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ACTIONS OF THE DERIVED GROUP OF A MAXIMAL UNIPOTENT SUBGROUP ON G-VARIETIES

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INTRODUCTION

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Let G be a semisimple simply-connected algebraic group over $\Bbbk$ and U a maximal unipotent subgroup of G. One of the fundamental invariant-theoretic facts, which goes back to Hadžiev [9], is that $\Bbbk[G/U]$ is a finitely generated $\Bbbk$-algebra and regarded as G-module it contains every finite-dimensional simple G-module exactly once. From this, one readily deduces that the algebra of U-invariants, $\Bbbk[G/U]^U$, is polynomial. More precisely, choose a maximal torus $T \subset \mathsf{Norm}_G(U)$. Let r be the rank of G, $\varpi_1, \dots, \varpi_r$ the fundamental weights of T corresponding to U, and $\alpha_1, \dots, \alpha_r$ the respective simple roots. Set $\mathfrak{X}_+ = \sum_{i=1}^r \mathbb{N}\varpi_i$, and let $\mathsf{R}(\lambda)$ denote the simple G-module with highest weight $\lambda \in \mathfrak{X}_+$. Then

$$\Bbbk[G/U] \simeq \bigoplus_{\lambda \in \mathfrak{X}_+} \mathsf{R}(\lambda).$$

Let f_i be a non-zero element of one-dimensional space $\mathsf{R}(\varpi_i)^U \subset \Bbbk[G/U]^U$. Then $\Bbbk[G/U]^U$ is freely generated by $f_1, \dots, f_r$.

For an affine G-variety X, the algebra of U-invariants, $\Bbbk[X]^U$, is multigraded (by T-weights). If $X = V$ is a G-module, then there is an integral formula for the corresponding Poincaré series [4, Theorem 1]. Using that formula, M. Brion discovered useful "symmetries" of the Poincaré series and applied them (in case G is simple) to obtaining the classification of simple G-modules with polynomial algebras $\Bbbk[V]^U$ [4, Ch. III]. Afterwards, I proved that similar "symmetries" of Poincaré series occur for conical factorial G-varieties with only rational singularities [16], [17, Ch. 5]. Since there is no integral formula for Poincaré series in general, another technique was employed. Namely, I used the transfer principle for U, "symmetries" of the Poincaré series of $\Bbbk[G/U]$, and results of F. Knop relating the canonical module of an algebra and a subalgebra of invariants [13].

Our objective is to extend these results to the derived group $U' = (U, U)$. In Section 1, we prove that $\mathsf{R}(\lambda)^{U'}$ is a cyclic U/U'-module for any $\lambda \in \mathfrak{X}_+$ and $\dim \mathsf{R}(\lambda)^{U'} = \prod_{i=1}^r ((\lambda, \alpha_i^\vee) + 1)$, where $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$, see Theorem 1.6. From these properties, we deduce that $\Bbbk[G/U]^{U'}$ is a polynomial algebra of Krull dimension $2r$. More precisely, we have $\dim \mathsf{R}(\varpi_i)^{U'} = 2$ for each i, and if $(f_i, \tilde{f}_i)$ is a basis in $\mathsf{R}(\varpi_i)^{U'}$, then $\{f_i, \tilde{f}_i \mid i = 1, \dots, r\}$

freely generate $\Bbbk[G/U]^{U'}$ (see Theorem 1.8). This fact seems to have remained unnoticed before. As a by-product, we show that the subgroup $TU' \subset G$ is epimorphic (i.e., $\Bbbk[G]^{TU'} = \Bbbk$) if and only if $G \ne SL_2, SL_3$.

Section 2 is devoted to general properties of U'-actions on affine G-varieties. We show that $\Bbbk[G/U']$ is generated by fundamental G-modules sitting in it, and using this fact we explicitly construct an equivariant affine embedding of G/U' with the boundary of codimension $\geqslant 2$ (Theorem 2.2). Since $\Bbbk[G/U']$ is finitely generated, $\Bbbk[X]^{U'}$ is finitely generated for any affine G-variety X [8]. Furthermore, $\mathsf{Spec}(\Bbbk[X]^{U'})$ inherits some other good properties of X (factoriality, rationality of singularities) (Theorem 2.3). We also give an algorithm for constructing a finite generating system of $\Bbbk[X]^{U'}$, if generators of $\Bbbk[X]^U$ are already known (Theorem 2.4). This appears to be very helpful in classifying simple G-modules with polynomial algebras of U'-invariants (for G simple).

In Section 3, we study the Poincaré series of multigraded algebras $\Bbbk[X]^{U'}$, where X is factorial affine G-variety with only rational singularities (e.g. X can be a G-module). Assuming that $G \ne SL_2, SL_3$, we obtain analogues of our results for Poincaré series of $\Bbbk[X]^U$. One of the practical outcomes concerns the case in which V is a G-module and $\Bbbk[V]^{U'}$ is polynomial. If $d_1, \dots, d_m$ (resp. $\mu_1, \dots, \mu_m$) are the degrees (resp. T-weights) of basic U'-invariants, then $\sum_i d_i \leqslant \dim V$ and $\sum_i \mu_i \leqslant 2\rho - \sum_{j=1}^r \alpha_j$, where $\rho = \sum_{j=1}^r \varpi_j$. The second inequality requires some explanations, though. Unlike the case of U-invariants, there is no natural free monoid containing the T-weights of all U'-invariants. But for $G \ne SL_2, SL_3$, these T-weights generate a convex cone. Therefore, such a free monoid does exist, and the above inequality for $\sum_i \mu_i$ is understood as componentwise inequality with respect to any such monoid and its basis. Moreover, $\sum_i d_i = \dim V$ if and only if $\sum_i \mu_i = 2\rho - \sum_{j=1}^r \alpha_j$. Again, these relations are to be useful for our classification of polynomial algebras $\Bbbk[V]^{U'}$, which is obtained in Section 5. Note that $2\rho - \sum_{j=1}^r \alpha_j$ is the sum of all positive non-simple roots, i.e., the roots of U'.

Section 4 is a kind of combinatorial digression. Let $\mathcal{C}$ be the cone generated by all T-weights occurring in $\Bbbk[G/U]^{U'}$. Our description of generators shows that $\mathcal{C}$ is actually generated by $\varpi_i, \varpi_i - \alpha_i$ ($i = 1, \dots, r$). We prove that the dual cone of $\mathcal{C}$ is generated by the non-simple positive roots (Theorem 4.2). We also obtain a partition of $\mathcal{C}$ in simplicial cones, which is parametrised by the disjoint subsets on the Dynkin diagram of G.

My motivation to consider U'-invariants arose from attempts to understand the structure of centralisers of certain nilpotent elements in simple Lie algebras. For applications to centralisers one needs Theorem 1.6 in case of SL_3, and this was the result initially proved. This application will be the subject of a subsequent article.

Notation. If an algebraic group Q acts on an irreducible affine variety X, then

- $Q_x = \{q \in Q \mid q{\cdot}x = x\}$ is the stabiliser of $x \in X$;

• $\Bbbk[X]^Q$ is the algebra of Q-invariant polynomial functions on X. If $\Bbbk[X]^Q$ is finitely generated, then $X/\!\!/Q := \mathsf{Spec}(\Bbbk[X]^Q)$, and the *quotient morphism* $\pi_{X,Q} : X \to X/\!\!/Q$ is the mapping associated with the embedding $\Bbbk[X]^Q \hookrightarrow \Bbbk[X]$.

• $\Bbbk(X)^Q$ is the field of Q-invariant rational functions;

Throughout, G is a semisimple simply-connected algebraic group and $r = \mathsf{rk}\, G$.

– Δ is the root system of (G, T), $\Pi = \{\alpha_1, \dots, \alpha_r\}$ are the simple roots corresponding to U, and $\varpi_1, \dots, \varpi_r$ are the corresponding fundamental weights.

– The character group of T is denoted by $\mathfrak{X}$. All roots and weights are regarded as elements of the r-dimensional vector space $\mathfrak{X} \otimes \mathbb{Q} =: \mathfrak{X}_{\mathbb{Q}}$. For any $\lambda \in \mathfrak{X}_+$, λ^* is the highest weight of the dual G-module. The μ-weight space of $\mathsf{R}(\lambda)$ is denoted by $\mathsf{R}(\lambda)_\mu$.

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1. The algebra of U'-invariants on G/U

For any $\lambda \in \mathfrak{X}_+$, we wish to study the subspace $\mathsf{R}(\lambda)^{U'}$. First of all, we notice that $B \subset \mathsf{Norm}_G(U')$ (actually, they are equal if G has no simple factors SL_2) and therefore $\mathsf{R}(\lambda)^{U'}$ is a B/U'-module. In particular, T normalises U' and hence $\mathsf{R}(\lambda)^{U'}$ is a direct sum of its own weight spaces. Let $\mathcal{P}(\lambda)$ be the set of weights of $\mathsf{R}(\lambda)$. It is a poset with respect to the *root order*. This means that μ covers ν if $\mu - \nu \in \Pi$. Then λ is the unique maximal element of $\mathcal{P}(\lambda)$. Let $e_i \in \mathfrak{g} = \mathrm{Lie}\,(G)$ be a root vector corresponding to $\alpha_i \in \Pi$.

Given a nonzero $x \in \mathsf{R}(\lambda)^{U'}$, consider

$$M_x = \{(n_1, \dots, n_r) \in \mathbb{N}^r \mid e_1^{n_1} \dots e_r^{n_r}(x) \neq 0\}.$$

We also write $\boldsymbol{n} = (n_1, \dots, n_r)$ and $\boldsymbol{e}^{\boldsymbol{n}} = e_1^{n_1} \dots e_r^{n_r}$. Notice that $\boldsymbol{e}^{\boldsymbol{n}}(x)$ does not depend on the ordering of e_i's since $[e_i, e_j] \in \mathrm{Lie}\,(U')$ for all i, j and $\mathsf{R}(\lambda)^{U'}$ is an U/U'-module. We regard M_x as poset with respect to the componentwise inequalities, i.e., $\boldsymbol{n} \succcurlyeq \boldsymbol{n'}$ if and only if $n_i \geqslant n_i'$ for all i. Clearly, M_x is finite and $(0, \dots, 0)$ is the unique minimal element of it.

Lemma 1.1. *Let $x \in \mathsf{R}(\lambda)^{U'}$ be a weight vector. The poset M_x contains a unique maximal element, say $\boldsymbol{m} = (m_1, \dots, m_r)$. Furthermore, $\boldsymbol{e}^{\boldsymbol{m}}(x)$ is a highest vector of $\mathsf{R}(\lambda)$.*

Proof. If $\boldsymbol{n} \in M_x$ is maximal, then $e_i(\boldsymbol{e}^{\boldsymbol{n}}(x)) = 0$ for each i. Hence $\boldsymbol{e}^{\boldsymbol{n}}(x)$ is a highest vector of $\mathsf{R}(\lambda)$. Next,

$$\text{the weight of } \boldsymbol{e}^{\boldsymbol{n}}(x) = (\text{the weight of } x) + \sum_{i=1}^{r} n_i \alpha_i \,.$$

Hence all nonzero vectors of the form $\boldsymbol{e}^{\boldsymbol{n}}(x)$ are linearly independent. This yields the uniqueness of a maximal element. □

Corollary 1.2. *M_x is a multi-dimensional array, i.e., $M_x = \{(n_1, \dots, n_r) \mid 0 \leqslant n_i \leqslant m_i \ \forall i\}$.*

Let I_λ denote the set of T-weights in $\mathsf{R}(\lambda)^{U'}$. It is a subset of $\mathcal{P}(\lambda)$.

