \qquad z \in P^1(F) \quad (50)$$

They were defined in Section 3.6 of [G12] using either a geometric construction of [G9], or a construction of the motivic fundamental torsor of path between the tangential base points given in [DG]. In particular one has $\mathrm{Li}_n^{\mathcal{M}}(0) = \mathrm{Li}_n^{\mathcal{M}}(\infty) = 0$. A natural construction of the elements (50) using the moduli space $\overline{\mathcal{M}}_{0,n+3}$ is given in Chapter 4.6 below.

iv) The Hodge realization of the element (50) is equivalent to the framed Hodge-Tate structure $\mathrm{Li}_n^{\mathcal{H}}(z)$ from Chapter 4.3. This fact is more or less straitforward if one uses the fundamental torsor of path on the punctured projective line to define the element $\mathrm{Li}_n^{\mathcal{M}}(z)$, and follows from the general specialization theorem proved in [G9] if one uses the approach of loc. cit. This implies that the Lie-period of $\mathrm{Li}_n^{\mathcal{H}}(z)$ equals to $\mathcal{L}_n(z)$. Indeed, for the Hodge-Tate structure $H(z)$ assigned to $\mathrm{Li}_n(z)$ this was shown in [BD1].

v) The crucial formula (Section 6.3 of [G12]):

$$\Delta\mathrm{Li}_n^{\mathcal{M}}(z) = \sum_{k=0}^{n} \mathrm{Li}_{n-k}^{\mathcal{M}}(z) \otimes \frac{\log^{\mathcal{M}}(z)^k}{k!} \tag{51}$$

where $\log^{\mathcal{M}}(z) \in \mathcal{A}_1(F)$ is the element corresponding to z under the isomorphism (48), and $\log^{\mathcal{M}}(z)^k \in \mathcal{A}_k(F)$ is its k-th power. It follows from formula (46) using the standard trick based on Borel's theorem to reduce a motivic claim to the corresponding Hodge one.

vi) The Borel regulator map on $K_{2n-1}(F) \otimes \mathbb{Q}$, which sits via (48) inside of $\mathcal{A}_n(F)$, is induced by the Hodge realization functor on the category $\mathcal{M}_T(F)$.

Having this background, we proceed as follows. Namely, let

$$\mathcal{L}_\bullet(F) := \frac{\mathcal{A}_{>0}(F)}{\mathcal{A}_{>0}(F)^2}$$

be the fundamental Lie coalgebra of $\mathcal{M}_T(F)$. Its cobracket δ is induced by Δ. Projecting the element (50) into $\mathcal{L}_n(F)$ we get an element $l_n^{\mathcal{M}}(z) \in \mathcal{L}_n(F)$ such that

$$\delta l_n^{\mathcal{M}}(z) = l_{n-1}^{\mathcal{M}}(z) \wedge z \tag{52}$$

Consider the following map

$$\widetilde{l}_n : \mathbb{Q}[P^1(F)] \longrightarrow \mathcal{L}_n(F), \qquad \{z\} \longmapsto l_n^{\mathcal{M}}(z)$$

Proposition 4.4 *The map $\widetilde{l}_n$ kills the subspace $\mathcal{R}_n(F)$, providing a well defined homomorphism*

$$l_n : \mathcal{B}_n(F) \longrightarrow \mathcal{L}_n(F); \qquad \{z\}_n \longrightarrow \widetilde{l}_n(z)$$

Proof. We proceed by the induction on n. The case $n = 1$ is self-obvious. Suppose we are done for $n-1$. Then there is the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{Q}[F] & \xrightarrow{\delta_n} & \mathcal{B}_{n-1}(F) \otimes F^* \\
\downarrow \widetilde{l}_n & & \downarrow l_{n-1} \wedge \mathrm{Id} \\
\mathcal{L}_n(F) & \xrightarrow{\delta} & \oplus_{k \leq n/2} \mathcal{L}_{n-k}(F) \wedge \mathcal{L}_k(F)
\end{array}$$

Indeed, its commutativity is equivalent to the basic formula (52).

Let $x \in P^1(F)$. Recall the specialization at x homomorphism

$$s_x : \mathcal{B}_n(F(T)) \longrightarrow \mathcal{B}_n(F), \qquad \{f(T)\}_n \longmapsto \{f(x)\}_n$$

It gives rise to the specialization homomorphism

$$s'_x : \mathcal{B}_{n-1}(F(T)) \otimes F(T)^* \longrightarrow \mathcal{B}_{n-1}(F) \otimes F^*,$$

$$\{f(T)\}_{n-1} \otimes g(T) \longmapsto \{f(x)\}_{n-1} \otimes \frac{g(T)}{(T-x)^{v_x(g)}}(x)$$

(Use the local parameter T^{-1} when $x = \infty$).

Now let

$$\alpha(T) \in \mathrm{Ker}\Big(\delta_n : \mathbb{Q}[F(T)] \rightarrow \mathcal{B}_{n-1}(F(T)) \otimes F(T)^*\Big)$$

Using $\mathrm{Li}_n^{\mathcal{M}}(0) = \mathrm{Li}_n^{\mathcal{M}}(\infty) = 0$, for any $x \in P^1(F)$ one has $\delta(\widetilde{l}_n(\alpha(x))) = 0$. Thus

$$\widetilde{l}_n(\alpha(x)) \in K_{2n-1}(F) \otimes \mathbb{Q} \subset \mathcal{L}_n(F)$$

Let us show that this element is zero. Given an embedding $\sigma : F \hookrightarrow \mathbb{C}$, write $\sigma(\alpha(T)) = \sum_i n_i\{f_i^\sigma(T)\}$. Applying the Lie-period map to this element and using v) we get $\sum_i n_i \mathcal{L}_n f_i^\sigma(z)$. By Theorem 1.13 in [G2] the condition on $\alpha(T)$ implies that this function is constant on $\mathbb{C}P^1$. Thus the difference of its values at $\sigma(x_1)$ and $\sigma(x_2)$, where $x_0, x_1 \in P^1(F)$, is zero. On the other hand thanks to v) and vi) it coincides with the Borel regulator map applied to the corresponding element of $K_{2n-1}(F) \otimes \mathbb{Q}$. Thus the injectivity of the Borel regulator map proves the claim. So $\widetilde{l}_n(\alpha(x_0) - \alpha(x_1)) = 0$. The proposition is proved.

Proposition 4.4 implies that we get a homomorphism of complexes

$$\begin{array}{ccc}
\mathcal{B}_n(F) & \xrightarrow{\delta} & \mathcal{B}_{n-1}(F) \otimes F^* \\
\downarrow l_n & & \downarrow l_{n-1} \wedge \mathrm{Id} \\
\mathcal{L}_n(F) & \xrightarrow{\delta} & \oplus_{k \leq n/2} \mathcal{L}_{n-k}(F) \wedge \mathcal{L}_k(F)
\end{array}$$

The theorem follows immediately from this. Indeed, it remains to check commutativity of the diagram (47), and it follows from v) and vi).