Proposition 1.3. *For any $\lambda \in \mathfrak{X}_+$, $\mathsf{R}(\lambda)^{U'}$ is a multiplicity free T-module. More precisely,*

$$\mathsf{R}(\lambda)^{U'} = \bigoplus_{\mu \in I_\lambda} \mathsf{R}(\lambda)^{U'}_\mu ,$$

where $\dim \mathsf{R}(\lambda)^{U'}_\mu = 1$ for each μ and $I_\lambda \subset \{\lambda - \sum_i a_i\alpha_i \mid 0 \leqslant a_i \leqslant (\lambda, \alpha_i^\vee)\}$.

Proof. If $x \in \mathsf{R}(\lambda)^{U'}_\mu$ and $(m_1, \dots, m_r)$ is the maximal element of M_x, then $\mu + \sum_i m_i\alpha_i = \lambda$ and $\mu + \sum_i n_i\alpha_i \in \mathcal{P}(\lambda)$ for any $(n_1, \dots, n_r) \in M_x$. In particular, $\lambda - m_i\alpha_i \in \mathcal{P}(\lambda)$. Whence $m_i \leqslant (\lambda, \alpha_i^\vee)$ and $I_\lambda \subset \{\lambda - \sum_i a_i\alpha_i \mid 0 \leqslant a_i \leqslant (\lambda, \alpha_i^\vee)\}$.

Assume that $x, y \in \mathsf{R}(\lambda)^{U'}_\mu$ are linearly independent. It follows from Lemma 1.1 that $M_x = M_y$. Since $\boldsymbol{e}^{\boldsymbol{m}}(x), \boldsymbol{e}^{\boldsymbol{m}}(y) \in \mathsf{R}(\lambda)_\lambda$, we have $\boldsymbol{e}^{\boldsymbol{m}}(x - cy) = 0$ for some $c \in \Bbbk^\times$. This means that $M_{x-cy} \neq M_x$, a contradiction! Thus, each $\mathsf{R}(\lambda)^{U'}_\mu$ is one-dimensional. □

Lemma 1.4. *I_λ is a connected subset in the Hasse diagram of $\mathcal{P}(\lambda)$ that contains λ.*

Proof. Indeed, suppose $0 \neq v \in \mathsf{R}(\lambda)^{U'}_\mu$. If $e_{\alpha_i}{\cdot}v = 0$ for all i, then v is a U-invariant and hence $\mu = \lambda$. Otherwise, we have $e_{\alpha_i}{\cdot}v \neq 0$ for some i and therefore $\mu + \alpha_i$ is also a weight of $\mathsf{R}(\lambda)^{U'}$. Then we argue by induction. □

Proposition 1.5. *For any fundamental weight ϖ_i, we have $\mathsf{R}(\varpi_i)^{U'} = \mathsf{R}(\varpi_i)_{\varpi_i} \oplus \mathsf{R}(\varpi_i)_{\varpi_i - \alpha_i}$. In particular, $I_{\varpi_i} = \{\varpi_i, \varpi_i - \alpha_i\}$ and $\dim \mathsf{R}(\varpi_i)^{U'} = 2$.*

Proof. Note that $\varpi_i - \alpha_i \in \mathcal{P}(\varpi_i)$ and $\dim \mathsf{R}(\varpi_i)_{\varpi_i - \alpha_i} = 1$, while $\varpi_i - 2\alpha_i \notin \mathcal{P}(\varpi_i)$. We obviously have $\mathsf{R}(\varpi_i)^{U'} \supset \mathsf{R}(\varpi_i)_{\varpi_i} \oplus \mathsf{R}(\varpi_i)_{\varpi_i - \alpha_i}$. Any weight of $\mathsf{R}(\varpi_i)$ covered by $\varpi_i - \alpha_i$ is of the form $\varpi_i - \alpha_i - \alpha_j$, where α_j is a simple root adjacent to α_i in the Dynkin diagram of G. Since $\varpi_i - \alpha_j \notin \mathcal{P}(\varpi_i)$, Kostant's weight multiplicity formula shows that $\dim \mathsf{R}(\varpi_i)_{\varpi_i - \alpha_i - \alpha_j} = 1$. Since $\alpha_i + \alpha_j$ is a root of U', we have $\mathsf{R}(\varpi_i)_{\varpi_i - \alpha_i - \alpha_j} \not\subset \mathsf{R}(\varpi_i)^{U'}$ and it follows from Lemma 1.4 that there cannot be anything else in $\mathsf{R}(\varpi_i)^{U'}$. □

Set $\tilde{X} = \mathsf{Spec}(\Bbbk[G]^U)$. It is an affine G-variety containing G/U as a dense open subset. Recall that $\tilde{X}$ has the following explicit model, see [25]. Let $v_{-\varpi_i}$ be a lowest weight vector in $\mathsf{R}(\varpi_i)^*$. Then the stabiliser of $(v_{-\varpi_1}, \dots, v_{-\varpi_r}) \in \mathsf{R}(\varpi_1)^* \oplus \ldots \oplus \mathsf{R}(\varpi_r)^*$ is the maximal unipotent subgroup that is opposite to U and

$$\tilde{X} \simeq \overline{G{\cdot}(v_{-\varpi_1}, \dots, v_{-\varpi_r})} \subset \mathsf{R}(\varpi_1)^* \oplus \ldots \oplus \mathsf{R}(\varpi_r)^* .$$

Let $p_i : \tilde{X} \to \mathsf{R}(\varpi_i)^*$ be the projection to the i-th component. Then the pull-back of the linear functions on $\mathsf{R}(\varpi_i)^*$ yields the unique copy of the G-module $\mathsf{R}(\varpi_i)$ in $\Bbbk[\tilde{X}]$. The additive decomposition $\Bbbk[\tilde{X}] = \bigoplus_{\lambda \in \mathfrak{X}_+} \mathsf{R}(\lambda)$ is a polygrading; i.e., if $f_i \in \mathsf{R}(\lambda_i) \subset \Bbbk[\tilde{X}]$, $i = 1, 2$, then $f_1 f_2 \in \mathsf{R}(\lambda_1 + \lambda_2)$.

Definition 1. Let Q be an algebraic group with Lie algebra $\mathfrak{q}$. A Q-module V is said to be *cyclic* if there is $v \in V$ such that $\mathcal{U}(\mathfrak{q}){\cdot}v = V$, where $\mathcal{U}(\mathfrak{q})$ is the enveloping algebra of $\mathfrak{q}$. Such v is called a *cyclic vector*.

Theorem 1.6. *For any $\lambda \in \mathfrak{X}_+$, we have*

(i) $I_\lambda = \{\lambda - \sum_{i=1}^r a_i\alpha_i \mid 0 \leqslant a_i \leqslant (\lambda, \alpha_i^\vee)\}$;
(ii) $\mathsf{R}(\lambda)^{U'}$ *is a cyclic U/U'-module of dimension $\prod_{i=1}^r((\lambda, \alpha_i^\vee) + 1)$. Up to a scalar multiple, there is a unique cyclic vector that is a T-eigenvector.*

Proof. In view of Lemma 1.1 and Proposition 1.3, it suffices to prove that $\mathsf{R}(\lambda)^{U'}$ contains a vector of weight $\lambda - \sum_{i=1}^r(\lambda, \alpha_i^\vee)\alpha_i$. This vector have to be cyclic, because applying the e_i's to it we obtain weight vectors with all weights from $\{\lambda - \sum_{i=1}^r a_i\alpha_i \mid 0 \leqslant a_i \leqslant (\lambda, \alpha_i^\vee)\}$, hence the whole of $\mathsf{R}(\lambda)^{U'}$.

Let $\tilde{f}_i$ be a nonzero vector in one-dimensional space $\mathsf{R}(\varpi_i)_{\varpi_i-\alpha_i}$. Using the unique copy of $\mathsf{R}(\varpi_i)$ inside $\Bbbk[\tilde{X}]$, we regard $\tilde{f}_i$ as U'-invariant polynomial function on $\tilde{X}$. Take the product (monomial) $F := \prod_{i=1}^r \tilde{f}_i^{(\lambda,\alpha_i^\vee)} \in \Bbbk[\tilde{X}]$. Since $\Bbbk[\tilde{X}]$ is a domain, $F \neq 0$. The multiplicative structure of $\Bbbk[\tilde{X}]$ shows that $F \in \mathsf{R}(\lambda)^{U'}$ and the weight of F equals $\sum_{i=1}^r(\lambda, \alpha_i^\vee)(\varpi_i - \alpha_i) = \lambda - \sum_{i=1}^r(\lambda, \alpha_i^\vee)\alpha_i$. □

Remark 1.7. For the group $TU' \subset B$, we have $\dim TU' = \dim U$. It is well known that TU' is a spherical subgroup of G (e.g. apply [5, Prop. 1.1]). The sphericity also follows from the fact $\mathsf{R}(\lambda)^{U'}$ is a multiplicity free T-module (Proposition 1.3). That $\mathsf{R}(\lambda)^{U'}$ is a multiplicity free T-module follows also from [10, Corollary 8]. However, we obtain the explicit description of the corresponding weights and the U/U'-module structure of $\mathsf{R}(\lambda)^{U'}$.

Theorem 1.8. *Let f_i (resp. $\tilde{f}_i$) be a nonzero vector in one-dimensional space $\mathsf{R}(\varpi_i)_{\varpi_i}$ (resp. $\mathsf{R}(\varpi_i)_{\varpi_i-\alpha_i}$). Then the algebra of U'-invariants, $\Bbbk[G/U]^{U'}$, is freely generated by $f_1, \tilde{f}_1, \dots, f_r, \tilde{f}_r$.*

Proof. It follows from (the proof of) Theorem 1.6 that the monomials $\prod_{i=1}^r f_i^{c_i}\tilde{f}_i^{(\lambda,\alpha_i^\vee)-c_i}$, $0 \leqslant c_i \leqslant (\lambda, \alpha_i^\vee)$, form a basis for $\dim \mathsf{R}(\lambda)^{U'}$ for each $\lambda \in \mathfrak{X}_+$. Hence $\Bbbk[G/U]^{U'}$ is generated by $f_1, \tilde{f}_1, \dots, f_r, \tilde{f}_r$. Since U' is unipotent and $\dim(G/U) - \dim U' = 2r$, the Krull dimension of $\Bbbk[G/U]^{U'}$ is at least $2r$. Hence there is no relations between the above generators. □

Recall that a closed subgroup $H \subset G$ is said to be *epimorphic* if $\Bbbk[G/H] = \Bbbk$ or, equivalently, $\mathsf{R}(\lambda)^H = \{0\}$ unless $\lambda = 0$, see e.g. [8, § 23B].

Proposition 1.9. *Suppose G is simple. The subgroup TU' is epimorphic if and only if $G \neq SL_2$ or SL_3.*

Proof. The case of SL_2 is obvious, so we assume that $r \geqslant 2$. In view of Theorem 1.8, we have to check that neither of the monomials $\prod_{i=1}^r (f_i^{c_i}\tilde f_i^{(\lambda,\alpha_i^\vee)-c_i})$, $0 \leqslant c_i \leqslant (\lambda,\alpha_i^\vee)$, has zero weight if $G \neq SL_3$. The weight in question equals

$$\mu := \sum_{i=1}^r (\lambda,\alpha_i^\vee)\varpi_i - \sum_{i=1}^r c_i\alpha_i\ .$$

Set $\rho^\vee = \frac12 \sum\limits_{\gamma\in\Delta^+}\gamma^\vee$. Then $(\mu,\rho^\vee) = \sum_{i=1}^r(\lambda,\alpha_i^\vee)(\varpi_i,\rho^\vee) - \sum_{i=1}^r c_i$. Notice that

$$2(\varpi_i,\rho^\vee) = \sum_{\gamma\in\Delta^+}(\varpi_i,\gamma^\vee) \geqslant \#\{\gamma\in\Delta^+ \mid (\gamma,\varpi_i)>0\}.$$

That is, $2(\varpi_i,\rho^\vee)$ is at least the dimension of the nilpotent radical of the maximal parabolic subalgebra corresponding to ϖ_i. This readily implies that $(\varpi_i,\rho^\vee)>1$ for all i whenever $\mathfrak g \neq \mathfrak{sl}_3$. Whence $(\mu,\rho^\vee)$ is positive.

For SL_3, the monomial $(\tilde f_1\tilde f_2)^a$ has zero weight. That is, $\mathsf{R}(a(\varpi_1+\varpi_2))^{TU'} \neq \{0\}$. $\square$

Remark 1.10. If $G = SL_3$, then TU' is a Borel subgroup of a reductive subgroup $GL_2 \subset SL_3$. Proposition 1.9 can also be deduced from a result of Pommerening [18, Korollar 3.6].

Example 1.11. Let U_n be a maximal unipotent subgroup of $G = SL_n$ and let U_{n-1} be a maximal unipotent subgroup of a standardly embedded group $SL_{n-1} \subset SL_n$. It is well known that $\Bbbk[SL_n/U_n]^{U_{n-1}}$ is a polynomial algebra of Krull dimension $2(n-1)$ and its generators have a simple description, see e.g. [1, Sect. 3]. The reason is that SL_n/U_n is a spherical SL_{n-1}-variety and the branching rule $SL_n \downarrow SL_{n-1}$ is rather simple. That is, $\Bbbk[SL_n/U_n]^{U_{n-1}}$ and $\Bbbk[SL_n/U_n]^{U_n'}$ are polynomial rings of the same dimension, and also $\dim U_{n-1} = \dim U_n'$. However, the subgroups $U_n', U_{n-1} \subset SL_n$ are essentially different unless $n = 2, 3$.

2. Some properties of algebras of U'-invariants

The main result of Section 1 says that $\Bbbk[G/U]^{U'}$ is a polynomial algebra of Krull dimension $2r$. This can also be understood in the other way around, since $\Bbbk[G/U]^{U'}$ and $\Bbbk[G/U']^U$ are canonically isomorphic. Indeed, for any closed subgroup $H \subset G$, we regard $\Bbbk[G/H]$ as subalgebra of $\Bbbk[G]$:

$$\Bbbk[G/H] = \{f \in \Bbbk[G] \mid f(gh) = f(g) \text{ for any } g\in G, h\in H\}.$$

Any subgroup of G acts on G/H by left translations. Therefore

$$\Bbbk[G/U]^{U'} \simeq \{f \in \Bbbk[G] \mid f(u_1gu_2) = f(g) \text{ for any } g\in G, u_1\in U', u_2\in U\},$$
$$\Bbbk[G/U']^{U} \simeq \{f \in \Bbbk[G] \mid f(u_2gu_1) = f(g) \text{ for any } g\in G, u_1\in U', u_2\in U\}.$$

The involutory mapping $(f \in \Bbbk[G]) \mapsto \hat f$, where $\hat f(g) = f(g^{-1})$, takes $\Bbbk[G/U]^{U'}$ to $\Bbbk[G/U']^U$, and vice versa.

One can deduce some properties of $\Bbbk[G/U']$ using the known structure of $\Bbbk[G/U']^U$. Set $\mathcal{A} = \Bbbk[G/U']$. It is a rational G-algebra, which can be decomposed as G-module:

$$\mathcal{A} = \bigoplus_{\lambda \in \mathfrak{X}_+} m_{\lambda,\mathcal{A}} \mathsf{R}(\lambda).$$

By Frobenius reciprocity, the multiplicity $m_{\lambda,\mathcal{A}}$ is equal to $\dim \mathsf{R}(\lambda^*)^{U'}$. Therefore, it is finite. In our situation,

$$\dim \mathsf{R}(\lambda^*)^{U'} = \dim \mathsf{R}(\lambda)^{U'} = \prod_{i=1}^{r}((\lambda, \alpha_i^\vee) + 1).$$

In particular, $m_{\varpi_i,\mathcal{A}} = 2$ for any i. One can also argue as follows.

The group $G \times G$ acts on G by left and right translations and the decomposition of $\Bbbk[G]$ as $G \times G$-module is of the form:

$$\Bbbk[G] = \bigoplus_{\lambda \in \mathfrak{X}_+} \mathsf{R}(\lambda^*) \otimes \mathsf{R}(\lambda)\ ,$$

where the first (resp. second) copy of G in $G \times G$ acts on the first (resp. second) factor of tensor product in each summand [14, Ch. 2, § 3, Theorem 3]. Then

$$\mathcal{A} = \Bbbk[G/U'] = \bigoplus_{\lambda \in \mathfrak{X}_+} \mathsf{R}(\lambda^*) \otimes \mathsf{R}(\lambda)^{U'}, \tag{2·1}$$

$$\mathcal{A}^U = \bigoplus_{\lambda \in \mathfrak{X}_+} \mathsf{R}(\lambda^*)^U \otimes \mathsf{R}(\lambda)^{U'}. \tag{2·2}$$

In this context, Theorem 1.8 asserts that any basis of the $2r$-dimensional vector space $\bigoplus_{i=1}^r \mathsf{R}(\varpi_i^*)^U \otimes \mathsf{R}(\varpi_i)^{U'}$ freely generates the polynomial algebra $\mathcal{A}^U$. It is known that $\Bbbk[G/U']$ is finitely generated (see [7, Theorem 7]). Below, we obtain a more precise assertion.

Lemma 2.1. *$\mathcal{A}$ is generated by the copies of fundamental G-modules, i.e., by the subspace $\bigoplus_{i=1}^r \mathsf{R}(\varpi_i^*) \otimes \mathsf{R}(\varpi_i)^{U'}$.*

Proof. We know that $\mathcal{A}^U = \Bbbk[G/U']^U$ is a polynomial algebra, generated by $2r$ functions. Using Equations (2·1) and (2·2), one sees that the generators of $\mathcal{A}^U$ are just the highest vectors of all fundamental G-module sitting in $\mathcal{A}$. It follows that the subalgebra of $\mathcal{A}$ generated by all fundamental G-modules is G-stable and contains the highest vectors of all simple G-modules inside $\mathcal{A}$. Hence it is equal to $\mathcal{A}$. □

For a quasi-affine G/H, it is known that $\Bbbk[G/H]$ is finitely generated if and only if there is a G-equivariant embedding $i : G/H \to V$, where V is a finite-dimensional G-module, such that the boundary of $i(G/H)$ is of codimension $\geqslant 2$ [8, § 4]. As U' is unipotent, G/U' is quasi-affine. Hence such an embedding of G/U' exists and, making use of Lemma 2.1, we explicitly construct it.

Recall that f_i and $\tilde{f}_i$ are nonzero weight vectors in $\mathsf{R}(\varpi_i)_{\varpi_i}$ and $\mathsf{R}(\varpi_i)_{\varpi_i-\alpha_i}$, respectively.

Theorem 2.2. *Let* $p = (f_1, \tilde{f}_1, \dots, f_r, \tilde{f}_r) \in 2\mathsf{R}(\varpi_1) \oplus \dots \oplus 2\mathsf{R}(\varpi_r)$. *Then*

(i) $G_p = U'$;
(ii) $\Bbbk[\overline{G{\cdot}p}] = \Bbbk[G/U']$ *and* $\overline{G{\cdot}p} \simeq \mathsf{Spec}(\mathcal{A})$ *is normal;*
(iii) $\operatorname{codim}(\overline{G{\cdot}p} \setminus G{\cdot}p) \geqslant 2$.

Proof. Part (i) is obvious. Then $G{\cdot}p \simeq G/U'$ and hence $\mathcal{B} := \Bbbk[\overline{G{\cdot}p}]$ is a subalgebra of $\mathcal{A}$. By the very construction, $m_{\varpi_i,\mathcal{B}} \geqslant 2$. (Consider different non-trivial projections $\overline{G{\cdot}p} \to \mathsf{R}(\varpi_i)$ for all i.) Since $m_{\varpi_i,\mathcal{B}} \leqslant m_{\varpi_i,\mathcal{A}} = 2$ and $\mathcal{A}$ is generated by the fundamental G-modules, we must have $\mathcal{B} = \mathcal{A}$. This yields the rest. □

Let X be an algebraic variety equipped with a regular action of G. Then X is said to be a G-variety. The "transfer principle" ([3, Ch. 1], [20, § 3], [8, § 9]) asserts that

$$\Bbbk[X]^H \simeq (\Bbbk[X] \otimes \Bbbk[G/H])^G$$

for any affine G-variety X and any subgroup $H \subset G$. In particular, if $\Bbbk[G/H]$ is finitely generated, then so is $\Bbbk[X]^H$. In view of Lemma 2.1, this applies to $H = U'$, hence $\Bbbk[X]^{U'}$ is always finitely-generated. Moreover, the polynomiality of $\Bbbk[G/U']^U$ implies that $\Bbbk[X]^{U'}$ inherits a number of other good properties from $\Bbbk[X]$. Recall that $\mathsf{Spec}(\Bbbk[X]^{U'})$ is denoted by $X/\!\!/U'$; hence $\Bbbk[X/\!\!/U']$ and $\Bbbk[X]^{U'}$ are the same objects.

We often use below the notion of a *variety with rational singularities*. Let us provide some relevant information for the affine case.

a) If $\phi : \tilde{X} \to X$ is a resolution of singularities, then X is said to have rational singularities if $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \Bbbk[X]$ and $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for $i \geqslant 1$. In particular, X is necessarily normal.

b) If X has only rational singularities and G is a reductive group acting on X, then $X/\!\!/G$ has only rational singularities (Boutot [2]).

c) If X has only rational singularities, then X is Cohen-Macaulay (Kempf [12]). It follows that if X is factorial and has rational singularities, then X is Gorenstein.

Theorem 2.3. *Let X be an irreducible affine G-variety. If X has only rational singularities, then so has $X/\!\!/U'$. Furthermore, if X is factorial, then $X/\!\!/U'$ is factorial, too.*

Proof. This is a straightforward consequence of known technique. Since $\Bbbk[G/U']^U$ is a polynomial algebra, $G/\!\!/U'$ has rational singularities by Kraft's theorem [3, Theorem 1.6], [20]. By the transfer principle for $H = U'$, we have $X/\!\!/U' \simeq \big(X \times (G/\!\!/U')\big)/\!\!/G$. Applying Boutot's theorem [2] to the right-hand side, we conclude that $X/\!\!/U'$ has rational singularities. The second assertion stems from the fact that U' has no non-trivial rational characters. □

We have $\Bbbk[X]^U \subset \Bbbk[X]^{U'}$, and both algebras are finitely generated. Assuming that generators of $\Bbbk[X]^U$ are known, we obtain a finite set of generators for $\Bbbk[X]^{U'}$, as follows.

Theorem 2.4. *Suppose that $f_1, \dots, f_m$ is a set of T-homogeneous generators of $\Bbbk[X]^U$ and the weight of f_i is λ_i. (That is, there is a G-submodule $\mathbb{V}_i \subset \Bbbk[X]$ such that $\mathbb{V}_i \simeq \mathsf{R}(\lambda_i)$ and $f \in (\mathbb{V}_i)^U$.) Then the union of bases of the spaces $(\mathbb{V}_i)^{U'}$, $i = 1, \dots, m$, generate $\Bbbk[X]^{U'}$. In particular, $\Bbbk[X]^{U'}$ is generated by at most $\sum_{i=1}^m \prod_{j=1}^r \bigl((\lambda_i, \alpha_j^\vee) + 1\bigr)$ functions.*

Proof. Let $\mathcal{B}$ be the algebra generated by the spaces $(\mathbb{V}_i)^{U'}$. Clearly, $\mathcal{B}$ is B/U'-stable and contains $\Bbbk[X]^U$. Hence it meets every simple G-submodule of $\Bbbk[X]$. Therefore, it is sufficient to prove that $\mathcal{B}$ contains U/U'-cyclic vectors of all simple G-submodules.