If we assume the existence of the hypothetical abelian category of mixed Tate motives over an arbitrary field F, the same argumentation as above (see Chapter 6.1) implies the following result: one should have canonical homomorphisms

$$H^i(\Gamma(F,n)) \otimes \mathbb{Q} \longrightarrow \mathrm{gr}_n^\gamma K_{2n-i}(F) \otimes \mathbb{Q}$$

The most difficult part of Conjecture 4.2 says that this maps are supposed to be isomorphisms.

In the next section we define a regulator map on the polylogarithmic motivic complexes. Combined with these maps, it should give an explicit construction of the regulator map.

5. A construction of the motivic ζ-elements (1). The formula (2) leads to the motivic extension $\zeta^{\mathcal{M}}(n)$ as follows ([GM]). Recall the moduli space $\overline{\mathcal{M}}_{n+3}$ parametrising stable curves of genus zero with $n+3$ marked points. It contains as an open subset the space $\mathcal{M}_{n+3}$ parametrising the $(n+3)$-tuples of distinct points on $\mathbb{P}^1$ modulo $\mathrm{Aut}(\mathbb{P}^1)$. Then the complement $\partial\mathcal{M}_{n+3} := \overline{\mathcal{M}}_{n+3} - \mathcal{M}_{n+3}$ is a normal crossing divisor, and the pair $(\mathcal{M}_{n+3}, \partial\mathcal{M}_{n+3})$ is defined over $\mathbb{Z}$. Let us identify sequences $(t_1, ..., t_n)$ of distinct complex numbers different from 0 and 1 with the points $(0, t_1, ..., t_n, 1, \infty)$ of $\mathcal{M}_{n+3}(\mathbb{C})$. Let us consider the integrand in (2) as a holomorphic form on $\mathcal{M}_{n+3}(\mathbb{C})$. Meromorphically extending it to $\overline{\mathcal{M}}_{n+3}$ we get a differential form with logarithmic singularities Ω_n. Let A_n be its divisor. Similarly, embed the integration simplex $0 < t_1 < ... < t_n < 1$ into the set of real points of $\overline{\mathcal{M}}_{n+3}(\mathbb{R})$, take its closure Δ_n there, and consider the Zariski closure B_n of its boundary $\partial\Delta_n$. Then one can show that the mixed motive

$$H^n(\overline{\mathcal{M}}_{n+3} - A_n, B_n - (A_n \cap B_n)) \tag{53}$$

is a mixed Tate motive over $\mathrm{Spec}(\mathbb{Z})$. Indeed, it is easy to prove that its l-adic realization is unramified outside l, and is glued from the Tate modules of different weights, and then refer to [DG]. The mixed motive (53) comes equipped with an additional data, framing, given by non-zero morphisms

$$[\Omega_n] : \mathbb{Z}(-n) \longrightarrow \mathrm{gr}_{2n}^W H^n(\overline{\mathcal{M}}_{n+3} - A_n, B_n - (A_n \cap B_n)), \tag{54}$$

$$[\Delta_n] : \mathrm{gr}_0^W H^n(\overline{\mathcal{M}}_{n+3} - A_n, B_n - (A_n \cap B_n)) \longrightarrow \mathbb{Z}(0), \tag{55}$$

There exists the minimal subquotient of the mixed motive (53) which inherits non-zero framing. It delivers the extension class $\zeta^{\mathcal{M}}(n)$. Leibniz formula (2) just means that $\zeta(n)$ is its period.

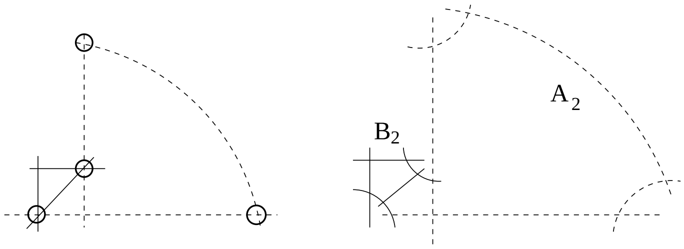


Figure 2: Constructing $\zeta^{\mathcal{M}}(2)$

Example. To construct $\zeta^{\mathcal{M}}(2)$, take the pair of triangles in P^2 shown on the left of Figure 2. The triangle shown by the punctured lines is the divisor of poles of the differential $d\log(1-t_1) \wedge d\log t_2$ in (2), and the second triangle is the algebraic closure of the boundary of the integration cycle $0 \leq t_1 \leq t_2 \leq 1$. The corresponding configuration of six lines is defined uniquely up to a projective equivalence. Blowing up the four points shown by little circles on Figure 2 (they are the triple intersection points of the lines), we get the moduli space $\overline{\mathcal{M}}_{0,5}$. Its boundary is the union of the two pentagons, A_2 and B_2, projecting to the two triangles on P^2. Then $\zeta^{\mathcal{M}}(2)$ is a subquotient of the mixed Tate motive $H^2(\overline{\mathcal{M}}_{0,5} - A_2, B_2 - (A_2 \cap B_2))$ over Spec($\mathbb{Z}$). Observe that an attempt to use a similar construction for the pair of triangles in P^2 fails since there is no non-zero morphism $[\Delta_2]$ in this case. Indeed, there are two vertices of the B-triangle shown on the left of Figure 2 lying at the sides of the A-triangle, and therefore the chain $0 \leq t_1 \leq t_2 \leq 1$ does not give rise to a relative class in $H_2(P^2(\mathbb{C}) - A, B - (A \cap B))$.

6. A geometric construction of the motivic classical polylogarithm $\mathrm{Li}_n^{\mathcal{M}}(z)$. The above construction is easily generalized to the case of the classical polylogarithm. Let $A_n(z)$ be the divisor of the meromorphic differential form

$$\Omega_n(z) := \frac{dt_1}{z^{-1} - t_1} \wedge \frac{dt_2}{t_2} \wedge ... \wedge \frac{dt_n}{t_n}$$

extended as a rational form to $\overline{\mathcal{M}}_{0,n+3}$. Suppose that F is a number field and $z \in F$. Then there is the following mixed Tate motive over F:

$$H^n(\overline{\mathcal{M}}_{n+3} - A_n(z), B_n - (A_n(z) \cap B_n)) \tag{56}$$

One checks that the vertices (that is the zero-dimensional strata) of the divisor B_n are disjoint with the divisor $A_n(z)$. Using this we define a framing $([\Omega_n(z)], [\Delta_n])$ on the mixed motive (56) similar to the one (54) - (55). The geometric condition on the divisor $A_n(z)$ is used to show that the framing morphism $[\Delta_n]$ is non-zero. We define $\mathrm{Li}_n^{\mathcal{M}}(z)$ as the framed mixed Tate motive (56) with the framing $([\Omega_n(z)], [\Delta_n])$. A similar construction for the multiple polylogarithms was worked out in the Ph. D. thesis of Q. Wang [Wa].

It follows from a general result in [G9] that the Hodge realization of $\mathrm{Li}_n^{\mathcal{M}}(z)$ is equivalent to the framed mixed Hodge structure $\mathrm{Li}_n^{\mathcal{H}}(z)$. If $n = 2$ the two corresponding mixed Hodge structures are isomorphic ([Wa]), while in general they are not isomorphic but equivalent as framed mixed Hodge structures.

5 Regulator maps on the polylogarithmic motivic complexes

In this chapter we define explicitly these regulator maps via classical polylogarithm functions following [G6] and [G2]. This implies how the special values of ζ-functions of algebraic varieties outside of the critical strip should be expressed using the classical polylogarithms.

1. The numbers $\beta_{k,p}$. Define for any integers $p \geq 1$ and $k \geq 0$ the numbers

$$\beta_{k,p} := \quad (-1)^p (p-1)! \sum_{0 \leq i \leq [\frac{p-1}{2}]} \frac{1}{(2i+1)!} \beta_{k+p-2i}$$

For instance $\beta_{k,1} = -\beta_{k+1}$; $\quad \beta_{k,2} = \beta_{k+2}$; $\quad \beta_{k,3} = -2\beta_{k+3} - \frac{1}{3}\beta_{k+1}$.

One has recursions

$$2p \cdot \beta_{k+1,2p} = -\beta_{k,2p+1} - \frac{1}{2p+1}\beta_{k+1}; \qquad (2p-1) \cdot \beta_{k+1,2p-1} = -\beta_{k,2p} \tag{57}$$

These recursions together with $\beta_{k,1} = -\beta_{k+1}$ determine the numbers $\beta_{k,p}$.

Let $m \geq 1$. Then one can show that

$$\beta_{0,2m} = \beta_{0,2m+1} = \frac{1}{2m+1}, \quad \beta_{1,2m-1} = -\frac{1}{(2m-1)(2m+1)}, \quad \beta_{1,2m} = 0$$

2. The regulator map on the polylogarithmic motivic complexes in the case $F = \mathbb{C}(X)$**, where** X **is a complex algebraic variety** X. Let us define differential 1-forms $\widehat{\mathcal{L}}_{p,q}$ on $\mathbb{C}P^1 \backslash \{0, 1, \infty\}$ for $q \geq 1$ as follows:

$$\widehat{\mathcal{L}}_{p,q}(z) := \widehat{\mathcal{L}}_p(z) \log^{q-1}|z| \cdot d\log|z|, \quad p \geq 2 \tag{58}$$

$$\widehat{\mathcal{L}}_{1,q}(z) := \alpha(1-z, z) \log^{q-1}|z|$$

For any rational function f on a complex variety X the 1-form $\widehat{\mathcal{L}}_{p,q}(f)$ provides a distribution on $X(\mathbb{C})$. Set

$$\mathcal{A}_m \left\{ \bigwedge_{i=1}^{2p} d\log|g_i| \wedge \bigwedge_{i=2p+1}^{m} di \arg g_j \right\} :=$$

$$\mathrm{Alt}_m \left\{ \frac{1}{(2p)!(m-2p)!} \bigwedge_{i=1}^{2p} d\log|g_i| \wedge \bigwedge_{i=2p+1}^{m} di \arg g_j \right\}$$

and

$$\mathcal{A}_m \left\{ \log|g_1| \cdot \bigwedge_{i=2}^{p} d\log|g_i| \wedge \bigwedge_{i=p+1}^{m} di \arg g_j \right\} :=$$

$$\mathrm{Alt}_m \left\{ \frac{1}{(p-1)!(m-p)!} \log|g_1| \cdot \bigwedge_{i=2}^{p} d\log|g_i| \wedge \bigwedge_{i=p+1}^{m} di \arg g_j \right\}$$

So $\mathcal{A}_m(F(g_1, .., g_m))$ is a weighted alternation (we divide by the order of the stabilizer of the term we alternate).

Let $f, g_1, ..., g_m$ be rational functions on a complex variety X. Set

$$r_{n+m}(m+1) : \{f\}_n \otimes g_1 \wedge ... \wedge g_m \longmapsto$$

$$\widehat{\mathcal{L}}_n(f) \cdot \mathcal{A}_m \left\{ \sum_{p \geq 0} \frac{1}{2p+1} \bigwedge_{i=1}^{2p} d\log|g_i| \wedge \bigwedge_{j=2p+1}^{m} di \arg g_j \right\} + \tag{59}$$

$$\sum_{k \geq 1} \sum_{1 \leq p \leq m} \beta_{k,p} \widehat{\mathcal{L}}_{n-k,k}(f) \wedge \mathcal{A}_m \left\{ \log|g_1| \bigwedge_{i=2}^{p} d\log|g_i| \wedge \bigwedge_{j=p+1}^{m} di \arg g_j \right\} \tag{60}$$

Proposition 5.1 *The differential form $r_{n+m}(m+1)(\{f\}_n \otimes g_1 \wedge ... \wedge g_m)$ defines a distribution on $X(\mathbb{C})$.*

Example 1. $r_n(1)(\{f\}_n) = \widehat{\mathcal{L}}_n(f)$.
Example 2. $r_n(n)(g_1 \wedge ... \wedge g_n) = r_{n-1}(g_1 \wedge ... \wedge g_n)$.
Example 3. $m = 1$, n is arbitrary. Then

$$r_{n+1}(2) : \{f\}_n \otimes g \longmapsto \quad \widehat{\mathcal{L}}_n(f) di \arg g - \sum_{k=1}^{n-1} \beta_{k+1} \widehat{\mathcal{L}}_{n-k,k}(f) \cdot \log |g|$$

Example 4. $m = 2$, n is arbitrary.

$$r_{n+2}(3) : \{f\}_n \otimes g_1 \wedge g_2 \longmapsto \widehat{\mathcal{L}}_n(f) \left\{ di \arg g_1 \wedge di \arg g_2 + \frac{1}{3} d \log |g_1| \wedge d \log |g_2| \right\}$$

$$- \sum_{k=1}^{n-1} \beta_{k+1} \widehat{\mathcal{L}}_{n-k,k}(f) \wedge (\log |g_1| di \arg g_2 - \log |g_2| di \arg g_1)$$

$$+ \sum_{k \geq 1} \beta_{k+2} \widehat{\mathcal{L}}_{n-k,k}(f) \wedge (\log |g_1| d \log |g_2| - \log |g_2| d \log |g_1|)$$

Let $\mathcal{A}^i(\eta_X)$ be the space of real smooth i-forms at the generic point $\eta_X := \mathrm{Spec}\mathbb{C}(X)$ of a complex variety X. Let $\mathcal{D}$ be the de Rham differential on distributions on $X(\mathbb{C})$, and d the de Rham differential on $\mathcal{A}^i(\eta_X)$. For example:

$$d\Big(di \arg z\Big) = 0; \quad \mathcal{D}\Big(di \arg z\Big) = 2\pi i \delta(z)$$

Recall the residue homomorphisms defined in Chapter 4.2.