We argue by induction on the root order '$\preccurlyeq$' on the set of dominant weights. Let $c_i \in (\mathbb{V}_i)^{U'}$ be the unique U/U'-cyclic weight vector. By definition, $c_i \in \mathcal{B}$. We normalise f_i and c_i such that $E(\lambda_i)(c_i) = f_i$, where the operator $E(\lambda)$, $\lambda \in \mathfrak{X}_+$, is defined by $E(\lambda) := \prod_{i=1}^r e_i^{(\lambda, \alpha_i^\vee)}$. Assume that for any simple G-module $\mathbb{W}$ of type $\mathsf{R}(\mu)$ occurring in $\Bbbk[X]$, with $\mu \prec \lambda$, the cyclic vector of $\mathbb{W}$ belong to $\mathcal{B}$. Consider an arbitrary simple submodule $\mathbb{V} \subset \Bbbk[X]$ of type $\mathsf{R}(\lambda)$. Take a polynomial P in m variables such that $f{=}P(f_1, \dots, f_m)$ is a highest vector of $\mathbb{V}$. Without loss of generality, we may assume that every monomial of P is of weight λ. We claim that $P(c_1, \dots, c_m) \neq 0$. Indeed, it is easily seen that $E(\lambda)P(c_1, \dots, c_m) = P\bigl(E(\lambda_1)(c_1), \dots, E(\lambda_m)(c_m)\bigr) = f$. The last equality does not guarantee us that $P(c_1, \dots, c_m) \in \mathbb{V}$. However, this means that the projection of this element to $\mathbb{V}$ is well-defined and it must be a U/U'-cyclic vector of $\mathbb{V}$, say c. More precisely, $P(c_1, \dots, c_m) = c + \tilde{c}$, where $\tilde{c}$ belong to a sum of simple submodules of types $\mathsf{R}(\nu_i)$ with $\nu_i \prec \lambda$. If P is a monomial, then this follows from the uniqueness of the Cartan component in tensor products. In our case, the Cartan component of the tensor product associated with every monomial of P is $\mathsf{R}(\lambda)$, which easily yields the general assertion. By definition, $P(c_1, \dots, c_m) \in \mathcal{B}$, and by the induction assumption, $\tilde{c} \in \mathcal{B}$. Thus, $c \in \mathcal{B}$. □

This theorem provides a good upper bound on the number of generators of $\Bbbk[X]^{U'}$. However, it is not always the case that a minimal generating system of $\Bbbk[X]^U$ is a part of a minimal generating system of $\Bbbk[X]^{U'}$. (See examples in Section 5.)

Since U' has no rational characters, $\dim X /\!\!/ U' = \operatorname{trdeg} \Bbbk(X)^{U'} = \dim X - \dim U' + \min_{x \in X} \dim(U')_x$. To compute the last quantity, we use the existence of a generic stabiliser for U-actions on irreducible G-varieties [6, Thm. 1.6].

Lemma 2.5. *Let $U_\star$ be a generic stabiliser for $(U : X)$. Then $\min_{x \in X} \dim(U')_x = \dim(U_\star \cap U')$.*

Proof. Let $\Psi \subset X$ be a dense open subset of generic points, i.e., U_x is U-conjugate to $U_\star$ for any $x \in \Psi$. Since U' is a normal subgroup, $U_x \cap U'$ is also U-conjugate to $U_\star \cap U'$. Thus, all U'-orbits in Ψ are of dimension $\dim U' - \dim(U_\star \cap U')$. □

Remark 2.6. 1) If X is (quasi)affine, then one can choose $U_\star$ in a canonical way. Let $\mathcal{M}(X)$ be the monoid of highest weight of all simple G-modules occurring in $\Bbbk[X]$. Then $U_\star$ is the product of all root unipotent subgroup U^μ ($\mu \in \Delta^+$) such that $(\mu, \mathcal{M}(X)) = 0$ [17, Ch. 1, § 3]. Equivalently, $U_\star$ is generated by the simple root unipotent subgroups U^{α_i} such that $(\alpha_i, \mathcal{M}(X)) = 0$. It follows that $U_\star \cap U' = (U_\star, U_\star)$. This also means that if $\mathcal{M}(X)$ is known, then $\min_{x\in X} \dim(U')_x$ can effectively be computed.

2) The group $U_\star$ is a maximal unipotent subgroup of a generic stabiliser for the diagonal G-action on $X \times X^*$ [17, Theorem 1.2.2]. Here X^* is the so-called *dual G-variety*. It coincides with the dual G-module, if X is a G-module. Using tables of generic stabilisers for representations of G, one can again compute $U_\star$ and $(U_\star, U_\star)$.

3. Poincaré series of multigraded algebras of U'-invariants

Let X be an irreducible affine G-variety. (Eventually, we impose other constraints on X.) Since T normalises U', it acts on $X/\!\!/U'$ and the algebra $\Bbbk[X]^{U'}$ acquires a multigrading (by T-weights). Our objective is to describe some properties of the corresponding Poincaré series. Before we stick to considering U'-invariants, let us give a brief outline of notation and results to be used below.

Let $\mathcal{R}$ be a finitely generated $\mathbb{N}^m$-graded $\Bbbk$-algebra such that $\Bbbk[\mathcal{R}]_0 = 0$. Set $X = \mathsf{Spec}(\mathcal{R})$.

- The Poincare series of $\mathcal{R}$ is (the Taylor expansion of) a rational function in $t_1, \dots, t_m$:
$$\mathcal{F}(\mathcal{R}; \underline{t}) = P(\underline{t})/Q(\underline{t})$$
for some polynomials P, Q.
- If $\mathcal{R}$ is Cohen-Macaulay, then $\Omega_{\mathcal{R}}$ (or Ω_X) stands for the canonical module of $\mathcal{R}$; $\Omega_{\mathcal{R}}$ is naturally $\mathbb{Z}^m$-graded such that the Poincaré series of $\Omega_{\mathcal{R}}$ is
$$\mathcal{F}(\Omega_{\mathcal{R}}; \underline{t}) = (-1)^{\dim X} \mathcal{F}(\mathcal{R}; \underline{t}^{-1}).$$
- If $\mathcal{R}$ is Gorenstein, then the rational function $\mathcal{F}(\mathcal{R}; \underline{t})$ satisfies the equality
$$\mathcal{F}(\mathcal{R}; \underline{t}^{-1}) = (-1)^{\dim X} \underline{t}^{q(X)} \mathcal{F}(\mathcal{R}; \underline{t}),$$
for some $q(X) = (q_1(X), \dots, q_m(X)) \in \mathbb{Z}^m$, and the degree of a homegeneous generator $\omega_{\mathcal{R}}$ of $\Omega_{\mathcal{R}}$ is $\deg(\omega_{\mathcal{R}}) = q(X)$ [22, Theorem 6.1], [23, 1.12].
- If X has only rational singularities, then $q_i(X) \geqslant 0$ and $q(X) \neq (0, \dots, 0)$ [3, Proposition 4.3]
- Let G be a semisimple group acting on X (of course, it is assumed that G preserves the $\mathbb{N}^m$-grading of $\mathcal{R}$). Then there is a relationship betweem $\Omega_{\mathcal{R}}$ and $\Omega_{\mathcal{R}^G}$ [13] and hence between $q(X)$ and $q(X/\!\!/G)$, see below.

We begin with the case of $X = G$, where G is regarded as G-variety with respect to right translations. That is, we are going to study the graded structure of $\mathcal{A} = \Bbbk[G/U']$. Since G is simply-connected, it is a factorial variety. Therefore, $\mathsf{Spec}(\mathcal{A}) = G/\!\!/U'$ is factorial (and has only rational singularities). In particular, $G/\!\!/U'$ is Cohen-Macaulay (= CM). There is the direct sum decomposition

$$\mathcal{A} = \bigoplus_{\gamma\in\mathfrak{X}} \mathcal{A}_\gamma,$$

where $\mathcal{A}_\gamma = \{f \in \mathcal{A} \mid f(gt) = \gamma(t)f(g) \text{ for any } g \in G, t \in T\}$. The weights γ such that $\mathcal{A}_\gamma \ne 0$ form a finitely generated monoid, which is denoted by Γ. Since $\mathsf{R}(\lambda)^{U'}$ is a multiplicity free T-module, it follows from Eq. (2·1) that, for any $\lambda \in \mathfrak{X}_+$, different copies of $\mathsf{R}(\lambda^*)$ lie in the different weight spaces $\mathcal{A}_\gamma$. More precisely, the corresponding set of weights is I_λ (see Section 1). In particular, two copies of $\mathsf{R}(\varpi_i^*)$ belong to $\mathcal{A}_{\varpi_i}$ and $\mathcal{A}_{\varpi_i-\alpha_i}$. Therefore, Γ is generated by the weights $\varpi_i, \varpi_i - \alpha_i$, $i = 1, \dots, r$. Note that the group generated by Γ coincides with $\mathfrak{X}$, since Γ contains all fundamental weights.

Lemma 3.1. *If G has no simple factors SL_2 or SL_3, then $\Gamma \setminus \{0\}$ lies in an open half-space of $\mathfrak{X}_\mathbb{Q}$, $\mathcal{A}_0 = \Bbbk$, and $\dim \mathcal{A}_\gamma < \infty$ for all $\gamma \in \Gamma$.*

Proof. It is shown in the proof of Proposition 1.9 that $(\rho^\vee, \varpi_i - \alpha_i) > 0$ for all i. Hence the half-space determined by $\rho^\vee$ will do. We have $\mathcal{A}_0 = \Bbbk[G/TU'] = \Bbbk$, since TU' is epimorphic. This also implies the last claim, because $\mathcal{A}$ is finitely generated. □

The algebra $\mathcal{A}$ is Γ-graded, and we are going to study the corresponding Poincaré series. Unfortunately, Γ is not always a free monoid. Therefore we want to embed Γ into a free monoid $\mathbb{N}^r$. This is always possible, if Γ generates a convex cone in $\mathfrak{X}_\mathbb{Q}$, see e.g. [15, Corollary 7.23]. For this reason, we assume below that G has no simple factors SL_2 or SL_3, and choose an embedding $\Gamma \hookrightarrow \mathbb{N}^r$. In other words, we find $v_1, \dots, v_r \in \mathfrak{X}$ such that $\mathfrak{X} = \bigoplus_{i=1}^r \mathbb{Z}v_1$ and $\Gamma \subset \bigoplus_{i=1}^r \mathbb{N}v_1$. Furthermore, one can achieve that $(v_i, \rho^\vee) > 0$ for all i. Then $(v_1, \dots, v_r)$ is said to be a Γ-*adapted basis* for $\mathfrak{X}$. Thus, every $\gamma \in \Gamma$ gains a unique expression of the form $\gamma = \sum_i k_i(\gamma)v_i,\ k_i(\gamma) \in \mathbb{N}$.

Now, we define the multigraded Poincaré series of $\mathcal{A}$ as the power series

$$\mathcal{F}(\mathcal{A}; t_1, \dots, t_r) = \mathcal{F}(\mathcal{A}; \underline{t}) = \sum_{\gamma\in\Gamma} (\dim \mathcal{A}_\gamma)\underline{t}^\gamma,$$

where $\underline{t}^\gamma = t_1^{k_1(\gamma)} \dots t_r^{k_r(\gamma)}$. As is well-known, $\mathcal{F}(\mathcal{A}; \underline{t})$ is a rational function. Since $\mathcal{A}$ is a factorial CM domain, it is Gorenstein. Therefore, there exists $\underline{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ such that

$$\mathcal{F}(\mathcal{A}; \underline{t}^{-1}) = (-1)^{\dim G/U'} \underline{t}^{\underline{a}} \mathcal{F}(\mathcal{A}; \underline{t}), \tag{3·1}$$

where $\underline{t}^{-1} = (t_1^{-1}, \dots, t_r^{-1})$ [22, § 6]. Moreover, since $G/\!\!/U'$ has only rational singularities, all a_i are actually non-negative, and $\underline{a} \ne (0, \dots, 0)$ [3, Proposition 4.3].