Theorem 5.2 *a) The maps $r_n(\cdot)$ provide a homomorphism of complexes*

$$\begin{array}{ccccccc}
\mathcal{B}_n(\mathbb{C}(X)) & \stackrel{\delta}{\rightarrow} & \mathcal{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \stackrel{\delta}{\rightarrow} & \ldots & \stackrel{\delta}{\rightarrow} & \bigwedge^n \mathbb{C}(X)^* \\
\downarrow r_n(1) & & \downarrow r_n(2) & & & & \downarrow r_n(n) \\
\mathcal{A}^0(\eta_X)(n-1) & \stackrel{d}{\rightarrow} & \mathcal{A}^1(\eta_X)(n-1) & \stackrel{d}{\rightarrow} & ... & \stackrel{d}{\rightarrow} & \mathcal{A}^{n-1}(\eta_X)(n-1)
\end{array}$$

b) The maps $r_n(m)$ are compatible with the residues:

$$\mathcal{D} \circ r_n(m) - r_n(m+1) \circ \delta = 2\pi i \cdot \sum_{Y \in X^{(1)}} r_{n-1}(m-1) \circ \partial_{v_Y}, \quad m < n$$

$$\mathcal{D} \circ r_n(n) - \pi_n(d \log f_1 \wedge ... \wedge d \log f_n) = 2\pi i \cdot \sum_{Y \in X^{(1)}} r_{n-1}(n-1) \circ \partial_{v_Y}$$

where v_Y is the valuation on the field $\mathbb{C}(X)$ defined by a divisor Y.

Let X be a regular variety over $\mathbb{C}$. Recall the n-th Beilinson-Deligne complex $\underline{\mathbb{R}}_{\mathcal{D}}(n)_X$ defined as a total complex associated with the following bicomplex of sheaves in classical topology on $X(\mathbb{C})$:

$$\begin{array}{l}
\Big(\underline{\mathcal{D}}_X^0 \stackrel{d}{\longrightarrow} \underline{\mathcal{D}}_X^1 \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} \underline{\mathcal{D}}_X^n \stackrel{d}{\longrightarrow} \underline{\mathcal{D}}_X^{n+1} \stackrel{d}{\longrightarrow} \ldots\Big) \otimes \mathbb{R}(n-1) \\
\qquad\qquad\qquad\qquad\qquad\quad \uparrow \pi_n \qquad\quad \uparrow \pi_n \\
\qquad\qquad\qquad\qquad\qquad\quad \Omega^n_{X,\log} \stackrel{\partial}{\longrightarrow} \Omega^{n+1}_{X,\log} \stackrel{\partial}{\longrightarrow}
\end{array}$$

Here $\underline{\mathcal{D}}^0_X$ is in degree 1 and $(\Omega^\bullet_{X,\log}, \partial)$ is the de Rham complex of holomorphic forms with logarithmic singularities at infinity. We will denote by $\underline{\mathbb{R}}_{\mathcal{D}}(n)(U)$ the complex of the global sections.

Theorem 5.2 can be reformulated as follows. Set $\widetilde{r}_n(i) := r_n(i)$ for $i < n$ and

$$\widetilde{r}_n(n) : \Lambda^n \mathbb{C}(X)^* \longrightarrow \mathcal{A}^{n-1}(\eta_X)(n-1) \oplus \Omega^n_{\log}(\eta_X)$$

$$f_1 \wedge ... \wedge f_n \longmapsto r_n(n)(f_1 \wedge ... \wedge f_n) + d\log f_1 \wedge ... \wedge d\log f_n \tag{61}$$

Theorem 5.3 *Let X be a complex algebraic variety. Then there is a homomorphism of complexes*

$$\widetilde{r}_n(\cdot) : \Gamma(\mathbb{C}(X); n) \longrightarrow \underline{\mathbb{R}}_{\mathcal{D}}(n)(\mathrm{Spec}\mathbb{C}(X)) \tag{62}$$

compatible with the residues as explained in the part b) of Theorem 5.2.

3. The general case. Let X be a regular projective variety over a field F. Let $d := \mathrm{dim}X$. Then the complex $\Gamma(X; n)$ *should be* defined as the total complex of the following bicomplex:

$$\Gamma(F(X); n) \longrightarrow \oplus_{Y_1 \in X^{(1)}} \Gamma(F(Y_1); n-1)[-1] \longrightarrow$$

$$\oplus_{Y_2 \in X^{(2)}} \Gamma(F(Y_2); n-2)[-1] \longrightarrow ... \longrightarrow \oplus_{Y_d \in X^{(d)}} \Gamma(F(Y_d); n-d)[-d]$$

where the arrows are provided by the residue maps, see [G1], p 239-240. The complex $\Gamma(X; n) \otimes \mathbb{Q}$ should be quasiisomorphic to the weight n motivic complex.

However there is a serious difficulty in the definition of the complex $\Gamma(X; n)$ for a general variety X and $n > 3$, ([G1], p. 240). It would be resolved if homotopy invariance of the polylogarithmic complexes were known (Conjecture 1.39 in [G1]). As a result we have an unconditional definition of the polylogarithmic complexes $\Gamma(X; n)$ only in the following cases:

a) $X = Spec(F)$, F is an arbitrary field.

b) X is an regular curve over any field, and n is arbitrary.

c) X is an arbitrary regular scheme, but $n \leq 3$.

Now let F be a subfield of $\mathbb{C}$. Having in mind applications to arithmetic, we will restrict ourself by the case when $F = \mathbb{Q}$. Assuming we are working with one of the above cases, or assuming the above difficulty has been resolved, let us define the regulator map

$$\Gamma(X; n) \longrightarrow \mathcal{C}_{\mathcal{D}}(X(\mathbb{C}); \mathbb{R}(n))$$

We specify it for each of $\Gamma(\mathbb{Q}(Y_k); n-k)[-k]$ where $k = 0, ..., d$. Namely, we take the homomorphism $r_{n-k}(\cdot)$ for Spec($\mathbb{Q}(Y_k)$) and multiply it by $(2\pi i)^{n-k}\delta_{Y_{n-k}}$. Notice that the distribution δ_Y depends only on the generic point of a subvariety Y. Then Theorem 5.2, and in particular its part b), providing *compatibility with the residues* property, guarantee that we get a homomorphism of complexes. Here are some examples.

4. The weight one. The regulator map on the weight one motivic complex looks as follows:

$$\begin{array}{ccc} \mathbb{Q}(X)^* & \longrightarrow & \oplus_{Y \in X^{(1)}} \mathbb{Z} \\ \downarrow r_1(1) & & \downarrow r_1(2) \\ \mathcal{D}^{0,0} & \stackrel{2\overline{\partial}\partial}{\longrightarrow} & \mathcal{D}^{1,1}_{cl,\mathbb{R}}(1) \end{array}$$

$$r_1(2) : Y_1 \longmapsto 2\pi i \cdot \delta_{Y_1}, \qquad r_1(1) : f \longmapsto \log|f|$$

Here the top line is the weight 1 motivic complex, sitting in degrees $[1, 2]$.

5. The weight two. The regulator map on the weight two motivic complexes looks as follows.

$$\begin{array}{ccccccc} \mathcal{B}_2(\mathbb{Q}(X)) & \xrightarrow{\delta} & \Lambda^2\mathbb{Q}(X)^* & \xrightarrow{\partial} & \oplus_{Y\in X^{(1)}}\mathbb{Q}(Y)^* & \xrightarrow{\partial} & \oplus_{Y\in X^{(2)}}\mathbb{Z} \\ \downarrow r_2(1) & & \downarrow r_2(2) & & \downarrow r_2(3) & & \downarrow r_2(4) \\ \mathcal{D}^{0,0}_{\mathbb{R}}(1) & \xrightarrow{D} & (\mathcal{D}^{0,1}\oplus\mathcal{D}^{1,0})_{\mathbb{R}}(1) & \xrightarrow{D} & \mathcal{D}^{1,1}_{\mathbb{R}}(1) & \xrightarrow{2\overline{\partial}\partial} & \mathcal{D}^{2,2}_{cl,\mathbb{R}}(2) \end{array}$$

where D is the differential in $\mathcal{C}_{\mathcal{D}}(X,\mathbb{R}(2))$

$$r_2(4): Y_2 \longmapsto (2\pi i)^2\cdot\delta_{Y_2}; \qquad r_2(3):(Y_1,f)\longmapsto 2\pi i\cdot\log|f|\delta_{Y_1}$$

$$r_2(2): f\wedge g\longmapsto -\log|f|di\arg g+\log|g|di\arg f; \qquad r_2(1):\{f\}_2\longmapsto \widehat{\mathcal{L}}_2(f)$$

To prove that we get a morphism of complexes we use Theorem 5.2. The following argument is needed to check the commutativity of the second square. The de Rham differential of the distribution $r_2(2)(f\wedge g)$ is

$$\mathcal{D}\Big(-\log|f|di\arg g+\log|g|di\arg f\Big)=$$

$$\pi_2(d\log f\wedge\log g)+2\pi i\cdot(\log|g|\delta(f)-\log|f|\delta(g))$$

This **does not** coincide with $r_2(3)\circ\partial(f\wedge g)$, but the difference is

$$(\mathcal{D}\circ r_2(2)-r_2(3)\circ\partial)(f\wedge g)=\pi_2(d\log f\wedge\log g)\in \quad (\mathcal{D}^{0,2}\oplus\mathcal{D}^{2,0})_{\mathbb{R}}(1)$$

Defining the differential D on the second group of the complex $\mathcal{C}_{\mathcal{D}}(X,\mathbb{R}(2))$ we take the de Rham differential and throw away from it precisely these components. Therefore the middle square is commutative.

5. The weight three. The weight three motivic complex $\Gamma(X;3)$ is the total complex of the following bicomplex: (the first group is in degree 1)

$$\begin{array}{ccccc} \mathcal{B}_3(\mathbb{Q}(X)) & \longrightarrow & \mathcal{B}_2(\mathbb{Q}(X))\otimes\mathbb{Q}(X)^* & \longrightarrow & \Lambda^3\mathbb{Q}(X)^* \\ & & \downarrow & & \downarrow \\ & & \oplus_{Y_1\in X^{(1)}}\mathcal{B}_2(\mathbb{Q}(Y_1)) & \longrightarrow & \oplus_{Y_1\in X^{(1)}}\Lambda^2\mathbb{Q}(Y_1)^* \\ & & & & \downarrow \\ & & & & \oplus_{Y_2\in X^{(2)}}\mathbb{Q}(Y_2)^* \\ & & & & \downarrow \\ & & & & \oplus_{Y_3\in X^{(3)}}\mathbb{Q}(Y_3)^* \end{array}$$

The Deligne complex $\mathcal{C}_{\mathcal{D}}(X,\mathbb{R}(3))$ looks as follows:

$$\begin{array}{cccccccc} & & & & & & & \mathcal{D}^{3,3}_{cl,\mathbb{R}}(3) \\ & & & & & & 2\overline{\partial}\partial\nearrow & \\ \mathcal{D}^{0,2} & \xrightarrow{\partial} & \mathcal{D}^{1,2} & \xrightarrow{\partial} & \mathcal{D}^{2,2} & & & \\ \uparrow\overline{\partial} & & \uparrow\overline{\partial} & & \uparrow\overline{\partial} & & & \\ \mathcal{D}^{0,1} & \xrightarrow{\partial} & \mathcal{D}^{1,1} & \xrightarrow{\partial} & \mathcal{D}^{1,2} & & & \\ \uparrow\overline{\partial} & & \uparrow\overline{\partial} & & \uparrow\overline{\partial} & & & \\ \mathcal{D}^{0,0} & \xrightarrow{\partial} & \mathcal{D}^{1,0} & \xrightarrow{\partial} & \mathcal{D}^{2,0} & & & \end{array}$$