Set $b(\mathcal{A}) := \sum_{i=1}^{r} a_i v_i \in \mathfrak{X}$. A priori, this element might depend on the choice of an embedding $\Gamma \hookrightarrow \mathbb{N}^r$. Fortunately, it doesn't. Roughly speaking, this can be explained via properties of the *canonical module* $\Omega_{\mathcal{A}}$, which is a free $\mathcal{A}$-module of rank one. However, even if we accurately accomplish this program, then we still do not find the very element $b(\mathcal{A}) \in \mathfrak{X}$. Therefore, we choose another path. Our plan consists of the following steps:

(1) $\mathcal{A}^U$ is a polynomial algebra and its Poincaré series can be written down explicitly;
(2) Using the formula for this Poincare series, we determine $b(\mathcal{A}^U) \in \mathfrak{X}$;
(3) Using results of [17, 5.4], we prove that $b(\mathcal{A}) = b(\mathcal{A}^U)$.

The algebra $\mathcal{A}^U$ is acted upon by $T \times T$. Two copies of T acts on $\mathcal{A}^U \subset \Bbbk[G]$ via left and right translations. For the presentation of Eq. (2·2), the first (resp. second) copy of T acts on the first (resp. second) factor in tensor products. Then

$$\mathcal{A}^U = \bigoplus_{\lambda \in \mathfrak{X}_+, \gamma \in \Gamma} \mathcal{A}^U_{\lambda,\gamma},$$

where $\mathcal{A}^U_{\lambda,\gamma} = \{f \in \mathcal{A}^U \subset \Bbbk[G] \mid f(tgt') = \lambda(t)^{-1}\gamma(t')f(g) \ \text{ for all } t, t' \in T\}$, and we set

$$\mathcal{F}(\mathcal{A}^U; \underline{s}, \underline{t}) = \sum_{\lambda,\gamma} (\dim \mathcal{A}^U_{\lambda,\gamma}) \underline{s}^{\lambda} \underline{t}^{\gamma}.$$

Here $\underline{s} = (s_1, \dots, s_r)$ and $\underline{s}^{\lambda} = s_1^{n_1} \dots s_r^{n_r}$ if $\lambda = \sum_i n_i \varpi_i$.

Proposition 3.2. *We have*

$$\mathcal{F}(\mathcal{A}^U; \underline{s}, \underline{t}) = \prod_{i=1}^{r} \frac{1}{(1 - s_{i^*} \underline{t}^{\varpi_i})(1 - s_{i^*} \underline{t}^{\varpi_i - \alpha_i})},$$

where i^* is defined by $(\varpi_i)^* = \varpi_{i^*}$.

Proof. This follows from the fact that $\mathcal{A}^U$ is freely generated by the space $R = \bigoplus_{i=1}^{r} \mathsf{R}(\varpi_i^*)^U \otimes \mathsf{R}(\varpi_i)^{U'}$, and the $(T \times T)$- weights of a bi-homogeneous basis of R are $(\varpi_i^*, \varpi_i), (\varpi_i^*, \varpi_i - \alpha_i)$, $i = 1, \dots, r$. $\square$

Of course, $\underline{t}^{\varpi_i}$ should be understood as $t_1^{k_1(\varpi_1)} \dots t_r^{k_r(\varpi_r)}$, and likewise for $\varpi_i - \alpha_i$. Since $\sum_i (\varpi_i + \varpi_i - \alpha_i) = 2\rho - |\Pi| = |\Delta^+ \setminus \Pi|$, we readily obtain

Corollary 3.3. $\mathcal{F}(\mathcal{A}^U; \underline{s}^{-1}, \underline{t}^{-1}) = (s_1 \dots s_r)^2 \underline{t}^{2\rho - |\Pi|} \mathcal{F}(\mathcal{A}^U; \underline{s}, \underline{t})$.

One can disregard (for a while) the $\mathfrak{X}_+$-grading of $\mathcal{A}^U$ and consider only the Γ-grading induced from $\mathcal{A}$. This amount to letting $s_i = 1$ for all i. Then we obtain $b(\mathcal{A}^U) = 2\rho - |\Pi|$, and, surely, this does not depend on the choice of $\Gamma \hookrightarrow \mathbb{N}^r$. Thus, we have completed steps (1) and (2) of the above plan.

Now, we recall a relationship between the multigraded Poincaré series of algebras $\Bbbk[X]$ and $\Bbbk[X]^U$. For G-modules, these results are due to M. Brion [3, Ch. IV], [4, Theorem 2].

A general version is found in [16], [17, Ch. 5]. We will consider two types of conditions imposed on G-varieties X:

$$(\mathfrak{C}_1) \qquad \begin{cases} X \text{ is an irreducible factorial } G\text{-variety with only rational singularities and} \\ \Bbbk[X]^G = \Bbbk. \end{cases}$$

$$(\mathfrak{C}_2) \qquad \begin{cases} X \text{ is an irreducible factorial } G\text{-variety with only rational singularities; } \Bbbk[X] \text{ is} \\ \mathbb{N}^m\text{-graded, } \Bbbk[X] = \bigoplus_{n\in\mathbb{N}^m} \Bbbk[X]_n, \text{ and } \Bbbk[X]_0 = \Bbbk. \end{cases}$$

In particular, X is Gorenstein in both cases. Suppose X satisfies $(\mathfrak{C}_2)$. The Poincaré series of the Gorenstein algebra $\Bbbk[X]$ satisfies an equality of the form

$$\mathcal{F}(\Bbbk[X];\underline{t}^{-1}) = (-1)^{\dim X}\underline{t}^{q(X)}\mathcal{F}(\Bbbk[X];\underline{t}), \tag{3·2}$$

where $\underline{t} = (t_1,\dots,t_m)$ and $q(X) = (q_1(X),\dots,q_m(X))$. The affine variety $X/\!\!/U$ inherits all good properties of X, i.e., it is irreducible, factorial, etc. Furthermore, $\Bbbk[X]^U$ is naturally $\mathfrak{X}_+\times\mathbb{N}^m$-graded, and one defines the Poincaré series

$$\mathcal{F}(\Bbbk[X]^U;\underline{s},\underline{t}) = \sum_{\lambda\in\mathfrak{X}_+,n\in\mathbb{N}^m} (\dim\Bbbk[X]^U_{\lambda,n})\underline{s}^{\lambda}\underline{t}^{n}.$$

Since $X/\!\!/U$ is again Gorenstein, this series satisfies an equality of the form

$$\mathcal{F}(\Bbbk[X]^U;\underline{s}^{-1},\underline{t}^{-1}) = (-1)^{\dim X/\!\!/U}\underline{s}^{\underline{b}}\,\underline{t}^{q(X/\!\!/U)}\mathcal{F}(\Bbbk[X]^U;\underline{s},\underline{t})$$

for some $\underline{b} = \underline{b}(X/\!\!/U) = (b_1,\dots,b_r)$ and $q(X/\!\!/U) = (q_1(X/\!\!/U),\dots,q_m(X/\!\!/U))$.

Theorem 3.4 (see [17, Theorem 5.4.26])**.** *Suppose that X satisfies condition $(\mathfrak{C}_2)$. Then*

1. $0 \leqslant b_i \leqslant 2$;
2. $0 \leqslant q_i(X/\!\!/U) \leqslant q_i(X)$ *for all* i;
3. *the following conditions are equivalent:*
   - $\underline{b} = (2,\dots,2)$;
   - *For* $D = \{z\in X \mid \dim U_z > 0\}$, *we have* $\operatorname{codim}_X D \geqslant 2$;
   - $q(X/\!\!/U) = q(X)$;

Let us apply this theorem to the G-variety $\mathsf{Spec}(\mathcal{A}) = G/\!\!/U'$. The algebra $\mathcal{A}$ is Γ-graded and hence suitably $\mathbb{N}^r$-graded, as explained before. Note that $\mathsf{Spec}(\mathcal{A})$ satisfies both conditions $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$. At the moment, we consider $X = \mathsf{Spec}(\mathcal{A})$ as variety satisfying condition $(\mathfrak{C}_2)$, with $m = r$. Comparing Eq. (3·1) and (3·2), we see that $\underline{a} = q(X)$. Proposition 3.2 and Corollary 3.3 show that here $\underline{b}(X/\!\!/U) = (2,\dots,2)$ and $q(X/\!\!/U)$ corresponds to $b(\mathcal{A}^U) = 2\rho - |\Pi|$. Now, Theorem 3.4(3) guarantee us that $q(X) = q(X/\!\!/U)$, i.e.,

$$b(\mathcal{A}) = b(\mathcal{A}^U) = 2\rho - |\Pi|. \tag{3·3}$$

This completes our computation of $b(\mathcal{A})$. Note that we computed $b(\mathcal{A})$ without knowing an explicit formula of the Poincaré series $\mathcal{F}(\mathcal{A};\underline{t})$.

Our next goal is to obtain analogues of results of [17, 5.4], where U is replaced with U', i.e., results on Poincaré series of algebras $\Bbbk[X]^{U'}$.

Suppose X satisfies $(\mathfrak{C}_1)$. The algebra $\Bbbk[X]^{U'}$ is Γ-graded, and we consider the Poincaré series

$$\mathcal{F}(\Bbbk[X]^{U'};\underline{t}) = \sum_{\gamma\in\Gamma} \dim \Bbbk[X]^{U'}_{\gamma}\, \underline{t}^{\gamma},$$

where $\Bbbk[X]^{U'}_{\gamma} = \{f \in \Bbbk[X]^{U'} \mid f(t.z) = \gamma(t)^{-1}f(z)\}$ and, as above, $\underline{t}^{\gamma}$ is determined via the choice of a Γ-adapted basis $(v_1,\dots,v_r)$ for $\mathfrak{X}$. The assumption $\Bbbk[X]^G = \Bbbk$ and the convexity of the cone generated by Γ guarantee us that $\Bbbk[X]^{U'}_0 = \Bbbk$ and all spaces $\Bbbk[X]^{U'}_{\gamma}$ are finite-dimensional. Since $X/\!\!/U'$ is again factorial, with only rational singularities (Theorem 2.3), it is Gorenstein and hence

$$\mathcal{F}(\Bbbk[X]^{U'};\underline{t}^{-1}) = (-1)^{\dim X/\!\!/U'}\underline{t}^{a}\,\mathcal{F}(\Bbbk[X]^{U'};\underline{t})$$

for some $a = (a_1,\dots,a_r) \in \mathbb{N}^r$. Using the basis $(v_1,\dots,v_r)$, we set $b(X/\!\!/U') = \sum_{i=1}^{r} a_iv_i \in \mathfrak{X}$.

Theorem 3.5. *Suppose that X satisfies $(\mathfrak{C}_1)$. Then*

(1) $0 \leqslant b(X/\!\!/U') \leqslant b(\mathcal{A}) = 2\rho - |\Pi|$
(componentwise, with respect to any Γ-adapted basis $v_1,\dots,v_r$);
(2) *the following conditions are equivalent:*
  a) $b(X/\!\!/U') = 2\rho - |\Pi|$;
  b) *For $D = \{x \in X \mid \dim(U')_x > 0\}$, we have $\operatorname{codim}_X D \geqslant 2$;*

Proof. Using our results on $\mathcal{A}$ and $\mathcal{A}^U$ obtained above, one can easily adapt the proof of [17, Theorem 5.4.21]. For the reader's convenience, we recall the argument.

(1) We have $0 \leqslant b(X/\!\!/U')$, since $X/\!\!/U'$ has rational singularities.
Set $Z = X \times (G/\!\!/U')$. It is a factorial G-variety with only rational singularities and $\Bbbk[Z] = \Bbbk[X] \otimes \mathcal{A}$. Define the Γ-grading of $\Bbbk[Z]$ by $\Bbbk[Z]_{\beta} = \Bbbk[X] \otimes \mathcal{A}_{\beta}$, $\beta \in \Gamma$. By the transfer principle, $\Bbbk[Z]^G \simeq \Bbbk[X]^{U'}$ and the Γ-grading of $\Bbbk[X]^{U'}$ corresponds under this isomorphism to the Γ-grading of $\Bbbk[Z]^G$ as subalgebra of $\Bbbk[Z]$.

In this situation (a semisimple group G acting on a factorial variety Z with only rational singularities), one can apply results of Knop to the quotient morphism $\pi_G : Z \to Z/\!\!/G$. Set $m = \max_{z\in Z} \dim G.z$. Recall that Ω_X is the canonical module of $\Bbbk[X]$. By Theorems 1,2 in [13], there is an injective G-equivariant homomorphism of degree 0 of graded $\Bbbk[Z]$-modules

$$\bar{\gamma} : \Omega_Z \to \wedge^m\mathfrak{g}^* \otimes \pi_G^*(\Omega_{Z/\!\!/G}).$$

Here $\Omega_Z = \Omega_X \otimes \Omega_{G/\!\!/U'}$ and grading of Ω_Z comes from the grading of $\Omega_{G/\!\!/U'}$. The injectivity of $\bar\gamma$ implies that

$$b(X/\!\!/U') = \left\{\begin{matrix}\text{degree of a homogeneous}\\ \text{generator of } \Omega_{X/\!\!/U'} \simeq \Omega_{Z/\!\!/G}\end{matrix}\right\} \leqslant \left\{\begin{matrix}\text{degree of a homogeneous}\\ \text{generator of } \Omega_{G/\!\!/U'}\end{matrix}\right\} = b(\mathcal{A}).$$

This yields the rest of part (1).

(2) To prove the equivalence of a) and b), we replace each of them with an equivalent condition stated in terms of Z:

a′) $\deg(\omega_{Z/\!\!/G}) = \deg(\omega_Z)$;

b′) $\mathrm{codim}_Z \tilde D \geqslant 2$, where $\tilde D = \{z \in Z \mid \dim G_z > 0\}$.

The argument in part (1) shows that a) and a′) are equivalent. The equivalence of b) and b′) follows from the fact that G/U' is dense in $G/\!\!/U'$ and the complement is of codimension $\geqslant 2$, see Theorem 2.2.

The injectivity and G-equivariance of $\bar\gamma$ means that there is $c \in (\wedge^m \mathfrak g^* \otimes \Bbbk[Z])^G$ such that $\bar\gamma(\omega_Z) = c{\cdot}\omega_{Z/\!\!/G}$. We can regard c as G-equivariant morphism $c' : Z \to \wedge^m \mathfrak g^*$. It is shown in [13] that if $\dim G.z = m$ and $z \in Z_{reg}$, then $c'(z)$ is nonzero and it yields (normalised) Plücker coordinates of the m-dimensional space $\mathfrak g_z^\perp \subset \mathfrak g^*$.

Assume a′), i.e., $\deg(\omega_{Z/\!\!/G}) = \deg(\omega_Z)$. Then $\deg c = 0$, i.e.,

$$c \in (\wedge^m \mathfrak g^* \otimes \Bbbk[Z]_0)^G = (\wedge^m \mathfrak g^* \otimes \Bbbk[X])^G.$$

This means that c' can be pushed through the projection to X:

$$Z = X \times (G/\!\!/U') \to X \to \wedge^m \mathfrak g^*.$$

Let $z = (x, v) \in X \times (G/\!\!/U')$ be a generic point, i.e., $x \in X_{reg}$, $v \in G/U'$, and $\dim G.z = m$. Since $c'(z)$ depends only on x, we see that $\mathfrak g_z$ does not depend on v. But this is only possible if $\dim \mathfrak g_z = 0$, that is, $m = \dim G$. This already proves that $\mathrm{codim}_Z \tilde D \geqslant 1$. If $\mathrm{codim}_Z \tilde D = 1$, then formulae (6), (7), (12) in [13] show that $\tilde D = \{z \in Z \mid c'(z) = 0\}$. However, $\wedge^m \mathfrak g^*$ is the trivial 1-dimensional G-module, hence $c \in \Bbbk[X]^G = \Bbbk$. That is, c' is a constant (nonzero) mapping. This contradiction shows that $\mathrm{codim}_Z \tilde D \geqslant 2$.

Conversely, if b′) holds, then $\tilde D$ is a proper subvariety of Z, i.e., $m = \dim G$ and $c \in (\wedge^{\dim G} \mathfrak g^* \otimes \Bbbk[Z])^G = \Bbbk[Z]^G$. Furthermore, since $\mathrm{codim}_Z \tilde D \geqslant 2$, c has no zeros on Z (because Z is normal and $c'(z) = c(z) \neq 0$ for any $z \in Z_{reg} \setminus \tilde D$.) It follows that c is constant, $\deg c = 0$ and hence $\deg(\omega_{Z/\!\!/G}) = \deg(\omega_Z)$. □

If X satisfies $(\mathfrak C_2)$, then the algebra $\Bbbk[X]^{U'}$ is naturally $\Gamma \times \mathbb N^m$-graded, and we consider the Poincaré series

$$\mathcal F(\Bbbk[X]^{U'}; \underline{s}, \underline{t}) = \sum_{n \in \mathbb N^m, \gamma \in \Gamma} \dim \Bbbk[X]^{U'}_{n,\gamma}\, \underline{s}^n \underline{t}^\gamma,$$

where $\Bbbk[X]^{U'}_{n,\gamma}=\{f\in\Bbbk[X]^{U'}_n \mid f(t.z)=\gamma(t)^{-1}f(z)\}$ and $\underline{t}^\gamma$ is as above. Since $X/\!\!/U'$ is again Gorenstein, we have

$$\mathcal F(\Bbbk[X]^{U'};\underline{s}^{-1},\underline{t}^{-1})=(-1)^{\dim X/\!\!/U'}\underline{t}^{a}\,\underline{s}^{q(X/\!\!/U')}\mathcal F(\Bbbk[X]^{U'};\underline{s},\underline{t})$$

for some $a=(a_1,\dots,a_r)\in\mathbb N^r$ and $q(X/\!\!/U')\in\mathbb N^m$. Using the basis $(v_1,\dots,v_r)$, we set $b(X/\!\!/U')=\sum_{i=1}^r a_iv_i\in\mathfrak X$. The following is a U'-analogue of Theorem 3.4.

Theorem 3.6. *Suppose that X satisfies $(\mathfrak C_2)$. Then*

(1) $0\leqslant b(X/\!\!/U')\leqslant b(\mathcal A)=2\rho-|\Pi|$
(componentwise, with respect to any Γ-adapted basis $v_1,\dots,v_r$);
(2) $0\leqslant q_i(X/\!\!/U')\leqslant q_i(X)$ *for all i;*
(3) *the following conditions are equivalent:*
  (i) $b(X/\!\!/U')=2\rho-|\Pi|$;
  (ii) *For $D=\{x\in X\mid \dim(U')_x>0\}$, we have* $\mathrm{codim}_X D\geqslant 2$;
  (iii) $q(X/\!\!/U')=q(X)$.

We leave it to the reader to adapt the proof of Theorem 5.4.26 in [17] to the U'-setting.

These results may (and will) be applied to describing G-varieties X with polynomial algebras $\Bbbk[X]^{U'}$. Suppose for simplicity that $\Bbbk[X]$ is $\mathbb N$-graded (i.e., $m=1$). If $f_1,\dots,f_s$ are algebraically independent homogeneous generators of $\Bbbk[X]^{U'}$, then $\sum\deg f_i=q(X/\!\!/U')\leqslant q(X)$. In particular, if X is a G-module with the usual $\mathbb N$-grading of $\Bbbk[X]$, then $\sum\deg f_i\leqslant\dim X$. Similarly, if ω_i is the T-weight of f_i, then $\sum_{i=1}^s\omega_i\leqslant 2\rho-|\Pi|$. The idea to use an *a priori* information on the Poincaré series for classifying group actions with polynomial algebras of invariants is not new. It goes back to T.A. Springer [21]. Since then it was applied many times to various group actions.

4. Some combinatorics related to U'-invariants

In previous sections, we have encountered some interesting objects in $\mathfrak X$ related to the study of U'-invariants. These are $b(\mathcal A)=2\rho-|\Pi|$, the set of T-weights in $\mathsf R(\lambda)^{U'}$ (denoted I_λ), and the monoid Γ generated by $\varpi_i,\varpi_i-\alpha_i$ for all $i\in\{1,\dots,r\}=:[r]$.

Proposition 4.1.

(i) *If G has no simple ideals SL_2, then $2\rho-|\Pi|$ is a strictly dominant weight;*
(ii) *For any $\lambda\in\mathfrak X_+$, the weight $|I_\lambda|$ is dominant. Furthermore, $(|I_\lambda|,\alpha_i)>0$ if and only if there is j such that $(\lambda,\alpha_j^\vee)>0$ and $(\alpha_i,\alpha_j)>0$.*

Proof. (i) is obvious.

(ii) Recall that $I_\lambda=\{\lambda-\sum_{i=1}^r c_i\alpha_i \mid 0\leqslant c_i\leqslant(\lambda,\alpha_i^\vee),\ i=1,\dots,r\}$. Choose $i\in[r]$ and slice I_λ into the layers, where all coordinates c_j with $j\ne i$ are fixed, i.e., consider

$$I_\lambda(c_1,\dots,\widehat{c_i},\dots,c_r)=\{\lambda-\sum_{j:j\ne i}c_j\alpha_j-c_i\alpha_i \mid 0\leqslant c_i\leqslant(\lambda,\alpha_i^\vee)\}.$$

Then one easily verifies that $\big(|I_\lambda(c_1,\dots,\widehat{c_i},\dots,c_r)|,\alpha_i^\vee\big)=((\lambda,\alpha_i^\vee)+1)(-\sum_{j\ne i}c_j\alpha_j,\alpha_i^\vee)\geqslant 0$. Hence $(|I_\lambda|,\alpha_i^\vee)\geqslant 0$, and the condition of positivity is also inferred. □

Let $\mathcal{C}$ be the cone in $\mathfrak{X}_\mathbb{Q}$ generated Γ, i.e., by all weights $\varpi_i,\varpi_i-\alpha_i$. Consider the dual cone $\check{\mathcal{C}}:=\{\eta\in\mathfrak{X}_\mathbb{Q}\mid(\eta,\varpi_i)\geqslant 0\ \&\ (\eta,\varpi_i-\alpha_i)\geqslant 0 \text{ for all } i\}$.

Theorem 4.2. *The cone $\check{\mathcal{C}}$ is generated by the non-simple positive roots.*

Proof. 1) Let $\mathcal{K}$ denote the cone in $\mathfrak{X}_\mathbb{Q}$ generated by $\Delta^+\setminus\Pi$. It is easily seen that $\mathcal{K}\subset\check{\mathcal{C}}$. Indeed, let $\delta\in\Delta^+\setminus\Pi$. Then $(\varpi_i,\delta)\geqslant 0$. If $s_i\in W$ is the reflection corresponding to $\alpha_i\in\Pi$, then $s_i(\varpi_i)=\varpi_i-\alpha_i$ and $s_i(\delta)\in\Delta^+$. Hence $(\varpi_i-\alpha_i,\delta)=(\varpi_i,s_i(\delta))\geqslant 0$.

2) Conversely, we prove that $\check{\mathcal{K}}\subset\mathcal{C}$. We construct a partition of $\check{\mathcal{K}}$ into finitely many simplicial cones, and show that each cone belong in $\mathcal{C}$.