We construct the regulator map $\Gamma(X;3)\longrightarrow\mathcal{C}_{\mathcal{D}}(X,\mathbb{R}(3))$ by setting

$$r_3(6): Y_3\longmapsto(2\pi i)^3\cdot\delta_{Y_3}; \qquad r_3(5):(Y_2,f)\longmapsto(2\pi i)^2\cdot\log|f|\delta_{Y_2}$$

$$r_3(4) : (Y_1, f \wedge g) \longmapsto 2\pi i \cdot (-\log|f| di \arg g + \log|g| di \arg f)\delta_{Y_1}$$

$$r_3(3) : (Y_1, \{f\}_2) \longmapsto 2\pi i \cdot \widehat{\mathcal{L}}_2(f)\delta_{Y_1}$$

$$r_3(3) : f_1 \wedge f_2 \wedge f_3 \longmapsto \mathrm{Alt}_3\Big(\frac{1}{6}\log|f_1| d\log|f_2| \wedge d\log|f_3| + \frac{1}{2}\log|f_1| di \arg f_2 \wedge di \arg f_3\Big)$$

$$r_3(2) : \{f\}_2 \otimes g \longmapsto \widehat{\mathcal{L}}_2(f) di \arg g - \frac{1}{3}\log|g| \cdot \Big(-\log|1-f| d\log|f| + \log|f| d\log|1-f|\Big)$$

$$r_3(1) : \{f\}_3 \longmapsto \widehat{\mathcal{L}}_3(f)$$

6. Classical polylogarithms and special values of ζ-functions of algebraic varieties. We conjecture that the polylogarithmic motivic complex $\Gamma(X;n) \otimes \mathbb{Q}$ should be quasiisomorphic to the weight n motivic complex, and Beilinson's regulator map under this quasiisomorphism should be equal to the defined above regulator map on $\Gamma(X;n)$. This implies that the special values of ζ-functions of algebraic varieties outside of the critical strip should be expressed via classical polylogarithms.

The very special case of this conjecture when $X = \mathrm{Spec}(F)$ where F is a number field is equivalent to Zagier's conjecture [Z1].

The next interesting case is when X is a curve over a number field. The conjecture in this case was elaborated in [G3], see also [G8] for a survey. For example if X is an elliptic curve this conjecture suggests that the special values $L(X,n)$ for $n \geq 2$ are expressed via certain generalized Eisenstein–Kronecker series. For $n = 2$ this was discovered of S. Bloch [Bl4], and for $n = 3$ it implies Deninger's conjecture [De].

A homomorphism from K-theory to $H^i(\Gamma(X;n)) \otimes \mathbb{Q}$ has been constructed in the following cases:

(1) F is an arbitrary field, $n \leq 3$ ([G1], [G2], [G5]]) and $n = 4$, $i > 1$ (to appear).

(2) X is a curve over a number field, $n \leq 3$ ([G5]) and $n = 4$, $i > 1$ (to appear).

In all these cases we proved that this homomorphism followed by the regulator map on polylogarithmic complexes (when $F = \mathbb{C}(X)$ in (1)) coincides with Beilinson's regulator. This proves the difficult "surjectivity" property: the image of the regulator map on these polylogarithmic complexes *contains* the image of Beilinson's regulator map in the Deligne cohomology. The main ingredient of the proof is an explicit construction of the motivic cohomology class

$$H^{2n}(BGL_{\bullet}, \Gamma(*;n))$$

The results (2) combined with the results of R. de Jeu [dJ1]- [dJ2] prove that the image of Beilinson's regulator map in the Deligne cohomology *coincides* with the image of the regulator map on polylogarithmic complexes in the case (2).

7. Coda: Polylogarithms on curves, Feynman integrals and special values of L-functions. The classical polylogarithm functions admit the following generalization. Let X be a regular complex algebraic curve. Let us assume first that X is projective. Let us choose a metric on $X(\mathbb{C})$. Denote by $G(x,y)$ the Green function provided by this metric. Then one can define real-valued functions $G_n(x,y)$ depending on a pair of points x, y of $X(\mathbb{C})$ with values in the complex vector space $S^{n-1}H_1(X(\mathbb{C}), \mathbb{C})(1)$, see [G10], Chapter 9.1. By definition $G_1(x,y)$ is the Green function $G(x,y)$. It has a singularity at the diagonal, but provides a generalized function. For $n > 1$ the function $G_n(x,y)$ is well-defined on $X(\mathbb{C}) \times X(\mathbb{C})$. Let $X = E$ be an elliptic curve. Then the functions $G_n(x,y)$ are translation invariant, and thus reduced to a signle variable functions: $G_n(x,y) = G_n(x-y)$. The function $G_n(x)$ is given by the classical Kronecker-Eisenstein series. Finally, adjasting this construction to the case when $X = \mathbb{C}^*$, we get a function $G_n(x,y)$ invariant under the shifts on the group $\mathbb{C}^*$, i.e. one has $G_n(x,y) = G_n(x/y)$. The function $G_n(z)$ is given by the single-valued version (39) of the classical n-logarithm function.

Let us extend the function $G_n(x,y)$ by linearity to a function $G_n(D_1, D_2)$ depending on a pair of divisors on $X(\mathbb{C})$. Although the function $G_n(x,y)$ depends on the choice of metric on $X(\mathbb{C})$, the restriction of the function $G_n(D_1, D_2)$ to the subgroups of the degree zero divisors is independent of the choice of the metric.

Let X be a curve over a number field F. In the Section 9.1 of [G10] we proposed a conjecture which allows to express the special values $L(\mathrm{Sym}^{n-1}H^1(X), n)$ via determinants whose entries are given by the values of the function $G_n(D_1, D_2)$, where D_1, D_2 are degree zero divisors on $(X(\overline{F}))$ invariant under the action of the Galois group $\mathrm{Gal}(\overline{F}/F)$. In the case when X is an elliptic curve it boils down to the so-called elliptic analog of Zagier's conjecture, see [Wil], [G11]. It has been proved for $n = 2$ at [GL], and a part of this proof (minus surjectivity) can be transformed to the case of arbitrary X.

It was conjectured in the Section 9.3 of [G10] that the special values $L(\mathrm{Sym}^{n-1}H^1(X), n + m - 1)$, where $m \geq 1$, should be expressed similarly via the special values of the depth m multiple polylogarithms on the curve X, defined in the Section 9.2 of loc. cit. One can show that in the case when $X = E$ is an elliptic curve, and $n = m = 2$, this conjecture reduces to Deninger's conjecture [De] on $L(E, 3)$, which has been proved in [G3] An interesting aspect of this story is that the multiple polylogarithms on curves are introduced via mathematically well defined Feynman integrals. This seems to be the first application of Feynman integrals in number theory.

6 Motivic Lie algebras and Grassmannian polylogarithms

1. Motivic Lie coalgebras and motivic complexes. Beilinson conjectured that for an arbitrary field F there exists an abelian $\mathbb{Q}$–category $\mathcal{M}_T(F)$ of mixed Tate motives over F. This category is supposed to be a mixed Tate category, see Section 8 of [G12] for the background. Then the Tannakian formalism implies that there exists a positively graded Lie coalgebra $\mathcal{L}_\bullet(F)$ such that the category of finite dimensional graded comodules over $\mathcal{L}_\bullet(F)$ is naturally equivalent to the category $\mathcal{M}_T(F)$.