Suppose that $\mu\in\mathfrak{X}$ and $(\mu,\delta)\geqslant 0$ for all $\delta\in\Delta^+\setminus\Pi$. Set $J=J_{(\mu)}=\{j\in[r]\mid(\mu,\alpha_j)<0\}$. We identify the elements of $[r]$ with the corresponding nodes of the Dynkin diagram of G. The obvious but crucial observation is that the nodes in J are disjoint on the Dynkin diagram. (Such subsets J are said to be *disjoint*.)

Claim. *The r vectors ϖ_i ($i\notin J$), $\varpi_j-\alpha_j$ ($j\in J$) form a basis for $\mathfrak{X}_\mathbb{Q}$.*
Proof. Since J is disjoint, $\prod_{j\in J}s_j\in W$ takes these r vectors to $\varpi_1,\dots,\varpi_r$.

Thus, we can uniquely write

$$\mu=\sum_{i\notin J}b_i\varpi_i+\sum_{j\in J}a_j(\varpi_j-\alpha_j),\quad b_i,a_j\in\mathbb{Q}.$$

By the assumption, $(\mu,\alpha_i)\geqslant 0$ if and only if $i\notin J$. For $j\in J$, we have $(\mu,\alpha_j^\vee)=-a_j<0$, i.e., $a_j>0$. It is therefore suffices to prove that all b_i are nonnegative. Choose any $i\notin J$. Let $J[i]$ denote the set of all nodes in J that are adjacent to i. Set $w_i=\prod_{j\in J[i]}s_j\in W$. (If $J[i]=\varnothing$, then $w=1$.) Then $w_i(\alpha_i)$ is either α_i or a non-simple positive root. In both cases, we know that $(\mu,w_i(\alpha_i))\geqslant 0$. On the other hand, this scalar product is equal to $(w_i(\mu),\alpha_i)=b_i(\varpi_i,\alpha_i)$. Thus, each b_i is nonnegative and $\mu\in\mathcal{C}$. □

Remark 4.3. The argument in the second part of proof shows that $\mathcal{C}$ is the union of simplicial cones parametrised by the disjoint subset of the Dynkin diagram. For any such set $J\subset[r]$, let $\mathcal{C}_J$ denote the simplicial cone generated by ϖ_i ($i\notin J$), $\varpi_j-\alpha_j$ ($j\in J$). Then

$$\mathcal{C}=\bigcup_{J\text{ disjoint}}\mathcal{C}_J.$$

Here $\mathcal{C}_\varnothing$ is the dominant Weyl chamber and $\mathcal{C}_J = (\prod_{j\in J} s_j)\mathcal{C}_\varnothing$. Furthermore, if $\mathcal{C}^o_J = \{\sum_{i\not\in J} b_i\varpi_i + \sum_{j\in J} a_j(\varpi_j-\alpha_j) \mid a_j>0,\ b_i\geqslant 0\}$, then

$$\mathcal{C} = \bigsqcup_{J \text{ disjoint}} \mathcal{C}^o_J.$$

Remark 4.4. It is a natural problem to determine the edges (one-dimensional faces) of the cone $\check{\mathcal{C}}$. We can prove that, for $\mathbf{A}_r$ and $\mathbf{C}_r$, the edges are precisely the roots of height 2 and 3. However, this is no longer true in the other cases, because a root of height 4 is needed.

5. Irreducible representations of simple Lie algebras with polynomial algebras of U'-invariants

In this section, we obtain the list of all irreducible representations of simple Lie algebras with polynomial algebras of U'-invariants. If $G = SL_2$, then U' is trivial and so is the classification problem. Therefore we assume that $\mathsf{rk}\, G \geqslant 2$.

Theorem 5.1. *Let G be a connected simple algebraic group with $\mathsf{rk}\, G \geqslant 2$ and $\mathsf{R}(\lambda)$ a simple G-module. The following conditions are equivalent:*

(i) $\Bbbk[\mathsf{R}(\lambda)]^{U'}$ *is generated by homogeneous algebraically independent polynomials;*
(ii) *Up to the symmetry of the Dynkin diagram of G, the weight λ occurs in Table 1.*

For each item in the table, the degrees and weights of homogeneous algebraically independent generators are indicated. We use the numbering of simple roots as in [24].

Table 1: The simple G-modules with polynomial algebras of U'-invariants

G	λ	Degrees and weights of homogeneous generators of $\Bbbk[\mathsf{R}(\lambda)]^{U'}$
$\mathbf{A}_r$ $(r\geqslant 2)$	ϖ_r	$(1,\varpi_1),(1,\varpi_1-\alpha_1)$
$\mathbf{A}_{2r-1}$ $(r\geqslant 2)$	ϖ_2^*	$(1,\varpi_2),(2,\varpi_4),\dots,(r-1,\varpi_{2r-2}),(r,\underline{0})$, $(1,\varpi_2-\alpha_2),(2,\varpi_4-\alpha_4),\dots,(r-1,\varpi_{2r-2}-\alpha_{2r-2})$
$\mathbf{A}_{2r}$ $(r\geqslant 2)$	ϖ_2^*	$(1,\varpi_2),(2,\varpi_4),\dots(r-1,\varpi_{2r-2}),(r,\varpi_{2r})$, $(1,\varpi_2-\alpha_2),(2,\varpi_4-\alpha_4),\dots,(r,\varpi_{2r}-\alpha_{2r})$
$\mathbf{B}_r$	ϖ_1	$(1,\varpi_1),(1,\varpi_1-\alpha_1),(2,\underline{0})$
$\mathbf{B}_3$	ϖ_3	$(1,\varpi_3),(1,\varpi_3-\alpha_3),(2,\underline{0})$
$\mathbf{B}_4$	ϖ_4	$(1,\varpi_4),(1,\varpi_4-\alpha_4),(2,\varpi_1),(2,\varpi_1-\alpha_1),(2,\underline{0})$
$\mathbf{B}_5$	ϖ_5	$(1,\varpi_5),(1,\varpi_5-\alpha_5),(2,\varpi_1),(2,\varpi_1-\alpha_1),(2,\varpi_2),(2,\varpi_2-\alpha_2)$, $(3,\varpi_5),(3,\varpi_5-\alpha_5),(4,\varpi_3-\alpha_3),(4,\varpi_4),(4,\varpi_4-\alpha_4),(4,\underline{0})$
$\mathbf{C}_r$	ϖ_1	$(1,\varpi_1),(1,\varpi_1-\alpha_1)$
$\mathbf{D}_r$ $(r\geqslant 4)$	ϖ_1	$(1,\varpi_1),(1,\varpi_1-\alpha_1),(2,\underline{0})$
$\mathbf{D}_5$	ϖ_5	$(1,\varpi_4),(1,\varpi_4-\alpha_4),(2,\varpi_1),(2,\varpi_1-\alpha_1)$

G	λ	Degrees and weights of homogeneous generators of $\Bbbk[\mathsf{R}(\lambda)]^{U'}$
$\mathbf{D}_6$	ϖ_6	$(1,\varpi_6),(1,\varpi_6{-}\alpha_6),(2,\varpi_2),(2,\varpi_2{-}\alpha_2),$ $(3,\varpi_6),(3,\varpi_6{-}\alpha_6),(4,\varpi_4{-}\alpha_4),(4,\underline{0})$
$\mathbf{E}_6$	ϖ_1	$(1,\varpi_5),(1,\varpi_5-\alpha_5),(2,\varpi_1),(2,\varpi_1-\alpha_1),(3,\underline{0})$
$\mathbf{E}_7$	ϖ_1	$(1,\varpi_1),(1,\varpi_1{-}\alpha_1),(2,\varpi_6),(2,\varpi_6{-}\alpha_6),$ $(3,\varpi_1),(3,\varpi_1{-}\alpha_1),(4,\varpi_2{-}\alpha_2),(4,\underline{0})$
$\mathbf{F}_4$	ϖ_1	$(1,\varpi_1),(1,\varpi_1{-}\alpha_1),(2,\varpi_1),(2,\varpi_1{-}\alpha_1),(3,\varpi_2{-}\alpha_2),(2,\underline{0}),(3,\underline{0})$
$\mathbf{G}_2$	ϖ_1	$(1,\varpi_1),(1,\varpi_1-\alpha_1),(2,\underline{0})$

Before starting the proof, we develop some more tools. Let V be a simple G-module. *A posteriori*, it appears to be true that if $\mathsf{rk}\, G>1$ and $\Bbbk[V]^{U'}$ is polynomial, then so is $\Bbbk[V]^U$. Therefore our list is contained in Brion's list of representations with polynomial algebras $\Bbbk[V]^U$ [4, p. 13]. However, we could not find a conceptual proof. The following is a reasonable substitute:

Proposition 5.2. *Suppose that $\Bbbk[V]^{U'}$ is polynomial and $G\ne SL_3$. Then $\Bbbk[V]^G$ is polynomial.*

Proof. As in Section 3, consider the Γ-grading $\Bbbk[V]^{U'}=\bigoplus_{\gamma\in\Gamma}\Bbbk[V]^{U'}_\gamma$.

If $G\ne SL_3$, then TU' is epimorphic and hence $\Bbbk[V]^{U'}_0=\Bbbk[V]^G$. Furthermore, since Γ generates a convex cone, $\bigoplus_{\gamma\ne 0}\Bbbk[V]^{U'}_\gamma$ is a complementary ideal to $\Bbbk[V]^G$. In this situation, a minimal system of homogeneous generators for $\Bbbk[V]^G$ is a part of a minimal system of homogeneous generators for $\Bbbk[V]^{U'}$. □

Remark 5.3. For $G=SL_3$, it is not hard to verify that the only representations with polynomial algebras of U'-invariants are $\mathsf{R}(\varpi_1)$ and $\mathsf{R}(\varpi_2)$. The reason is that U' is the maximal unipotent subgroup of $SL_2\subset SL_3$. Therefore, by classical Roberts' theorem, we have $\Bbbk[V]^{U'}\simeq\Bbbk[V\oplus\mathsf{R}_1]^{SL_2}$, where V is regarded as SL_2-module and R_1 is the tautological SL_2-module. All SL_2-modules with polynomial algebras of invariants are known [19, Theorem 4], and the restriction of the simple SL_3-modules to SL_2 are easily computed.

Let $U'_\star$ denote a U'-stabiliser of minimal dimension for points in $\mathsf{R}(\lambda)$. Recall that Lemma 2.5 and Remark 2.6 provide effective tools for computing $U'_\star$ and $\dim U'_\star$. If a ring of invariants $\mathfrak{A}$ is polynomial, then elements of a minimal generating system of $\mathfrak{A}$ are said to be *basic invariants*.

Proposition 5.4. *Suppose that $\Bbbk[\mathsf{R}(\lambda)]^{U'}$ is polynomial and $G\ne SL_3$. Then*

$$\dim\mathsf{R}(\lambda)\leqslant 2\dim(U'/U'_\star)+\prod_{i=1}^{r}((\lambda,\alpha_i^\vee)+1).$$

In particular, $\dim\mathsf{R}(\lambda)\leqslant 2\dim U'+\prod_{i=1}^{r}((\lambda,\alpha_i^\vee)+1)$.

Proof. We consider $\Bbbk[\mathsf{R}(\lambda)]$ with the usual $\mathbb{N}$-grading by the total degree of polynomial. Then $\Bbbk[\mathsf{R}(\lambda)]^{U'}$ is $\Gamma\times\mathbb{N}$-graded, and it has a minimal generating system that consists of (multi)homogeneous polynomials. Let $f_1,\dots,f_s$ be such a system. By Theorem 3.6(ii), we have

$$\sum \deg(f_i) = q(\mathsf{R}(\lambda)/\!\!/U') \leqslant q(\mathsf{R}(\lambda)) = \dim \mathsf{R}(\lambda).$$

On the other hand, $s = \dim\mathsf{R}(\lambda) - \dim(U'/U'_\star)$ and the number of basic invariants of degee 1 equals $a(\lambda) := \prod_{i=1}^r((\lambda,\alpha_i^\vee)+1)$. All other basic invariants are of degree $\geqslant 2$, and we obtain

$$a(\lambda)+2(\dim\mathsf{R}(\lambda)-\dim(U'/U'_\star)-a(\lambda)) = a(\lambda)+2(s-a(\lambda)) \leqslant q(\mathsf{R}(\lambda)/\!\!/U') \leqslant \dim\mathsf{R}(\lambda).$$

Hence $\dim\mathsf{R}(\lambda) \leqslant 2\dim(U'/U'_\star)+a(\lambda)$. □

Proof of Theorem 5.1.

(i) ⇒ (ii). The list of irreducible representations of simple Lie algebras with polynomial algebras $\Bbbk[V]^G$ is obtained in [11]. By Proposition 5.2, it suffices to prove that the representations in [11, Theorem 1] that do not appear in Table 1 cannot have a polynomial algebra of U'-invariants. The list of representation in question is the following:

I) $(\mathbf{A}_r,\varpi_3)$, $r=6,7,8$; $(\mathbf{A}_7,\varpi_4)$; $(\mathbf{A}_2,3\varpi_1)$; $(\mathbf{B}_r,2\varpi_1)$, $r\geqslant 2$; $(\mathbf{D}_r,2\varpi_1)$, $r\geqslant 4$; $(\mathbf{B}_6,\varpi_6)$; $(\mathbf{D}_8,\varpi_8)$; $(\mathbf{C}_r,\varpi_2)$, $r\geqslant 4$; $(\mathbf{C}_4,\varpi_4)$; the adjoint representations.

II) $(\mathbf{A}_5,\varpi_3)$; $(\mathbf{C}_3,\varpi_2)$; $(\mathbf{C}_3,\varpi_3)$; $(\mathbf{D}_7,\varpi_7)$; $(\mathbf{A}_r,2\varpi_r)$.

• For list I), a direct application of Proposition 5.4 yields the conclusion. For instance, consider $\mathsf{R}(\varpi_3)$ for $\mathbf{A}_r$ and $r=6,7,8$. Here $a(\varpi_3)=2$ and the second inequality in Proposition 5.4 becomes

$$(r+1)r(r-1)/6 \leqslant r(r-1)+2,$$

which is wrong for $r=6,7,8$. The same argument applies to all representations in I), except $(\mathbf{A}_2,3\varpi_1)$. (The SL_3-case is explained in Remark 5.3.)

• For list II), the inequality of Proposition 5.4 is true, and more accurate estimates are needed.

Consider the case $(\mathbf{A}_5,\varpi_3)$. Here $\dim\mathsf{R}(\varpi_3)=20$, $\dim U'=10$ and $U'_\star=\{1\}$. Hence $\dim\mathsf{R}(\varpi_3)/\!\!/U'=10$. Assume that $\mathsf{R}(\varpi_3)/\!\!/U'\simeq\mathbb{A}^{10}$. The number of basic invariants of degree 1 equals $a(\varpi_3)=2$. It is known that $\Bbbk[\mathsf{R}(\varpi_3)]^G$ is generated by a polynomial of degree 4. This is our third basic invariant. Since we must have $\sum_{i=1}^{10}\deg f_i\leqslant\dim\mathsf{R}(\varpi_3)=20$, the only possibility is that the other 7 basic invariants are of degree 2. However, $\mathcal{S}^2(\mathsf{R}(\varpi_3))=\mathsf{R}(2\varpi_3)\oplus\mathsf{R}(\varpi_1{+}\varpi_5)$, which shows that the number of basic invariants of degree 2 is at most $\dim\mathsf{R}(\varpi_1{+}\varpi_5)^{U'}=4$. This contradiction shows that $\Bbbk[\mathsf{R}(\varpi_3)]^{U'}$ cannot be polynomial. Such an argument also works for $(\mathbf{C}_3,\varpi_2)$, $(\mathbf{C}_3,\varpi_3)$, and $(\mathbf{D}_7,\varpi_7)$.

For $(\mathbf{A}_r,2\varpi_r)$, $r\geqslant 2$, we argue as follows. Here the algebra of U-invariants is polynomial, and the degrees and weights of basic U-invariants are $(1,2\varpi_1),(2,2\varpi_2),\dots,(r,2\varpi_r)$,

$(r+1,0)$ [4]. Using Theorem 2.4, we conclude that $\Bbbk[\mathsf{R}(2\varpi_r)]^{U'}$ can be generated by $3r+1$ polynomials whose degrees are $1,1,1;2,2,2;\dots;r,r,r;r+1$. This set of polynomials can be reduced somehow to a minimal generating system. Here $\dim \mathsf{R}(2\varpi_r)/\!\!/U' = \dim \mathsf{R}(2\varpi_r) - \dim U' = 2r+1$. Assume that $\mathsf{R}(2\varpi_r)/\!\!/U' \simeq \mathbb{A}^{2r+1}$. Then we can remove r polynomials from the above (non-minimal) generating system such that the sum of degrees of the remaining polynomials is at most $\dim \mathsf{R}(2\varpi_r) = (r+1)(r+2)/2$. This means that the sum of degrees of the r removed polynomials must be at least $r(r+1)$. Clearly, this is impossible.

(ii) $\Rightarrow$ (i). All representations in Table 1 have a polynomial algebra of U-invariants whose structure is well-understood. Therefore, using Theorem 2.4 we obtain an upper bound on the number of generators of $\Bbbk[\mathsf{R}(\lambda)]^{U'}$. On the other hand, we can easily compute $\dim \mathsf{R}(\lambda)/\!\!/U'$. In many cases, these two numbers coincide, which immediately proves that $\Bbbk[\mathsf{R}(\lambda)]^{U'}$ is polynomial. In the remaining cases, we use a simple procedure that allows us to reduce the non-minimal generating system provided by Theorem 2.4. This appears to be sufficient for our purposes.

- For $G=\mathbf{D}_5$, the algebra $\Bbbk[\mathsf{R}(\varpi_5)]^U$ has two generators whose degrees and weights are $(1,\varpi_4)$ and $(2,\varpi_1)$. By Theorem 2.4, $\Bbbk[\mathsf{R}(\varpi_5)]^{U'}$ can be generated by polynomials of degrees and weights $(1,\varpi_4),(1,\varpi_4-\alpha_4),(2,\varpi_1),(2,\varpi_1-\alpha_1)$. On the other hand, the monoid $\mathcal{M}(\mathsf{R}(\varpi_5))$ is generated by ϖ_1,ϖ_4. Therefore a generic stabiliser $U_\star$ is generated by the root unipotent subgroups U^{α_2}, U^{α_3}, and U^{α_5} (see Remark 2.6). Hence $\dim U_\star = 6$ and $\dim U'_\star = 3$. Thus $\dim \mathsf{R}(\varpi_5)/\!\!/U' = 16-15+3=4$ and the above four polynomials freely generate $\Bbbk[\mathsf{R}(\varpi_5)]^{U'}$.

The same method works for $(\mathbf{A}_r,\varpi_r)$; $(\mathbf{A}_r,\varpi_{r-1})$; $(\mathbf{B}_r,\varpi_1)$; $(\mathbf{C}_r,\varpi_1)$; $(\mathbf{D}_r,\varpi_1)$; $(\mathbf{B}_r,\varpi_r)$, $r=3,4$; $(\mathbf{E}_6,\varpi_1)$.

There still remain four cases, where this method yields the number of generators that is *one more* than $\dim \mathsf{R}(\lambda)/\!\!/U'$. Therefore, we have to prove that one of the functions provided by Theorem 2.4 can safely be removed. The idea is the following. Suppose that $\Bbbk[\mathsf{R}(\lambda)]^U$ contains two basic invariants of the same fundamental weight ϖ_i, say $p_1\sim(d_1,\varpi_i)$, $p_2\sim(d_2,\varpi_i)$. Consider the corresponding U'-invariant functions p_1,q_1,p_2,q_2, where $q_j\sim(d_j,\varpi_i-\alpha_i)$, $j=1,2$. Assuming that p_j,q_j are normalised such that $e_i{\cdot}q_j=p_j$, the polynomial $p_1q_2-p_2q_1\in\Bbbk[\mathsf{R}(\lambda)]$ appears to be U-invariant, of degree d_1+d_2 and weight $2\varpi_i-\alpha_i$. If we know somehow that there is a unique U-invariant of such degree and weight, then this U-invariant is not required for the minimal generating system of $\Bbbk[\mathsf{R}(\lambda)]^{U'}$. For instance, consider the case $(\mathbf{F}_4,\varpi_1)$. According to Brion [4], the free generators of $\Bbbk[\mathsf{R}(\varpi_1)]^{U(\mathbf{F}_4)}$ are $(1,\varpi_1),(2,\varpi_1),(3,\varpi_2),(2,\underline{0}),(3,\underline{0})$. Theorem 2.4 provides a generating system for $\Bbbk[\mathsf{R}(\varpi_1)]^{U'(\mathbf{F}_4)}$ that consists of eight polynomials, namely:

$$(1,\varpi_1),(1,\varpi_1-\alpha_1),(2,\varpi_1),(2,\varpi_1-\alpha_1),(3,\varpi_2),(3,\varpi_2-\alpha_2),(2,\underline{0}),(3,\underline{0}).$$

Here the weight ϖ_1 occurs twice and $2\varpi_1 - \alpha_1 = \varpi_2$. Therefore the polynomial $(3, \varpi_2)$ can be removed form this set. Since $\dim \mathsf{R}(\varpi_1) = 26$, $\dim U' = 20$, and $\dim U'_\star = 1$, we have $\dim \mathsf{R}(\varpi_1)/\!\!/U' = 7$. The other three cases, where it works, are $(\mathbf{B}_5, \varpi_5)$, $(\mathbf{D}_6, \varpi_6)$, $(\mathbf{E}_7, \varpi_1)$.

This completes the proof of Theorem 5.1. □

Remark 5.5. For a G-module V, let $\mathsf{ed}(\Bbbk[V]^{U'})$ denote the *embedding dimension* of $\Bbbk[V]^{U'}$, i.e., the minimal number of generators. Since $\Bbbk[V]^{U'}$ is Gorenstein, $\mathsf{ed}(\Bbbk[V]^{U'}) - \dim V/\!\!/U' = \mathsf{hd}(\Bbbk[V]^{U'})$ is the homological dimension of $\Bbbk[V]^{U'}$ (see [19]). The same argument as in the proof of (ii) $\Rightarrow$ (i) shows that for $(\mathbf{C}_3, \varpi_2)$, $(\mathbf{C}_3, \varpi_3)$, and $(\mathbf{A}_5, \varpi_3)$, we have $\mathsf{hd}(\Bbbk[V]^{U'}) \leqslant 2$. Hence these Gorenstein algebras of U'-invariants are complete intersections. We can also prove that $\Bbbk[\mathsf{R}(2\varpi_r)]^{U'(\mathbf{A}_r)}$ is a complete intersection, of homological dimension $r-1$. This means that *a postreriori* the following is true: *If G is simple, V is irreducible, and $\Bbbk[V]^U$ is polynomial, then $\Bbbk[V]^{U'}$ is a complete intersection.* It would be interesting to realise whether it is true in a more general situation.

Remark 5.6. There is a unique item in Table 1, where the sum of degrees of the basic invariants equals $\dim \mathsf{R}(\lambda)$ or, equivalently, the sum of weights equals $2\rho - |\Pi|$. This is $(\mathbf{B}_5, \varpi_5)$. By Theorem 3.6(iii), this is also the only case, where the set of points in $\mathsf{R}(\lambda)$ with non-trivial U'-stabiliser does not contain a divisor.

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