Moreover $\mathcal{L}_\bullet(F)$ depends functorially on F. This combined with Beilinson's conjectural formula for the Ext groups in the category $\mathcal{M}_T(F)$ imply that the cohomology of this Lie coalgebra are computed by the formula

$$H^i_{(n)}(\mathcal{L}_\bullet(F)) = \mathrm{gr}^\gamma_n K_{2n-i}(F) \otimes \mathbb{Q} \tag{63}$$

where $H^i_{(n)}$ means the degree n part of H^i. This conjecture provides a new point of view on algebraic K-theory of fields, suggesting and explaining several conjectures and results, see ch. 1 [G2].

The degree n part of the standard cochain complex

$$\mathcal{L}_\bullet(F) \stackrel{\delta}{\longrightarrow} \Lambda^2\mathcal{L}_\bullet(F) \stackrel{\delta\wedge\mathrm{Id}-\mathrm{Id}\wedge\delta}{\longrightarrow} \Lambda^3\mathcal{L}_\bullet(F) \longrightarrow ...$$

of the Lie coalgebra $\mathcal{L}_\bullet(F)$ is supposed to be quasiisomorphic to the weight n motivic complex of F, providing yet another point of view on motivic complexes. For example the first four of the motivic complexes should look as follows:

$$\mathcal{L}_1(F); \qquad \mathcal{L}_2(F) \longrightarrow \Lambda^2\mathcal{L}_1(F); \qquad \mathcal{L}_3(F) \longrightarrow \mathcal{L}_2(F) \otimes \mathcal{L}_1(F) \longrightarrow \Lambda^3\mathcal{L}_1(F)$$

$$\mathcal{L}_4(F) \longrightarrow \mathcal{L}_3(F) \otimes \mathcal{L}_1(F) \oplus \Lambda^2\mathcal{L}_2(F) \longrightarrow \mathcal{L}_2(F) \otimes \Lambda^2\mathcal{L}_1(F) \longrightarrow \Lambda^4\mathcal{L}_1(F)$$

Comparing this with the formula (63), the shape of complexes (10) and (13), and the known information relating their cohomology with algebraic K-theory we conclude the following. One must have

$$\mathcal{L}_1(F) = F^* \otimes \mathbb{Q}; \quad \mathcal{L}_2(F) = B_2(F) \otimes \mathbb{Q}$$

and we expect to have an isomorphism

$$B_3(F) \otimes \mathbb{Q} \stackrel{\sim}{\longrightarrow} \mathcal{L}_3(F)$$

Moreover the motivic complexes (10) and (13) are simply the degree n parts of the standard cochain complex of the Lie coalgebra $\mathcal{L}_\bullet(F)$. More generally, the very existence for an arbitrary field F of the motivic classical polylogarithms (50) implies that one should have canonical homomorphisms

$$l_n : \mathcal{B}_n(F) \longrightarrow \mathcal{L}_n(F)$$

These homomorphisms are expected to satisfy the basic relation 52. Therefore the maps l_n give rise to a canonical homomorphism from the weight n polylogarithmic complex of F to the degree n part of the standard cochain complex of $\mathcal{L}_\bullet(F)$.

$$\begin{array}{ccccc} \mathcal{B}_n(F) & \longrightarrow & \mathcal{B}_{n-1}(F) \otimes F^* & \longrightarrow ... \longrightarrow & \Lambda^n F^* \\ \downarrow l_n & & \downarrow l_{n-1} \wedge l_1 & & \downarrow = \\ \mathcal{L}_n(F) & \longrightarrow & \Lambda^2_{(n)} \mathcal{L}_\bullet(F) & \longrightarrow ... \longrightarrow & \Lambda^n \mathcal{L}_1(F) \end{array} \quad (64)$$

where $\Lambda^2_{(n)}$ denotes the degree n part of $\Lambda^2_{(n)}$. For $n = 1, 2, 3$, these maps, combined with the ones (43), lead to the maps above. For $n > 3$ the map of complexes (64) will not be an isomorphism. The conjecture that it is a quisiisomorphism is equivalent to the Freeness conjecture for the Lie coalgebra $\mathcal{L}_\bullet(F)$, see [G1]-[G2].

Therefore we have two different points of view on the groups $B_n(F)$ for $n = 1, 2, 3$: according to one of them they are the particular cases of the groups $\mathcal{B}_n(F)$, and according to the other they provide an explicit computation of the first three of the groups $\mathcal{L}_n(F)$. It would be very interesting to find an explicit construction of the groups $\mathcal{L}_n(F)$ for $n > 3$ generalizing the definition of the groups $B_n(F)$. More specifically, we would like to have a "finite dimensional" construction of all vector spaces $\mathcal{L}_n(F)$, i.e. for every n there should exist a finite number of finite dimensional algebraic varieties X^i_n and R^j_n such that

$$\mathcal{L}_n(F) = \text{Coker}\left(\oplus_j \mathbb{Q}[R^j_n(F)] \longrightarrow \oplus_i \mathbb{Q}[X^i_n(F)]\right)$$

Such a construction would be provided by the scissor congruence motivic Hopf algebra of F ([BMS], [BGSV]). However so far the problem in the definition of the coproduct in loc. cit. for non generic generators has not been resolved. A beautiful construction of the motivic Lie coalgebra of a field F was suggested by Bloch and Kriz in [BKr]. However it is not finite dimensional in the above sense.

2. Grassmannian approach of the motivic Lie coalgebra. We suggested in [G5] that there should exist a construction of the Lie coalgebra $\mathcal{L}_\bullet(F)$ such that the variety X_n is the variety of configurations of $2n$ points in P^{n-1} and the relations varieties R^j_n are provided by the functional equations for the *motivic Grassmannian n-logarithm.* Let us explain this in more detail.

Let $\widetilde{\mathcal{L}}_n(F)$ be the free abelian group generated by 2n-tuples of points $(l_1, ..., l_{2n})$ in $P^{n-1}(F)$ subject to the following relations:

1)$(l_1, ..., l_{2n}) = (gl_1, ..., gl_{2n})$ for any $g \in PGL_n(F)$.

2) $(l_1, ..., l_{2n}) = (-1)^{|\sigma|}(l_{\sigma(1)}, ..., l_{\sigma(2n)})$ for any permutation $\sigma \in S_{2n}$.

3) for any $2n + 1$ points $(l_0, ..., l_{2n})$ in $P^{n-1}(F)$ one has

$$\sum_{i=0}^{2n} (-1)^i (l_0, ...\hat{l}_i, ..., , l_{2n}) = 0$$

4) for any $2n + 1$ points $(l_0, ..., l_{2n})$ in $P^n(F)$ one has

$$\sum_{i=0}^{2n} (-1)^i (l_i | l_0, ...\hat{l}_i, ..., , l_{2n}) = 0$$

We conjecture that $\mathcal{L}_n(F)$ is a quotient of $\widetilde{\mathcal{L}}_n(F)$. It is a nontrivial quotient already for $n = 3$. Then to define the Lie coalgebra $\mathcal{L}_\bullet(F)$ one needs to produce a cobracket

$$\delta : \mathcal{L}_n(F) \longrightarrow \oplus_i \mathcal{L}_i(F) \wedge \mathcal{L}_{n-i}(F)$$

Here is how to do this in the first nontrivial case, $n = 4$.

Let us define a homomorphism

$$\widetilde{\mathcal{L}}_4(F) \quad \stackrel{\delta}{\longrightarrow} \quad B_3(F) \otimes F^* \quad \oplus \quad B_2(F) \wedge B_2(F)$$

by setting $\delta = (\delta_{3,1}, \delta_{2,2})$ where

$$\delta_{3,1}(l_1, ..., l_8) := -\frac{1}{9}\mathrm{Alt}_8\Big(\Big(r_3(l_1|, l_2, l_3, l_4; l_5, l_6, l_7) + \{r(l_1 l_2|l_3, l_6, l_4, l_5)\}_3 -$$

$$-\{r(l_1 l_2|l_3, l_5, l_4, l_6)\}_3\Big) \wedge \Delta(l_5, l_6, l_7, l_8)\Big) \in B_3(F) \wedge F^*$$

$$\delta_{2,2}(l_1, ..., l_8) :=$$

$$\frac{1}{7} \cdot \mathrm{Alt}_8\Big(\{r_2(l_1, l_2|l_3, l_4, l_5, l_6)\}_2 \wedge \{r_2(l_3, l_4|l_1, l_2, l_5, l_7)\}_2\Big) \in \Lambda^2 B_2(F)$$

These formulae are obtained by combining the definitions at p. 136-137 and p 156 of [G5]. The following key result is Theorem 5.1 in loc. cit.

Theorem 6.1 *a) The homomorphism δ kills the relations 1) - 4).*

b) The following composition equals to zero:

$$\widetilde{\mathcal{L}}_4(F) \stackrel{\delta}{\longrightarrow} B_3(F) \otimes F^* \quad \oplus \quad B_2(F) \wedge B_2(F) \stackrel{\delta}{\longrightarrow} B_2(F) \otimes \Lambda^2 F^*$$

Here the second differential is defined by the Leibniz rule, using the differentials in complexes (10) and (13).

Taking the "connected component" of zero in Kerδ (like we did in s. 1 ch. 4 above, or in [G1]) we should get the set of defining relations for the group $\mathcal{L}_4(F)$. An explicit construction of them is not known.

4. The motivic Grassmannian tetralogarithm. Let F be a function field on a complex variety X. There is a morphism of complexes

$$\begin{array}{ccccc}
B_3(F) \otimes F^* \quad \oplus \quad B_2(F) \wedge B_2(F) & \stackrel{\delta}{\longrightarrow} & B_2(F) \otimes \Lambda^2 F^* & \stackrel{\delta}{\longrightarrow} & \Lambda^4 F^* \\
\downarrow R_4(2) & & \downarrow R_4(3) & & \downarrow R_4(4) \\
\mathcal{A}^1(SpecF) & \stackrel{d}{\longrightarrow} & \mathcal{A}^2(SpecF) & \stackrel{d}{\longrightarrow} & \mathcal{A}^3(SpecF)
\end{array}$$

extending the homomorphism $r_4(*)$ from Chapter 5. Namely, $R_4(*) = r_4(*)$ for $* = 3, 4$ and $R_4(2) = (r_4(2), r_4'(2))$ where

$$r_4'(2) : \Lambda^2 B_2(F) \longrightarrow \mathcal{A}^1(SpecF)$$

$$\{f\}_2 \wedge \{g\}_2 \mapsto \frac{1}{3} \cdot \Big(\widehat{\mathcal{L}}_2(g) \cdot \alpha(1-f, f) - \widehat{\mathcal{L}}_2(f) \cdot \alpha(1-g, g)\Big)$$

It follows from Theorem 6.1 that the composition $R_4(2) \circ \delta(l_1, ..., l_8)$ is a closed 1-form on the configuration space of 8 points in $\mathbb{C}P^3$. One can show (Proposition 5.3 in [G5]) that integrating this 1-form we get a single valued function on the configuration space, denoted $\mathcal{L}_4^M$ and called the *motivic Grassmannian tetralogarithm*. It would be very interesting to compute the difference $\mathcal{L}_4^M - \mathcal{L}_4^{\mathcal{L}}$, similarly to the formula (40) in the case $n = 3$. We expect that is expressed as a sum of products of functions $\mathcal{L}_3$, $\mathcal{L}_2$ and $\log | * |$.

Theorem 6.2 *a) There exists a canonical map*

$$K_7^{[3]}(F) \otimes \mathbb{Q} \quad \longrightarrow \quad \mathrm{Ker}\Big(\widetilde{\mathcal{L}}_4(F) \stackrel{\delta}{\longrightarrow} B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F)\Big)_{\mathbb{Q}}$$

b) In the case $F = \mathbb{C}$ the composition $K_7(\mathbb{C}) \longrightarrow \widetilde{\mathcal{L}}_4(\mathbb{C}) \stackrel{\mathcal{L}_4^M}{\longrightarrow} \mathbb{R}$ coincides with a nonzero rational multiple of the Borel regulator map.

The generalization of the above picture to the case $n > 4$ is unknown. It would be very interesting at least to define the motivic Grassmannian polylogarithms via the Grassmannian polylogarithms.

For $n = 3$ the motivic Grassmannian trilogarithm $\mathcal{L}_3^M$ is given by the first term of the formula (40). It is known ([GZ]) that $\mathcal{L}_3^{\mathcal{L}}$ does not provide a homomorphism $\mathcal{L}_3(\mathbb{C}) \longrightarrow \mathbb{R}$ since the second term in (40) does not have this property. Indeed, the second term in (40) vanishes on the special configuration of 6 points in P^2, but does not vanish at the generic configuration. On the other hand the defining relations in $\mathcal{L}_3(F)$ allow to express any configuration as a linear combination of the special ones.

We expect the same situation in general: for $n > 2$ the Grassmannian n–logarithms should not satisfy all the functional equations for the motivic Grassmannian n–logarithms. It would be very interesting to find a conceptual explanation of this surprising phenomena.

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